On Positive Harmonic Functions in Cones and Cylinders

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Abstract. We first consider a question raised by Alexander Eremenko and show that if Ω is an arbitrary connected open cone in $\mathbb{R}^d$, then any two positive harmonic functions in Ω that vanish on $\partial\Omega$ must be proportional -an already known fact when Ω has a Lipschitz basis or more generally a John basis. It is also shown however that when $d \geq 4$, there can be more than one Martin point at infinity for the cone though non-tangential convergence to the canonical Martin point at infinity always holds. In contrast, when $d \leq 3$, the Martin point at infinity is unique for every cone. These properties connected with the dimension are related to well-known results of M. Cranston and T. R. McConnell about the lifetime of conditioned Brownian motions in planar domains and also to subsequent results by R. Bañuelos and B. Davis. We also investigate the nature of the Martin points arising at infinity as well as the effects on the Martin boundary resulting from the existence of John cuts in the basis of the cone or from other regularity assumptions. The main results together with their proofs extend to cylinders $\mathscr{C}_Y(\Sigma) = \mathbb{R} \times \Sigma$ –where Σ is a relatively compact region of a manifold M–, equipped with a suitable second order elliptic operator.

1 Introduction. Main results.

We consider the cone $\mathscr{C}_o(\Sigma)$ of $\mathbb{R}^d$, $d \geq 2$, generated by a region Σ of the unit sphere S_{d-1}, i.e., $\mathscr{C}_o(\Sigma) = \{r\omega\,;\, r > 0,\ \omega \in \Sigma\}$, and study the positive harmonic functions in $\mathscr{C}_o(\Sigma)$ (where $\mathscr{C}_o$ is for cone). Recently Alexander Eremenko asked whether it is always true that any two such functions that moreover vanish on $\partial\mathscr{C}_o(\Sigma)$ must be proportional. Our first main result, Theorem 1.1 below (see also Theorem 2.13), answers this question by the positive. A generalization to a large class of cylinders is described in section 7.

To deal with non necessarily Dirichlet-regular Σ, we say, following a usual convention, that a function w in Σ vanishes on the open subset T of $\partial\Sigma$ (or, more precisely, that w vanishes in the weak sense on T) if w is bounded in a neighborhood of each $\xi \in T$ and if $A := \{\xi \in T\,;\, \limsup_{\Sigma \ni x \to \xi} |w(x)| > 0\}$ is polar in S_{d-1}. By definition, $A \subset S_{d-1}$ is polar in S_{d-1} if for each $\xi \in A$ there is a chart of S_{d-1},

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$\chi : V \to W \subset \mathbb{R}^{d-1}$, $V \ni \xi$, such that $\chi(V \cap A)$ is polar in $\mathbb{R}^{d-1}$. Note that A is polar in S_{d-1} if and only if $\{t\xi\,;\, \xi \in A, t > 0\}$ is polar in $\mathbb{R}^d$.

For a function w defined in a region Ω of $\mathbb{R}^d$, the relation $w = 0$ on $T \subset \partial\Omega$ is defined similarly. If $w : \Omega \to \mathbb{R}$ is harmonic with respect to a second order uniformly elliptic operator in divergence form with bounded measurable coefficients in Ω and if $w = 0$ in an open subset W of ∂V then $\lim_{x\to\xi} w(x) = 0$ for every Dirichlet-regular boundary point $\xi \in W$.

Theorem 1.1 *The nonnegative harmonic functions in $\mathscr{C}_o(\Sigma)$ which vanish (in the weak sense) on the boundary of $\mathscr{C}_o(\Sigma)$ are the functions h in the form $h(r\omega) = c\, r^{\alpha_\Sigma}\varphi_0(\omega)$, $\omega \in \Sigma$, $r > 0$, where c is a nonnegative constant, $\alpha_\Sigma = \frac{-(d-2)+\sqrt{(d-2)^2+4\lambda_1(\Sigma)}}{2}$ and φ_0 is a positive solution of $\Delta_{S_{d-1}}\varphi_0 + \lambda_1(\Sigma)\,\varphi_0 = 0$ in Σ.*

Here $\Delta_{S_{d-1}}$ is the spherical Laplacian –denoted also Δ_S in the rest of the paper– and $\lambda_1(\Sigma)$ (later denoted λ_1) is the first eigenvalue of the opposite of the Dirichlet Laplacian in Σ. As well-known $\lambda_1(\Sigma)$ coincides with the Raleigh constant of Σ, i.e. $\lambda_1(\Sigma) = \inf\{\int |\nabla u|^2\, d\sigma_{S_{d-1}};\, u \in C_c^1(\Sigma),\ \int |u|^2\, d\sigma_{S_{d-1}} \geq 1\}$ –where $\sigma_{S_{d-1}}$ is the standard Riemannian spherical measure in S_{d-1}– and $\lambda_1(\Sigma)$ is > 0 if and only if $S_{d-1} \setminus \Sigma$ is not polar in S_{d-1} (see e.g. [19]). In this case, $\frac{1}{\lambda_1(\Sigma)}$ is also the largest eigenvalue of the (nonnegative self-adjoint compact) Green's operator in $L^2(\Sigma; \sigma_{S_{d-1}})$, $\varphi \mapsto G(\varphi) = (-\Delta_S)^{-1}(\varphi)$. It is known (see e.g. [5]) that $\lambda_1(\Sigma)$ is also the greatest real λ for which there is a positive $(\Delta_S + \lambda I)$-superharmonic function in Σ (distinct from the constant $+\infty$). For $\lambda = \lambda_1(\Sigma)$ such a function is unique –up to multiplication by a constant– and there is a unique positive solution $\varphi_0 \in H_0^1(\Sigma)$ of $\Delta_S\varphi_0 + \lambda_1(\Sigma)\varphi_0 = 0$ with $\|\varphi_0\|_{L^2(\Sigma)} = 1$. In particular $\varphi_0 = 0$ in $\partial\Sigma$. Since, as well-known, the function $H_0(x) = r^{\alpha_\Sigma}\varphi_0(\omega)$ $-r = |x|$, $\omega = x/|x|$– is harmonic, Theorem 1.1 means that any two positive harmonic functions in $\mathscr{C}_o(\Sigma)$ vanishing on $\partial\mathscr{C}_o(\Sigma)$ are proportional. Note also that Theorem 1.1 implies that H_0 is a positive *minimal* harmonic function in $\mathscr{C}_o(\Sigma)$.

Section 2 is devoted to a proof of Theorem 1.1. See Theorem 2.11 and an improvement in Theorem 2.13. It relies in particular on the study of minimal Martin functions arising at infinity in $\mathscr{C}_0(\Sigma)$ and the study of the convergence in the Martin topology towards such Martin points (for Martin's theory, see [24], [29], [16], or [5]).

When Σ is sufficiently regular Theorem 1.1 is well-known. See [23] for the NTA case. The recent paper of K. Hirata [22] establishes the result when Σ is John. These papers rely on (and provide) Harnack boundary inequalities which do not hold in the general case.

In section 6 we show that –in contrast with the case where Σ is John – another question which might seem at first to be another formulation of A. Eremenko's question has a negative answer for a general Σ, at least in higher dimensions.

Theorem 1.2 *For $d \geq 4$, there exists a domain Σ such that the Martin boundary of $\mathscr{C}_o(\Sigma)$ contains a one parameter family of minimal points which are limits of sequences $\{P_n\}$ in $\mathscr{C}_o(\Sigma)$ going to infinity in $\mathbb{R}^d$ (and whose all defining sequences go to infinity in $\mathbb{R}^d$).*

The class of examples provided to prove Theorem 1.2 is strongly related to the construction by Cranston and McConnell of a bounded domain $\mathbb{D}$ in $\mathbb{R}^3$ with a positive harmonic function h in $\mathbb{D}$ such that the lifetime of the h-Brownian motion is almost surely infinite [14]. As shown in [14] this

cannot happen in a bounded planar domain. There is a corresponding result here given by the next statement. The only interesting case is $d = 3$.

Theorem 1.3 *If $d \leq 3$ and Σ is a domain in S_{d-1}, every sequence $\{P_n\}$ in $\mathscr{C}_o(\Sigma)$ going to infinity in $\mathbb{R}^d$ converges in the Martin topology towards the canonical Martin function H_0.*

The proof is given in section 5 and relies on a result of Bañuelos and Davis [10]. A similar proof shows that for all $d \geq 2$, every sequence $\{P_n\}$ in $\mathscr{C}_o(\Sigma)$ going non-tangentially to infinity in $\mathscr{C}_o(\Sigma)$ converges in the Martin topology to H_0. See Theorem 3.1 and its proof in section 5. Extensions based on [8] to $d \geq 4$ of Theorem 1.3 are also mentioned there.

In section 6.4 we supplement Theorem 1.2 by showing that for $d \geq 4$, $\mathscr{C}_0(\Sigma)$ may admit plenty of minimal as well as non-minimal Martin points associated to sequences $\{P_n\}$ in $\mathscr{C}_o(\Sigma)$ going to infinity in $\mathbb{R}^d$ (and whose all defining sequences go to infinity in $\mathbb{R}^d$). The corresponding Martin functions are of the form $H(x) = r^\alpha \psi(\frac{x}{|x|})$ with $-\frac{d-2}{2} < \alpha < \alpha_\Sigma$. This supplementary construction is based on the following fact (see Theorem 4.6) proved also in section 6.4 and valid for all $d \geq 2$.

Proposition 1.4 *If Σ contains a hemisphere Σ_+ of S_{d-1}, then every sequence $\{P_n\}$ such that $\frac{P_n}{|P_n|} \in \Sigma_+$ and $|P_n| \to +\infty$ converges towards the canonical Martin point H_0.*

In section 4, some implications on the Martin boundary that follow from regularity conditions are observed. For example, if M is a closed John regular subset of Σ (see section 4) then every sequence $\{P_n\}$ in $\mathscr{C}_o(M)$ such that $|P_n| \to \infty$ converges towards the canonical minimal point H_0. This generalizes Hirata's main result in [22].

To prove the above mentioned results, it seems more convenient (and natural) to work with a cylinder model of $\mathscr{C}_o(\Sigma)$ given by $\mathbb{R} \times \Sigma$ equipped with a suitable elliptic operator L. See section 2.1. We note here that there is a large literature dealing with Martin's boundaries of product structures (e.g., [25], [18], [27],[28] or [20]).

Almost all the results (and their proofs) extend to the framework of a cylinder $\mathscr{C}_Y(\Sigma) := \mathbb{R} \times \Sigma$ where Σ is a relatively compact region in a manifold M, the cylinder being equipped with an elliptic operator L in the form $L = (\partial_t \circ \partial_t + \beta\, \partial_t) \oplus \mathscr{L}$ where $\beta \in \mathbb{R}$, ∂_t is differentiation with respect to the first variable and $\mathscr{L}$ is a second order uniformly elliptic operator in M. See section 7.

2 Proof of Theorem 1.1.

We will assume once for all that $S_{d-1} \setminus \Sigma$ is non polar in S_{d-1}. Otherwise, by a standard extension Theorem and the Liouville property, positive harmonic functions in $\mathscr{C}_0(\Sigma)$ are constant, $\lambda_1(\Sigma) = 0$ and Theorem 1.1 is obvious in this case. Thus, in what follows, 0 is a Dirichlet-regular boundary point for $\mathscr{C}_o(\Sigma)$ -using e.g. Wiener's test-, and $\lim_{x\to 0} h(x) = 0$ for the functions h under consideration in Theorem 1.1.

2.1 Reduction and some preliminary remarks

As a first step, using a simple change of coordinates we reduce the study of positive harmonic functions in the cone $\mathscr{C}_o(\Sigma)$ to the study of positive solutions in the cylinder $\mathscr{C}_Y(\Sigma) = \mathbb{R}\times\Sigma$ of some natural elliptic operator – the notation $\mathscr{C}_Y(\Sigma)$ will be used all along the paper–. Recall first the following classical expression of the Laplacian in polar coordinates: if f is C^2-smooth in the open set U of $\mathbb{R}^d$, $d\geq 2$,

$$\Delta f(x) = \frac{\partial^2 g}{\partial r^2}(r,\omega) + \frac{d-1}{r}\frac{\partial g}{\partial r}(r,\omega) + \frac{1}{r^2}(\Delta_S)_\omega\, g(r,\omega) \tag{2.1}$$

for $x = r\omega$, $(r,\omega)\in\tilde{U} := \{(r,\omega)\in\mathbb{R}_+^*\times S_{d-1};\, r\omega\in U\}$ and $g(r,\omega) = f(r\omega)$ for $(r,\omega)\in\tilde{U}$. (Recall that Δ_S is the Laplace-Beltrami operator on the Riemannian sphere S_{d-1})

Consider then the change of coordinates: $\Phi(r,\omega) = (\log(r),\omega)$, or $\Phi^{-1}(u,\omega) = (e^u,\omega)$. Obviously Φ defines a C^∞ diffeomorphism of $\mathbb{R}_+^*\times\Sigma$ onto the cylinder $\mathscr{C}_Y(\Sigma) = \mathbb{R}\times\Sigma$. By an elementary calculation, if $g\in C^2(\mathbb{R}_+^*\times\Sigma)$ and $h = g\circ\Phi^{-1}$, we have

$$(\Delta_S)_\omega g(r,\omega) + (d-1)\,r\frac{\partial g}{\partial r}(r,\omega) + r^2\frac{\partial^2 g}{\partial r^2} = (\Delta_S)_\omega h(u,\omega) + \frac{\partial^2 h}{\partial u^2}(u,\omega) + (d-2)\frac{\partial h}{\partial u}(u,\omega) \tag{2.2}$$

Using these two formulas we are reduced to the study of the Martin boundary of the cylindrical region $\mathscr{C}_Y(\Sigma) = \mathbb{R}\times\Sigma$ of the manifold $X := \mathbb{R}\times S_{d-1}$ with respect to the second order elliptic operator $L := \Delta_S + (d-2)\frac{\partial}{\partial u} + \frac{\partial^2}{\partial u^2}$ (where u denotes the $\mathbb{R}$-component).

There is a well known explicit (and elementary) expression of the heat semi-group $\{Q_t\}_{t\geq 0}$ associated to the component of L acting in $\mathbb{R}$, i.e., $L_{\mathbb{R}} = \partial_u^2 + (d-2)\partial_u$. *By heat semi-group associated to* $L_{\mathbb{R}}$ we will mean that for each $\varphi\in C_0^+(\mathbb{R})$ the function $f:(t,x)\mapsto Q_t(\varphi)(x)$ is the minimal positive solution of the Cauchy problem : $\partial_t f(t,x) = \partial_x^2 f(t,x) + (d-2)\,\partial_x f(t,x)$ for $t>0$ and $f(0,x) = \varphi(x)$, $x\in\mathbb{R}$.

Lemma 2.1 *For $t>0$ the kernel Q_t on $\mathbb{R}$ is given by the convolution with the density*

$$q_t(u) = \frac{1}{\sqrt{4\pi t}}\exp\Big(-\frac{(u+(d-2)t)^2}{4t}\Big),\ \ u\in\mathbb{R} \tag{2.3}$$

This means that $Q_t(\varphi)(u) = \int_{\mathbb{R}} q_t(u-v)\,\varphi(v)\,dv$ for $\varphi\in C_0(\mathbb{R})$. The product structure of $\mathbb{R}\times\Sigma$ is exploited through the next standard fact.

Lemma 2.2 *In $\mathbb{R}\times\Sigma$, the heat semi-group $\{H_t\}_{t>0}$ associated to $L = \frac{\partial^2}{\partial u^2} + (d-2)\frac{\partial}{\partial u} + \Delta_S$ is given by the densities*

$$h(t;u,x;v,y) = q_t(u-v)\pi_t(x,y),\ \ x,y\in\Sigma,\ \ u,v\in\mathbb{R}, t>0.$$

Here π_t, $t>0$, is the standard heat kernel density in Σ (with respect to Δ_S, the usual Riemannian measure σ_S in S_{d-1} and the Dirichlet boundary condition).

In other words $H_t(\varphi)(u,x) = \int_{-\infty}^{\infty}\int_\Sigma h(t;u,x;v,y)\,\varphi(v,y)\,dv\,d\sigma_S(y)$ if $\varphi\in C_0(\mathbb{R}\times\Sigma;\mathbb{R})$ and $(u,x)\in\mathscr{C}_Y(\Sigma)$.

2.2 Some inequalities satisfied by Green's function and their consequences

The Green's function with pole (v,y) and with respect to L in $\mathbb{R}\times\Sigma$ (and the measure $\mu(d(u,x)) = du\,d\sigma(x)$) is the function:

$$(u,x)\mapsto G(u,x;v,y)=\int_0^\infty h(t;u,x;v,y)\,dt,\ \ (u,x)\in\mathbb{R}\times\Sigma. \tag{2.4}$$

But $h(t;u,x;v,y)=\frac{1}{\sqrt{4\pi t}}\,e^{-\frac{(u-v+(d-2)t)^2}{4t}}\,\pi_t(x,y)$ and for $\rho>0$ we have

$$\begin{aligned} e^{-\frac{(u+\rho-v+(d-2)t)^2}{4t}} &= e^{-\frac{(u-v+(d-2)t)^2}{4t}}e^{-\frac{\rho}{2t}(\frac{\rho}{2}+(d-2)t+u-v)} \\ &\geq e^{-(d-2)\rho/2}e^{-\frac{(u-v+(d-2)t)^2}{4t}} \end{aligned} \tag{2.5}$$

if $v\geq u+\frac{\rho}{2}$.

On the other hand, if $\rho>0$, and $v\leq u+\frac{\rho}{2}$

$$e^{-\frac{(u+\rho-v+(d-2)t)^2}{4t}} = e^{-\frac{(u-v+(d-2)t)^2}{4t}}e^{\frac{\rho}{2t}(-\frac{\rho}{2}-(d-2)t+v-u)} \leq e^{-(d-2)\rho/2}\,e^{-\frac{(u-v+(d-2)t)^2}{4t}} \tag{2.6}$$

Upon integrating with respect to t these inequalities, we obtain the next proposition.

Proposition 2.3 *The Green's function G for L in $\mathbb{R}\times\Sigma$ satisfies the following relations:*

$$G(u,x;v,y)\leq e^{(d-2)\rho/2}\,G(u+\rho,x;v,y),\quad \text{for } u,\,v\in\mathbb{R},\ \text{and } x,\,y\in\Sigma,$$

if $\rho>0$ and $v\geq\frac{\rho}{2}+u$. And $G(u,x;v,y)\geq e^{(d-2)\rho/2}\,G(u+\rho,x;v,y)$ if $\rho>0$ and $v\leq\frac{\rho}{2}+u$.

Moreover, we have the following symmetry identities : for $x,y\in\Sigma, u,v,v_0,v_1\in\mathbb{R}$

$$G(v_0-u,x;v_0-v,y)=e^{(d-2)(u-v)}\,G(v_1+u,x;v_1+v,y), \tag{2.7}$$

Note that (2.7) follows immediately from (2.3), lemma 2.2 and (2.4). Of course these identities can be understood in terms of Kelvin's transformation if one returns to the cone $\mathscr{C}_o(\Sigma)$ equipped with the usual Laplacian.

The above leads to the following properties of the L-Martin functions associated to L in the cylinder $\mathscr{C}_Y(\Sigma)$. We choose and fix once for all a reference point $x_0\in\Sigma$ and take $(0,x_0)$ as the normalization point for Martin's functions in $\mathscr{C}_Y(\Sigma)$. Occasionally, we use the standard notations $\boldsymbol{\Delta}$ (resp. $\boldsymbol{\Delta}_1$) to denote the Martin boundary (resp. the minimal Martin boundary) of $(\mathscr{C}_Y(\Sigma),L)$, and $\widehat{\mathscr{C}_Y}(\Sigma)$ its Martin compactification (ref. [24], [29], [5]).

Proposition 2.4 *If K is an L-Martin function in $\mathscr{C}_Y(\Sigma)=\mathbb{R}\times\Sigma$ defined by a sequence (v_j,y_j) with $v_j\to+\infty$, $y_j\in\Sigma$, (i.e., $K(u,x)=\lim_{j\to\infty}K_{(v_j,y_j)}(u,x)$ where $K_{(v_j,y_j)}$ is the Martin kernel $K_{(v_j,y_j)}:=G(.,.;v_j,y_j)/G(0,x_0;v_j,y_j)$) then*

$$K(u+\rho,x)\geq e^{-(d-2)\rho/2}K(u,x),\quad (u,x)\in\mathbb{R}\times\Sigma \tag{2.8}$$

for every $\rho > 0$. In particular if K is minimal there exists $\alpha \geq -\frac{d-2}{2}$ such that $K(u,x) = e^{\alpha u}K(0,x)$, $(u,x) \in \mathbb{R}\times\Sigma$, and the function $s(x) = K(0,x)$ is a proper function in Σ, namely $\Delta_S(s) + \lambda s = 0$ for $\lambda = \alpha^2 + (d-2)\alpha$. So $\lambda \geq -\frac{(d-2)^2}{4}$, $\alpha = \frac{2-d+\sqrt{(d-2)^2+4\lambda}}{2}$ and s is a minimal positive $(\Delta_S + \lambda I)$–harmonic function in Σ.

The first statement follows from Proposition 2.3 and the definition of Martin functions. If K is minimal, then $(u,x) \mapsto K(u+\rho,x)$ is also L-minimal; by (2.8), it follows that $K(u+\rho,x) = c(\rho)K(u,x)$ for some function $c \in C(\mathbb{R};\mathbb{R}^*_+)$ and all $(\rho,u,x) \in \mathbb{R}\times\mathbb{R}\times\Sigma$ (assuming first $\rho > 0$). We have $c(\rho+\rho') = c(\rho)c(\rho')$ for ρ, $\rho' \in \mathbb{R}$ and so $c(\rho) = e^{\alpha\rho}$ for some $\alpha \geq -\frac{d-2}{2}$. The other claims are then immediate. □

Note that $\alpha \in [\alpha_0, \alpha_{\max}]$ where $\alpha_{\max} := \alpha_\Sigma = \frac{2-d+\sqrt{(d-2)^2+4\lambda_1}}{2}$ and $\alpha_0 = -\frac{d-2}{2}$. We set $\alpha_{\min} = \frac{2-d-\sqrt{(d-2)^2+4\lambda_1}}{2}$.

Remark 2.5 *Similarly if K is an L-Martin function related to a sequence (v_j,y_j), with $v_j \to -\infty$, $y_j \in \Sigma$, then $K(u+\rho,x) \leq e^{-(d-2)\rho/2}K(u,x)$ when $(u,x) \in \mathbb{R}\times\Sigma$ and $\rho \geq 0$. If K is minimal w.r. to L, then $K(u,x) = e^{\alpha u}s(x)$, for some $\alpha \in [\alpha_{\min}, \alpha_0]$ and some λ-proper function s in Σ, i.e., $\Delta_S(s) + \lambda s = 0$ in Σ. Here $\lambda = \alpha^2 + (d-2)\alpha \geq -\frac{(d-2)^2}{4}$, $\alpha = \frac{2-d-\sqrt{(d-2)^2+4\lambda}}{2}$ and s is $(\Delta_S + \lambda I)$-minimal in Σ.*

Observe that if $\mathbf{\Delta}^+$ (resp. $\mathbf{\Delta}^-$) is the set of the Martin functions arising from a sequence (v_j,y_j) with $\lim y_j = +\infty$ (resp. $\lim y_j = -\infty$) the identities (2.7) lead to a natural bijection $K \mapsto \tilde{K}$ from $\mathbf{\Delta}^+$ onto $\mathbf{\Delta}^-$, where $\tilde{K}(u,x) := e^{-(d-2)u}K(-u,x)$.

We note two other observations which complement Proposition 2.4 and Remark 2.5.

Proposition 2.6 *If $\zeta \in \mathbf{\Delta}$ is a Martin boundary point for $(\mathscr{C}_Y(\Sigma), L)$ which as a point of the Martin compactification is in the closure of both $\{(u,x) \in \mathscr{C}_Y(\Sigma)\,;\, u \geq \rho\}$ and $\{(u,x) \in \mathscr{C}_Y(\Sigma)\,;\, u \leq \rho'\}$ for some ρ, $\rho' \in \mathbb{R}$, $\rho' < \rho$, then K_ζ is in the form $K_\zeta(u,x) = e^{-\frac{d-2}{2}u}f(x)$ where f is a positive solution of $\Delta_S f - \frac{(d-2)^2}{4}f = 0$ in Σ.*

The same conclusion holds for $\zeta \in \mathbf{\Delta}_1$, if ζ is in the closure (for the Martin topology) of a set $\mathscr{C}^R_Y = \{(v,y)\,;\, |v| \leq R, y \in \Sigma\}$, $R > 0$, and if K_ζ is of the form $K_\zeta(u,x) = e^{\alpha u}f(x)$. Moreover f is a positive minimal solution of $\Delta_S f - \frac{(d-2)^2}{4}f = 0$ in Σ.

In the first case, by the above estimates of Green's function we must have $K_\zeta(u,x) = e^{-\frac{d-2}{2}u}f(x)$ for $\rho' < u < \rho$, $x \in \Sigma$, where $f(x) = e^{\frac{d-2}{2}\rho'}K_\zeta(\rho',x) = e^{\frac{d-2}{2}\rho}K_\zeta(\rho,x)$. Then f is necessarily as in the statement and by the Green's function estimates, we see that $K_\zeta(u,x) \geq e^{-\frac{d-2}{2}u}f(x)$ for all $u > \rho'$.

But a nonnegative solution w of $L(w) = 0$ in a domain $\Omega \subset \mathscr{C}_Y(\Sigma)$ that vanishes at some point vanishes everywhere (by Harnack inequalities). Thus $K_\zeta(u,x) = e^{-\frac{d-2}{2}u}f(x)$ when $u > \rho'$. A similar argument extends this equality to $u < \rho$.

In the second case, if $\zeta =: \lim(v_j,y_j)$, we must also have $\zeta = \lim(v_j+s,y_j)$ for every $s \in \mathbb{R}$. So the result follows from the first part of the proposition (the minimality of f being necessary for the minimality of K_ζ). □

2.3 Positive L-harmonic functions vanishing on $\mathbb{R}\times\partial\Sigma$ (a)

Let f denote a nonnegative L-harmonic function in $\mathscr{C}_Y(\Sigma):=\mathbb{R}\times\Sigma$ such that $f=0$ on $\mathbb{R}\times\partial\Sigma$ in the weak sense.

By the Martin boundary theory there is a unique integral representation of f in the form

$$f(u,x)=\int K_\zeta(u,x)\,d\mu(\zeta),\quad (u,x)\in\mathbb{R}\times\Sigma \tag{2.9}$$

where μ is a finite positive Borel measure on the minimal Martin boundary $\boldsymbol{\Delta}_1$ of $(\mathscr{C}_Y(\Sigma),L)$ and where K is the L-Martin kernel associated to L and the reference point $(0,x_0)\in\Sigma$.

Denote $\boldsymbol{\Delta}_1^\infty$ the trace on $\boldsymbol{\Delta}_1$ of the intersection of the closures –w. r. to Martin's topology– of the sets $X_N:=\{(-\infty,-N]\cup[N,+\infty)\}\times\Sigma$, $N\geq 1$. In other words, a point $\zeta\in\boldsymbol{\Delta}_1$ is in $\boldsymbol{\Delta}_1^\infty$ if and only if there is a sequence of points $\xi_j=(u_j,x_j)\in\mathbb{R}\times\Sigma$ with $|u_j|\to+\infty$ converging to ζ. In particular K_ζ is in the form given by either Proposition 2.3 or remark 2.5.

Lemma 2.7 *The measure μ_f is supported by $\boldsymbol{\Delta}_1^\infty$.*

Proof. Let $N\geq 1$. In $\mathscr{C}_Y(\Sigma)$, the function f is equal to $R_f^{X_N}$, its réduite (cf. [11], [13], [16]) on X_N (w.r. to L). This follows from the assumption $f=0$ in $\partial\Sigma\times\mathbb{R}$ and from (a standard extended form of) the maximum principle. So by the Martin boundary theory, the measure μ_f is supported by the set of the points $\zeta\in\boldsymbol{\Delta}_1$ such that X_N is not minimally thin at ζ. Such a point ζ is necessarily in the closure of X_N in $\widehat{\mathscr{C}_Y}(\Sigma)$, the L-Martin compactification of $\mathscr{C}_Y(\Sigma)$. Whence the result. □

Next we will observe a simple condition for μ_f to be concentrated on

$$\boldsymbol{\Delta}_1^{+\infty}:=\boldsymbol{\Delta}_1\cap\{\zeta\in\widehat{\mathscr{C}_Y}(\Sigma);\,\zeta=\lim_j(u_j,x_j)\text{ with }\ x_j\in\Sigma\text{ and }u_j\to+\infty\,\}. \tag{2.10}$$

Define similarly $\Delta_1^{-\infty}$ using sequences $\{(u_j,x_j)\}_{j\geq1}$ with $\lim u_j=-\infty$.

Lemma 2.8 *If $f(-t,x_1)=o(e^{\frac{d-2}{2}t})$ as $t\to+\infty$ for some (or all) $x_1\in\Sigma$, then the measure μ_f is supported by $\boldsymbol{\Delta}_1^{+\infty}$.*

This is because for $u\leq 0$ and $\zeta\in\boldsymbol{\Delta}_1^{-\infty}$ we have $K_\zeta(u,x_0)\geq e^{-\frac{d-2}{2}u}K_\zeta(0,x_0)=e^{-\frac{d-2}{2}u}$ (see Remark 2.5) and

$$f(u,x_0)=\int K_\zeta(u,x_0)\,d\mu(\zeta)\geq e^{-\frac{d-2}{2}u}\mu(\boldsymbol{\Delta}_1^\infty\setminus\boldsymbol{\Delta}_1^{+\infty})=e^{-\frac{d-2}{2}u}\mu(\boldsymbol{\Delta}_1^{-\infty}).$$

Thus $\mu(\boldsymbol{\Delta}_1^{-\infty})=0$. Note the special case where $f(t,x_0)=O(1)$ as $t\to-\infty$. □

2.4 End of proof of Theorem 1.1

In this subsection it is assumed that *moreover $f(u_n,x_0)=O(1)$ for a real sequence $u_n\to-\infty$*, that is $\liminf_{u\to-\infty} f(u,x)<\infty$ for each $x\in\Sigma$. We will show that f is unique up to a multiplication by a constant

and will thus prove Theorem 1.1. Examining the proof, it will be seen later that this assumption on $\liminf_{u\to-\infty} f(u,x_0)$ can be notably relaxed.

For any given real a_0 let us denote $\mathbf{\Delta}_1^{+\infty}(a_0)$ the set of all points $\zeta \in \mathbf{\Delta}_1^{+\infty}$ such that K_ζ is in the form : $K_\zeta(u,x) = \varphi(x)\,e^{\alpha u}$, $(u,x) \in \mathscr{C}_Y(\Sigma)$ for some real $\alpha \geq a_0$ and some minimal positive solution of $\Delta_S\varphi + \lambda\varphi = 0$ in Σ where $\lambda = \alpha^2 + (d-2)\alpha$. We set $\alpha = \alpha(\zeta)$ and $\lambda = \lambda(\zeta)$.

Repeating the argument in lemma 2.8 we first note the following.

Lemma 2.9 *The measure μ_f is concentrated in $\mathbf{\Delta}_1^{+\infty}(0)$.*

This is again immediate since given $\varepsilon > 0$ we have for $u \leq 0$,

$$f(u,x_0) \geq \int_{\mathbf{\Delta}_1^{+\infty}\setminus\mathbf{\Delta}_1^{+\infty}(-\varepsilon)} K_\zeta(u,x_0)\,d\mu_f(\zeta) \geq e^{-\varepsilon u}\mu_f(\mathbf{\Delta}_1^{\infty}\setminus\mathbf{\Delta}_1^{\infty}(-\varepsilon))$$

and hence $\mu_f(\mathbf{\Delta}_1^{+\infty}\setminus\mathbf{\Delta}_1^{+\infty}(-\varepsilon)) = 0$ since $\liminf_{u\to-\infty} f(u,x_0) < \infty$. □

Recall that $\lambda_1 = \lambda_{\max}$ is the greatest λ for which the equation $\Delta_S\varphi + \lambda\varphi = 0$ admits a positive solution (or even a positive supersolution) in Σ.

Lemma 2.10 *Set $A = \{\zeta \in \mathbf{\Delta}_1^{+\infty}(0)\,;\,\lambda(\zeta) < \lambda_{\max}(\Sigma)\}$. Then $\mu_f(A) = 0$.*

Assume that $\mu_f(A) > 0$. Then for some λ_1', $0 < \lambda_1' < \lambda_1$, the set A' of all $\zeta \in \mathbf{\Delta}_1^{+\infty}(0)$ for which $0 \leq \lambda(\zeta) \leq \lambda_1'$ has strictly positive measure: $\mu_f(A') > 0$. Note that A' is the set of the minimal points $\zeta \in \mathbf{\Delta}_1$ such that for some λ, $0 \leq \lambda \leq \lambda_1'$,

$$K_\zeta(u,x) = e^{\alpha u}k_\zeta^\lambda(x) \tag{2.11}$$

where $\alpha = \frac{-(d-2)+\sqrt{(d-2)^2+4\lambda}}{2}$ and k_ζ^λ is a $\Delta_S + \lambda I$–minimal function in Σ.

The function

$$\varphi(x) = \int_{A'} k_\zeta^\lambda(x)\,d\mu_f(\zeta),\ \ x \in \Sigma, \tag{2.12}$$

is a (strictly) positive Δ_S-superharmonic function in Σ which satisfies the following:

(i) φ vanishes weakly on the boundary of Σ (note that $\varphi(x) \leq f(u,x)$ for $x \in \Sigma$, $u \geq 0$ and that by assumption $f = 0$ on $\mathbb{R}\times\partial\Sigma$ in the weak sense),

(ii) the positive measure $-\Delta_S(\varphi)$ admits the density $\psi(x) = \int_{A'} \lambda k_\zeta^\lambda(x)\,d\mu_f(x)$ with respect to σ_S.

By (i) and a well-known form of the maximum principle φ is a potential in Σ with respect to the spherical Laplacian Δ_S in Σ, i.e. its greatest subharmonic minorant h in Σ is zero . Indeed there exists a positive Δ_S-superharmonic function s going to infinity at each $\zeta \in \partial\Sigma$ where φ has a non-zero upper limit since the set of these points ζ is polar. So by a standard form of the maximum principle $h - \varepsilon s \leq 0$ for each $\varepsilon > 0$.

It follows that φ is a Green's potential in Σ and so using first (ii) and then (2.12) and (ii) again, we get

$$G_\Sigma(\psi) = \varphi \geq \frac{1}{\lambda_1'}\psi \tag{2.13}$$

in Σ, where G_Σ denotes the Green's function in Σ w.r. to Δ_S.

The function ψ is in $L^2(\Sigma)$ (it is bounded in Σ -in fact ψ is also an element of $H_0^1(\Sigma)$-). Thus from $\psi \neq 0$ and $\|G_\Sigma(\psi)\|_2 \geq \frac{1}{\lambda_1}\|\psi\|_2$ we infer that $\frac{1}{\lambda_1} = \|G_\Sigma\|_{L(L^2,L^2)} \geq \frac{1}{\lambda_1}$ which is absurd. □

On the other hand, we know that every positive eigenfunction φ with respect to (Δ_S, Σ) and the eigenvalue $\lambda_1 = \lambda_1(\Sigma)$ is proportional to φ_0 (see e.g. [5]). Moreover φ_0 vanishes at every Dirichlet-regular boundary point $\xi \in \partial_S\Sigma$ and so vanishes quasi-everywhere on $\partial_S\Sigma$.

Thus we have proved the following which contains Theorem 1.1. See also Theorem 2.13.

Theorem 2.11 *There is a unique positive L-harmonic function F_+ on $\mathscr{C}_Y(\Sigma) = \mathbb{R}\times\Sigma$ vanishing (in the weak sense) on the boundary $\mathbb{R}\times\partial\Sigma$ and such that $\liminf_{u\to-\infty} F_+(u,x_0) < +\infty$ and $F_+(0,x_0) = 1$. This function is a minimal Martin function and $F_+(u,x) = e^{\alpha_{\max}u}\frac{\varphi_0(x)}{\varphi_0(x_0)}$ if $\alpha_{\max} = \frac{2-d+\sqrt{(d-2)^2+4\lambda_1}}{2}$ (i.e. $\alpha_{\max} = \alpha_\Sigma$).*

In what follows we will keep the notation F_+ for this "canonical" minimal function and let $F_-(u,x) = e^{\alpha_{\min}u}\frac{\varphi_0(x)}{\varphi_0(x_0)}$, where $\alpha_{\min} = \frac{2-d-\sqrt{(d-2)^2+4\lambda_1}}{2}$ for the similar minimal function related to the end "$u\to-\infty$" of the cylinder. We set $\alpha_0 = -\frac{d-2}{2}$.

In the course of the proof of Theorem 2.11 we have also essentially shown the following facts. Distinguish three classes of minimal Martin functions K w.r. to $(\mathscr{C}_Y(\Sigma), L)$: (i) the functions in the form $K_\zeta(u,x) = k(x)e^{\alpha u}$ with $0 < |\alpha-\alpha_0| \leq \frac{\sqrt{(d-2)^2+4\lambda_1}}{2}$, (ii) those in the form $K_\zeta(u,x) = k(x)e^{\alpha_0 u}$, (iii) the class of all other minimal functions.

Proposition 2.12 *If K_ζ is in the third class there is a unique $v_\zeta \in \mathbb{R}$ such that $\lim v_j = v_\zeta$ for any sequence $\{(v_j,y_j)\}$ in $\mathscr{C}_Y(\Sigma)$ converging to ζ and for such a sequence $\lim_j(w_j,y_j) = \zeta$ whenever $w_j \to v_\zeta$ in $\mathbb{R}$. If K_ζ is in the first class, every corresponding sequence $\{(v_j,y_j)\}_{j\geq1}$ is such that $\lim v_j = +\infty$ or $\lim v_j = -\infty$ depending whether $\alpha > \alpha_0$ or $\alpha < \alpha_0$. Finally if K_ζ is in the class (ii) there are sequences (v,y_j) (with a fixed first coordinate) converging to ζ and for any such sequence $\lim_j(v_j,y_j) = \zeta$ for every real bounded sequence $\{v_j\}$; moreover there are sequences $\{v_j\}$ such that $\lim v_j = +\infty$ and $\lim_j(v_j,y_j) = \zeta$.*

A minimal function in the class (iii) will be said to be of the finite type.

Of course if Σ is smooth, the first class reduces to $\{F_+,F_-\}$ and the second class is empty. We shall see later that there may exist minimal as well as non minimal Martin points ζ in the form $K_\zeta(u,x) = e^{\alpha u}k(x)$, for all α such that $|\alpha-\alpha_0| < \frac{\sqrt{(d-2)^2+4\lambda_1}}{2}$. See 6.4.

Proof. To establish the last claim let ζ be in the second class. If $v\in\mathbb{R}$ and if $(v_j,y_j)\to\zeta$ then $(2v-v_j,y_j)\to\zeta$ (by the identities (2.7) with $v_0 = v_1 = 0$). But a minimal Martin point has a neighborhood basis $\{U_j\}$ in $\widehat{\mathscr{C}}_Y(\Sigma)$ with $U_j\cap\mathscr{C}_Y(\Sigma)$ connected (by the general theory, see e.g. [29] p. 223) and so we can find points $z_j\in\Sigma$ with $(v,z_j)\to\zeta$. If $\{v_j\}$ is bounded it follows at once from the (local) Harnack inequalities and the translation invariance with respect to the first coordinate that $(v_j,z_j)\to\zeta$. It is then obvious that if $v_j\to+\infty$ sufficiently slowly $(v_j,y_j)\to\zeta$.

It also follows immediately from translation invariance that if a sequence $\{(v_j,y_j)\}_{j\geq1}$ converges to a point $\zeta\in\Delta$ then $\lim_j(w_j,y_j) = \zeta$ for $\{w_j\}$ such that $|v_j-w_j|\to0$. □

2.5 Positive L-harmonic functions vanishing on $\mathbb{R}\times\partial\Sigma$ (b)

The proof of Lemma 2.10 can be extended so as to use a much weaker assumption on the behavior of $f(u,x_0)$ for $u\to-\infty$. This leads to a description of the positive L-solution in $\mathbb{R}\times\Sigma$ vanishing on $\mathbb{R}\times\partial\Sigma$ which also improves Theorem 2.11.

Theorem 2.13 *If f is L-harmonic in $\mathscr{C}_Y(\Sigma)$ and vanishes on $\mathbb{R}\times\partial\Sigma$, then f is a linear combination of F_+ and F_-. Thus if $\liminf_{u\to-\infty} e^{\alpha_{\min}u} f(u,x_0)=0$ then f is proportional to F_+.*

Using the Martin disintegration of F, we may write $F=aF_++bF_-+F_1+F_2$ with

$$F_1(u,x)=\int_A e^{\alpha u}k_\zeta(x)\,d\mu(\zeta)\ ,\ F_2(u,x)=\int_B e^{\alpha u}k_\zeta(x)\,d\nu(\zeta)$$

where $A=\{\zeta\in\boldsymbol{\Delta}_1\,;\,\exists\alpha_\zeta,\ \alpha_0\le\alpha_\zeta<\alpha_{\max}$ and $k_\zeta\in C_+(\Sigma)$ s.t. $K_\zeta(u,x)\equiv e^{\alpha_\zeta u}k_\zeta(x)\ \}$, $B=\{\zeta\in\boldsymbol{\Delta}_1\,;\,\exists\alpha_\zeta,\ \alpha_{\min}<\alpha_\zeta<\alpha_0$ and $k_\zeta\in C_+(\Sigma)$ with $K_\zeta(u,x)\equiv e^{\alpha_\zeta u}k_\zeta(x)\ \}$ and where μ and ν are finite Borel measures supported by A and B respectively.

We claim that $\mu(A)=0$. If not there exists $\alpha_1'\in(\alpha_0,\alpha_{\max})$ such that $\mu(A')>0$ if $A'=\{\zeta\in A;\,\alpha_0\le\alpha_\zeta\le\alpha_1'\}$ and repeating the argument in Lemma 2.10 we may conclude using now $\varphi(x):=\int_{A'}k_\zeta(x)\,d\mu(\zeta)$ and the potential theory w. r. to the operator $\Delta_S-\frac{(d-2)^2}{4}I=\Delta_S-\lambda_0 I$ in S_{d-1} (in particular the related Green kernel $G_\Sigma^{\lambda_0}$). Note that (2.13) becomes : $G_\Sigma^{\lambda_0}(\psi)=\varphi\ge\frac{1}{(\lambda_1'-\lambda_0)}\psi$ where λ_1' is the eigenvalue corresponding to α'.

In the same way (or using the observation after remark 2.5) it is shown that $F_2=0$. □

3 Nontangential convergence to F_+ or H_0

The next statement is about how Martin's topology relates to the canonical minimal F_+ in the cylinder $\mathscr{C}_Y(\Sigma)$ (or to the minimal H_0 in the cone $\mathscr{C}_0(\Sigma)$). It says that nontangential convergence of the current point $(u,x)\in\mathscr{C}_Y(\Sigma)$ (resp. $x=r\omega\in\mathscr{C}_o(\Sigma)$) to the end "$u=+\infty$" (or "$r=+\infty$") implies its convergence to the canonical Martin point at infinity.

Theorem 3.1 *For every sequence $\zeta_j:=(u_j,x_j)$ in $\mathbb{R}\times\Sigma$ such that $u_j\to+\infty$ and $\{x_j\}$ is relatively compact in Σ, it holds that $K_{\zeta_j}(u,x)\to e^{\alpha_{\max}u}\varphi_0(x)/\varphi_0(x_0)$ (i.e. $\{\zeta_j\}$ converges to the Martin function F_+). In fact, the following Harnack boundary inequalities hold*

$$C^{-1}G(u,x_0;v,x_0)\,G(v,x_0;w,x_0)\le G(u,x_0;w,x_0)\le C\,G(u,x_0;v,x_0)\,G(v,x_0;w,x_0)\qquad(3.1)$$

for $u,v,w\in\mathbb{R}$, $u+1\le v$, $v+1\le w$ and some constant $C=C(d,\Sigma,x_0)\ge1$. The inequalities obtained by replacing G by its transposed kernel in (3.1) also hold.

A proof will be given at the end of section 5. We note here that inequalities (3.1) imply by themselves that for $t\to+\infty$ the point (t,x_0) converges to a minimal point in the Martin boundary (see [2] Théorème 2 or [3] p. 516).

4 Martin boundary and subsets of Σ

In this section we collect some properties of the Martin boundary of $\mathscr{C}_Y(\Sigma)$ resulting from regularity conditions on a subset M of Σ. The results will not be used before section 6.3.

4.1 John regular subsets

Let M be a closed subset of Σ. For $\eta > 0$, let $M_\eta := \{x \in \Sigma; d_a(x,M) \le \eta\}$ where $d_a(x,y) := \inf\{\mathrm{diam}(C); C \subset \Sigma \text{ connected}, x, y \in \Sigma\}$ - S_{d-1} is equipped with the usual metric in $\mathbb{R}^d$-. For $0 < c_0 \le 1$ we say that M is c_0-John in Σ if there are points $A_1, \dots, A_N$ in Σ such that (i) $d(A_j, \partial\Sigma) \le c_0^{-1} d(A_k, \partial\Sigma)$ for $1 \le j,k \le N$, (ii) for $\eta := c_0 \max\{d(A_j,\partial\Sigma); 1 \le j \le N\}$, each $a \in M_\eta$ can be connected to one A_j by a c_0-John arc in Σ (see [6] Définition 1.1 and Théorème 5.3). $\{A_j\}_{1\le j\le N}$ is then called a c_0-admissible set of poles for M (note that N can always be chosen smaller than a constant $N_0(d,c_0)$).

The next statement generalizes Hirata's main result in [22]. We rely on Theorem 3.1 and a boundary Harnack principle given in [6] ([1] for $N \le 2$). Note that this statement may be easily reduced to the $N = 1$ case.

Theorem 4.1 *Let M be a closed and c_0-John subset of Σ with poles A_j, $1 \le j \le N$. Then $\lim_{u\to+\infty}(v,y) = F_+$ in the Martin topology, uniformly with respect to $y \in M$.*

Proof of Theorem 4.1 Denote K the Martin kernel in $\widehat{\mathscr{C}_Y}(\Sigma)$ with respect to the normalization point $(0,x_0)$. Applying Théorème 5.3 and Remarque 5.4 in [6] to $M \times [v-1, v+1]$ as a subset of $\mathscr{C}_Y(\Sigma)$ (or rather -so as to deal with the classical Laplacian- to the corresponding situation in $\mathscr{C}_0(\Sigma)$) we obtain a constant $C = C(d,c_0) \ge 1$ such that

$$K_{(v,y)}(u,x) \le C \sum_{j=1}^{N} \frac{K_{(v,y)}(v,A_j)}{K_{(v,A_j)}(v,A_j')} K_{(v,A_j)}(u,x) \tag{4.1}$$

whenever $(v,y) \in \mathbb{R} \times M$, $|v| \ge 1$, and $(u,x) \in \mathscr{C}_Y(\Sigma)$ satisfies $|u-v| \ge 1$ (or $d_a(x,M) \ge c_0$). Here A_j' is arbitrarily chosen in $\partial B(A_j, \frac{c_0}{100}\mathrm{dist}(A_j; S_{d-1} \setminus \Sigma))$ and we restrict to y such that $|y - A_j| \ge 2|A_j - A_j'|$ for all j.

By Harnack inequalities

$$K_{(v,y)}(u,x) \ge c \frac{K_{(v,y)}(v,A_j)}{K_{(v,A_j)}(v,A_j')} K_{(v,A_j)}(u,x)$$

when $(u,x) \in \partial B((v,A_j), |A_j - A_j'|)$ and hence –by the maximum principle–, also for $(u,x) \in \mathscr{C}_Y(\Sigma) \setminus B((v,A_j), |A_j - A_j'|)$. Taking $(u,x) = (0,x_0)$ we see that $c \frac{K_{(v,y)}(v,A_j)}{K_{(v,A_j)}(v,A_j')} \le 1$. So it follows from (4.1) that

$$K_{(v,y)}(u,x) \le C' \sum_{j=1}^{N} K_{(v,A_j)}(u,x) \tag{4.2}$$

when $(v,y) \in \mathbb{R} \times M$, $|v| \geq 1$, and $(u,x) \in \mathscr{C}_Y(\Sigma)$ satisfies $|u-v| \geq 1$. Since by Theorem 3.1 $K_{(v,A_j)} \to F_+$ for $j \to \infty$ and since F_+ is minimal, the result follows. □

The proof also yields the following more general statement. Here the results of [6] for John subsets with more than one pole are effectively used.

Theorem 4.1′ *Let $\{M_n\}$ be a sequence of closed John regular subsets of Σ with a common John constant c_0. Assume that $v_n \to +\infty$ in $\mathbb{R}$ and that $(v_n, A_j^{(n)}) \to \zeta_j \in \boldsymbol{\Delta}_1$, $1 \leq j \leq N$, where for each n, $\{A_j^{(n)}\}_{1 \leq j \leq N}$ is a c_0-admissible set of poles for M_n. Then if $y_n \in M_n$, every Martin cluster function of $\{(v_n, y_n)\}$ is a linear combination of the K_{ζ_j}.*

In particular if the points ζ_j all coincide with a minimal boundary point ζ then $(v_n, y_n) \to \zeta$.

4.2 John cuts

Assume now that M is a John regular closed subset of Σ and that $\Sigma \setminus M$ is the disjoint union of two open subsets U_0 and U_1. Fix $\delta > 0$ and set $U_j^\delta = \{x \in U_j \,;\, d_a(x,M) \geq \delta\}$ and $V_j^\delta = \mathbb{R} \times U_j^\delta$, $j = 1, 2$. Let $V_j = \mathbb{R} \times U_j$.

Proposition 4.2 *If $h = K_\mu$ is the positive superharmonic function in $\mathscr{C}_Y(\Sigma)$ generated by a probability measure μ supported on the closure of V_0 in the Martin compactification $\widehat{\mathscr{C}}_Y(\Sigma)$ and not charging $(0, x_0)$, we have*

$$h(u,x) \leq C\,[F_+(u,x) + F_-(u,x)], \qquad (u,x) \in V_1^\delta \tag{4.3}$$

for some constant $C = C(\Sigma, M, U_0, x_0, \delta)$.

Proof. We may assume that $x_0 \in U_1^\delta$ (using Harnack and changing the reference point) and it suffices to prove (4.3) for each $h = K_{(v,y)}$, $y \in V_0$ with a constant $C > 0$ as in the statement. Reducing h on V_1, it suffices to prove the result for $K_{(v,y)}$, $y \in M$, $v \in \mathbb{R}$.

For such a pole (v,y), with say $v > 0$, it follows from (4.2) and Theorem 3.1 that for $x \in U_1^\delta$ such that $d(x, A_j) \geq \frac{1}{2} d(A_j, \partial\Sigma)$ for $j = 1, \ldots, N$, (we use the same notations as above)

$$\begin{aligned} K_{(v,y)}(u,x) &\leq C' \sum_{j=1}^{N} K_{(v,A_j)}(u,x) \\ &\leq C'' \sum_j K_{(v,A_j)}(v, A_j')\, e^{\alpha_{\max}(u-v)}\, \varphi_0(x) \\ &\leq C''' e^{\alpha_{\max} u} \varphi_0(x). \end{aligned} \tag{4.4}$$

In the second line we have used the maximum principle (as above in the proof of Theorem 4.1) to compare the positive L-harmonic function $e^{\alpha_{\max} u}\varphi_0(x)$ with the Green function with pole at (v, A_j). In the last line we have used the inequalities given by Theorem 3.1 which imply that $K_{(v,A_j)}(v, A_j') \simeq K_{(v,A_j)}(v-1, A_j) \simeq F_+(v-1, A_j)$. Using the similar inequality for $v \leq 0$ we get the desired conclusion. □

Remark 4.3 For $h = K_{(v,y)}$, $y \in M$, the proof shows that $h(u,x) \leq C\,[F_+(u,x) + F_-(u,x)]$ if $x \in U_1$, $|u-v| \geq 1$. *If moreover $v \geq 0$, then $h(u,x) \leq C F_+(u,x)$, for these points (u,x).*

Corollary 4.4 *Under the assumptions of Proposition 4.2 two sequences* $\{(v_j,y_j)\}$ *and* $\{(w_j,z_j)\}$ *such that* $v_j, w_j \to +\infty$, $y_j \in U_0$ *and* $z_j \in U_1$, *have at most one common cluster point in* $\boldsymbol{\Delta}$ *which can only be* F_+.

Corollary 4.5 *If* h_0 *and* h_1 *are positive harmonic in* $\mathscr{C}_Y(\Sigma)$, *if* $h_j = K_{\mu_j}$ *with* μ_j *supported by* $\overline{U_j \times \mathbb{R}}$ *(closure in* $\widehat{\mathscr{C}_Y}(\Sigma)$*),* $j = 0, 1$, *if* $\lim_{u\to+\infty} e^{-\alpha_{max}u}(h_0 \wedge h_1)(u,x_0) = \lim_{u\to+\infty} e^{\alpha_{min}u}(h_0 \wedge h_1)(-u,x_0) = 0$ *then* $h_0 \wedge h_1$ *is a potential (that is, has no positive L-harmonic minorant).*

4.3 Inner ball property

If we have a boundary point $z \in \partial\Sigma$ and an open ball (or cap) $B(a,r) \subset \Sigma$ with $z \in \partial B(a,r)$, $r < 2$, the results in [4] (see also [2]) tell us (using Proposition 2.12) that *as* $v \to v_0$ *in* $\mathbb{R}$ *and* $y \to z$ *non-tangentially in* $B(a,r)$, *the point* (v,y) *tends to a finite type minimal boundary point* $\xi = \xi(v_0;(z,a))$ *in* $\widehat{\mathscr{C}_Y}(\Sigma)$. Moreover the minimal K_ξ is bounded away from (v_0,z) and vanishes on $\partial\mathscr{C}_Y(\Sigma) \setminus \{(v_0,z)\}$.

There is a parallel statement for the behavior of (v,y) for $v \to +\infty$. But here the inner ball should be large. This will be used later for an example's construction in section 6.4.

Theorem 4.6 *Assume that* Σ *contains an open hemisphere* Σ_+ *in* S_{d-1}. *Then if* $\{y_j\}$ *is a sequence in* Σ^+ *and if* $v_j \to +\infty$ *in* $\mathbb{R}$, *the sequence* (v_j,y_j) *converges in the Martin compactification of* $\mathscr{C}_Y(\Sigma)$ *to the canonical Martin point* F_+, *i.e.,* $\lim_j K_{(v_j,y_j)} = F_+$.

The proof is deferred to section 6.4.

5 Uniqueness of the Martin point at infinity for $d = 3$.

We now prove Theorem 1.3 (rather its cylinder version), using in an essential way a result of R. Bañuelos and B. Davis on the heat kernels in planar domains ([10], [9]). This result says that given the point $x_1 \in \Sigma$ there is a $t_0 > 0$ and for each $t \geq t_0$ a constant $C(t) > 1$ such that $\lim_{t\uparrow\infty} C_t = 1$ and

$$C(t)^{-1}\, e^{-\lambda_1 t}\, \varphi_0(x_1)\, \varphi_0(y) \leq \pi_t(x_1,y) \leq C(t)\, e^{-\lambda_1 t}\, \varphi_0(x_1)\, \varphi_0(y) \tag{5.1}$$

when $t \geq t_0$ and $y \in \Sigma$ (see in [10] Theorem 1 and section 4). Recall $\{\pi_t\}$ is the heat semi-group generated by the Laplacian in Σ and φ_0 is normalized by the condition $\|\varphi_0\|_{L^2(\Sigma)} = 1$.

Theorem 5.1 *If* $d = 3$, *every sequence* $\{\xi_j\}_{j\geq1} = \{(v_j,y_j)\}_{j\geq1}$, *in* $\mathscr{C}_Y(\Sigma)$ *such that* $v_j \to +\infty$ *converges to* F_+, *i.e.,* $K_{\xi_j}(u,x) \to K_{\zeta_\infty}(u,x) := e^{\alpha_{\max} u}\, \varphi_0(x)/\varphi_0(x_0)$ *for* $j \to \infty$.

The following simple lemma (valid for all $d \geq 2$) deals with times in $(0,t_0]$.

Lemma 5.2 *Given* $\delta_0 > 0$ *and* $x_1 \in \Sigma$, *there is a constant* $C = C(\delta_0;\Sigma,x_1) \geq 1$ *such that*

$$\pi_t(x_1,y) \leq Ce^{-\lambda_1 t}\varphi_0(x_1)\varphi_0(y) \tag{5.2}$$

for all $y \in \Sigma$ *such that* $|y - x_1| \geq \delta_0$ *and all* $t \geq 0$

Assuming as we may that $\delta_0 < d(x_0, S_{d-1} \setminus \Sigma)$, this is a simple consequence of the parabolic maximum principle applied in the region $\{(x,t)\,;\, t>0, x\in\Sigma\}\setminus\{(x,t)\,;\,|x-x_0|\le\delta_0, 0\le t\le 1\}$ (the two members of (5.2) are $(\partial_t - L)-$harmonic in (y,t) and the first has by definition minimal growth at infinity in $\Sigma\times\mathbb{R}_+$).

Lemma 5.3 *Assume $d=3$ and let T, $\mu>0$, $x_1\in\Sigma$ be given. Then, as $a\to+\infty$,*

$$\int_0^T e^{-\frac{a^2}{4t}-\mu t}\,\pi_t(x_1,y)\,\frac{dt}{\sqrt{t}} = o\Big(\int_T^\infty e^{-\frac{a^2}{4t}-\mu t}\,\pi_t(x_1,y)\,\frac{dt}{\sqrt{t}}\Big) \tag{5.3}$$

uniformly in $y\in\Sigma$, $\|y-x_1\|\ge\delta_0$.

Proof. We may assume $T\ge t_0$. By (5.1) and lemma 5.2, it suffices to prove the relation obtained from (5.3) when the terms $\pi_t(x_1,y)$ are removed from the integrals.

Now $\int_0^T e^{-\frac{a^2}{4t}-\mu t}\,\frac{dt}{\sqrt{t}} \le \frac{1}{2}\sqrt{T}\,e^{-\frac{a^2}{4T}}$ and, if $\theta\ge 1$, $\int_\theta^\infty e^{-\frac{a^2}{4t}-\mu t}\,\frac{dt}{\sqrt{t}} \ge e^{-\frac{a^2}{4\theta}}\int_\theta^\infty e^{-(\mu+\frac{1}{2})t}\,dt$, where for the last inequality we use the observation that $t\mapsto\frac{e^{\frac{t}{2}}}{\sqrt{t}}$ is increasing in $(1,\infty)$.

So $\int_\theta^\infty e^{-\frac{a^2}{4t}-\mu t}\,\frac{dt}{\sqrt{t}} \ge \frac{1}{\nu}\,e^{-\frac{a^2}{4\theta}}\,e^{-\nu\theta}$, where $\nu=\mu+\frac{1}{2}$, and we may conclude since as $a\to+\infty$, $\frac{\nu}{2}\sqrt{T}\,e^{-\frac{a^2}{4}(\frac{1}{T}-\frac{1}{\theta})+\nu\theta}\to 0$ for any fixed $\theta>T$. □

Proof of Theorem 5.1. Assume as we may that $y_j\to y_\infty\in\overline{\Sigma}$. Using the Bañuelos-Davis Theorem and the above lemma and its proof, we have if $y_\infty\ne x_1$ and $j\to\infty$,

$$\begin{aligned}
\sqrt{4\pi}\,G(u,x_1;v_j,y_j) &= \int_0^\infty e^{-\frac{(u+(d-2)t-v_j)^2}{4t}}\,\pi_t(x_1,y_j)\,\frac{dt}{\sqrt{t}} \\
&= e^{\frac{(d-2)(v_j-u)}{2}}\int_0^\infty e^{-\frac{(u-v_j)^2}{4t}}e^{-\frac{(d-2)^2}{4}t}\,\pi_t(x_1,y_j)\,\frac{dt}{\sqrt{t}} \\
&\sim e^{\frac{(d-2)(v_j-u)}{2}}\quad \varphi_0(x_1)\varphi_0(y_j)\int_0^\infty e^{-\frac{(u-v_j)^2}{4t}-[\frac{(d-2)^2}{4}+\lambda_1]t}\,\frac{dt}{\sqrt{t}}.
\end{aligned} \tag{5.4}$$

Thus, for $x_1,x_2\in\Sigma$, we see that $G(u,x_2;v_j,y_j)/G(u,x_1;v_j,y_j)\to\varphi_0(x_2)/\varphi_0(x_1)$ for $j\to\infty$ (assuming first $y_\infty\ne x_1$, $y_\infty\ne x_2$). This shows that a cluster function K of the Martin kernels K_{v_j,y_j}, as $j\to\infty$, is in the form $K(u,x)=g(u)\varphi_0(x)$ and hence must be $K=F_+$.

In fact,

$$\int_0^\infty e^{-\frac{(u-v_j)^2}{4t}-[\frac{(d-2)^2}{4}+\lambda_1]t}\,\frac{dt}{\sqrt{4\pi t}} = \frac{1}{2\sqrt{\mu}}\,e^{-\sqrt{A_j\mu}} \tag{5.5}$$

where $A_j=|u-v_j|^2$, $\mu=\frac{(d-2)^2}{4}+\lambda_1$ (note that the left member of (5.5) is the Green's function in $\mathbb{R}$ with pole at the origin and with respect to $\mathscr{L}=\frac{d^2}{dx^2}-\mu$ evaluated at $\sqrt{A_j}$). It follows that $G(u,x;v_j,y_j)/G(0,x_0;v_j,y_j)\to F_+(u,x)$ when $j\to\infty$.

This shows that $G((u,x_1);(v,y))\sim\frac{1}{\sqrt{(d-2)^2+4\lambda_1}}\varphi_0(x_1)\varphi_0(y)e^{-\alpha_{\max}(v-u)}$ uniformly with respect to y, as $v-u\to+\infty$. Similarly, $G((u,x_1);(v,y))\sim\frac{1}{\sqrt{(d-2)^2+4\lambda_1}}\varphi_0(x_1)\varphi_0(y)e^{-\alpha_{\min}(u-v)}$ as $v-u\to-\infty$, uniformly w. r. to y. □

Proof of Theorem 3.1. The proof of Theorem 3.1 (where $d \geq 2$) is completely similar. In fact as well known (see e.g. [12], [31]) for every compact $K \subset \Sigma$, $e^{\lambda_1 t}\pi_t(x,y) \to \varphi_0(x)\varphi_0(y)$ as $t \to +\infty$, uniformly w. r. to $y \in K$. So an obvious adaptation of the above gives the convergence to F_+. Inequalities 3.1 follow from the fact that now $G(u,x_0;0,x_0) \sim C' e^{\alpha_{\min} u}$ for $u \to +\infty$.

Remark 5.4 The proof also shows that Theorem 5.1 extends to $d \geq 3$ if the base Σ is intrisically ultracontractive with respect to Δ_S, (see [15], [9], [8]). For example, by Bañuelos results in [8] this is the case if for some $p > \frac{n}{2}$ the base Σ is L^p-averaging (that is $\rho_\Sigma \in L^p(\Sigma)$ where $\rho_\Sigma(x)$ is the pseudo-hyperbolic distance to x_0). See [8] for other examples.

6 Examples for $d \geq 4$

In this section we show that for $d \geq 4$ there are cones in $\mathbb{R}^d$ with a host of Martin points at infinity. See sections 6.2 and 6.3 (another example described in section 6.4 shows that these points can be minimal as well as non minimal). This is closely connected with the existence –established by Cranston and McConnell in [14]– of a bounded domain D in $\mathbb{R}^3$ with an h-Brownian motion in D with an infinite lifetime. In fact we use a variant of the construction in [14] section 3.

As before we work with the model of the cone $\mathscr{C}_0(\Sigma)$ given by the cylinder $(\mathscr{C}_Y(\Sigma), L)$.

6.1 Preliminary lemmas

Fix $\lambda_0 > 0$, $d \geq 2$, and consider a cap $B = B(a,r) := \{x \in S;\, |x-a| < r\}$ in $S := S_{d-1}$, $r \leq 1$, with two given points ξ, $\xi' \in \partial B$, symmetric in S with respect to a. Let $T = B(\xi; r/100) \cap \partial B$, $T' = B(\xi', r/100) \cap \partial B$, and let $M_B := \{x \in B;\, \|x-\xi\| = \|x-\xi'\|\}$.

Let Ω be a region in S_{d-1} such that $B \subset \Omega \subset S_{d-1} \setminus (\partial B \setminus T)$, $\overline{T'} \cap \overline{\Omega \setminus B} = \emptyset$. Set $\tilde{\Omega} = \Omega \times \mathbb{R}$.

Lemma 6.1 *Let $v = H_f$ solves $\Delta_S v - \lambda v - \partial_t v = 0$ in $\tilde{\Omega}$ and $v(y,t) = f(y,t)$ in $\partial\tilde{\Omega}$ where $f(y,t)$ is bounded measurable in $\partial\tilde{\Omega}$, nondecreasing in t and $f(y,t) = 0$ for $y \notin T'$. Then, given $\eta \in (0,1)$, there exists $\varepsilon_1 = \varepsilon_1(d,\lambda_0,\eta) > 0$ such that for $0 < \varepsilon \leq \varepsilon_1$ and $0 \leq \lambda \leq \lambda_0$,*

$$H_f(x,t) \leq \int_{\partial\Omega} [\eta\, f(y,t) + (1-\eta)\, f(y, t-\varepsilon r^2)]\, d\mu_x^\Omega(y),\ (x,t) \in M_B \times \mathbb{R} \tag{6.1}$$

Moreover $H_f(x,t)$ is nondecreasing in t. Here μ_x^Ω is the harmonic measure of x in Ω w. r. to $\Delta_S - \lambda I$.

Proof. The last claim follows from the parabolic maximum principle and the translation invariance in t of $\Delta_S - \lambda I - \partial_t$.

To prove the first, observe that by the monotonicity assumption we may assume that $f(y,s) = \varphi(y)$ for $s > t - \varepsilon r^2$ and $f(y,s) = \psi(y)$ when $s \leq t - \varepsilon r^2$. Since the inequality is an identity when $f(y,s)$ is independent of s we may assume $\psi = 0$ and also that $t = \varepsilon r^2$ by time translation invariance.

Then $\int_{\partial\Omega}[\eta\, f(y,\varepsilon r^2)+(1-\eta)f(y,0)]\, d\mu_x^{\Omega}(y) = \eta\Phi(x)$ where Φ solves $\Delta\Phi-\lambda\Phi=0$ in Ω and $\Phi = f$ on $\partial\Omega$. We want to show that $H_f(x,\varepsilon r^2)\le \eta\Phi(x)$ provided $\varepsilon<\frac{1}{8}$ is sufficiently small. Let N be the integer part of $\frac{4}{\varepsilon}$ and set $w(x,s)=H_f(x,s+\varepsilon r^2)-H_f(x,s)$.

By the parabolic Harnack inequalities [26], $Cw(a,r^2(1-k\varepsilon))\ge w(a,\frac{r^2}{2})$ for $1\le k\le N$ and a constant $C=C(d)$. Thus, on summing up, $Nw(a,r^2/2)\le CH_f(a,r^2)\le C\Phi(a)$.

Applying next the parabolic boundary Harnack principle in $\Omega\times\mathbb{R}$ ([17], [21]) to $w(x,s)$ and Φ (viewed as functions of (x,s)) we obtain $w(x,\varepsilon r^2)\le c_1\frac{w(a,r^2/2)}{\Phi(a)}\Phi(x)\le\frac{c_1C}{N}\Phi(x)$ for $x\in M_B$ with $c_1=c_1(C,d,\lambda_0)$. The result follows. □

We will use lemma 6.1 in conjunction with the following lemma.

Lemma 6.2 *Let $\{m_k\}_{1\le k\le N}$ be a finite sequence of probability measures in $\mathbb{R}$ of the form $m_k=\frac{1}{2}\delta_0+\frac{1}{2}\delta_{-a_k}$ for $1\le k\le N$, where $0\le a_k\le 1$. Let L and ε be given positive numbers. There is an $A=A(L,\varepsilon)>0$ such that if $\sum_{k=1}^N a_k\ge A$, the measure $\nu_N=m_1*\cdots*m_N$ satisfies : $\nu_N([-L,0])\le\varepsilon$.*

The probability ν_N is the law of the random variable $Z:=-\sum_{j=1}^N a_jX_j$ if $X_1,\dots,X_N$ are independent random variables such that $P(X_j=0)=P(X_j=1)=\frac{1}{2}$. For $\beta>0$, we have

$$P(-Z\le L)=P(e^{\beta Z}\ge e^{-\beta L})\le e^{\beta L}E(e^{\beta Z})=e^{\beta L}\prod_{j=1}^N E(e^{-\beta a_jX_j})=e^{\beta L}\prod_{j=1}^N(1-\frac{1-e^{-\beta a_j}}{2}).$$

Thus, using $a_k\le 1$, $P(Z\ge -L)\le e^{\beta L}\prod_{j=1}^N(1-\frac{\beta}{2}e^{-\beta}a_j)\le e^{\beta L}\exp(-\frac{\beta}{2}e^{-\beta}\sum_{j=1}^N a_j)$. The lemma follows. □

6.2 A class of cylinders.

We now consider domains $\Sigma\subset S_{d-1}$, $d\ge 4$, similar to examples introduced in [14]: there are disjoint open balls $B_j=B(x_j,r_j)$, $j\ge 0$, in S_{d-1} such that (i) $\sum_{j\ge 0}r_j^2=+\infty$, (ii) $B_j\subset\Sigma$, (iii) for $N\ge 1$, $\Sigma\setminus B_N$ has two components Σ_N^+, Σ_N^- with disjoint closures and $\Sigma_N^-\supset\bigcup_{j<N}B_j$, $\Sigma_N^+\supset\bigcup_{j>N}B_j$, (iv) there are caps T_j, T_j' in ∂B_j, $j\ge 0$, symmetric with respect to x_j, of radius $\rho_j\le r_j/10$ and such that $\Sigma_j^-\cap\overline{B}_j\subset T_j$, $\Sigma_j^+\cap\overline{B}_j\subset T_j'$.

Remark 6.3 There is an $\varepsilon_d>0$ such that whenever $r_j>0$, $j\ge 1$, satisfy $\sum r_j^2=\infty$ and $\sum r_j^{d-1}\le\varepsilon_d$, there exists a corresponding Σ such that moreover: (a) $|x_j-x_k|\ge 4\max\{r_k,r_j\}$ for $j\ne k$, (b) the centers x_j have a limit P_0 in S_{d-1} (c) Σ is locally Lipschitz in $S_{d-1}\setminus\{P_0\}$ and is Dirichlet-regular in S_{d-1}. The proof is left to the reader.

Set $\Sigma_N=\Sigma_N^-\cup B_N$ for $N\ge 1$ and fix $\lambda_0>0$. Let k be a bounded positive solution of $\Delta k-\lambda k=0$, $k=0$ in $\partial\Sigma_N\setminus T_N'$, $0\le\lambda\le\lambda_0$. For $\ell>0$, let $h=h_\ell$ solves: $\partial_t h(x,t)-\Delta_x h(x,t)+\lambda h(x,t)=0$ in $\Sigma_N\times\mathbb{R}$, $h(t,x)=\mathbf{1}_{t\ge-\ell}k(y)$ in $\partial\Sigma_N\times\mathbb{R}$.

Proposition 6.4 *Let ℓ and ε be positive reals and let $x\in\Sigma_q^-$, $1\le q<N$. There is an integer $N_\Sigma(q,\varepsilon,\ell,\lambda_0)$ such that whenever $N\ge N_\Sigma(q,\varepsilon,\ell,\lambda_0)$,*

$$h_\ell(x,0)\le\varepsilon\, k(x). \tag{6.2}$$

Proof. Fix $\eta = \frac{1}{2}$ and a corresponding $\varepsilon_0 > 0$ as given by lemma 6.1. Let $p \in \{q, q+1, \ldots\}$, let $f(y,t)$ be a bounded Borel function in $T'_p \times \mathbb{R}$ which is non decreasing in t and let $v = H_f$ denote the solution of $\Delta_S v - \lambda v - \partial_t v = 0$ in $\Sigma_p \times \mathbb{R}$ with $v = \mathbf{1}_{T'_p \times \mathbb{R}} f$ on the boundary. We show by induction on $n = p - q$, that

$$H_f(x,t) \leq \int_{T'_p} \Big(\int_{-\infty}^{\infty} f(y, t+s)\, d\nu_{q,p}(s) \Big)\, d\mu_x^{\Sigma_p}(y), \;\; x \in \Sigma_q \setminus B_q \tag{6.3}$$

where $\nu_{q,p} = \overset{p}{\underset{j=q}{\star}} (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{-\varepsilon_0 r_j^2})$ and where $\mu_x^{\Sigma_p}$ is the harmonic measure of x in Σ_p w.r. to $\Delta_S - \lambda I$. Denote $\nu_j := \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{-\varepsilon_0 r_j^2}$.

For $n = 0$ this is lemma 6.1. Assuming that the property holds for $n - 1 \geq 0$ and viewing H_f as a solution in $\Sigma_q \times \mathbb{R}$ of a Dirichlet problem for $\Delta_S - \lambda I - \partial_t$ we get by lemma 6.1 and maximum principle

$$\begin{aligned}
H_f(x,t) &\leq \int_{T'_q} \Big(\int_{-\infty}^{+\infty} H_f(y, s+t)\, d\nu_q(s) \Big)\, d\mu_x^{\Sigma_q}(y) \\
&\leq \int_{T'_q} \Big(\int_{-\infty}^{+\infty} \Big[\int_{T'_p} \int_{-\infty}^{+\infty} f(z, s+t+\tau)\, d\nu_{q+1,p}(\tau)\, d\mu_y^{\Sigma_p}(z) \Big] d\nu_q(s) \Big)\, d\mu_x^{\Sigma_q}(y) \\
&= \int_{T'_q} \Big(\int_{T'_p} \Big[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(z, s+t+\tau)\, d\nu_{q+1,p}(\tau)\, d\nu_q(s) \Big] d\mu_y^{\Sigma_p}(z) \Big)\, d\mu_x^{\Sigma_q}(y) \\
&= \int_{T'_q} \Big(\int_{T'_p} \Big[\int_{-\infty}^{+\infty} f(z, t+\theta)\, d\nu_{q,p}(\theta) \Big] d\mu_y^{\Sigma_p}(z) \Big)\, d\mu_x^{\Sigma_q}(y) \\
&= \int_{T'_p} \Big(\int_{-\infty}^{+\infty} f(z, t+\theta)\, d\nu_{q,p}(\theta) \Big] d\mu_x^{\Sigma_p}(z), \;\; x \in \Sigma_q^-,
\end{aligned}$$

where we have used in the second line the induction assumption, in the third the fact that integration with respect to s and integration with respect to z commute and –in the last line– the formula $\mu_x^{\Sigma_p} = \int \mu_y^{\Sigma_p}\, d\mu_x^{\Sigma_q}(y)$ (for $x \in \Sigma_q^-$). This proves (6.3).

From (6.3) it follows that for $x \in \Sigma_q^- = \Sigma_q \setminus B_q$,

$$h_\ell(x,0) \leq \int_{T'_N} \Big(\int_{-\ell}^{0} k(y)\, d\nu_{q,N}(s) \Big)\, d\mu_x^{\Sigma_N}(y) = k(x)\, \nu_{q,N}([-\ell, 0)), \tag{6.4}$$

and the proposition follows from lemma 6.2 and the condition $\sum_{j \geq 1} r_j^2 = +\infty$. □

We now take for k the Green's function $k = G_y^\lambda$ in Σ with pole at some point $y \in \Sigma \setminus \overline{\Sigma}_N$ and with respect to $\Delta_S - \lambda I$. It is easily checked that for $-\ell \leq s \leq 0$, $x \in \Sigma_N$, $h_\ell(s,x) \geq \int_0^{\ell - |s|} e^{-\lambda t}\, \pi_t(x,y)\, dt$. Recall that $k(x) = \int_0^\infty e^{-\lambda t} \pi_t(x,y)\, dt$ and that the parabolic Green's function with pole at (y_0, t_0) in Σ –and w.r. to $\Delta_S - \lambda - \partial_t$– is $\Gamma(x,t;y_0,t_0) : (x,t) \mapsto \mathbf{1}_{t>t_0}\, e^{-\lambda(t-t_0)} \pi_{t-t_0}(x,y_0)$; thus $u(x,s) := \int_{-\infty}^{+\infty} \mathbf{1}_{\{t > -\ell\}}\, \Gamma(x,s;y,t)\, dt = \int_0^{\ell - |s|} e^{-\lambda t}\, \pi_t(x,y)\, dt$ is bounded by $h_\ell(x,s)$ in $\Sigma_N \cap \{-\ell < s < 0\}$ by the parabolic maximum principle.

Thus, the previous result can be read as follows.

Lemma 6.5 *For any sequence $\{y_j\}$ converging in Σ to the end $\mathscr{E}$ of Σ defined by the cuts B_N (a basis for $\mathscr{E}$ is provided by the sets Σ_N^+) and every $\lambda \geq 0$, we have*

$$\lim_{j\to\infty} \frac{\int_0^{t_0} e^{-\lambda t}\, \pi_s(x,y_j)\, ds}{\int_0^{\infty} e^{-\lambda t}\, \pi_s(x,y_j)\, ds} = 0$$

for every fixed $t_0 \geq 0$ and every $x \in \Sigma$.

So, $\lim_{j\to\infty} \frac{\int_{t_0}^{\infty} e^{-\lambda t}\pi_s(x,y_j)\, ds}{\int_0^{\infty} e^{-\lambda t}\pi_s(x,y_j)\, ds} = 1$ for $t_0 \geq 0$, $x \in \Sigma$ and $\lambda \geq 0$.

Lemma 6.6 *Let $\{y_j\}$ be as in lemma 6.5. For $j \to \infty$ and for given reals ρ, ρ' the ratio*

$$\frac{[\int_0^{\infty} t^{-\frac{1}{2}} e^{-\frac{(\rho+(d-2)t)^2}{4t}} \pi_t(x,y_j)\, dt]}{[\int_0^{\infty} t^{-\frac{1}{2}} e^{-\frac{(\rho'+(d-2)t)^2}{4t}} \pi_t(x,y_j)\, dt]} \tag{6.5}$$

converges towards $e^{-\frac{d-2}{2}(\rho-\rho')}$ for each $x \in \Sigma$.

We have $\int_0^{\infty} t^{-\frac{1}{2}} e^{-\frac{(\rho+(d-2)t)^2}{4t}} \pi_t(x,y_j)\, dt = e^{-\frac{(d-2)\rho}{2}} \int_0^{\infty} t^{-\frac{1}{2}} e^{-\frac{\rho^2}{4t}} e^{-t\frac{(d-2)^2}{4}} \pi_t(x,y_j)\, dt$. If we set $\varphi(t) = t^{-\frac{1}{2}} e^{-\frac{\rho^2}{4t}} e^{-t\frac{(d-2)^2}{4}}$, then for every $t_0 > 0$ and for $j \to \infty$

$$\int_0^{t_0} \varphi(t)\pi_t(x,y_j)\, dt = o(\int_{t_0}^{\infty} \varphi(t)\pi_t(x,y_j)\, dt) \tag{6.6}$$

In fact, with $A > \frac{(d-2)^2}{4}$, we have for $t_1 > 0$ large enough

$$\int_{t_1}^{\infty} \varphi(t)\pi_t(x,y_j)\, dt \geq C(t_1,A,d) \int_{t_1}^{\infty} e^{-At}\pi_t(x,y_j)\, dt \tag{6.7}$$

(note that $\frac{\varphi(t)}{\varphi(t_1)} \geq \frac{e^{-At}}{e^{-At_1}}$ for $t \geq t_1$, because $\varphi(t)e^{tA}$ is increasing for t large enough). On the other hand for such a fixed t_1, we have $\int_0^{t_1} \varphi(t)\pi_t(x,y_j)\, dt \leq C'(t_1,A,\rho,d) \int_0^{t_1} e^{-At}\, \pi_t(x,y_j)\, dt$ and (6.6) follows by lemma 6.5.

Since $e^{-\frac{\rho^2}{4t}} \to 1$ for $t \to \infty$ (ρ being fixed) we see now that as $j \to +\infty$,

$$\int_0^{\infty} t^{-\frac{1}{2}} e^{-\frac{\rho^2}{4t}} e^{-t\frac{(d-2)^2}{4}} \pi_t(x,y_j)\, dt \sim \int_0^{\infty} t^{-\frac{1}{2}} e^{-t\frac{(d-2)^2}{4}} \pi_t(x,y_j)\, dt \tag{6.8}$$

Using also this result for ρ' the lemma follows. □

6.3 The Martin boundary of the first example

Using Lemma 6.6 we get a (partial) description of the Martin boundary of the cylinder $\mathscr{C}_Y(\Sigma) = \mathbb{R} \times \Sigma$ with respect to $L := \partial_{uu}^2 + \frac{d-2}{2}\partial_u + \Delta_S$. In particular it will be seen that there are Martin boundary

points related to sequences $(u_j,y_j) \in \mathscr{C}_Y(\Sigma)$ with $\lim u_j = +\infty$ and distinct from the canonical point F_+ given by Theorem 2.11.

Denote $\widetilde{\partial}\Sigma := \partial\Sigma \cap \{\bigcup_{n\geq 1} \partial\Sigma_N^-\}$ the set of points in $\partial\Sigma$ "away from the end $\mathscr{E}$" (see Lemma 6.5). Because $\mathscr{E}$ is defined by a "smooth" system of cuts (the balls B_N or the mediators M_N of T_N and T_N' in B_N) it follows from standard forms of the boundary Harnack principle (see e.g. [4], [6]) that for $\lambda < \lambda_1(\Sigma)$, the end $\mathscr{E}$ is the trace on Σ of the neighborhoods system –in the Martin compactification of $(\Sigma, \Delta_S + \lambda I)$– of a minimal $\Delta_S + \lambda I$–harmonic function $k_{\mathscr{E}}^{\lambda}$ (normalized at x_0) which vanishes on $\widetilde{\partial}\Sigma$ ([4] Théorème 2.5).

For $\alpha \in [\alpha_0, \alpha_{\max})$ –recall $\alpha_0 := -\frac{d-2}{2}$ and $\alpha_{\max} := \frac{-(d-2)+\sqrt{(d-2)^2+4\lambda_1(\Sigma)}}{2}$–, define $K_{\mathscr{E}}^{\alpha}(u,x) := e^{\alpha u} k_{\mathscr{E}}^{\lambda(\alpha)}(x)$, $(u,x) \in \Sigma \times \mathbb{R}$. Here $\lambda(\alpha) = \alpha^2 + (d-2)\alpha$ (thus $\lambda(\alpha_0) = -\frac{(d-2)^2}{4}$). Recall $\mathbf{\Delta}$ denotes the Martin boundary of $\mathscr{C}_Y(\Sigma)$ w.r. to L, and $\mathbf{\Delta}_1$ its minimal part.

Theorem 6.7 *The function $K_{\mathscr{E}}^{\alpha_0}$ is L-minimal (so $K_{\mathscr{E}}^{\alpha_0} = K_\xi$ for some $\xi \in \mathbf{\Delta}_1$) and there exists $\Phi : \Sigma \to \mathbb{R}_+$ going to $+\infty$ along $\mathscr{E}$ and such that $(u_j,y_j) \to \xi$ when $|u_j| \leq \Phi(y_j)$ and $\{y_j\} \to \mathscr{E}$. Moreover for every $\alpha \in (\alpha_0, \alpha_{\max})$, $K_{\mathscr{E}}^{\alpha}$ is minimal L-harmonic in $\mathscr{C}_y(\Sigma)$ and each associated sequence $\{u_j,y_j\}$ in $\mathscr{C}_Y(\Sigma)$ satisfies : (i) $\{y_j\} \to \mathscr{E}$ and (ii) $u_j \to +\infty$.*

Similarly, for $\alpha \in (\alpha_{\min}, \alpha_0)$, the function $K_{\mathscr{E}}^{\alpha}$ is L-minimal in $\mathscr{C}_y(\Sigma)$ and every associated sequence $\{u_j,y_j\}$ satisfies : (i) $\{y_j\}$ converges to $\mathscr{E}$ and (ii) $u_j \to -\infty$.

Proof. (a) By lemma 6.6, if $\{y_j\}$ is a sequence in Σ converging towards $\mathscr{E}$ and such that $\{(0,y_j)\}$ converges to some $\xi \in \mathbf{\Delta}$, the Martin function K_ξ satisfies: $K_\xi(\rho,x)/K_\xi(\rho',x) = e^{-\frac{d-2}{2}(\rho-\rho')}$ for $\rho, \rho' \in \mathbb{R}$. Thus $K_\xi(u,x) = e^{-\frac{d-2}{2}u} k(x)$ where k is independent of u and necessarily a positive solution of $\Delta_S k + \lambda_0 k = 0$ in Σ, $\lambda_0 = \lambda(\alpha_0)$.

Using the John cuts M_n (the mediator in B_n between T_n and T_n'), $n \geq 1$, and Proposition 4.2 we see that h_ξ vanishes on $\tilde{\partial}\Sigma \times \mathbb{R}$. So $k_\xi = 0$ in $\tilde{\partial}\Sigma$ and as mentioned before k must be the $(\Delta_{S_{d-1}} + \lambda_0 I)$-minimal function corresponding to $\mathscr{E}$, i.e $k = k_{\mathscr{E}}^{\lambda(\alpha_0)}$.

(b) It follows that $(u_j,y_j) \to \xi$ when $\{u_j\}$ is bounded and $\{y_j\}$ as before (see Proposition 2.12). And for ρ_y growing sufficiently slowly to $+\infty$ as $y \to \mathscr{E}$, the point (u,y) tends to ξ for $y \to \mathscr{E}$ and $|u| \leq \rho_y$ (the convergence holds in the Martin space of $(\mathscr{C}_Y(\Sigma), L)$). In particular there is no minimal boundary point $\zeta = \lim(u_j,y_j)$, with $y_j \to \mathscr{E}$ of the finite type (i.e. non exponential in the vertical variable) described in Proposition 2.12 (iii).

(c) We now show that h_ξ is minimal L-harmonic in $\mathscr{C}_Y(\Sigma)$ and more generally that for each $\alpha \in [\alpha_0, \alpha_{\max})$ the function $h_0(u,x) = e^{\alpha u} k_0(x)$, where $k_0 = k_{\mathscr{E}}^{\lambda(\alpha)}$, is minimal harmonic for $(\mathscr{C}_Y(\Sigma), L)$. Consider its Martin's disintegration into L-minimal functions. This disintegration does not charge the set of minimal functions in the form $e^{\beta u} k(x)$ with $\beta \neq \alpha$ (this would contradict the behavior of h_0 as $u \uparrow +\infty$ or $u \downarrow -\infty$). Since $h_0 = 0$ on $\mathbb{R} \times \widetilde{\partial}\Sigma$, it is supported by the set of minimal points ζ such that $\zeta = \lim(u_j,y_j)$, $|u_j| \to \infty$ and $y_j \to \mathscr{E}$ (using Proposition 4.2 and (b) above) and hence $K_\zeta(u,x) = e^{\beta u} k(x)$ with k vanishing on $\tilde{\partial}\Sigma$. Thus $k = k_{\mathscr{E}}^{\lambda(\beta)}$ and the disintegration is supported by a minimal point ζ such that $K_\zeta = h_0$.

The remaining assertion clearly follows from Corollary 4.4 and the proof is complete. □

6.4 Proof of Theorem 4.6 and a second example

In this section we construct a second example –based on the first– for which there are for each $\alpha \in (\alpha_{\min}, \alpha_{\max})$ corresponding minimal and non minimal Martin points ζ with $K_\zeta(u,x) = e^{\alpha u} k(x)$. We first establish –in the spirit of [2]– Theorem 4.6.

A. Preliminaries. Denote Σ_+ the hemisphere $\{t \in S_{d-1}; t_1 > 0\}$ of S_{d-1}, $x_0 = (1,0,\dots,0)$ its center, and σ the reflexion $x = (t_1,\dots,t_d) \mapsto (-t_1,t_2,\dots,t_d)$.

Proposition 6.8 *Assume that $\Sigma \supset \Sigma_+$ and denote G the Green's function of $\mathbb{R} \times \Sigma$. Given $\rho > 0$, there is a $C = C(d,\rho)$ such that whenever $y \in \Sigma_+$, $x \in \Sigma$, and $u \le v - 2\rho$*

$$G_{(v,y)}(u,x) := G(u,x;v,y) \le C\, G_{(v,y)}(u,x_0) \tag{6.9}$$

Proof. a) By a known general estimate (see [2], [4]) of the Green's function of a domain containing a C^2-ball (here $\{(t,x) \in \mathscr{C}_Y(\Sigma); |x-x_0|^2 + |t-w|^2 \le 2\}$) -together with Harnack inequalities and elementary observations- we have when $z \in \Sigma_+$, $z' \in \Sigma$, $|w'-w| \ge \rho$:

$$G(w',z';w,z) \le C_{d,\rho}\; G(w \pm \rho, x_0; w, z) \tag{6.10}$$

b) Denote $\tilde\sigma(t,x) = (t,\sigma(x))$. Applying the maximum principle in $\mathbb{R} \times \Sigma_+$ to the functions $G_{(v,y)}$ and $G_{(v,y)} \circ \tilde\sigma$ in $\mathscr{C}_Y(\sigma(\Sigma_+))$ -extend the second by zero outside $\mathscr{C}_Y(\sigma(\Sigma))$- we have

$$G_{(v,y)}(w,z) \le G_{(v,y)}(w,\sigma(z)) \le \sup\{G_{(v,y)}(w,z'); z' \in \Sigma_+\} \tag{6.11}$$

for $z \in \Sigma_- = \Sigma \setminus \Sigma_+$, $w \ne v$. In particular we need only consider $x \in \Sigma_+$ (take $w = u$).

Similarly, if $\mu^t_{(u,x)}$, $t \in \mathbb{R}$, is the L-harmonic measure of (u,x), $u \le t$, in the truncated cylinder $C^t_- := \mathscr{C}_Y(\Sigma) \cap \{(w,z); w < t\}$ we have for $x \in \Sigma_+$

$$\mu^t_{(u,x)} \le \sigma(\mu^t_{(u,x)}) \qquad \text{in } \{t\} \times \sigma(\Sigma_-) \subset \partial C^t_- \tag{6.12}$$

This is because the adjoint Green's function $\overset{*}{G}{}^t_{(u,x)}$ with respect to C^t_- is larger than $\overset{*}{G}{}^t_{(u,x)} \circ \tilde\sigma$ in $C^t_- \cap \{(w,z); z \in \Sigma_+\}$, and $\mu^t_{(u,x)}(t,dz)$ is $\partial_w \overset{*}{G}{}^t_{(u,x)}\, d\sigma_S(z)$.

c) Now write the réduite (w. r. to the cylinder $\mathscr{C}_Y(\Sigma)$) of $G_{(v,y)}$ over $\Sigma^{u+\rho}_+ := \{u+\rho\} \times \Sigma_+$, i.e., $p = R^{\Sigma^{u+\rho}_+}_{G_{(v,y)}}$ ([11], [13]), as a potential G_μ of a positive measure μ on $\Sigma^{u+\rho}_+$. Then

$$\begin{aligned} G_\mu(u,x) &= \int_{\Sigma^{u+\rho}_+} G(u,x;\zeta)\, d\mu(\zeta) \\ &\le C \int_{\Sigma^{u+\rho}_+} G(u,x_0;\zeta)\, d\mu(\zeta) \\ &= C\, G_\mu(u,x_0) \le C\, G_{(v,y)}(u,x_0), \end{aligned} \tag{6.13}$$

using (6.10).

d) Finally $q = G_{(v,y)} - p$ is majorized in $C^{u+\rho}_-$ by the solution h to the Dirichlet problem in $C^{u+\rho}_-$, with the boundary condition $h = G_{(v,y)}$ in $\Sigma_- \times \{u+\rho\}$ and $h = 0$ on the rest of ∂U. By (6.11) and (6.12), $q(u,x) \le p(u,x)$ and by (6.13) $q(u,x) \le C\, G_{(v,y)}(u,x_0)$. □

Corollary 6.9 *Suppose $\Sigma_+ \subset \Sigma$. Let $\{y_j\}$ be a sequence in Σ^+ and let $v_j \to +\infty$ in $\mathbb{R}$. Then $\{(v_j, y_j)\}$ converges to the canonical Martin point at $+\infty$, i.e., $\lim_j K_{(v_j,y_j)} = F_+$.*

This is Theorem 4.6. Similarly of course, if $v_j \to -\infty$, $y_j \in \Sigma_+$, we have $\lim_j K_{(v_j,y_j)} = F_-$.

Proof. If ζ is a cluster Martin point for $\{(v_j,y_j)\}_{j\geq 1}$ it immediately follows from Proposition 6.8 that if h denotes the L-harmonic measure of $\{0\}\times\Sigma$ in $\mathscr{C}_Y(\Sigma)$,

$$K_\zeta(u,x) \leq C K_\zeta(u_0,x_0)\, h(u-u_0,x)$$

for $u \leq u_0$, $x \in \Sigma$. This shows that K_ζ vanishes on $\mathbb{R}\times\partial\Sigma$ and that K_ζ is bounded for $u \leq u_0$. Thus $K_\zeta = F_+$ by Theorem 2.11. □

B. Assuming $d \geq 4$ we construct Σ as follows. We start with the hemisphere Σ_+, a point $P_0 \in \partial\Sigma_+$ and a sequence of points $P_n \in \partial\Sigma_+$ such that $|P_n - P_0| = 4^{-n}$, $n \geq 1$. For each $n \geq 1$, let Ω_n be a domain in S_{d-1}, $\overline{\Omega}_n \subset B_{S_{d-1}}(P_n, \frac{1}{4^{n+1}}) \setminus \overline{\Sigma}_+$ of the type considered in remark 6.3, (starting with disjoint balls $\overline{B}_{n,j}$, $j \geq 1$, in S_{d-1} converging to some point Q_n in $B(P_n, \frac{1}{4^{n+1}}) \setminus \overline{\Sigma}_+$ and such that the sum of the squares of their radii diverges). Let U_n be a region in $B(P_n, \frac{1}{4^{n+1}}) \setminus (\overline{\Sigma}_+ \cup \overline{\Omega}_n)$ such that $\overline{U}_n \cap \overline{\Sigma}_+$ is a closed ball $\overline{\Delta}_n$ in $\partial\Sigma_+$ of center P_n and $\overline{U}_n \cap \overline{\Omega}_n$ a cap $\overline{\Delta}'_n \subset \partial B_{n,1} \cap \partial\Omega_n$ (where Δ_n and Δ'_n are the relative interiors).

The domain Σ is the union of Σ_+, the joining regions U_n and the disks Δ_n and Δ'_n, $n \geq 1$.

Let $\alpha \in (\alpha_{\min}, \alpha_{\max})$ and $n \geq 1$. As before, by [4] there is a unique positive $(\Delta + \lambda(\alpha)I)$-harmonic function k_n in Σ vanishing in $\partial\Sigma \setminus \{Q_n\}$ and such that $k_n(x_0) = 1$ (x_0 is the center of the hemisphere Σ). Moreover k_n is minimal Δ-harmonic in Σ and by Theorem 6.7 the function $K^{(\alpha)}_{Q_n}(u,x) = e^{\alpha u} k_n(x)$ is minimal L-harmonic in $\mathscr{C}_Y(\Sigma)$. Denote $h^{(\alpha)}(u,x) = e^{\alpha u} k(x)$ the similar L-harmonic function in $\mathscr{C}_Y(\Sigma)$ with a pole at P_0.

Proposition 6.10 *The function $h^{(\alpha)}$ is a non minimal Martin function for $(\mathscr{C}_Y(\Sigma), L)$.*

Proof. It follows from standard arguments that the k_n vanish uniformly on the boundary of Σ away from P_0 (as $n \to \infty$), so that $k = \lim k_n$. Thus $h^{(\alpha)} = \lim K^{(\alpha)}_{Q_n}$ and $h^{(\alpha)}$ is an L-Martin function in $\mathscr{C}_Y(\Sigma)$ associated to a point $\zeta \in \mathbf{\Delta}$. If $\alpha \geq \alpha_0$ (resp. $\alpha \leq \alpha_0$), there is a sequence $\{(v_j,y_j)\}$ with $y_j \in \Omega_j$, $\lim v_j = +\infty$ (resp. $\lim v_j = -\infty$) converging to ζ.

By Corollary 6.9, the point ζ –as a Martin boundary point– is not in the closure of $\mathbb{R}\times\Sigma_+$. Thus every sufficiently small neighborhood V of ζ meet $\mathbb{R}\times\Omega_n$ for all large n, but not $\mathbb{R}\times\Sigma_+$. And $V \cap \mathscr{C}_Y(\Sigma)$ is not connected. But (by a general property) each neighborhood of a *minimal* Martin boundary point contains another whose trace in $\mathscr{C}_Y(\Sigma)$ is connected (see e.g. [29] p. 223). Hence ζ is not minimal. □

7 Extensions to more general cylinders

The argument in sections 2 and 3 can be extended to more general second order elliptic operators in cylinders. We describe here a simple generalization and state the corresponding results. Assume Σ

is a *relatively compact* region in a C^1 Riemannian manifold M of dimension $d-1$ ($d \geq 2$) equipped with a second order elliptic operator L_M in the form

$$L_M(\varphi) = \operatorname{div}(A\nabla\varphi) + B.\nabla\varphi + \gamma\varphi \tag{7.1}$$

where A is a measurable, bounded and uniformly elliptic section of $\operatorname{End}(T(M)$, B a bounded measurable vector field in M and γ a nonpositive bounded measurable function in M (ref. [33]). We also assume that $M \setminus \Sigma$ is non polar. Thus L_M admits a Green function in Σ.

We consider now a differential operator in the cylinder $\mathscr{C}_Y(\Sigma) = \mathbb{R} \times \Sigma$ which is a direct sum $L = L_{\mathbb{R}} \oplus L_M$ where $L_{\mathbb{R}} = \frac{d^2}{du^2} + b\frac{d}{du}$ is translation invariant in $\mathbb{R}$ (i.e., b is a real constant). Again we fix some $x_0 \in \Sigma$ and take $(0, x_0)$ as the normalization point in $\mathscr{C}_Y(\Sigma)$ for the Martin functions.

We define $\lambda_1 := \lambda_1(L_M;\Sigma)$ as the supremum of all real t such that $L_M + tI$ admits a Green's function in Σ (or such that the cone of nonnegative $L_M + tI$-superharmonic functions has a dimension > 1). It is well known that $0 < \lambda_1 < \infty$ and that for $t := \lambda_1$ all nonnegative $(L_M + tI)$–superharmonic functions in Σ are proportional to the unique (up to scalar multiplication) positive $L_M + tI$ positive solution φ_0 in Σ. This solution φ_0 is bounded, vanishes in the weak sense on the boundary $\partial\Sigma$ and $\varphi_0 \in H_0^1(\Sigma)$. As also well-known we have similar properties for the formal adjoint operator L_M^*, and moreover $\lambda_1(L_M^*;\Sigma) = \lambda_1(L_M;\Sigma)$. We denote φ_0^* a positive $(L^* + \lambda_1 I)$-superharmonic function in Σ. Again φ_0^* is unique up to multiplication by a constant, vanishes on $\partial\Sigma$ and is $(L_M^* + \lambda_1 I)$-harmonic in Σ.

Repeating the argument used in section 2 we obtain a similar description of the L-minimal Martin function associated with the end $u \to +\infty$ in $\mathscr{C}_Y(\Sigma)$.

Proposition 7.1 *If K is a minimal L-Martin function in $\mathscr{C}_Y(\Sigma) = \mathbb{R} \times \Sigma$ associated to a sequence (u_j, x_j) with $u_j \to +\infty$, $x_j \in \Sigma$, then K is in the form*

$$K(u,x) = e^{\alpha u} K(0,x), \quad (u,x) \in \mathbb{R} \times \Sigma \tag{7.2}$$

for some $\alpha \geq -\frac{b}{2}$ and $s(x) = K(0,x)$ is a proper function in Σ: $L_M s + \lambda s = 0$, $\lambda = \alpha^2 + b\alpha$. So $\alpha = \frac{-b+\sqrt{b^2+4\lambda}}{2}$ and $-\frac{b^2}{4} \leq \lambda \leq \lambda_1$; moreover s is $(L_M + \lambda I)$–minimal in Σ.

As before there is a natural bijection $K \mapsto \tilde{K}$ between the set $\mathbf{\Delta}^{+\infty}$ of the Martin function arising from some sequence (v_j, y_j) in $\mathscr{C}_Y(\Sigma)$ with $\lim v_j = +\infty$ and the analogue set $\mathbf{\Delta}^{-\infty}$ (related to the condition $\lim v_j = -\infty$) by letting $\tilde{K}(u,x) = e^{-bu}K(-u,x)$.

Theorem 2.13 can also be extended to the present framework, but a slight modification is required in the proof. Set $F_+(u,x) = e^{\alpha_{\max} u}\frac{\varphi_0(x)}{\varphi_0(x_0)}$ and $F_-(u,x) = e^{\alpha_{\min} u}\frac{\varphi_0(x)}{\varphi_0(x_0)}$ where $\alpha_{\max} := \frac{-b+\sqrt{b^2+4\lambda_1}}{2}$ and $\alpha_{\min} := \frac{-b-\sqrt{b^2+4\lambda_1}}{2}$.

Theorem 7.2 *If f is L-harmonic in $\mathscr{C}_Y(\Sigma)$ and vanishes in weak sense on $\mathbb{R} \times \partial\Sigma$, then f is a linear combination of F_+ and F_-. In particular, if moreover $\liminf_{u\to-\infty} e^{\alpha_{\min} u} f(u,x_0) = 0$ then f is proportional to F_+. Thus F_+ and F_- are L-minimal in $\mathscr{C}_Y(\Sigma)$.*

As in the proof of Theorem 2.13, we may reduce ourselves to show the following. A function F in $\mathscr{C}_Y(\Sigma)$ which vanishes on $\mathbb{R} \times \partial\Sigma$ and which is in the form $F(u,x) = \int_{A'} K_\zeta(u,x)\, d\mu(\zeta)$ where

$A' = \{\zeta \in \mathbf{\Delta}^{+\infty} \cap \mathbf{\Delta}_1\,;\, -\frac{b}{2} \le \alpha(\zeta) \le \alpha_1\}$, $\alpha_1 < \alpha_{\max}$, and where μ is a finite positive Borel measure on A, must be the zero function. Denote $\lambda_1' = \alpha_1^2 + b\alpha_1$.

As before the function $\varphi(x) = \int_{A'} K_\zeta(0,x)\, d\mu(\zeta)$ is positive superharmonic with respect to $L_0 = L_M + \lambda_0 I$, $\lambda_0 = -\frac{b^2}{4}$ and vanishes in the weak sense on $\partial\Sigma$. The measure $-L_0(\varphi)$ is given by the density $\psi(x) = \int_A (\lambda - \lambda_0)\, k_\zeta^\lambda(x)\, d\mu_f(x)$. It follows that φ is the L_0-Green's potential in Σ of ψ and again, $G_\Sigma^{L_0}(\psi) = \varphi \ge \frac{1}{\lambda_1' - \lambda_0}\psi$ in Σ, where $G_\Sigma^{L_0}$ is Green's function in Σ w.r. to L_0. To conclude we then slightly modify the argument in section 2 using now the minimal heat semi-group P_t generated by L in Σ.

$$\int_\Sigma G_\Sigma^{L_0}(\psi)\, \varphi_0^*\, d\sigma_M = \int_0^{+\infty} \int_\Sigma e^{\lambda_0 t} P_t(\psi)\, \varphi_0^*\, d\sigma_M\, dt = \int_0^{+\infty} \int_\Sigma e^{-(\lambda_1 - \lambda_0)t}\, \psi\, \varphi_0^*\, d\sigma\, dt$$

because $P_t^*(\varphi_0) = e^{-\lambda_1 t}\varphi_0^*$. Thus $\int_\Sigma G_\Sigma^{L_0}(\psi)\, \varphi_0^*\, d\sigma_M = \frac{1}{\lambda_1 - \lambda_0} \int \psi\, \varphi_0^*\, d\sigma_M$.

But on the other hand from $G_\Sigma^{L_0}(\psi) \ge \frac{1}{\lambda_1' - \lambda_0}\psi$ it follows that $\int_\Sigma G_\Sigma^{L_0}(\psi)\, \varphi_0^*\, d\sigma_M$ is larger than $\frac{1}{\lambda_1' - \lambda_0} \int \psi\, \varphi_0^*\, d\sigma$. Thus $\int \psi\, \varphi_0^*\, d\sigma = 0$ and $\psi = 0$ in Σ. So $\varphi = 0$. □

Corollary 7.3 *Every positive L-harmonic function $f(u,x)$ on $\mathscr{C}_Y(\Sigma)$ vanishing (in the weak sense) on $\mathbb{R} \times \partial\Sigma$ and such that $\limsup_{u \to -\infty} f(u,x_0) < \infty$ –$x_0 \in \Sigma$– is proportional to F_+.*

Theorems 3.1 and 5.1 extend as follows.

Theorem 7.4 *Let $\xi_j = (v_j, y_j)$, $j \ge 1$, be a sequence of points in $\mathscr{C}_Y(\Sigma)$ such that $v_j \to +\infty$. If $d \le 3$ or if $d \ge 4$ and $\{y_j\}$ is relatively compact in Σ, the functions $K_{\xi_j}(u,x)$ converge to $K_{\zeta_\infty}(u,x) := e^{\alpha_\Sigma u}\, \psi_0(x)$. In particular when $d \le 3$, ζ_∞ is the only Martin point at infinity.*

The proofs are the same as above in section 5 using the natural extensions of (5.1) (with y in a relatively compact subset of Σ when $d \ge 4$) to our present setting. Denote $\{\pi_t\}$ the heat semi-group generated by the L in Σ and as above φ_0^* any positive eigenfunction of the adjoint elliptic operator $-L^*$ in Σ for the eigenvalue λ_1. Then we have :

(i) if $d = 3$, there is a $t_0 > 0$ and a function $C : [t_0, \infty) \to (1, +\infty)$ such that $\lim_{t\to\infty} C(t) = 1$ and –if $C_0 = (\int_\Sigma \varphi_0(y)\varphi_0^*(y)\, d\sigma(y))^{-1}$,

$$C_0 C(t)^{-1}\, e^{-\lambda_1 t}\, \varphi_0(x_0)\, \varphi_0^*(y) \le \pi_t(x_0,y) \le C_0 C(t)\, e^{-\lambda_1 t}\, \varphi_0(x_0)\, \varphi_0^*(y),\ \ t \ge t_0 \qquad (7.3)$$

for all $y \in \Sigma$. The proof in [10] Theorem 1 for the Laplacian can be adapted after one shows that the Cranston-McConnell inequalities [14] [7] (see also [9]) hold for all subdomains ω of Σ: i.e., there is a constant $C = C(\Sigma)$ such that for every ω and every positive L-harmonic function h in ω one has $G^\omega(h) \le C\,|\omega| h$.

(ii) for all $d \ge 3$, it is well-known that (7.3) holds provided y is restricted to a relatively compact subset A of Σ (see [30] Theorem 1.2 (iii) with a class of elliptic operators slightly different from ours, see also [31], [32]).

Let us finally also mention that the results in sections 4.1 and 4.2 extend to the present setting if we restrict to John conditions with $N = 1$, where N is the number of poles –recall the needed results in

[6] require for $N \geq 2$ the symmetry of the underlying operator. When $B = 0$ and A is symmetric, the restriction $N = 1$ can be removed since $L = L_{\mathbb{R}} + \partial^2_{uu} + b\partial_u$ is then symmetric with respect to the reference measure $\mu(du,dx) = e^{bu}\, du\, d\sigma_M(x)$ (i.e. L is symmetric in $M \times \mathbb{R}$ equipped with the riemannian metric $g_{(u,x)}(du,dx) = e^{bu}\,(du)^2\, g_M(dx)$).

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