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SEIDEL COMPLEMENTATION ON $(P_5, House, Bull)$ -FREE GRAPHS

J.L. FOUQUET AND J.M. VANHERPE

ABSTRACT.

1. SEIDEL COMPLEMENTATION

Notation 1.1. $N(v)$ denotes the neighborhood of the vertex v , $N[v] = N(v) \cup \{v\}$ and $\overline{N}(v) = V - N(v)$.

Given a graph $G = (V, E)$ and a vertex v of G , the seidel complement of G on v inverses all edges between $N(v)$ and $V - N[v]$. More formally

Definition 1.2. Let $G = (V, E)$ be an undirected graph and let v be a vertex of G . The Seidel complement of G on v , denoted $G * v$ is defined as follows :
 $G * v = (V, E_1 \cup E_2 \cup E_3)$ where $E_1 = \{xy \mid xy \in E, x \in N[v], y \in N[v]\}$, $E_2 = E \cap \overline{N}(v)^2$ and $E_3 = \{xy \mid xy \notin E, x \in N(v), y \in \overline{N}(v)\}$.

Remark 1.3. [3]

- $G * v * v = G$.
- If G is a cograph and v is a vertex of G then $G * v$ is a cograph.
- G is prime with respect to modular decomposition if and only if $G * v$ is prime with respect to modular decomposition
- $\overline{G * v} = \overline{G} * v$

2. SEIDEL COMPLEMENTATION ON $(P_5, \overline{P_5}, Bull)$ -FREE GRAPHS

Theorem 2.1. A graph G is $(P_5, House, Bull)$ -free if and only if for all vertex v of G , $G * v$ is $(P_5, House, Bull)$ -free.

Proof Let v be a vertex of G . Assume that $G * v$ contains an induced subgraph, say H , which is isomorphic to either a P_5 or a $House$ or a $Bull$.

Claim 2.1.1. The vertex v does not belong to H .

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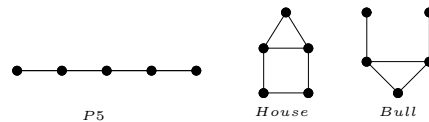
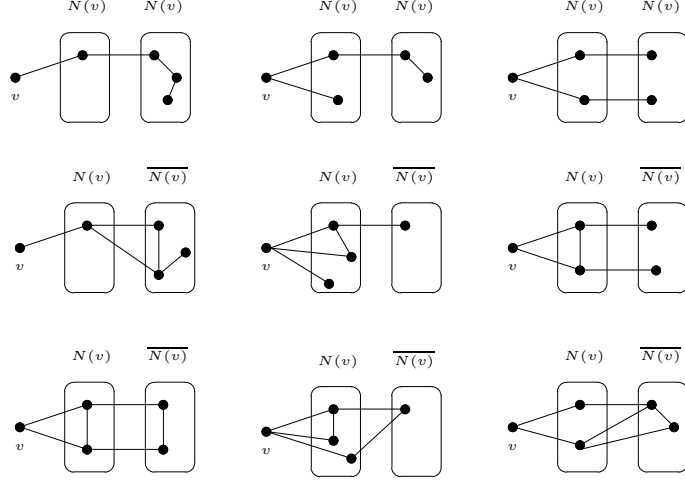


FIGURE 1. Forbidden configurations for $(P_5, House, Bull)$ -free graphs.

FIGURE 2. The possible cases when v is a vertex of H .

Proof Assume not. Figure 2 describes all cases that can occur whenever v is a vertex of H . It is not difficult to check that, in all cases, $(G * v) * v$ would contain a subgraph isomorphic to P_5 or to $House$ or to $Bull$, a contradiction since $G * v * v = G$ and G is assumed to be $(P_5, House, Bull)$ -free. \square

In the following we suppose that the vertices of H are in $N(v) \cup \overline{N(v)}$, moreover H has vertices in both sets $N(v)$ and $\overline{N(v)}$, otherwise H would be an induced subgraph of $G * v * v = G$, a contradiction.

Figure 3 (resp Figure 4, Figure 5) describes all cases that can occur whenever H is isomorphic to a P_5 (resp. a $Bull$, a $House$) and has at most two vertices in $N(v)$. In all cases we get a contradiction with Claim 2.1.1 or a forbidden configuration appears in $G * v * v$. When H has more than 2 vertices in $N(v)$ we get a similar contradiction in considering \overline{G} . \square

3. SEIDEL COMPLEMENTATION ON THE MODULAR DECOMPOSITION TREE OF $(P_5, \overline{P_5}, Bull)$ -FREE GRAPHS

3.1. On $(P_5, \overline{P_5})$ -free graphs. We shall say that an induced subgraph of a $(P_5, \overline{P_5})$ -free graph is a *buoy*[1] whenever we can find a partition of its vertex set into 5 subsets $A_i, i = 1, \dots, 5$ (subscript i is to be taken modulo 5, such that A_i and A_{i+1} are joined by every possible edge, while no possible edge are allowed between A_i and A_j when $j \neq i + 1 \pmod{5}$, and such that the A_i 's are maximal for these properties.

Theorem 3.1. [1, 2] *Let G be a connected $(P_5, \overline{P_5})$ -free graph. If G contains an induced C_5 then every C_5 of G is contained into a buoy, and this buoy is either equal to G or is an homogeneous set of G .*

Corollary 3.2. *Every prime $(P_5, \overline{P_5})$ -free graph is either a C_5 or is C_5 -free*

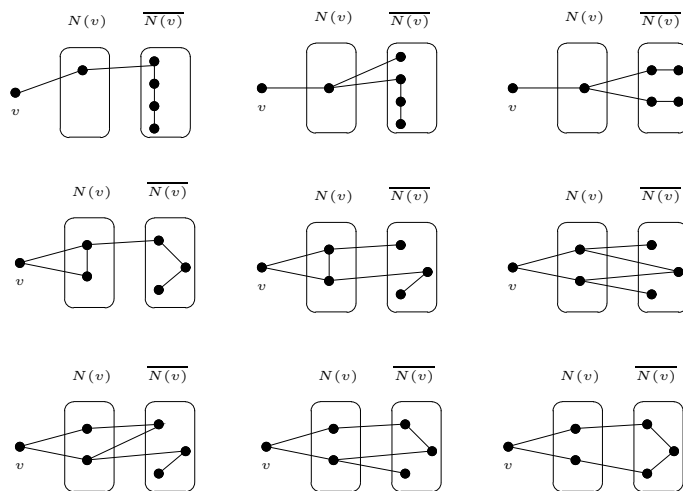


FIGURE 3. Cases : H is a P_5 and has at most 2 vertices in $N(v)$.

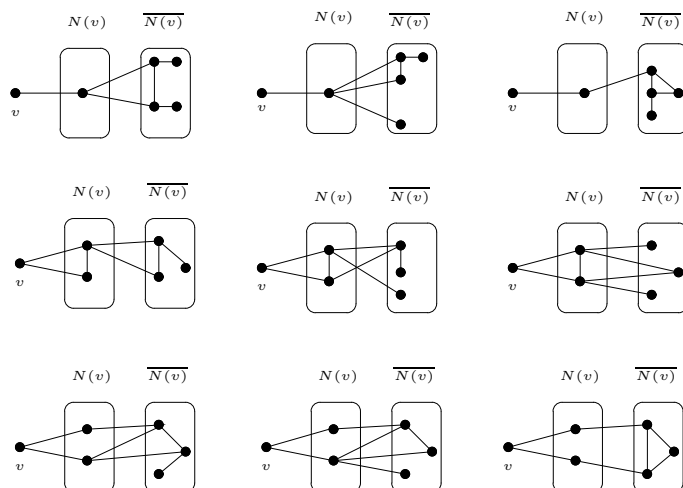


FIGURE 4. Cases : H is a *Bull* and has at most 2 vertices in $N(v)$.

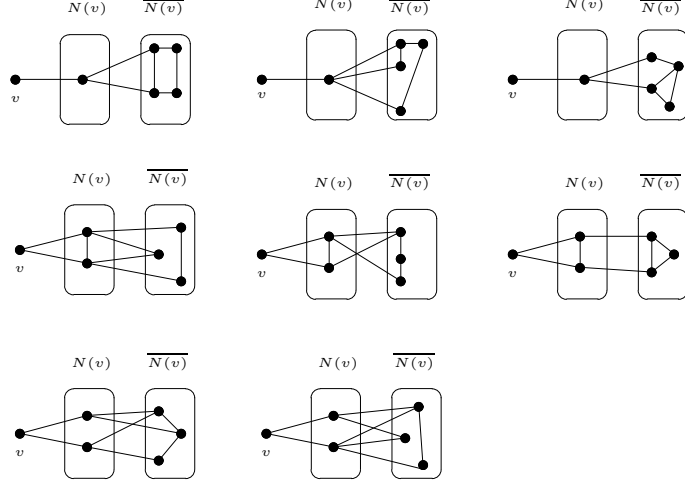
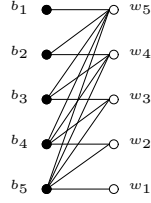
3.2. On $(P_5, \overline{P_5}, \text{Bull})$ -free graphs.

Theorem 3.3. [1] *Let G be a prime graph. G is a P_5HB -free graph if and only if one of the following conditions is satisfied*

- (i) G is isomorphic to a C_5 ;
- (ii) G is bipartite and P_5 -free;
- (iii) \overline{G} is bipartite and P_5 -free.

In a prime P_5 -free bipartite graph the neighborhoods of two distinct vertices cannot overlap properly, thus :

Proposition 3.4. *Let $G = (V, E)$ be a prime graph of n vertices. G is bipartite and P_5 -free iff the following conditions are verified:*

FIGURE 5. Cases : H is a *House* and has at most 2 vertices in $N(v)$.FIGURE 6. A Prime P_5 -free bipartite graph.

- (i) There exists a partition of $V(G)$ into two stable sets $B = \{b_1, b_2, \dots, b_{\frac{n}{2}}\}$ and $W = \{w_1, w_2, \dots, w_{\frac{n}{2}}\}$.
- (ii) The neighbors of b_i ($i = 1 \dots \frac{n}{2}$) are precisely $w_1, \dots, w_{\frac{n}{2}-i+1}$.

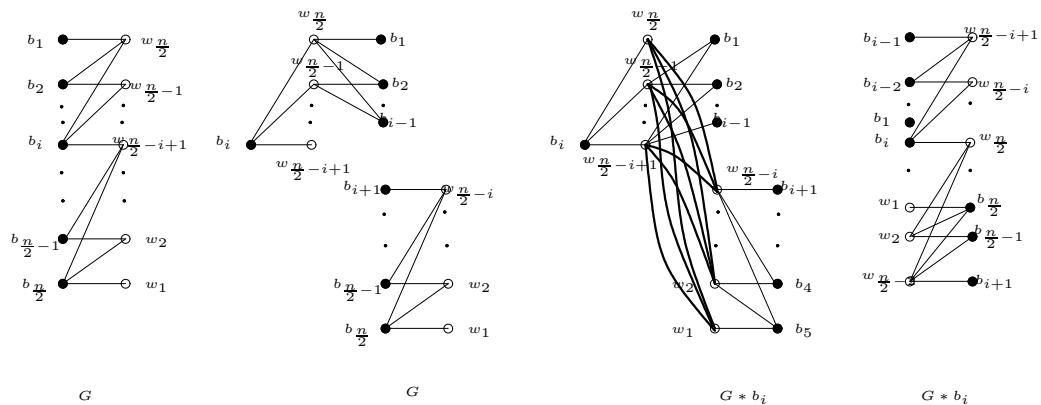
3.3. Seidel complementation of a Prime P_5 -free bipartite graph.

Proposition 3.5. Let $G = (B \cup W, E)$ be a prime bipartite P_5 -free graph such that $B = \{b_1, \dots, b_{\frac{n}{2}}\}$ and $W = \{w_1, \dots, w_{\frac{n}{2}}\}$, then $G * b_i$ is a prime P_5 -free bipartite graph together with the bipartition :

$$B' = \{b_{i-1}, b_{i-2}, \dots, b_i, w_1, w_2, \dots, w_{\frac{n}{2}}\}, W' = \{b_{i+1}, \dots, b_{\frac{n}{2}}, w_{\frac{n}{2}}, w_{\frac{n}{2}-1}, \dots, w_{\frac{n}{2}-i+1}\}$$

Proof We have $N(b_i) = \{w_{\frac{n}{2}}, w_{\frac{n}{2}-1}, \dots, w_{\frac{n}{2}-i+1}\}$ and $\overline{N(b_i)} = \{b_1, b_2, \dots, b_{i-1}, w_{\frac{n}{2}-i}, \dots, w_1\}$. It must be pointed out that the subgraphs induced by the sets $S_1 = \{w_{\frac{n}{2}}, w_{\frac{n}{2}-1}, \dots, w_{\frac{n}{2}-i+1}, b_1, b_2, \dots, b_{i-1}\}$ and $S_2 = \{b_{i+1}, \dots, b_{\frac{n}{2}}, w_{\frac{n}{2}-i}, \dots, w_1\}$ are prime P_5 -free bipartite graphs as well as $S_3 = \{w_{\frac{n}{2}}, w_{\frac{n}{2}-1}, \dots, w_{\frac{n}{2}-i+1}, b_{i+1}, \dots, b_{\frac{n}{2}}\}$ induces a complete bipartite graph. The result follows when considering the Seidel complementation on b_i (see Figure 7). \square

Corollary 3.6. For a prime P_5 -free bipartite graph (or its complement) Seidel complementation at any vertex can be performed in constant time.

FIGURE 7. Seidel complementation on b_2 .

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