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# Closed Set Based Discovery of Small Covers for Association Rules

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*ABSTRACT. In this paper, we address the problem of the usefulness of the set of discovered association rules. This problem is important since real-life databases yield most of the time several thousands of rules with high confidence. We propose new algorithms based on Galois closed sets to reduce the extraction to small covers (or bases) for exact and approximate rules, adapted from lattice theory and data analysis domain. Once frequent closed itemsets – which constitute a generating set for both frequent itemsets and association rules – have been discovered, no additional database pass is needed to derive these bases. Experiments conducted on real-life databases show that these algorithms are efficient and valuable in practice.*

*RÉSUMÉ. Nous traitons dans cet article du problème de l'utilisabilité des règles d'association découvertes. Ce problème est primordial car, dans la plupart des cas, les jeux de données réels conduisent à plusieurs milliers de règles d'association dont la mesure de confiance est élevée. Nous proposons de nouveaux algorithmes, basés sur l'utilisation de la fermeture de la connexion de Galois, permettant d'extraire des couvertures réduites (ou bases) pour les règles d'association exactes et partielles, adaptées du domaine de la théorie des treillis et de l'analyse de données. L'approche proposée consiste à extraire les itemsets fermés fréquents – qui constituent un ensemble générateur pour les itemsets fréquents et les règles d'association – et générer ensuite ces bases sans autre accès à la base de données. Les expérimentations menées sur des bases de données réelles montrent l'efficacité et l'utilité de ces algorithmes.*

*KEYWORDS: data mining, Galois closure operator, frequent closed itemsets, bases for association rules, algorithms.*

*MOTS-CLÉS : extraction de connaissances dans les bases de données, fermeture de la connexion de Galois, itemsets fermés fréquents, bases pour les règles d'association, algorithmes.*

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## 1. Introduction and Motivation

Data mining has been extensively addressed for the last years, specially the problem of discovering association rules. The aim when discovering association rules is to exhibit relationships between data items (or attributes) and compute the precision of each relationship in the database. Usual precision measures are support and confidence [AGR 93] that point the proportion of database transactions (or objects) upholding each rule out. When an association rule has support and confidence exceeding some user-defined minimum thresholds, the rule is considered as relevant and the extracted knowledge would likely be used for supporting decision making. A classical example of association rules fits in the context of market basket data analysis and highlights a particular feature in customers behavior: 80% of customers who buy cereals and sugar also buy milk and 20% of customers buy both three items.

Since the problem was stated [AGR 93], various approaches have been proposed for an increased efficiency of rule discovery [AGR 94, BAY 98, BRI 97b, LIN 98, PAS 98, PAS 99b, PAS 99a, SAV 95, TOI 96, ZAK 97]. However, fully taking advantage of exhibited knowledge means capabilities to handle such a knowledge. In fact, by using a synthetic dataset containing 100,000 objects, each of which encompassing around 10 items, our experiments yield more than 16,000 rules with confidence outcoming 90%. The problem is much more critical when collected data is highly correlated or dense, like in statistical or medical databases. For instance, when applied to a census dataset of 10,000 objects, each of which characterized by values of 73 attributes, experiments result in more than 2,000,000 rules with support and confidence outcoming 90%.

Thus the talked issue could be rephrased as follows: which relevant knowledge can be learned from several thousands of rules highly redundant? Which aid could be offered to users for handling countless rules and focusing on useful ones? Before explaining how our approach answers the previous questions, let us examine proposed solutions for meeting such needs.

### 1.1. *Related Work: an Outline*

Among approaches addressing the described issue, two main trends can be distinguished. The former provides users with mechanisms for filtering rules. In [BAR 97, KLE 94], the user defines templates, and rules not matching with them are discarded. In [NG 98, SRI 97], boolean operators are introduced for selecting rules including (or not) given items. In [SRI 96, TOI 95], methods for pruning rules with weak measures of improvement, that characterize the difference between supports and confidences of a rule and its sub-rules, i.e. with smaller antecedent and same consequent, are proposed. A similar approach expanded with boolean operators for selecting rules is proposed in [BAY 99b]. In [MEO 96], an SQL-like operator called MINE RULE, allowing the specification of general extraction criteria, is defined. The use of the user's domain knowledge for selecting unexpected rules, using measures of distance

between rules called deviation measures, is proposed in [HEC 96, PIA 91, SIL 96]. In [BAY 99a], the proposed approach consists in selecting rules with maximal antecedent, called A-maximal rules, that are rules for which the addition of an item to the antecedent reduces the population concerned by the rule. The quoted approaches operate “a posteriori”, i.e. once huge amount of rules are extracted, querying facilities make it possible to handle rule subsets selected according to the user preferences.

In contrast, the second trend addresses the problem with an “a priori” vision, by attempting to minimize the number of exhibited rules. In [HAN 95, SRI 95], information about taxonomies are used to define criteria of interest which apply for pruning redundant rules. The use of statistical measures, such as Pearson’s correlation, chi-squared test, conviction, interest, entropy gain, gini or lift, instead of the confidence measure is studied in [BRI 97a, MOR 98, SIL 98].

## 1.2. Contribution: an Overview

The approach presented in this paper belongs to the second trend since it aims to extract not all possible rules but a subset called small cover or basis for association rules. When computing such a basis, redundant rules are discarded since they do not vehicule relevant knowledge. Such a pruning operation is a key-step during rule extraction, and significantly reduces the resulting set. Moreover, since rules unexpected by the user are important [LIU 97, SIL 96], presenting a list of rules covering all the frequent items in the dataset is also needed. The approach proposed in this paper meets this requirement.

First, using the closure operator of the Galois connection [BIR 67], we characterize frequent closed itemsets introduced in [PAS 98]. Then, we show that frequent closed itemsets represent a generating set for both frequent itemsets and association rules. The underlying theorem states the foundations of our approach since it makes it possible to generate the bases from frequent closed itemsets by avoiding handling of large sets of rules. We propose two new algorithms: the former achieves frequent closed itemsets from frequent itemsets without accessing the dataset, and the latter, called Apriori-Close, extends the Apriori algorithm [AGR 94] by discovering simultaneously frequent itemsets and frequent closed itemsets without additional execution time.

Then, using the frequent closed itemsets and the pseudo-closed itemsets defined by Duquenne and Guigues in lattice theory [BUR 98, DUQ 86], we define the *Duquenne-Guigues basis for exact association rules* (rules with a 100% confidence). Rules in this basis are non-redundant exact rules. Besides, using the frequent closed itemsets and results proposed by Luxenburger in lattice theory [LUX 91], we define the *proper basis* and the *structural basis for approximate association rules*. The proper basis is a small set containing non-redundant approximate association rules. The structural basis can be viewed as an abstract of all approximate rules that hold and can be useful when the proper basis is large. We propose three algorithms intended for yielding

these three bases. Using the set of frequent closed itemsets, generating the evoked bases is performed without any access to the dataset.

An algorithm discovering closed and pseudo-closed itemsets has been proposed in [GAN 91] and implemented in CONIMP [BUR 98]. However, this algorithm does not consider the support of itemsets and, since it works only in main memory, it cannot be applied when the number of objects exceeds some hundreds and the number of items some tens. From the results presented in [LUX 91], no algorithm was proposed. In [PAS 98, PAS 99a], the association rule framework based on the Galois connection is defined. Fitting in this groundwork, two efficient algorithms that discover frequent closed itemsets for association rules are defined: the Close algorithm [PAS 98, PAS 99a] for correlated data and the A-Close algorithm [PAS 99b] for weakly correlated data. The work presented in this paper differs from [PAS 98, PAS 99b, PAS 99a] in the following points:

1. It shows that frequent closed itemsets constitute a generating set for frequent itemsets and association rules.
2. It extends the Apriori algorithm and algorithms for discovering maximal frequent itemsets to generate frequent closed itemsets.
3. It adapts the Duquenne-Guigues basis and Luxenburger results for exact and partial implications to the context of association rules. This adaptation is based on 1. (generating set).
4. It presents new algorithms for generating bases for exact and approximate association rules using frequent closed itemsets.
5. It shows that the algorithms proposed are efficient for both improving the usefulness of extracted association rules and decreasing the execution time of the association rule extraction.

### **1.3. Paper Organization**

In Section 2, we present the association rule framework based on the Galois connection. Section 3 addresses the concept of basis for both exact and approximate association rules. New algorithms for discovering frequent and frequent closed itemsets are described in Section 4 and the following section presents algorithms computing the bases for association rules from the frequent closed itemsets. Experimental results achieved from various datasets are given in Section 6. Finally, as a conclusion, we evoke further work in Section 7.

## **2. Association Rule Framework**

In this section, we present the association rule framework based on the Galois connection, primarily introduced in [PAS 98].

**Definition 1 (Data mining context).** A data mining context<sup>1</sup> is defined as  $\mathcal{D} = (\mathcal{O}, \mathcal{I}, \mathcal{R})$ , where  $\mathcal{O}$  and  $\mathcal{I}$  are finite sets of objects and items respectively.  $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{I}$  is a binary relation between objects and items. Each couple  $(o, i) \in \mathcal{R}$  denotes the fact that the object  $o \in \mathcal{O}$  is related to the item  $i \in \mathcal{I}$ .

Depending on the target system, a data mining context can be a relation, a class, or the result of an SQL/OQL query.

**Example 1.** An example data mining context  $\mathcal{D}$  consisting of 5 objects (identified by their OID) and 5 items is illustrated in Table 1.

OID	Items			
1	A	C	D	
2	B	C	E	
3	A	B	C	E
4	B	E		
5	A	B	C	E

**Table 1.** The example data mining context  $\mathcal{D}$ .

**Definition 2 (Galois connection).** Let  $\mathcal{D} = (\mathcal{O}, \mathcal{I}, \mathcal{R})$  be a data mining context. For  $O \subseteq \mathcal{O}$  and  $I \subseteq \mathcal{I}$ , we define:

$$f: 2^{\mathcal{O}} \rightarrow 2^{\mathcal{I}} \quad g: 2^{\mathcal{I}} \rightarrow 2^{\mathcal{O}}$$

$$f(O) = \{i \in \mathcal{I} \mid \forall o \in O, (o, i) \in \mathcal{R}\} \quad g(I) = \{o \in \mathcal{O} \mid \forall i \in I, (o, i) \in \mathcal{R}\}$$

$f(O)$  associates with  $O$  the items common to all objects  $o \in O$  and  $g(I)$  associates with  $I$  the objects related to all items  $i \in I$ . The couple of applications  $(f, g)$  is a Galois connection between the power set of  $\mathcal{O}$  ( $2^{\mathcal{O}}$ ) and the power set of  $\mathcal{I}$  ( $2^{\mathcal{I}}$ ). The following properties hold for all  $I, I_1, I_2 \subseteq \mathcal{I}$  and  $O, O_1, O_2 \subseteq \mathcal{O}$ :

$$(1) I_1 \subseteq I_2 \Rightarrow g(I_1) \supseteq g(I_2) \quad (1') O_1 \subseteq O_2 \Rightarrow f(O_1) \supseteq f(O_2)$$

$$(2) O \subseteq g(I) \iff I \subseteq f(O)$$

**Definition 3 (Frequent itemsets).** Let  $I \subseteq \mathcal{I}$  be a set of items from  $\mathcal{D}$ . The support count of the itemset  $I$  in  $\mathcal{D}$  is:

$$\text{supp}(I) = \frac{|g(I)|}{|\mathcal{O}|}$$

$I$  is said to be frequent if the support of  $I$  in  $\mathcal{D}$  is at least  $\text{minsupp}$ . The set  $L$  of frequent itemsets in  $\mathcal{D}$  is:

$$L = \{I \subseteq \mathcal{I} \mid \text{supp}(I) \geq \text{minsupp}\}$$

1. By extension, we will call dataset a data mining context.

**Definition 4 (Association rules).** An association rule is an implication between two itemsets, with the form  $I_1 \rightarrow I_2$  where  $I_1, I_2 \subseteq \mathcal{I}$ ,  $I_1, I_2 \neq \emptyset$  and  $I_1 \cap I_2 = \emptyset$ .  $I_1$  and  $I_2$  are called respectively the antecedent and the consequent of the rule. The support  $supp(r)$  and confidence  $conf(r)$  of an association rule  $r : I_1 \rightarrow I_2$  are defined using the Galois connection as follows:

$$supp(r) = \frac{|g(I_1 \cup I_2)|}{|\mathcal{O}|}, \quad conf(r) = \frac{supp(I_1 \cup I_2)}{supp(I_1)}$$

Association rules holding in the context are those that have support and confidence greater than or equal to the  $minsupp$  and  $minconf$  thresholds respectively. We define the set  $AR$  of association rules holding in  $\mathcal{D}$  given  $minsupp$  and  $minconf$  thresholds as follows:

$$AR = \{r : I_1 \rightarrow I_2 \setminus I_1 \mid I_1 \subset I_2 \subseteq \mathcal{I} \wedge supp(I_2) \geq minsupp \\ \wedge conf(r) \geq minconf\}$$

If  $conf(r)=1$  then  $r$  is called an exact association rule or implication rule, otherwise  $r$  is called approximate association rule.

**Example 2.** Exact and approximate association rules extracted from  $\mathcal{D}$  for  $minsupp = 2/5$  and  $minconf = 1/2$  are given in Table 2.

Exact rule	Supp	Approximate rule	Supp	Conf	Approximate rule	Supp	Conf
ABC $\Rightarrow$ E	2/5	BCE $\rightarrow$ A	2/5	2/3	B $\rightarrow$ AE	2/5	2/4
ABE $\Rightarrow$ C	2/5	AC $\rightarrow$ BE	2/5	2/3	E $\rightarrow$ AB	2/5	2/4
ACE $\Rightarrow$ B	2/5	BE $\rightarrow$ AC	2/5	2/4	A $\rightarrow$ CE	2/5	2/3
AB $\Rightarrow$ CE	2/5	CE $\rightarrow$ AB	2/5	2/3	C $\rightarrow$ AE	2/5	2/4
AE $\Rightarrow$ BC	2/5	AC $\rightarrow$ B	2/5	2/3	E $\rightarrow$ AC	2/5	2/4
AB $\Rightarrow$ C	2/5	BC $\rightarrow$ A	2/5	2/3	B $\rightarrow$ CE	3/5	3/4
AB $\Rightarrow$ E	2/5	BE $\rightarrow$ A	2/5	2/4	C $\rightarrow$ BE	3/5	3/4
AE $\Rightarrow$ B	2/5	AC $\rightarrow$ E	2/5	2/3	E $\rightarrow$ BC	3/5	3/4
AE $\Rightarrow$ C	2/5	CE $\rightarrow$ A	2/5	2/3	A $\rightarrow$ B	2/5	2/3
BC $\Rightarrow$ E	3/5	BE $\rightarrow$ C	3/5	3/4	B $\rightarrow$ A	2/5	2/4
CE $\Rightarrow$ B	3/5	A $\rightarrow$ BCE	2/5	2/3	C $\rightarrow$ A	3/5	3/4
A $\Rightarrow$ C	3/5	B $\rightarrow$ ACE	2/5	2/4	A $\rightarrow$ E	2/5	2/3
B $\Rightarrow$ E	4/5	C $\rightarrow$ ABE	2/5	2/4	E $\rightarrow$ A	2/5	2/4
E $\Rightarrow$ B	4/5	E $\rightarrow$ ABC	2/5	2/4	B $\rightarrow$ C	3/5	3/4
		A $\rightarrow$ BC	2/5	2/3	C $\rightarrow$ B	3/5	3/4
		B $\rightarrow$ AC	2/5	2/4	C $\rightarrow$ E	3/5	3/4
		C $\rightarrow$ AB	2/5	2/4	E $\rightarrow$ C	3/5	3/4
		A $\rightarrow$ BE	2/5	2/3			

**Table 2.** Association rules extracted from  $\mathcal{D}$  for  $minsup = 2/5$  and  $minconf = 1/2$ .

### 3. Bases for Association Rules

In this section, we first demonstrate that the frequent closed itemsets constitute a generating set for frequent itemsets and association rules. Then, we characterize the *Duquenne-Guigues basis for exact association rules* and the *proper* and *structural bases for approximate association rules*. These bases are adaptations of the bases defined by Duquenne and Guigues [DUQ 86] and Luxenburger [LUX 91] in Lattice Theory and Data Analysis to the context of association rules. This adaptation is not trivial since additional constraints related to the specificity of association rules have to be considered. Theorem 2 states that the union of the Duquenne-Guigues basis for exact association rules and the proper basis or the structural basis for approximate association rules constitutes a basis for all valid association rules. The proof of this theorem is straightforward from Theorem 1 and [DUQ 86, LUX 91]. Interested readers could refer to [BIR 67, GAN 99, WIL 92] for further details on closed sets.

#### 3.1. Generating Set

**Definition 5 (Galois closure operators).** *The operators  $h = f \circ g$  in  $2^{\mathcal{I}}$  and  $h' = g \circ f$  in  $2^{\mathcal{O}}$  are Galois closure operators<sup>2</sup>. Given the Galois connection  $(f, g)$ , the following properties hold for all  $I, I_1, I_2 \subseteq \mathcal{I}$  and  $O, O_1, O_2 \subseteq \mathcal{O}$  [BIR 67]:*

*Extension :* (3)  $I \subseteq h(I)$  (3')  $O \subseteq h'(O)$   
*Idempotency :* (4)  $h(h(I)) = h(I)$  (4')  $h'(h'(O)) = h'(O)$   
*Monotonicity :* (5)  $I_1 \subseteq I_2 \Rightarrow h(I_1) \subseteq h(I_2)$  (5')  $O_1 \subseteq O_2 \Rightarrow h'(O_1) \subseteq h'(O_2)$

**Definition 6 (Frequent closed itemsets).** *An itemset  $I \subseteq \mathcal{I}$  in  $\mathcal{D}$  is a closed itemset iff  $h(I) = I$ . A closed itemset  $I$  is said to be frequent if the support of  $I$  in  $\mathcal{D}$  is at least  $minsupp$ . The smallest (minimal) closed itemset containing an itemset  $I$  is  $h(I)$ , the closure of  $I$ . The set  $FC$  of frequent closed itemsets in  $\mathcal{D}$  is defined as follows:*

$$FC = \{I \subseteq \mathcal{I} \mid I = h(I) \wedge \text{supp}(I) \geq \text{minsupp}\}$$

**Example 3.** A frequent closed itemset is a maximal set of items common to a set of objects, for which support is at least  $minsupp$ . The frequent closed itemsets in the context  $\mathcal{D}$  for  $minsupp=2/5$  are presented in Table 3. The itemset  $BCE$  is a frequent closed itemset since it is the maximal set of items common to the objects  $\{2, 3, 5\}$ . The itemset  $BC$  is not a frequent closed itemset since it is not a maximal set of items common to some objects: all objects in relation with the items  $B$  and  $C$  (objects 2, 3 and 5) are also in relation with the item  $E$ .

Hereafter, we demonstrate that the set of frequent closed itemsets with their support is the smallest collection from which frequent itemsets with their support and association rules can be generated (it is a generating set).

2. Here, we use the following notation:  $f \circ g(I) = f(g(I))$  and  $g \circ f(O) = g(f(O))$ .



Frequent closed itemset	Support
$\{\emptyset\}$	5/5
$\{C\}$	4/5
$\{AC\}$	3/5
$\{BE\}$	4/5
$\{BCE\}$	3/5
$\{ABCE\}$	2/5

**Table 3.** Frequent closed itemsets extracted from  $\mathcal{D}$  for  $\text{minsupp} = 2/5$ .

**Lemma 1.** [PAS 98, PAS 99a] The support of an itemset  $I$  is equal to the support of the smallest closed itemset containing  $I$ :  $\text{supp}(I) = \text{supp}(h(I))$ .

**Lemma 2.** [PAS 98, PAS 99a] The set of maximal frequent itemsets  $M = \{I \in L \mid \nexists I' \in L \text{ where } I \subset I'\}$  is identical to the set of maximal frequent closed itemsets  $MC = \{I \in FC \mid \nexists I' \in FC \text{ where } I \subset I'\}$ .

**Theorem 1 (Generating set).** The set  $FC$  of frequent closed itemsets with their support is a generating set for all frequent itemsets and their support, and for all association rules holding in the dataset, their support and their confidence.

*Proof.* Based on Lemma 2, all frequent itemsets can be derived from the maximal frequent closed itemsets. Based on Lemma 1, the support of each frequent itemset can be derived from the support of frequent closed itemsets. Then, the set of frequent closed itemsets  $FC$  is a generating set for both the set of frequent itemsets  $L$  and the set of association rules  $AR^3$ .  $\square$

### 3.2. Duquenne-Guigues Basis for Exact Association Rules

**Definition 7 (Frequent pseudo-closed itemsets).** An itemset  $I \subseteq \mathcal{I}$  in  $\mathcal{D}$  is a pseudo-closed itemset iff  $h(I) \neq I$  and  $\forall I' \subset I$  such as  $I'$  is a pseudo-closed itemset, we have  $h(I') \subset I$ . The set  $FP$  of frequent pseudo-closed itemsets in  $\mathcal{D}$  is defined as

$$FP = \{I \subseteq \mathcal{I} \mid \text{supp}(I) \geq \text{minsupp} \wedge I \neq h(I) \wedge \forall I' \in FP \text{ such as } I' \subset I \text{ we have } h(I') \subset I\}$$

**Definition 8 (Duquenne-Guigues basis for exact association rules).** Let  $FP$  be the set of frequent pseudo-closed itemsets in  $\mathcal{D}$ . The Duquenne-Guigues basis for exact association rules is defined as:

$$DG = \{r : I_1 \Rightarrow h(I_1) \setminus I_1 \mid I_1 \in FP \wedge I_1 \neq \emptyset\}$$

3. Furthermore,  $FC$  is the smallest generating set for  $L$  and  $AR$ . Hence, even if frequent itemsets can be derived from the maximal frequent itemsets, passes over the dataset are still needed to compute the frequent itemset supports.

The Duquenne-Guigues basis is minimal with respect to the number of rules since there can be no complete set with fewer rules than there are frequent pseudo-closed itemsets [DEM 92, GAN 99].

**Example 4.** A frequent pseudo-closed itemset  $I$  is a frequent non-closed itemset that includes the closures of all frequent pseudo-closed itemsets included in  $I$ . The set  $FP$  of frequent pseudo-closed itemsets and the Duquenne-Guigues basis for exact association rules extracted from  $\mathcal{D}$  for  $minsupp=2/5$  and  $minconf=1/2$  are presented in Table 4. The itemset  $AB$  is not a frequent pseudo-closed itemset since the closures of  $A$  and  $B$  (respectively  $AC$  and  $BE$ ) are not included in  $AB$ .  $ABCE$  is not a frequent pseudo-closed itemset since it is closed.

Frequent pseudo-closed itemset	Support	Exact rule	Support
{A}	3/5	$A \Rightarrow C$	3/5
{B}	4/5	$B \Rightarrow E$	4/5
{E}	4/5	$E \Rightarrow B$	4/5

**Table 4.** Frequent pseudo-closed itemsets and Duquenne-Guigues basis extracted from  $\mathcal{D}$  for  $minsupp = 2/5$ .

### 3.3. Proper Basis for Approximate Association Rules

**Definition 9 (Proper basis for approximate association rules).** Let  $FC$  be the set of frequent closed itemsets in  $\mathcal{D}$ . The proper basis for approximate association rules is:

$$PB = \{r : I_1 \rightarrow I_2 \setminus I_1 \mid I_1, I_2 \in FC \wedge I_1 \neq \emptyset \wedge I_1 \subset I_2 \wedge conf(r) \geq minconf\}$$

Association rules in  $PB$  are proper approximate association rules.

**Example 5.** The proper basis for approximate association rules extracted from  $\mathcal{D}$  for  $minsupp=2/5$  and  $minconf=1/2$  is presented in Table 5.

Approximate rule	Support	Confidence
$BCE \rightarrow A$	2/5	2/3
$AC \rightarrow BE$	2/5	2/3
$BE \rightarrow AC$	2/5	2/4
$BE \rightarrow C$	3/5	3/4
$C \rightarrow ABE$	2/5	2/4
$C \rightarrow BE$	3/5	3/4
$C \rightarrow A$	3/5	3/4

**Table 5.** Proper basis extracted from  $\mathcal{D}$  for  $minsupp = 2/5$  and  $minconf = 1/2$ .

### 3.4. Structural Basis for Approximate Association Rules

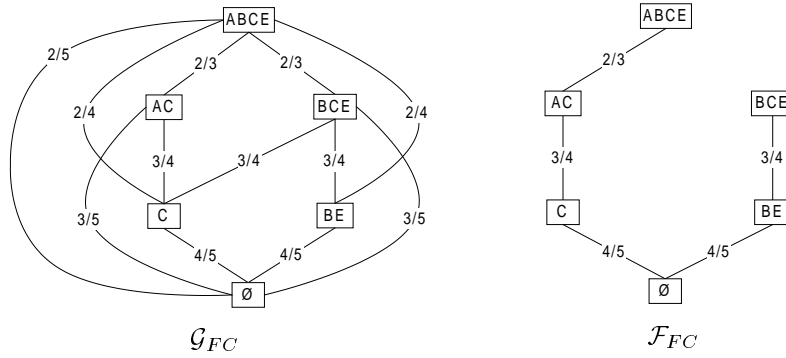
**Definition 10 (Undirected graph  $\mathcal{G}_{FC}$ ).** Let  $FC$  be the set of frequent closed itemsets in  $\mathcal{D}$ . We define  $\mathcal{G}_{FC} = (V, E)$  as the undirected graph associated with  $FC$  where the set of vertices  $V$  and the set of edges  $E$  are defined as follows:

$$V = \{I \subseteq \mathcal{I} \mid I \in FC\}$$

$$E = \{(I_1, I_2) \in V \times V \mid I_1 \subset I_2 \wedge \text{supp}(I_2)/\text{supp}(I_1) \geq \text{minconf}\}$$

With each edge in  $\mathcal{G}_{FC}$  between two vertices  $I_1$  and  $I_2$  with  $I_1 \subset I_2$  is associated the confidence =  $\text{supp}(I_2) / \text{supp}(I_1)$  of the proper approximate association rule  $I_1 \rightarrow I_2 \setminus I_1$  represented by the edge.

**Definition 11 (Maximal confidence spanning forest  $\mathcal{F}_{FC}$ ).** Let  $\mathcal{F}_{FC} = (V, E')$  be the maximal confidence spanning forest associated with  $FC$ .  $\mathcal{F}_{FC}$  is obtained from the undirected graph  $\mathcal{G}_{FC} = (V, E)$  by suppressing transitive edges and cycles. Cycles are removed by deleting some edges that enter the last vertex  $I$  (maximal vertex with respect to the inclusion) of the cycle. Among all edges entering in  $I$ , those with confidence less than the maximal confidence value associated with an edge with the form  $(I', I) \in E$  are deleted. If more than one edge have the maximal confidence value, the first one in lexicographic order is kept.



**Figure 1.** Undirected graph  $\mathcal{G}_{FC}$  and maximal confidence spanning forest  $\mathcal{F}_{FC}$  (a tree in this example) derived from  $\mathcal{D}$  for  $\text{minsupp} = 2/5$  and  $\text{minconf} = 1/2$ .

**Definition 12 (Structural basis for approximate association rules).** Let  $SB$  be the set of association rules represented by edges in  $\mathcal{F}_{FC}$  except rules from the vertex  $\{\emptyset\}$ . The structural basis for approximate association rules is:

$$SB = \{r : I_1 \rightarrow I_2 \setminus I_1 \mid I_1, I_2 \in V \wedge I_1 \subset I_2 \wedge I_1 \neq \emptyset \wedge (I_1, I_2) \in E'\}$$

In this basis, each frequent closed itemset is the consequent of at most one approximate association rule.

**Example 6.** The structural basis for approximate association rules extracted from  $\mathcal{D}$  for  $\text{minsupp}=2/5$  and  $\text{minconf}=1/2$  is presented in Table 6.

Approximate rule	Support	Confidence
$AC \rightarrow BE$	2/5	2/3
$BE \rightarrow C$	3/5	3/4
$C \rightarrow A$	3/5	3/4

**Table 6.** Structural basis extracted from  $\mathcal{D}$  for  $\text{minsupp} = 2/5$  and  $\text{minconf} = 1/2$ .

### 3.5. Basis for all Valid Association Rules

**Theorem 2 (Basis for valid association rules).** *The union of the Duquenne-Guigues basis for exact association rules and the proper basis or the structural basis for approximate association rules is a basis for all valid association rules, their support and their confidence.*

*Proof.* The proof of this theorem is straightforward from Theorem 1 and results presented in [DUQ 86, LUX 91]. All frequent closed itemsets and their support can be derived from the union of the Duquenne-Guigues basis and the proper or the structural basis since for each frequent closed itemset  $I_2$  there exists at least one association rule of the form  $r : I_1 \rightarrow I_2 \setminus I_1$  with  $\text{supp}(I_2) = \text{supp}(r)$  and  $\text{supp}(I_1) = \text{supp}(r)/\text{conf}(r)$ . Moreover, all valid association rules can be derived with their support from the set  $FC$  of frequent closed itemsets (Theorem 1). Then, obviously, all valid association rules, their support and their confidence can be derived from this union.  $\square$

## 4. Discovering Frequent and Frequent Closed Itemsets

In Section 4.1, we propose a new algorithm to achieve frequent closed itemsets from frequent itemsets without accessing the dataset. This algorithm discovers frequent closed itemsets while for instance an algorithm for discovering maximal frequent itemsets [BAY 98, LIN 98, ZAK 97] is used. In Section 4.2, we present an extension of the Apriori algorithm [AGR 94] called Apriori-Close for discovering frequent and frequent closed itemsets without additional computation time. Like in the Apriori algorithm, we assume in the following that items are sorted in lexicographic order and that  $k$  is the size of the largest frequent itemsets. Based on Lemma 2,  $k$  is also the size of the largest frequent closed itemsets.

### 4.1. Computing Frequent Closed Itemsets from Frequent Itemsets

Many efficient algorithms for mining frequent itemsets and their support have been proposed. Well-known proposals are presented in [AGR 94, BRI 97b, SAV 95, TOI 96]. Efficient algorithms for discovering the maximal frequent itemsets and then achieve all frequent itemsets have also been proposed [BAY 98, LIN 98, ZAK 97]. All these algorithms give as result the set  $L = \bigcup_{i=1}^{i=k} L_i$  where  $L_i$  contains all frequent

$i$ -itemsets (itemsets of size  $i$ ). Based on Proposition 1 and Lemma 2 (Section 3.1), the frequent closed itemsets and their support can be computed from the frequent itemsets and their support without any dataset access.

The pseudo-code to determine frequent closed itemsets among frequent itemsets is given in Algorithm 1. Notations are given in Table 7. The input of the algorithm are sets  $L_i$ ,  $1 \leq i \leq k$ , containing all frequent itemsets in the dataset. It recursively generates the sets  $FC_i$ ,  $0 \leq i \leq k$ , of frequent closed  $i$ -itemsets from  $FC_k$  to  $FC_0$ .

$L_i$	Set of frequent $i$ -itemsets and their support.
$FC_i$	Set of frequent closed $i$ -itemsets and their support.
$isclosed$	Variable indicating if the considered itemset is closed or not.

**Table 7.** Notations.

**Proposition 1.** *The support of a closed itemset is greater than the supports of all its supersets.*

*Proof.* Let  $l$  be a closed  $i$ -itemset and  $s$  a superset of  $l$ . We have  $l \subset s \Rightarrow g(l) \supseteq g(s)$  (Property (1) of the Galois connection). If  $g(l) = g(s)$  then  $h(l) = h(s) \Rightarrow l = h(s) \Rightarrow s \subseteq l$  (absurd). It follows that  $g(l) \supset g(s) \Rightarrow supp(l) > supp(s)$ .  $\square$

---

**Algorithm 1** Deriving frequent closed itemsets from frequent itemsets.

---

```

1)  $FC_k \leftarrow L_k$ ;
2) for ( $i \leftarrow k-1$ ;  $i \neq 0$ ;  $i--$ ) do begin
3)    $FC_i \leftarrow \{\}$ ;
4)   forall itemsets  $l \in L_i$  do begin
5)      $isclosed \leftarrow true$ ;
6)     forall itemsets  $l' \in L_{i+1}$  do begin
7)       if ( $l \subset l'$ ) and ( $l.support = l'.support$ ) then  $isclosed \leftarrow false$ ;
8)     end
9)     if ( $isclosed = true$ ) then  $FC_i \leftarrow FC_i \cup \{l\}$ ;
10)  end
11) end
12)  $FC_0 \leftarrow \{\emptyset\}$ ;
13) forall itemsets  $l \in L_1$  do begin
14)   if ( $l.support = |\mathcal{O}|$ ) then  $FC_0 \leftarrow \{l\}$ ;
15) end

```

---

First, the set  $FC_k$  is initialized with the set of largest frequent itemsets  $L_k$  (step 1). Then, the algorithm iteratively determines which  $i$ -itemsets in  $L_i$  are closed from  $L_{k-1}$  to  $L_1$  (steps 2 to 11). At the beginning of the  $i^{th}$  iteration the set  $FC_i$  of frequent closed  $i$ -itemsets is empty (step 3). In steps 4 to 10, for each frequent itemset  $l$  in  $L_i$ , we verify that  $l$  has the same support as a frequent  $(i+1)$ -itemset  $l'$  in  $L_{i+1}$  in which it is included. If so, we have  $l' \subseteq h(l)$  and then  $l \neq h(l)$ :  $l$  is not closed (step 7). Otherwise,  $l$  is a frequent closed itemset and is inserted in  $FC_i$  (step 9). During the

last phase, the algorithm determines if the empty itemset is closed by first initializing  $FC_0$  with the empty itemset (step 12) and then considering all frequent 1-itemsets in  $L_1$  (steps 13 to 15). If a 1-itemset  $l$  has a support equal to the number of objects in the context, meaning that  $l$  is common to all objects, then the itemset  $\emptyset$  cannot be closed (we have  $supp(\{\emptyset\}) = |\mathcal{O}| = supp(l)$ ) and is removed from  $FC_0$  (step 14). Thus, at the end of the algorithm, each set  $FC_i$  contains all frequent closed  $i$ -itemsets.

**Correctness** Since all maximal frequent itemsets are maximal frequent closed itemsets (Lemma 2), the computation of the set  $FC_k$  containing the largest frequent closed itemsets is correct. The correctness of the computation of sets  $FC_i$  for  $i < k$  relies on Proposition 1. This proposition enables to determine if a frequent  $i$ -itemset  $l$  is closed by comparing its support and the supports of the frequent  $(i+1)$ -itemsets in which  $l$  is included. If one of them has the same support as  $l$ , then  $l$  cannot be closed.

#### 4.2. Apriori-Close Algorithm

In this section, we present an extension of the Apriori algorithm [AGR 94] computing simultaneously frequent and frequent closed itemsets. The pseudo-code is given in Algorithm 2 and notations in Table 8. The algorithm iteratively generates the sets  $L_i$  of frequent  $i$ -itemsets from  $L_1$  to  $L_k$ . Besides, during the  $i^{th}$  iteration, all frequent closed  $(i-1)$ -itemsets in  $FC_{i-1}$  are determined. The set  $FC_k$  is determined during the last step of the algorithm.

$L_i$	Set of frequent $i$ -itemsets, their support and marker <i>isclosed</i> indicating if closed or not.
$FC_i$	Set of frequent closed $i$ -itemsets and their support.

**Table 8.** Notations.

First, the variable  $k$  is initialized to 0 (step 1). Then, the set  $L_1$  of frequent 1-itemsets is initialized with the list of items in the context (step 2) and one pass is performed to compute their support (step 3). The set  $FC_0$  is initialized with the empty itemset (step 4) and the supports of itemsets in  $L_1$  are considered (steps 5 to 8). All infrequent 1-itemsets are removed from  $L_1$  (step 6) and if a frequent 1-itemset has a support equal to the number of objects in the context then the empty itemset is removed from  $FC_0$  (step 7). During each of the following iterations (steps 9 to 28), frequent itemsets of size  $i+1$ ,  $k > i \geq 1$ , and frequent closed itemsets of size  $i$  are computed as follows. For all frequent  $i$ -itemsets in  $L_i$ , the marker *isclosed* is initialized to *true* (step 10). A set  $L_{i+1}$  of possible frequent  $(i+1)$ -itemsets is created by applying the Apriori-Gen function to the set  $L_i$  (step 11). For each of these possible frequent  $(i+1)$ -itemsets, we check that all its subsets of size  $i$  exist in  $L_i$  (steps 12 to 16). One pass is performed to compute the supports of the remaining itemsets in  $L_{i+1}$  (step 17). Then, for each  $(i+1)$ -itemsets  $l \in L_{i+1}$  (steps 18 to 25), if  $l$  is infrequent then it is discarded from  $L_{i+1}$  (step 19). Otherwise for all  $i$ -subsets  $l'$  of  $l$ , we verify that supports of

**Algorithm 2** Discovering frequent and frequent closed itemsets with Apriori-Close.

---

```

1)  $k \leftarrow 0$ ;
2) itemsets in  $L_1 \leftarrow \{1\text{-itemsets}\}$ ;
3)  $L_1 \leftarrow \text{Support-Count}(L_1)$ ;
4)  $FC_0 \leftarrow \{\emptyset\}$ ;
5) forall itemsets  $l \in L_1$  do begin
6)   if ( $l.\text{support} < \text{minsupp}$ ) then  $L_1 \leftarrow L_1 \setminus \{l\}$ ;
7)   else if ( $l.\text{support} = |\mathcal{O}|$ ) then  $FC_0 \leftarrow \{l\}$ ;
8) end
9) for ( $i \leftarrow 1$ ;  $L_i \neq \{\}$ ;  $i++$ ) do begin
10)  forall itemsets  $l' \in L_i$  do  $l'.\text{isclosed} \leftarrow \text{true}$ ;
11)   $L_{i+1} \leftarrow \text{Apriori-Gen}(L_i)$ ;
12)  forall itemsets  $l \in L_{i+1}$  do begin
13)    forall  $i$ -subsets  $l'$  of  $l$  do begin
14)      if ( $l' \notin L_i$ ) then  $L_{i+1} \leftarrow L_{i+1} \setminus \{l\}$ ;
15)    end
16)  end
17)   $L_{i+1} \leftarrow \text{Support-Count}(L_{i+1})$ ;
18)  forall itemsets  $l \in L_{i+1}$  do begin
19)    if ( $l.\text{support} < \text{minsupp}$ ) then  $L_{i+1} \leftarrow L_{i+1} \setminus \{l\}$ ;
20)    else do begin
21)      forall  $i$ -subsets  $l' \in L_i$  of  $l$  do begin
22)        if ( $l.\text{support} = l'.\text{support}$ ) then  $l'.\text{isclosed} \leftarrow \text{false}$ ;
23)      end
24)    end
25)  end
26)   $FC_i \leftarrow \{l \in L_i \mid l.\text{isclosed} = \text{true}\}$ ;
27)   $k \leftarrow i$ ;
28) end
29)  $FC_k \leftarrow L_k$ ;

```

---

$l'$  and  $l$  are equal; if so, then  $l'$  cannot be a closed itemset and its marker *isclosed* is set to false (steps 20 to 24). Then, all frequent  $i$ -itemsets in  $L_i$  for which marker *isclosed* is *true* are inserted in the set  $FC_i$  of frequent closed  $i$ -itemsets (step 26) and the variable  $k$  is set to the value of  $i$  (step 27). Finally, the set  $FC_k$  is initialized with the frequent  $k$ -itemsets in  $L_k$  (step 29).

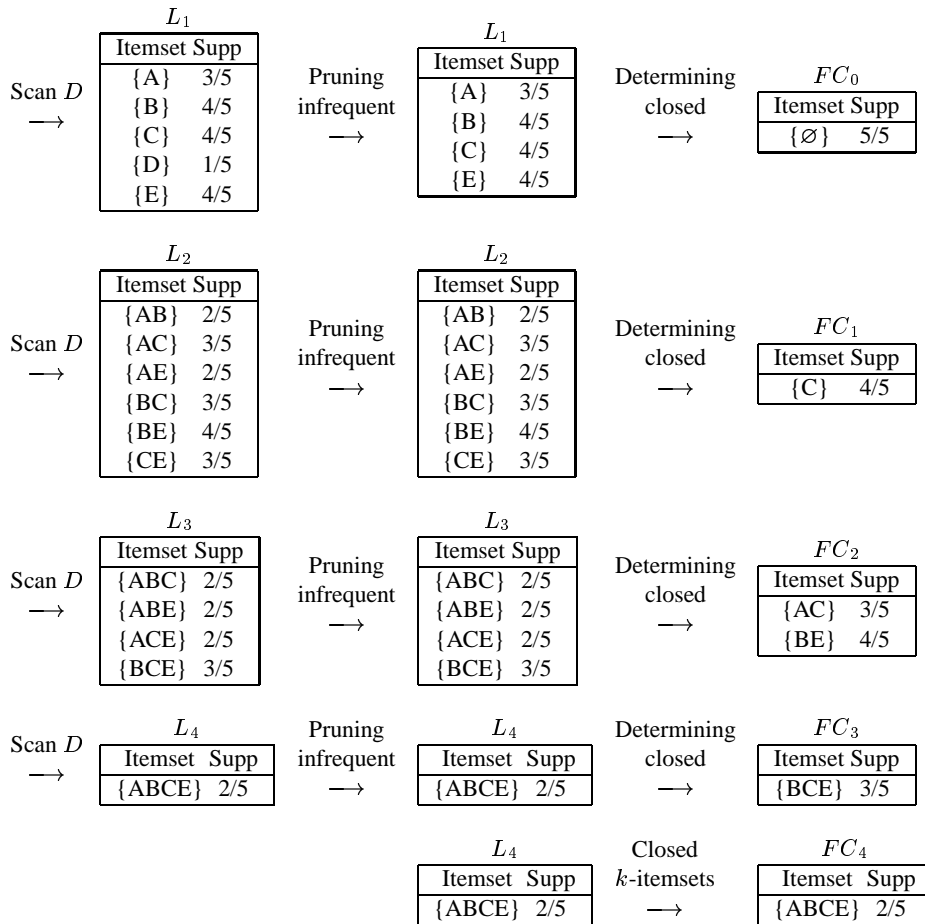
**Apriori-Gen function** The Apriori-Gen function [AGR 94] applies to a set  $L_i$  of frequent  $i$ -itemsets. It returns a set  $L_{i+1}$  of potential frequent  $(i+1)$ -itemsets. A new itemset in  $L_{i+1}$  is created by joining two itemsets in  $L_i$  sharing common first  $i-1$  items.

**Support-Count function** The Support-Count function takes a set  $L_i$  of  $i$ -itemsets as argument. It efficiently computes the supports of all itemsets  $l \in L_i$ . Only one dataset pass is required: for each object  $o$  read, the supports of all itemsets  $l \in L_i$  that are included in the set of items associated with  $o$ , i.e.  $l \subseteq f(\{o\})$ , are incremented. The

subsets of  $f(\{o\})$  are quickly found using the Subset function described in Section 5.2.

**Correctness** Since the support of a frequent closed itemset  $l$  is different from the support of all its supersets (Proposition 1), the computation of sets  $FC_i$  for  $i < k$  is correct. Hence, a frequent  $i$ -itemset  $l' \in L_i$  is determined closed or not by comparing its support with the supports of all frequent  $(i + 1)$ -itemsets  $l \in L_{i+1}$  for which  $l' \subset l$ . Lemma 2 ensures the correctness of the computation of the set  $FC_k$  containing the largest frequent closed itemsets.

**Example 7.** Figure 2 illustrates the execution of the Apriori-Close algorithm with the context  $\mathcal{D}$  for a minimum support of  $2/5$ .



**Figure 2.** Discovering frequent and frequent closed itemsets with Apriori-Close.



## 5. Generating Bases for Association Rules

In Section 5.1, we present an algorithm to generate the Duquenne-Guigues basis for exact association rules. In Sections 5.2 and 5.3 are described algorithms achieving the proper basis and the structural basis for approximate association rules respectively.

### 5.1. Generating Duquenne-Guigues Basis for Exact Association Rules

The pseudo-code generating the Duquenne-Guigues basis for exact association rules is given in Algorithm 3. Notations are given in Table 9. The algorithm takes as input the sets  $L_i$ ,  $1 \leq i \leq k$ , containing the frequent itemsets and their support, and the sets  $FC_i$ ,  $0 \leq i \leq k$ , containing the frequent closed itemsets and their support. It first computes the frequent pseudo-closed itemsets iteratively (steps 2 to 17) and then uses them to generate the Duquenne-Guigues basis for exact association rules  $DG$  (steps 18 to 22).

$L_i$	Set of frequent $i$ -itemsets and their support.
$FC_i$	Set of frequent closed $i$ -itemsets and their support.
$FP_i$	Set of frequent pseudo-closed $i$ -itemsets, their closure and their support.
$DG$	Duquenne-Guigues basis for exact association rules.

**Table 9.** Notations.

First, the set  $DG$  is initialized to the empty set (step 1). If the empty itemset is not a closed itemset (it is then necessarily a pseudo-closed itemset), it is inserted in  $FP_0$  (step 2). Otherwise  $FP_0$  is empty (step 3). Then, the algorithm recursively determines which  $i$ -itemsets in  $L_i$  are pseudo-closed from  $L_1$  to  $L_k$  (steps 4 to 16). At each iteration, the set  $FP_i$  is initialized with the list of frequent  $i$ -itemsets that are not closed (step 5) and each frequent  $i$ -itemsets  $l$  in  $FP_i$  is considered as follows (steps 6 to 15). The variable *pseudo* is set to *true* (step 7). We verify for each frequent pseudo-closed itemset  $p$  previously discovered (i.e. in  $FP_j$  with  $j < i$ ) if  $p$  is contained in  $l$  (steps 8 to 13). In that case and if the closure of  $p$  is not included in  $l$ , then  $l$  is not pseudo-closed and is removed from  $FP_i$  (steps 9 to 12). Otherwise, the closure of  $l$  (i.e. the smallest frequent closed itemset containing  $l$ ) is determined (step 14). Once all frequent pseudo-closed itemsets  $p$  and their closure are computed, all rules with the form  $r : p \Rightarrow (p.\text{closure} \setminus p)$  are generated (steps 17 to 21). The algorithm results in the set  $DG$  containing all rules in the Duquenne-Guigues basis for exact association rules.

**Correctness** Since the itemset  $\emptyset$  has no subset, if it is not a closed itemset then it is by definition a pseudo-closed itemset and the computation of the set  $FP_0$  is correct. The correctness of the computation of frequent pseudo-closed  $i$ -itemsets in  $FP_i$  for  $1 \leq i \leq k$  relies on Definition 7. All frequent  $i$ -itemsets  $l$  in  $L_i$  that are not closed, i.e. not in  $FC_i$ , are considered. Those  $l$  containing the closures of all frequent pseudo-closed itemsets that are subsets of  $l$  are inserted in  $FP_i$ . According to Definition 7,

**Algorithm 3** Generating Duquenne-Guigues basis for exact association rules.

---

```

1)  $DG \leftarrow \{\}$ ;
2) if ( $FC_0 = \{\}$ ) then  $FP_0 \leftarrow \{\emptyset\}$ ;
3) else  $FP_0 \leftarrow \{\}$ ;
4) for ( $i \leftarrow 1; i \leq k; i++$ ) do begin
5)    $FP_i \leftarrow L_i \setminus FC_i$ ;
6)   forall itemsets  $l \in FP_i$  do begin
7)      $pseudo \leftarrow true$ ;
8)     forall itemsets  $p \in FP_j$  with  $j < i$  do begin
9)       if ( $p \subset l$ ) and ( $p.closure \not\subseteq l$ ) then do begin
10)         $pseudo \leftarrow false$ ;
11)         $FP_i \leftarrow FP_i \setminus \{l\}$ ;
12)      end
13)    end
14)    if ( $pseudo = true$ ) then  $l.closure \leftarrow Min_{\subseteq}(\{c \in FC_{j>i} \mid l \subseteq c\})$ ;
15)  end
16) end
17) forall sets  $FP_i$  where  $FP_i \neq \{\}$  do begin
18)  forall pseudo-closed itemsets  $p \in FP_i$  do begin
19)     $DG \leftarrow DG \cup \{r : p \Rightarrow (p.closure \setminus p), p.support\}$ ;
20)  end
21) end

```

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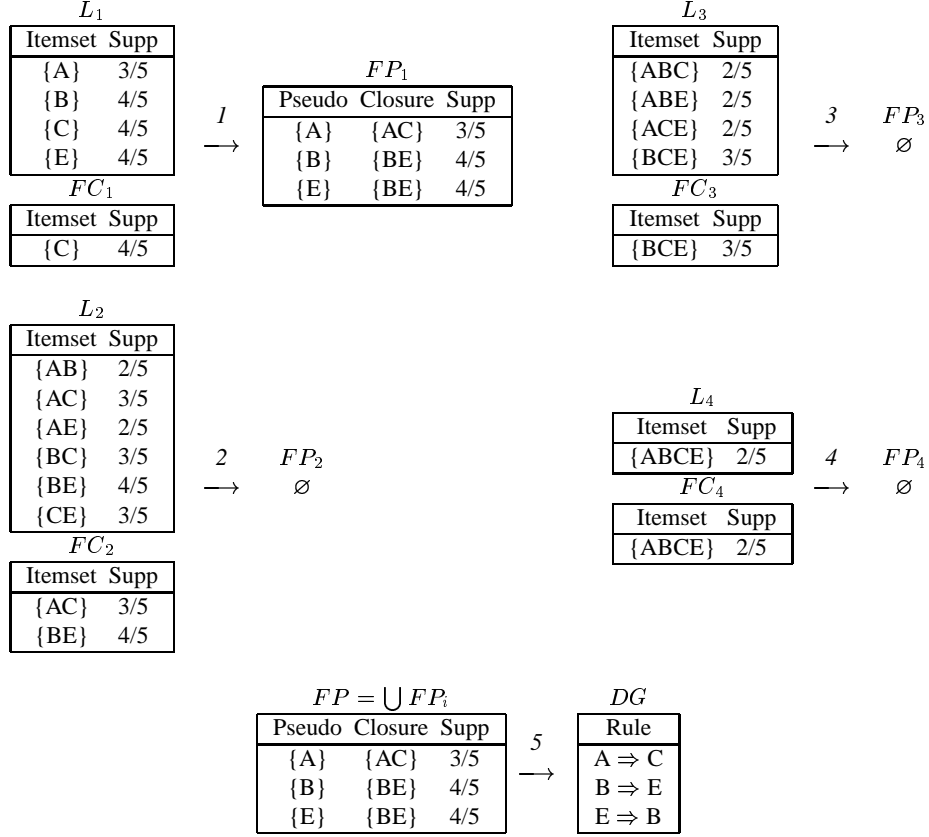
these  $i$ -itemsets are all frequent pseudo-closed  $i$ -itemsets and the sets  $FP_i$  are correct. The association rules generated in the last phase of the algorithm are all rules with a frequent pseudo-closed itemset in the antecedent. Then, the resulting set  $DG$  corresponds to the rules in the Duquenne-Guigues basis for exact association rules characterized in Definition 8.

**Example 8.** Figure 3 shows the generation of the Duquenne-Guigues basis for exact association rules from the context  $\mathcal{D}$  for a minimum support of  $2/5$ .

## 5.2. Generating Proper Basis for Approximate Association Rules

The pseudo-code generating the proper basis for approximate association rules is presented in Algorithm 4. Notations are given in Table 10. The algorithm takes as input the sets  $FC_i$ ,  $1 \leq i \leq k$ , containing the frequent closed non-empty itemsets and their support. The output of the algorithm is the proper basis for approximate association rules  $PB$ .

The set  $PB$  is first initialized to the empty set (step 1). Then, the algorithm iteratively considers all frequent closed itemsets  $l \in FC_i$  for  $2 \leq i \leq k$ . It determines which frequent closed itemsets  $l' \in FC_{j<i}$  are subsets of  $l$  and generates association rules with the form  $l' \rightarrow l \setminus l'$  that have sufficient confidence (steps 2 to 12) as follows. During the  $i^{th}$  iteration, each itemset  $l$  in  $FC_i$  is considered (steps 3 to 11). For each



**Figure 3.** Generating Duquenne-Guigues basis for exact association rules.

$FC_i$	Set of frequent closed $i$ -itemsets and their support.
$S_j$	Set of $j$ -itemsets that are subsets of the considered itemset.
$PB$	Proper basis for approximate association rules.

**Table 10.** Notations.

set  $FC_j$ ,  $1 \leq j < i$ , a set  $S_j$  containing all frequent closed  $j$ -itemsets in  $FC_j$  that are subsets of  $l$  is created (step 5). Then, for each of these subsets  $l' \in S_j$  (steps 6 to 9), we compute the confidence of the proper approximate association rule  $r : l' \rightarrow l \setminus l'$  (step 7). If the confidence of  $r$  is sufficient then  $r$  is inserted in  $PB$  (steps 8 to 9). At the end of the algorithm, the set  $PB$  contains all rules of the proper basis for approximate association rules.

**Subset function** The subset function takes a set  $X$  of itemsets and an itemset  $y$  as arguments. It determines all itemsets  $x \in X$  that are subsets of  $y$ . In algorithm imple-

---

**Algorithm 4** Generating proper basis for approximate association rules.

---

```

1)  $PB \leftarrow \{\}$ 
2) for ( $i \leftarrow 2; i \leq k; i++$ ) do begin
3)   forall itemsets  $l \in FC_i$  do begin
4)     for ( $j \leftarrow i-1; j > 0; j--$ ) do begin
5)        $S_j \leftarrow \text{Subsets}(FC_j, l)$ ;
6)       forall itemsets  $l' \in S_j$  do begin
7)          $conf(r) \leftarrow l.\text{support} / l'.\text{support}$ ;
8)         if ( $conf(r) \geq minconf$ )
9)           then  $PB \leftarrow PB \cup \{r : l' \rightarrow l \setminus l', l.\text{support}, conf(r)\}$ ;
10)        end
11)     end
12)   end
13) end

```

---

mentation, frequent and frequent closed itemsets are stored in a *prefix-tree* structure [PAS 98, PAS 99a] in order to improve efficiency of the subset search.

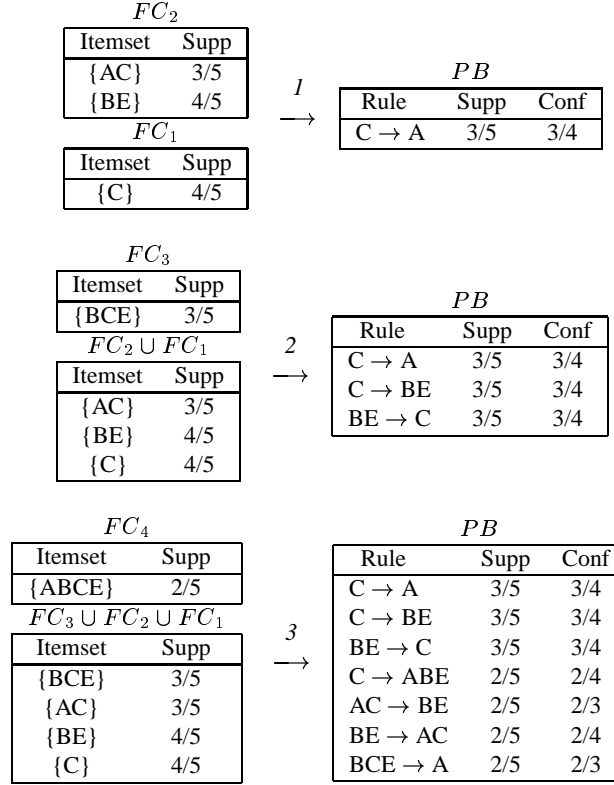
**Correctness** The correctness of the algorithm relies on the fact that we inspect all proper approximate association rules holding in the dataset. For each frequent closed itemset, the algorithm computes, among its subsets, all other frequent closed itemsets. Then, the generation of all rules between two frequent closed itemsets having sufficient confidence is ensured. These rules are all proper approximate association rules holding in the dataset, and the resulting set  $PB$  is the proper basis for approximate association rules defined in Theorem 3.

**Example 9.** Figure 4 shows the generation of the proper basis for approximate association rules in the context  $\mathcal{D}$  for a minimum support of  $2/5$  and a minimum confidence of  $1/2$ .

### 5.3. Generating Structural Basis for Approximate Association Rules

The pseudo-code generating the structural basis for approximate association rules is given in Algorithm 5. Notations are given in Table 11. The algorithm takes as input the sets  $FC_i$ ,  $1 \leq i \leq k$ , of frequent closed non-empty itemsets and their support. It generates the structural basis for approximate association rules  $SB$  represented by the maximal confidence spanning forest  $\mathcal{F}_{FC}$  associated with  $FC = \bigcup_{i=1}^{i=k} FC_i$  (without the empty itemset).

The set  $SB$  is first initialized to the empty set (step 1). Then, the algorithm iteratively considers all frequent closed itemsets  $l \in FC_i$  for  $2 \leq i \leq k$ . It determines which frequent closed itemsets  $l' \in FC_{j < i}$  are covered by  $l$ , i.e. are direct predecessors of  $l$ , and then generates the maximal confidence association rules with the form  $l \rightarrow l' \setminus l$  that hold (steps 2 to 25). During the  $i^{th}$  iteration, each itemset  $l$  in  $FC_i$



**Figure 4.** Generating proper basis for approximate association rules.

$FC_i$	Set of frequent closed $i$ -itemsets and their support.
$S_j$	Set of $j$ -itemsets that are subsets of the itemset considered.
$CR$	Set of candidate approximate association rules.
$SB$	Structural basis for approximate association rules.

**Table 11.** Notations.

is considered (steps 3 to 24) as follows. The set  $CR$  of candidate association rules with  $l$  in the consequent is initialized to the empty set (step 4). For  $1 \leq j < i$ , sets  $S_j$  containing all frequent closed  $j$ -itemsets in  $FC_j$  that are subsets of  $l$  are created (steps 5 to 7). Then, all these subsets of  $l$  are considered in decreasing order of their sizes (steps 8 to 18). For each of these subsets  $l' \in S_j$ , the confidence of the proper approximate association rule  $r : l' \rightarrow l \setminus l'$  is computed (step 10). If the confidence of  $r$  is sufficient,  $r$  is inserted in  $CR$  (step 12) and all subsets  $l''$  of  $l'$  are removed from  $S_{n < j}$  (steps 13 to 15). This because rules with the form  $l'' \rightarrow l \setminus l''$  with  $l'' \in S_{n < j}$  are transitive proper approximate rules. Finally, the candidate proper approximate rules with  $l$  in the consequent that are in  $CR$  are pruned (steps 19 to 23): the maximum con-

confidence value  $maxconf$  of rules in  $CR$  is determined (step 20) and the first rule with such a confidence is inserted in  $SB$  (steps 21 and 22). At the end of the algorithm, the set  $SB$  thus contains all rules in the structural basis for approximate association rules.

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**Algorithm 5** Generating structural basis for approximate association rules.

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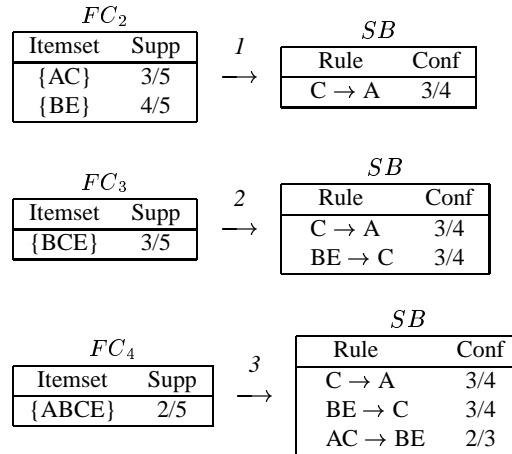
1)  $SB \leftarrow \{\}$ ;
2) for ( $i \leftarrow 2; i \leq k; i++$ ) do begin
3)   forall itemsets  $l \in FC_i$  do begin
4)      $CR \leftarrow \{\}$ ;
5)     for ( $j \leftarrow i-1; j > 0; j--$ ) do begin
6)        $S_j \leftarrow \text{Subsets}(FC_j, l)$ ;
7)     end
8)     for ( $j \leftarrow i-1; j > 0; j--$ ) do begin
9)       forall itemsets  $l' \in S_j$  do begin
10)         $conf(r) \leftarrow l.\text{support} / l'.\text{support}$ ;
11)        if ( $conf(r) \geq minconf$ ) then do begin
12)           $CR \leftarrow CR \cup \{r : l' \rightarrow l \setminus l', l.\text{support}, conf(r)\}$ ;
13)          for ( $n \leftarrow j-1; n > 0; n--$ ) do begin
14)             $S_n \leftarrow S_n \setminus \text{Subsets}(S_n, l')$ ;
15)          end
16)        end
17)      end
18)    end
19)    if ( $CR \neq \{\}$ ) then do begin
20)       $maxconf \leftarrow \text{Max}_{r \in CR}(conf(r))$ ;
21)      find first  $\{r \in CR \mid conf(r) = maxconf\}$ ;
22)       $SB \leftarrow SB \cup \{r\}$ ;
23)    end
24)  end
25) end

```

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**Correctness** The algorithm considers all association rules  $l' \rightarrow l \setminus l'$  with confidence  $\geq minconf$  between two frequent closed itemsets  $l$  and  $l'$  where  $l$  covers  $l'$ . These rules are all proper non-transitive approximate association rules that hold and can be represented by the edges of the graph  $\mathcal{G}_{FC}$  (Definition 8) without transitive edges. Moreover, among all rules with the form  $X \rightarrow l \setminus X$  (generated from  $l$ ), we keep only the first one with confidence equal to the maximal confidence of rules  $X \rightarrow l \setminus X$ . Only preserving this rule is equivalent to the cycle removing in the graph  $\mathcal{G}_{FC}$  in the same manner as explained in Definition 9. Then, the resulting set  $SB$  can be represented as the maximal confidence spanning forest  $\mathcal{F}_{FC}$  without edges from the empty itemset.  $SB$  contains all rules in the structural basis for approximate association rules defined in Theorem 4.

**Example 10.** Figure 5 depicts the generation of the structural basis for approximate association rules in the context  $\mathcal{D}$  for a minimum support of  $2/5$  and a minimum confidence of  $1/2$ .



**Figure 5.** Generating structural basis for approximate association rules.

## 6. Experimental Results

Experiments were performed on a Pentium II PC with a 350 Mhz clock rate, 128 MBytes of RAM, running the Linux operating system. Algorithms were implemented in C++. Characteristics of the datasets used are given in Table 12. These datasets are the T10I4D100K<sup>4</sup> synthetic dataset that mimics market basket data, the C20D10K and the C73D10K census datasets from the PUMS sample file<sup>5</sup>, and the MUSHROOMS<sup>6</sup> dataset describing mushroom characteristics. In all experiments, we attempted to choose significant minimum support and confidence threshold values: we observed threshold values used in other papers for experiments on similar data types and inspected rules extracted in the bases.

Name	Number of objects	Average size of objects	Number of items
T10I4D100K	100,000	10	1,000
MUSHROOMS	8,416	23	127
C20D10K	10,000	20	386
C73D10K	10,000	73	2,177

**Table 12.** Datasets.

### 6.1. Relative Performance of Apriori and Apriori-Close

We conducted experiments to compare response times obtained with Apriori and Apriori-Close on the four datasets. Results for the T10I4D100K and MUSHROOMS

4. <http://www.almaden.ibm.com/cs/quest/syndata.html>

5. <ftp://ftp2.cc.ukans.edu/pub/ippbr/census/pums/pums90ks.zip>

6. <ftp://ftp.ics.uci.edu/~cmerz/mldb.tar.Z>

datasets are presented in Table 13. We can observe that execution times are identical for the two algorithms: adding the frequent closed itemset derivation to the frequent itemset discovery does not induce additional computation time. Similar results were obtained for C20D10K and C73D10K datasets.

Minsupp	Apriori	Apriori-Close	Minsupp	Apriori	Apriori-Close
2.0%	1.99s	1.97s	90%	0.28s	0.28s
1.0%	3.47s	3.46s	70%	0.73s	0.73s
0.5%	9.62s	9.70s	50%	2.40s	2.70s
0.25%	15.02s	14.92s	30%	18.22s	17.93s

T10I4D100K MUSHROOMS

**Table 13.** Execution times of Apriori and Apriori-Close.

## 6.2. Number of Rules and Execution Times of the Rule Generation

Table 14 shows the total number of exact association rules and their number in the Duquenne-Guigues basis for exact rules. Table 15 shows the total number of approximate association rules, their number in the proper basis and in the structural basis for approximate rules, and the number of non-transitive rules in the proper basis for approximate rules ( $5^{th}$  column). For example in the context  $\mathcal{D}$ , rules  $C \rightarrow A$  and  $AC \rightarrow BE$  are extracted, as well as the rule  $C \rightarrow ABE$  which is clearly transitive. Since by construction, its confidence – retrieved by multiplying the confidences of the two former – is less than theirs, this rule is the less interesting among the three. Reducing the extraction to non-transitive rules in the proper basis for approximate rules can also be interesting. Such rules are generated by a variant of Algorithm 5 with the last pruning strategy (steps 20 and 21) removed: all candidate rules in  $CR$  are inserted in  $SB$ .

Table 16 shows for the four datasets the average relative size of bases compared with the sets of all rules obtained. In the case of weakly correlated data (T10I4D100K), no exact rule is generated and the proper basis for approximate rules contains all approximate rules that hold. The reason is that, in such data, all frequent itemsets are frequent closed itemsets. In the case of correlated data (MUSHROOMS, C20D10K and C73D10K), the number of extracted rules in bases is much smaller than the total number of rules that hold.

Figure 6 shows for each dataset the execution times of the computation of all rules (using the algorithm described in [AGR 94]) and bases. Execution times of the derivation of the Duquenne-Guigues basis for exact rules and the proper basis for non-transitive approximate rules are not presented since they are identical to those of the derivation of the Duquenne-Guigues basis for exact rules and the structural basis for approximate rules (*Duquenne-Guigues and structural bases*).



Dataset	Minsupp	Exact rules	Duquenne-Guigues basis
T10I4D100K	0.5%	0	0
MUSHROOMS	30%	7,476	69
C20D10K	50%	2,277	11
C73D10K	90%	52,035	15

**Table 14.** Number of exact association rules extracted.

Dataset (Minsupp)	Minconf	Approximate rules	Proper basis	Non-transitive basis	Structural basis
T10I4D100K (0.5%)	90%	16,260	16,260	3,511	916
	70%	20,419	20,419	4,004	1,058
	50%	21,686	21,686	4,191	1,140
	30%	22,952	22,952	4,519	1,367
MUSHROOMS (30%)	90%	12,911	806	563	313
	70%	37,671	2,454	968	384
	50%	56,703	3,870	1,169	410
C20D10K (50%)	30%	71,412	5,727	1,260	424
	90%	36,012	4,008	1,379	443
	70%	89,601	10,005	1,948	455
C73D10K (90%)	50%	116,791	13,179	1,948	455
	30%	116,791	13,179	1,948	455
	95%	1,606,726	23,084	4,052	939
C73D10K (90%)	90%	2,053,896	32,644	4,089	941
	85%	2,053,936	32,646	4,089	941
	80%	2,053,936	32,646	4,089	941

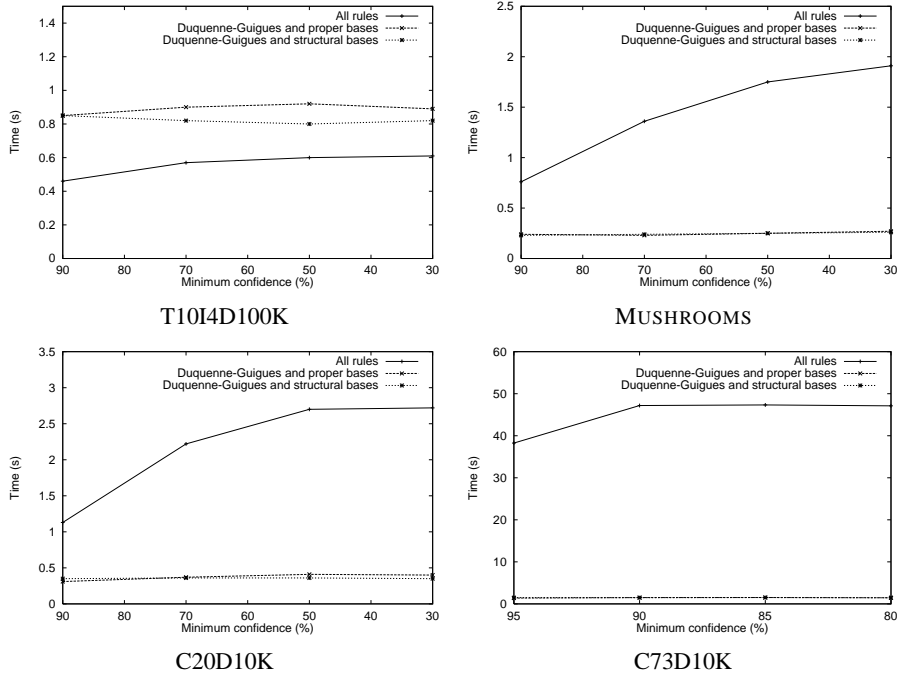
**Table 15.** Number of approximate association rules extracted.

Dataset	Duquenne-Guigues basis	Proper basis	Non-transitive basis	Structural basis
T10I4D100K	-	100.00%	20.05%	5.49%
MUSHROOMS	0.92%	6.90%	2.69%	1.19%
C20D10K	0.48%	11.21%	2.33%	0.63%
C73D10K	0.03%	1.55%	0.21%	0.05%

**Table 16.** Average relative size of bases.

## 7. Conclusion

In this paper, we present new algorithms for efficiently generating bases for association rules. A basis is a set of non-redundant rules from which all association rules can be derived, thus it captures all useful information. Moreover, its size is significantly reduced compared with the set of all possible rules because redundant, and thus useless, rules are discarded. Our approach has a twofold advantage: on one hand, the user is provided with a smaller set of resulting rules, easier to handle, and vehicul-



**Figure 6.** Execution times of the association rule derivation.

ing information of improved quality. On the other hand, execution times are reduced compared with the discovering of all association rules. Such results are proved (in the groundwork of lattice theory and data analysis) and illustrated by experiments, achieved from real-life datasets.

**Integrating reduction methods** Templates, as defined in [BAR 97, KLE 94], can directly be used for extracting from the bases all association rules matching some user specified patterns. Information in taxonomies associated with the dataset can also be integrated in the process as proposed in [HAN 95, SRI 95] for extracting bases for generalized (multi-level) association rules. Integrating item constraints and statistical measures, such as described in [BAY 99b, NG 98, SRI 97] and [BRI 97a, PIA 91] respectively, in the generation of bases requires further work.

**Functional and approximate dependencies** Algorithms presented in this paper can be adapted to generate bases for functional and approximate dependencies. In [HUH 98, LOP 00, MAN 94], such bases and algorithms for generating them were proposed. However, the Duquenne-Guigues basis is smaller than the basis for functional dependencies constituted of minimal non-trivial functional dependencies. Hence, the number of rules in the Duquenne-Guigues basis is minimal [DEM 92, GAN 99]. Further-

more, the proper and structural bases for approximate rules are also smaller than the basis for approximate dependencies defined in [HUH 98]. Adapting our algorithms to the discovery of functional and approximate dependencies is an ongoing research.

**Minimal non-redundant association rules** The bases for association rules defined in this paper significantly reduce the number of extracted rules and give a high quality non-redundant summary of valid association rules. However, they are not constituted of the non-redundant rules with minimal antecedent and maximal consequent, called *minimal non-redundant association rules*. Such rules are the most informatives, since they provide a minimal set that maximizes the information conveyed, and can be characterized using frequent closed itemsets and their generators. This is demonstrated in [BAS 00, PAS 00] and algorithms for generating them using frequent closed itemsets and their generators, such as extracted by the Close or the A-Close algorithm, or using the frequent itemsets, for extending an existing implementation, are proposed. Results of experiments conducted on real-life datasets are exhibited and show that this generation is both efficient and useful.

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