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# On a family of cubic graphs containing the flower snarks 

Jean-Luc Fouquet and Henri Thuillier and Jean-Marie Vanherpe

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#### Abstract

We consider cubic graphs formed with $k \geq 2$ disjoint claws $C_{i} \sim K_{1,3}(0 \leq i \leq k-1)$ such that for every integer $i$ modulo $k$ the three vertices of degree 1 of $C_{i}$ are joined to the three vertices of degree 1 of $C_{i-1}$ and joined to the three vertices of degree 1 of $C_{i+1}$. Denote by $t_{i}$ the vertex of degree 3 of $C_{i}$ and by $T$ the set $\left\{t_{1}, t_{2}, \ldots, t_{k-1}\right\}$. In such a way we construct three distinct graphs, namely $F S(1, k), F S(2, k)$ and $F S(3, k)$. The graph $F S(j, k)(j \in\{1,2,3\})$ is the graph where the set of vertices $\cup_{i=0}^{i=k-1} V\left(C_{i}\right) \backslash T$ induce $j$ cycles (note that the graphs $F S(2,2 p+1), p \geq 2$, are the flower snarks defined by Isaacs [8]). We determine the number of perfect matchings of every $F S(j, k)$. A cubic graph $G$ is said to be 2-factor hamiltonian if every 2 -factor of $G$ is a hamiltonian cycle. We characterize the graphs $F S(j, k)$ that are 2-factor hamiltonian (note that $F S(1,3)$ is the "Triplex Graph" of Robertson, Seymour and Thomas [15]). A strong matching $M$ in a graph $G$ is a matching $M$ such that there is no edge of $E(G)$ connecting any two edges of $M$. A cubic graph having a perfect matching union of two strong matchings is said to be a Jaeger's graph. We characterize the graphs $F S(j, k)$ that are Jaeger's graphs.


Key words: cubic graph; perfect matching; strong matching; counting; hamiltonian cycle; 2-factor hamiltonian

## 1 Introduction

The complete bipartite graph $K_{1,3}$ is called, as usually, a claw. Let $k$ be an integer $\geq 2$ and let $G$ be a cubic graph on $4 k$ vertices formed with $k$ disjoint claws $C_{i}=\left\{x_{i}, y_{i}, z_{i}, t_{i}\right\}(0 \leq i \leq k-1)$ where $t_{i}$ (the center of $\left.C_{i}\right)$ is joined to the three independent vertices $x_{i}, y_{i}$ and $z_{i}$ (the external vertices of $C_{i}$ ). For every integer $i$ modulo $k C_{i}$ has three neighbours in $C_{i-1}$ and three neighbours in $C_{i+1}$. For any integer $k \geq 2$ we shall denote the set of integers modulo $k$ as $\mathbf{Z}_{k}$. In the sequel of this paper indices $i$ of claws $C_{i}$ belong to $\mathbf{Z}_{k}$.


Fig. 1. Four consecutive claws
By renaming some external vertices of claws we can suppose, without loss of generality, that $\left\{x_{i} x_{i+1}, y_{i} y_{i+1}, z_{i} z_{i+1}\right\}$ are edges for any $i$ distinct from $k-1$. That is to say the subgraph induced on $X=\left\{x_{0}, x_{1}, \ldots, x_{k-1}\right\}$ (respectively $Y=\left\{y_{0}, y_{1}, \ldots, y_{k-1}\right\}, Z=\left\{z_{0}, z_{1}, \ldots, z_{k-1}\right\}$ ) is a path or a cycle (as induced subgraph of $G$ ). Denote by $T$ the set of the internal vertices $\left\{t_{0}, t_{1}, \ldots, t_{k-1}\right\}$.

Up to isomorphism, the matching joining the external vertices of $C_{k-1}$ to those of $C_{0}$ (also called, for $k \geq 3$, edges between $C_{k-1}$ and $C_{0}$ ) determines the graph G. In this way we construct essentially three distinct graphs, namely $F S(1, k), F S(2, k)$ and $F S(3, k)$. The graph $F S(j, k)(j \in\{1,2,3\})$ is the graph where the set of vertices $\cup_{i=0}^{i=k-1}\left\{C_{i} \backslash\left\{t_{i}\right\}\right\}$ induces $j$ cycles. For $k \geq 3$ and any $j \in\{1,2,3\}$ the graph $F S(j, k)$ is a simple cubic graph. When $k$ is odd, the $F S(2, k)$ are the graphs known as the flower snarks [8]. We note that $F S(3,2)$ and $F S(2,2)$ are multigraphs, and that $F S(1,2)$ is isomorphic to the cube. For $k=2$ the notion of "edge between $C_{k-1}$ and $C_{0}$ " is ambiguous, so we must define it precisely. For two parallel edges having one end in $C_{0}$ and the other in $C_{1}$, for instance two parallel edges having $x_{0}$ and $x_{1}$ as endvertices, we denote one edge by $x_{0} x_{1}$ and the other by $x_{1} x_{0}$. An edge in $\left\{x_{1} x_{0}, x_{1} y_{0}, x_{1} z_{0}, y_{1} x_{0}, y_{1} y_{0}, y_{1} z_{0}, z_{1} x_{0}, z_{1} y_{0}, z_{1} z_{0}\right\}$, if it exists, is an edge between $C_{1}$ and $C_{0}$. We will say that $x_{0} x_{1}, y_{0} y_{1}$ and $z_{0} z_{1}$ are edges between $C_{0}$ and $C_{1}$.

By using an ad hoc translation of the indices of claws (and of their vertices) and renaming some external vertices of claws, we see that for any reasoning about a sequence of $h \geq 3$ consecutive claws $\left(C_{i}, C_{i+1}, C_{i+2}, \ldots, C_{i+h-1}\right)$ there is no loss of generality to suppose that $0 \leq i<i+h-1 \leq k-1$. For a sequence of claws $\left(C_{p}, \ldots, C_{r}\right)$ with $0 \leq p<r \leq k-1$, since 0 is a possible value for subscript $p$ and since $k-1$ is a possible value for subscript $r$, it will be useful from time to time to denote by $x_{p-1}^{\prime}$ the neighbour in $C_{p-1}$ of the vertex $x_{p}$ of $C_{p}$ (recall that $x_{p-1}^{\prime} \in\left\{x_{k-1}, y_{k-1}, z_{k-1}\right\}$ if $p=0$ ), and to denote by $x_{r+1}^{\prime}$ the neighbour in $C_{r+1}$ of the vertex $x_{r}$ of $C_{r}$ (recall that $x_{r+1}^{\prime} \in\left\{x_{0}, y_{0}, z_{0}\right\}$ if $r=k-1$ ). We shall make use of analogous notations for neighbours of $y_{p}, z_{p}$, $y_{r}$ and $z_{r}$.

We shall prove in the following lemma that there are essentially two types of perfect matchings in $F S(j, k)$.

Lemma 1 Let $G \in\{F S(j, k), j \in\{1,2,3\}, k \geq 2$ and let $M$ be a perfect matching of $G$. Then the 2 -factor $G \backslash M$ induces a path of length 2 and an isolated vertex in each claw $C_{i}\left(i \in \mathbf{Z}_{k}\right)$ and $M$ fulfils one (and only one) of

## the three following properties :

i) For every $i$ in $\mathbf{Z}_{k} M$ contains exactly one edge joining the claw $C_{i}$ to the claw $C_{i+1}$,
ii) For every even i in $\mathbf{Z}_{k} \quad M$ contains exactly two edges between $C_{i}$ and $C_{i+1}$ and none between $C_{i-1}$ and $C_{i}$,
iii) For every odd $i$ in $\mathbf{Z}_{k} \quad M$ contains exactly two edges between $C_{i}$ and $C_{i+1}$ and none between $C_{i-1}$ and $C_{i}$.

Moreover, when $k$ is odd $M$ satisfies only item $i$ ).

Proof Let $M$ be a perfect matching of $G=F S(j, k)$ for some $j \in\{1,2,3\}$. Since $M$ contains exactly one edge of each claw, it is obvious that $G \backslash M$ induces a path of length 2 and an isolated vertex in each claw $C_{i}$.

For each claw $C_{i}$ of $G$ the vertex $t_{i}$ must be saturated by an edge of $M$ whose end (distinct from $t_{i}$ ) is in $\left\{x_{i}, y_{i}, z_{i}\right\}$. Hence there are exactly two edges of $M$ having one end in $C_{i}$ and the other in $C_{i-1} \cup C_{i+1}$.

If there are two edges of $M$ between $C_{i}$ and $C_{i+1}$ then there is no edge of $M$ between $C_{i-1}$ and $C_{i}$. If there are two edges of $M$ between $C_{i-1}$ and $C_{i}$ then there is no edge of $M$ between $C_{i}$ and $C_{i+1}$. Hence, we get $\left.i i\right)$ or $i i i$ ) and we must have an even number $k$ of claws in $G$.

Assume now that there is only one edge of $M$ between $C_{i-1}$ and $C_{i}$. Then there exists exactly one edge between $C_{i}$ and $C_{i+1}$ and, extending this trick to each claw of $G$, we get $i$ ) when $k$ is even or odd.

Definition 2 We say that a perfect matching $M$ of $F S(j, k)$ is of type 1 in Case i) of Lemma 1 and of type 2 in Cases ii) and iii). If neccessary, to distinguish Case ii) from Case iii) we shall say type 2.0 in Case ii) and type 2.1 in Case iii). We note that the numbers of perfect matchings of type 2.0 and of type 2.1 are equal.

Notation : The length of a path $P$ (respectively a cycle $\Gamma$ ) is denoted by $l(P)$ (respectively $l(\Gamma)$ ).

## 2 Counting perfect matchings of $F S(j, k)$

We shall say that a vertex $v$ of a cubic graph $G$ is inflated into a triangle when we construct a new cubic graph $G^{\prime}$ by deleting $v$ and adding three new vertices inducing a triangle and joining each vertex of the neighbourhood $N(v)$ of $v$ to
a single vertex of this new triangle. We say also that $G^{\prime}$ is obtained from $G$ by a triangular extension. The converse operation is the contraction or reduction of the triangle. The number of perfect matchings of $G$ is denoted by $\mu(G)$.

Lemma 3 Let $G$ be a bipartite cubic graph and let $\left\{V_{1}, V_{2}\right\}$ be the bipartition of its vertex set. Assume that each vertex in some subset $W_{1} \subseteq V_{1}$ is inflated into a triangle and let $G^{\prime}$ be the graph obtained in that way. Then $\mu(G)=$ $\mu\left(G^{\prime}\right)$.

Proof Note that $\left\{V_{1}, V_{2}\right\}$ is a balanced bipartition and, by Knig's Theorem, the graph $G$ is a cubic 3 -edge colourable graph. So, $G^{\prime}$ is also a cubic 3-edge colourable graph (hence, $G$ and $G^{\prime}$ have perfect matchings). Let $M$ be a perfect matching of $G^{\prime}$. Each vertex of $V_{1} \backslash W_{1}$ is saturated by an edge whose second end vertex is in $V_{2}$. Let $A \subseteq V_{2}$ be the set of vertices so saturated in $V_{2}$. Assume that some triangle of $G^{\prime}$ is such that the three vertices are saturated by three edges having one end in the triangle and the second one in $V_{2}$. Then we need to have at least $\left|W_{1}\right|+2$ vertices in $V_{2} \backslash A$, a contradiction. Hence, $M$ must have exactly one edge in each triangle and the contraction of each triangle in order to get back $G$ transforms $M$ in a perfect matching of $G$. Conversely, each perfect matching of $G$ leads to a unique perfect matching of $G^{\prime}$ and we obtain the result.

Let us denote by $\mu(j, k)$ the number of perfect matchings of $F S(j, k), \mu_{1}(j, k)$ its number of perfect matchings of type 1 and $\mu_{2}(j, k)$ its number of perfect matchings of type 2 .

Lemma 4 We have

- $\mu(1,3)=\mu_{1}(1,3)=9$
- $\mu(2,3)=\mu_{1}(2,3)=8$
- $\mu(3,3)=\mu_{1}(3,3)=6$
- $\mu(1,2)=9, \mu_{1}(1,2)=3$
- $\mu(2,2)=10, \mu_{1}(2,2)=4$
- $\mu(3,2)=12, \mu_{1}(3,2)=6$

Proof The cycle containing the external vertices of the claws of the graph $F S(1,3)$ is $x_{0}, x_{1}, x_{2}, y_{0}, y_{1}, y_{2}, z_{0}, z_{1}, z_{2}, x_{0}$. Consider a perfect matching $M$ containing the edge $t_{0} x_{0}$. There are two cases: i) $x_{1} x_{2} \in M$ and ii) $x_{1} t_{1} \in M$. In Case $i$ ) we must have $y_{0} y_{1}, t_{1} z_{1}, t_{2} z_{2}, z_{0} y_{2} \in M$. In Case $i i$ ) there are two sub-cases: ii). $a x_{2} y_{0} \in M$ and ii). $b x_{2} t_{2} \in M$. In Case $\left.i i\right) . a$ we must have $y_{1} y_{2}, t_{2} z_{2}, z_{0} z_{1} \in M$ and in Case $\left.i i\right) . b$ we must have $y_{0} y_{1}, y_{2} z_{0}, z_{1} z_{2} \in M$. Thus, there are exactly 3 distinct perfect matching containing $t_{0} x_{0}$. By symmetry, there are 3 distinct perfect matchings containing $t_{0} y_{0}$, and 3 distinct matchings containing $t_{0} z_{0}$, therefore $\mu(1,3)=9$.

It is well known that the Petersen graph has exactly 6 perfect matchings. Since $F S(2,3)$ is obtained from the Petersen graph by inflating a vertex into a triangle these 6 perfect matchings lead to 6 perfect matchings of $F S(2,3)$. We have two new perfect matchings when considering the three edges connected to this triangle (we have two ways to include these edges into a perfect matching). Hence $\mu(2,3)=8$.
$F S(3,3)$ is obtained from $K_{3,3}$ by inflating three vertices in the same colour of the bipartition. Since $K_{3,3}$ has six perfect matchings, applying Lemma 3 we get immediately the result for $\mu(3,3)$.

Is is a routine matter to obtain the values for $F S(j, 2) \quad(j \in\{1,2,3\})$.

Theorem 5 The numbers $\mu(i, k)$ of perfect matchings of $F S(i, k)(i \in\{1,2,3\})$ are given by:

- $\mu(2, k)=2^{k}$

When $k$ is odd • $\mu(1, k)=2^{k}+1$

- $\mu(3, k)=2^{k}-2$
- $\mu(2, k)=2 \times 3^{\frac{k}{2}}+2^{k}$

When $k$ is even - $\mu(1, k)=2 \times 3^{\frac{k}{2}}+2^{k}-1$

- $\mu(3, k)=2 \times 3^{\frac{k}{2}}+2^{k}+2$

Proof We shall prove this result by induction on $k$ and we distinguish the case " $k$ odd" and the case " $k$ even".

The following trick will be helpful. Let $i \neq 0$ and let $C_{i-2}, C_{i-1}, C_{i}$ and $C_{i+1}$ be four consecutive claws of $F S(j, k)(j \in\{1,2,3\})$. We can delete $C_{i-1}$ and $C_{i}$ and join the three external vertices of $C_{i-2}$ to the three external vertices of $C_{i+1}$ by a matching in such a way that the resulting graph is $F S\left(j^{\prime}, k-2\right)$. We have three distinct ways to reduce $F S(j, k)$ into $F S\left(j^{\prime}, k-2\right)$ when deleting $C_{i-1}$ and $C_{i}$.

Case 1: We add the edges $\left\{x_{i-2} x_{i+1}, y_{i-2} y_{i+1}, z_{i-2} z_{i+1}\right\}$ and we get $G_{1}=$ $F S\left(j_{1}, k-2\right)$

Case 2: We add the edges $\left\{x_{i-2} y_{i+1}, y_{i-2} z_{i+1}, z_{i-2} x_{i+1}\right\}$ and we get $G_{2}=$ $F S\left(j_{2}, k-2\right)$.

Case 3: We add the edges $\left\{x_{i-2} z_{i+1}, y_{i-2} x_{i+1}, z_{i-2} y_{i+1}\right\}$ and we get $G_{3}=$ $F S\left(j_{3}, k-2\right)$.

Following the cases, we shall precise the values of $j_{1}, j_{2}$ and $j_{3}$.
It is an easy task to see that each perfect matching of type 1 of $F S(j, k)$ leads to a perfect matching of either $G_{1}$ or $G_{2}$ or $G_{3}$ and, conversely, each perfect matching of type 1 of $G_{1}$ allows us to construct 2 distinct perfect matchings of type 1 of $F S(j, k)$, while each perfect matching of type 1 of $G_{2}$ and $G_{3}$ allows us to construct 1 perfect matching of type 1 of $F S(j, k)$.

We have

$$
\begin{equation*}
\mu_{1}(j, k)=2 \mu_{1}\left(G_{1}\right)+\mu_{1}\left(G_{2}\right)+\mu_{1}\left(G_{3}\right) \tag{1}
\end{equation*}
$$

Claim $1 \mu_{1}(2, k)=2^{k}$

Proof Since the result holds for $F S(2,3)$ and $F S(2,2)$ by Lemma 0 , in order to prove the result by induction on the number $k$ of claws, we assume that the property holds for $F S(2, k-2)$ with $k-2 \geq 2$.

In that case $G_{1}, G_{2}$ and $G_{3}$ are isomorphic to $F S(2, k-2)$. Using Equation [] we have, as claimed

$$
\mu_{1}(2, k)=4 \mu_{1}(2, k-2)=2^{k}
$$

CLAIM $2 \mu_{1}(1, k)=2^{k}-(-1)^{k}$ and $\mu_{1}(3, k)=2^{k}+2(-1)^{k}$

Proof Since the result holds for $F S(1,3), F S(1,2), F S(3,3)$, and $F S(3,2)$, by Lemma $\pi^{\pi}$, in order to prove the result by induction on the number $k$ of claws, we assume that the property holds for $F S(1, k-2)$, and $F S(3, k-2)$ with $k-2 \geq 2$.

When considering $F S(1, k), G_{1}$ is isomorphic to $F S(1, k-2)$, and among $G_{2}$ and $G_{3}$ one of them is isomorphic to $F S(3, k-2)$ and the other to $F S(1, k-2)$. In the same way, when considering $F S(3, k), G_{1}$ is isomorphic to $F S(3, k-2)$, and $G_{2}$ and $G_{3}$ are isomorphic to $F S(1, k-2)$.

Using Equation 1 we have,

$$
\mu_{1}(1, k)=2 \mu_{1}(1, k-2)+\mu_{1}(1, k-2)+\mu_{1}(3, k-2)
$$

and

$$
\mu_{1}(1, k)=2\left(2^{k-2}+1\right)+2^{k-2}+1+2^{k-2}-2=2^{k}+1
$$

$$
\mu_{1}(3, k)=2\left(2^{k-2}-2\right)+2^{k-2}+1+2^{k-2}+1=2^{k}-2
$$

When $k$ is odd, we have $\mu_{2}(j, k)=0$ by Lemma 1 and hence $\mu(j, k)=\mu_{1}(j, k)$
When $k$ is even it remains to count the number of perfect matchings of type 2. From Lemma 1, for every two consecutive claws $C_{i}$ and $C_{i+1}$, we have either two edges of $M$ joining the external vertices of $C_{i}$ to those of $C_{i+1}$ or none. We have 3 ways to choose 2 edges between $C_{i}$ and $C_{i+1}$, each choice of these two edges can be completed in a unique way in a perfect matching of the subgraph $C_{i} \cup C_{i+1}$. Hence we get easily that the number of perfect matchings of type 2 in $F S(j, k)(j \in\{1,2,3\})$ is

$$
\begin{equation*}
\mu_{2}(j, k)=2 \times 3^{\frac{k}{2}} \tag{2}
\end{equation*}
$$

Using Claims 1 and 2 and Equation 2 we get the results for $\mu(j, k)$ when $k$ is even.

## 3 Some structural results about perfect matchings of FS(j, k)

### 3.1 Perfect matchings of type 1

Lemma 6 Let $M$ be a perfect matching of type 1 of $G=F S(j, k)$. Then the 2 -factor $G \backslash M$ has exactly one or two cycles and each cycle of $G \backslash M$ has at least one vertex in each claw $C_{i}\left(i \in \mathbf{Z}_{k}\right)$.

Proof Let $M$ be a perfect matching of type 1 in $G$. Let us consider the claw $C_{i}$ for some $i$ in $\mathbf{Z}_{k}$. Assume without loss of generality that the edge of $M$ contained in $C_{i}$ is $t_{i} x_{i}$. The cycle of $G \backslash M$ visiting $x_{i}$ comes from $C_{i-1}$, crosses $C_{i}$ by using the vertex $x_{i}$ and goes to $C_{i+1}$. By Lemma 11, the path $y_{i} t_{i} z_{i}$ is contained in a cycle of $G \backslash M$. The two edges incident to $y_{i}$ and $z_{i}$ joining $C_{i}$ to $C_{i-1}$ (as well as those joining $C_{i}$ to $C_{i+1}$ ) are not contained both in $M$ (since $M$ has type 1). Thus, the cycle of $G \backslash M$ containing $y_{i} t_{i} z_{i}$ comes from $C_{i-1}$, crosses $C_{i}$ and goes to $C_{i+1}$. Thus, we have at most two cycles in $G \backslash M$, as claimed, and we can note that each claw must be visited by these cycles.

Definition 7 Let us suppose that $M$ is a perfect matching of type 1 in $G=$ $F S(j, k)$ such that the 2 -factor $G \backslash M$ has exactly two cycles $\Gamma_{1}$ and $\Gamma_{2}$. A claw $C_{i}$ intersected by three vertices of $\Gamma_{1}$ (respectively $\Gamma_{2}$ ) is said to be $\Gamma_{1}$-major (respectively $\Gamma_{2}$-major).

Lemma 8 Let $M$ be a perfect matching of type 1 of $G=F S(j, k)$ such that the 2 -factor $G \backslash M$ has exactly two cycles. Then, the lengths of these two cycles have the same parity as $k$, and those lengths are distinct when $k$ is odd.

Proof Let $\Gamma_{1}$ and $\Gamma_{2}$ be the two cycles of $G \backslash M$. By Lemma 6, for each $i$ in $\mathbf{Z}_{k}$ these two cycles must cross the claw $C_{i}$. Let $k_{1}$ be the number of $\Gamma_{1}$-major claws and let $k_{2}$ be the number of $\Gamma_{2}$-major claws. We have $k_{1}+k_{2}=k$, $l\left(\Gamma_{1}\right)=3 k_{1}+k_{2}$ and $l\left(\Gamma_{2}\right)=3 k_{2}+k_{1}$. When $k$ is odd, we must have either $k_{1}$ odd and $k_{2}$ even, or $k_{1}$ even and $k_{2}$ odd. Then $\Gamma_{1}$ and $\Gamma_{2}$ have distinct odd lengths. When $k$ is even, we must have either $k_{1}$ and $k_{2}$ even, or $k_{2}$ and $k_{1}$ odd. Then $\Gamma_{1}$ and $\Gamma_{2}$ have even lengths.

Lemma 9 Let $M$ be a perfect matching of type 1 of $G=F S(j, k)$ such that the 2 -factor $G \backslash M$ has exactly two cycles $\Gamma_{1}$ and $\Gamma_{2}$. Suppose that there are two consecutive $\Gamma_{1}$-major claws $C_{j}$ and $C_{j+1}$ with $j \in \mathbf{Z}_{k} \backslash\{k-1\}$. Then there is a perfect matching $M^{\prime}$ of type 1 such that the $2-$ factor $G \backslash M^{\prime}$ has exactly two cycles $\Gamma_{1}^{\prime}$ and $\Gamma_{2}^{\prime}$ having the following properties:
a) for $i \in \mathbf{Z}_{k} \backslash\{j, j+1\} C_{i}$ is $\Gamma_{2}^{\prime}$-major if and only if $C_{i}$ is $\Gamma_{2}$-major,
b) $C_{j}$ and $C_{j+1}$ are $\Gamma_{2}^{\prime}$-major,
c) $l\left(\Gamma_{1}^{\prime}\right)=l\left(\Gamma_{1}\right)-4$ and $l\left(\Gamma_{2}^{\prime}\right)=l\left(\Gamma_{2}\right)+4$.

Proof Consider the claws $C_{j}$ and $C_{j+1}$. Since $C_{j}$ is a $\Gamma_{1}$-major claw suppose without loss of generality that $t_{j} z_{j}$ belongs to $M$ and that $\Gamma_{1}$ contains the path $x_{j-1}^{\prime} x_{j} t_{j} y_{j} y_{j+1}$ where $x_{j-1}^{\prime}$ denotes the neighbour of $x_{j}$ in $C_{j-1}$ (then $x_{j} x_{j+1}$ belongs to $M$ ). Since $C_{j+1}$ is $\Gamma_{1}$-major and $\Gamma_{2}$ goes through $C_{j}$ and $C_{j+1}$, the cycle $\Gamma_{1}$ must contain the path $y_{j+1} t_{j+1} x_{j+1} x_{j+2}^{\prime}$ where $x_{j+2}^{\prime}$ denotes the neighbour of $x_{j+1}$ in $C_{j+2}$ (then $M$ contains $t_{j+1} z_{j+1}$ and $y_{j+1} y_{j+2}^{\prime}$ ). Denote by $P_{1}$ the path $x_{j-1}^{\prime} x_{j} t_{j} y_{j} y_{j+1} t_{j+1} x_{j+1} x_{j+2}^{\prime}$. Note that $\Gamma_{2}$ contains the path $P_{2}=z_{j-1}^{\prime} z_{j} z_{j+1} z_{j+1}^{\prime}$ where $z_{j-1}^{\prime}$ and $z_{j+1}^{\prime}$ are defined similarly. See to the left part of Figure

Let us perform the following local transformation: delete $x_{j} x_{j+1}, t_{j} z_{j}$ and $t_{j+1} z_{j+1}$ from $M$ and add $z_{j} z_{j+1}, t_{j} x_{j}$ and $t_{j+1} x_{j+1}$. Let $M^{\prime}$ be the resulting perfect matching. Then the subpath $P_{1}$ of $\Gamma_{1}$ is replaced by $P_{1}^{\prime}=x_{j-1}^{\prime} x_{j} x_{j+1} x_{j+2}^{\prime}$ and the subpath $P_{2}$ of $\Gamma_{2}$ is replaced by $P_{2}^{\prime}=z_{j-1}^{\prime} z_{j} t_{j} y_{j} y_{j+1} t_{j+1} z_{j+1} z_{j+2}^{\prime}$ (see Figure (2). We obtain a new 2-factor containing two new cycles $\Gamma_{1}^{\prime}$ and $\Gamma_{2}^{\prime}$. Note that $C_{j}$ and $C_{j+1}$ are $\Gamma_{2}^{\prime}$-major claws and for $i$ in $\mathbf{Z}_{k} \backslash\{j, j+1\} C_{i}$ is $\Gamma_{2}^{\prime}$-major (respectively $\Gamma_{1}^{\prime}$-major) if and only if $C_{i}$ is $\Gamma_{2}$-major (respectively
$\Gamma_{1}$-major). The length of $\Gamma_{1}$ (now $\Gamma_{1}^{\prime}$ ) decreases of 4 units while the length of $\Gamma_{2}$ (now $\Gamma_{2}^{\prime}$ ) increases of 4 units.


Fig. 2. Local transformation of type 1
The operation depicted in Lemma 9 above will be called a local transformation of type 1 .

Lemma 10 Let $M$ be a perfect matching of type 1 of $G=F S(j, k)$ such that the 2 -factor $G \backslash M$ has exactly two cycles $\Gamma_{1}$ and $\Gamma_{2}$. Suppose that there are three consecutive claws $C_{j}, C_{j+1}$ and $C_{j+2}$ with $j$ in $\mathbf{Z}_{k} \backslash\{k-1, k-2\}$ such that $C_{j}$ and $C_{j+2}$ are $\Gamma_{1}$-major and $C_{j+1}$ is $\Gamma_{2}$-major. Then there is a perfect matching $M^{\prime}$ of type 1 such that the 2 -factor $G \backslash M^{\prime}$ has exactly two cycles $\Gamma_{1}^{\prime}$ and $\Gamma_{2}^{\prime}$ having the following properties:
a) for $i \in \mathbf{Z}_{k} \backslash\{j, j+1, j+2\} C_{i}$ is $\Gamma_{2}^{\prime}$-major if and only if $C_{i}$ is $\Gamma_{2}$-major,
b) $C_{j}$ and $C_{j+2}$ are $\Gamma_{2}^{\prime}$-major and $C_{j+1}$ is $\Gamma_{1}^{\prime}$-major,
c) $l\left(\Gamma_{1}^{\prime}\right)=l\left(\Gamma_{1}\right)-2$ and $l\left(\Gamma_{2}^{\prime}\right)=l\left(\Gamma_{2}\right)+2$.

Proof Since $C_{j}$ is $\Gamma_{1}$-major, as in the proof of Lemma 8 suppose that $\Gamma_{1}$ contains the path $x_{j-1}^{\prime} x_{j} t_{j} y_{j} y_{j+1}$ (that is edges $t_{j} z_{j}$ and $x_{j} x_{j+1}$ belong to $M)$. Since $C_{j+1}$ is $\Gamma_{2}$-major the cycle $\Gamma_{1}$ contains the edge $y_{j+1} y_{j+2}$. Then we see that $\Gamma_{1}$ contains the path $Q_{1}=x_{j-1}^{\prime} x_{j} t_{j} y_{j} y_{j+1} y_{j+2} t_{j+2} z_{j+2} z_{j+3}^{\prime}$ and that $\Gamma_{2}$ contains the path $Q_{2}=z_{j-1}^{\prime} z_{j} z_{j+1} t_{j+1} x_{j+1} x_{j+2} x_{j+3}^{\prime}$. Note that $y_{j+1} t_{j+1}$, $z_{j+1} z_{j+2}$ and $t_{j+2} x_{j+2}$ belong to $M$.

Let us perform the following local transformation: delete $t_{j} z_{j}, x_{j} x_{j+1}, z_{j+1} z_{j+2}$ and $x_{j+2} t_{j+2}$ from $M$ and add $x_{j} t_{j}, z_{j} z_{j+1}, x_{j+1} x_{j+2}$ and $z_{j+2} t_{j+2}$ to $M$. Let $M^{\prime}$ be the resulting perfect matching. Then the subpath $Q_{1}$ of $\Gamma_{1}$ is replaced by $Q_{1}^{\prime}=x_{j-1}^{\prime} x_{j} x_{j+1} t_{j+1} z_{j+1} z_{j+2} z_{j+3}^{\prime}$ and the subpath $Q_{2}$ of $\Gamma_{2}$ is replaced by $Q_{2}^{\prime}=z_{j-1}^{\prime} z_{j} t_{j} y_{j} y_{j+1} y_{j+2} t_{j+2} x_{j+2} x_{j+3}^{\prime}$ (see Figure (3). We obtain a new 2 -factor containing two new cycles named $\Gamma_{1}^{\prime}$ and $\Gamma_{2}^{\prime}$. Note that $C_{j}$ and $C_{j+2}$ are now $\Gamma_{2}^{\prime-}$ major claws and $C_{j+1}$ is $\Gamma_{1}^{\prime}$-major. The length of $\Gamma_{1}$ decreases of 2 units while the length of $\Gamma_{2}$ increases of 2 units. It is clear that for $i \in \mathbf{Z}_{k} \backslash\{j, j+1, j+2\}$ $C_{i}$ is $\Gamma_{2}^{\prime}$-major (respectively $\Gamma_{1}^{\prime}$-major) if and only if $C_{i}$ is $\Gamma_{2}$-major (respectively $\Gamma_{1}$-major).


Fig. 3. Local transformation of type 2
The operation depicted in Lemma 10 above will be called a local transformation of type 2.

Lemma 11 Let $M$ be a perfect matching of type 1 of $G=F S(j, k)$ such that the 2 -factor $G \backslash M$ has exactly two cycles $\Gamma_{1}$ and $\Gamma_{2}$. Suppose that there are three consecutive claws $C_{j}, C_{j+1}$ and $C_{j+2}$ with $j$ in $\mathbf{Z}_{k} \backslash\{k-1, k-2\}$ such that $C_{j+1}$ and $C_{j+2}$ are $\Gamma_{2}$-major and $C_{j}$ is $\Gamma_{1}$-major. Then there is a perfect matching $M^{\prime}$ of type 1 such that the 2 -factor $G \backslash M^{\prime}$ has exactly two cycles $\Gamma_{1}^{\prime}$ and $\Gamma_{2}^{\prime}$ having the following properties:
a) for $i \in \mathbf{Z}_{k} \backslash\{j, j+1, j+2\} C_{i}$ is $\Gamma_{2}^{\prime}$-major if and only if $C_{i}$ is $\Gamma_{2}$-major,
b) $C_{j}$ and $C_{j+1}$ are $\Gamma_{2}^{\prime}$-major and $C_{j+2}$ is $\Gamma_{1}^{\prime}$-major,
c) $l\left(\Gamma_{1}^{\prime}\right)=l\left(\Gamma_{1}\right)$ and $l\left(\Gamma_{2}^{\prime}\right)=l\left(\Gamma_{2}\right)$.

Proof Since $C_{j}$ is $\Gamma_{1}$-major, as in the proof of Lemma 9 suppose that $\Gamma_{1}$ contains the path $x_{j-1}^{\prime} x_{j} t_{j} y_{j} y_{j+1}$ (that is edges $t_{j} z_{j}$ and $x_{j} x_{j+1}$ belong to $M)$. Since $C_{j+1}$ and $C_{j+2}$ are $\Gamma_{2}$-major, the unique vertex of $C_{j+1}$ (respectively $C_{j+2}$ ) contained in $\Gamma_{1}$ is $y_{j+1}$ (respectively $y_{j+2}$ ). Note that the perfect matching $M$ contains the edges $t_{j} z_{j}, x_{j} x_{j+1}, t_{j+1} y_{j+1}, z_{j+1} z_{j+2}$ and $t_{j+2} y_{j+2}$. Then the path $R_{1}=x_{j-1}^{\prime} x_{j} t_{j} y_{j} y_{j+1} y_{j+2} y_{j+3}^{\prime}$ is a subpath of $\Gamma_{1}$ and the path $R_{2}=z_{j-1}^{\prime} z_{j} z_{j+1} t_{j+1} x_{j+1} x_{j+2} t_{j+2} z_{j+2} z_{j+3}^{\prime}$ is a subpath of $\Gamma_{2}$. See to the left part of Figure 0 .

Let us perform the following local transformation: delete $t_{j} z_{j}, x_{j} x_{j+1}, t_{j+1} y_{j+1}$, $z_{j+1} z_{j+2}$ and $t_{j+2} y_{j+2}$ from $M$ and add $x_{j} t_{j}, z_{j} z_{j+1}, t_{j+1} x_{j+1}, y_{j+1} y_{j+2}$ and $t_{j+2} z_{j+2}$. Let $M^{\prime}$ be the resulting perfect matching. Then the subpath $R_{1}$ of $\Gamma_{1}$ is replaced by $R_{1}^{\prime}=x_{j-1}^{\prime} x_{j} x_{j+1} x_{j+2} t_{j+2} y_{j+2} y_{j+3}^{\prime}$ and the subpath $R_{2}$ of $\Gamma_{2}$ is replaced by $R_{2}^{\prime}=z_{j-1}^{\prime} z_{j} t_{j} y_{j} y_{j+1} t_{j+1} z_{j+1} z_{j+2} z_{j+3}^{\prime}$. We obtain a new 2factor containing two new cycles named $\Gamma_{1}^{\prime}$ and $\Gamma_{2}^{\prime}$ such that $l\left(\Gamma_{1}^{\prime}\right)=l\left(\Gamma_{1}\right)$ and $l\left(\Gamma_{2}^{\prime}\right)=l\left(\Gamma_{2}\right)$ (see Figure (4). It is clear that for $i \in \mathbf{Z}_{k} \backslash\{j, j+1, j+2\}$ $C_{i}$ is $\Gamma_{2}^{\prime}$-major (respectively $\Gamma_{1}^{\prime}$-major) if and only if $C_{i}$ is $\Gamma_{2}$-major (respectively $\Gamma_{1}$-major). Note that $C_{j}$ and $C_{j+1}$ are $\Gamma_{2}^{\prime}$-major and $C_{j+2}$ is $\Gamma_{1}^{\prime}$-major.


Fig. 4. Local transformation of type 3
The operation depicted in Lemma 11 above will be called a local transformation of type 3.

Lemma 12 Let $M$ be a perfect matching of type 1 of $G=F S(j, k)$ such that the 2 -factor $G \backslash M$ has exactly two cycles $\Gamma_{1}$ and $\Gamma_{2}$ such that $l\left(\Gamma_{1}\right) \leq l\left(\Gamma_{2}\right)$ and $l\left(\Gamma_{2}\right)$ is as great as possible. Then there exists at most one $\Gamma_{1}$-major claw.

Proof Suppose, for the sake of contradiction, that there exist at least two $\Gamma_{1}$-major claws. Since $l\left(\Gamma_{2}\right)$ is maximum, by Lemma 9 these claws are not consecutive. Then consider two $\Gamma_{1}$-major claws $C_{i}$ and $C_{i+h+1}$ (with $h \geq 1$ ) such that the $h$ consecutive claws $\left(C_{i+1}, \ldots, C_{i+h}\right)$ are $\Gamma_{2}$-major. Since $l\left(\Gamma_{2}\right)$ is maximum, by Lemma 10 the number $h$ is at least 2 . Then by applying $r=\left\lfloor\frac{h}{2}\right\rfloor$ consecutive local transformations of type 3 (Lemma 11) we obtain a perfect matching $M^{(r)}$ such that the 2-factor $G \backslash M^{(r)}$ has exactly two cycles $\Gamma_{1}^{(r)}$ and $\Gamma_{2}^{(r)}$ with $l\left(\Gamma_{1}^{(r)}\right)=l\left(\Gamma_{1}\right)$ and $l\left(\Gamma_{2}^{(r)}\right)=l\left(\Gamma_{2}\right)$ and such that $C_{i+2\left\lfloor\frac{h}{2}\right\rfloor}$ and $C_{i+h+1}$ are $\Gamma_{1}^{(r)}$-major. Since $l\left(\Gamma_{2}^{(r)}\right)$ is maximum, we can conclude by Lemma 9 and by Lemma 10 that $h$ is neither even nor odd, a contradiction.

### 3.2 Perfect matchings of type 2

We give here a structural result about perfect matchings of type 2 in $G=$ $F S(j, k)$.

Lemma 13 Let $M$ be a perfect matching of type 2 of $G=F S(j, k)$ (with $k \geq 4$ ). Then the 2 -factor $G \backslash M$ has exactly one cycle of even length $l \geq k$ and a set of $p$ cycles of length 6 where $l+6 p=4 k$ (with $0 \leq p \leq \frac{k}{2}$ ).

Proof Let $M$ be a perfect matching of type 2 in $G$. By Lemma 1 the number $k$ of claws is even. Let $i$ in $\mathbf{Z}_{k}$ such that there are two edges of $M$ between $C_{i-1}$ and $C_{i}$. There are no edges of $M$ between $C_{i}$ and $C_{i+1}$ and two edges of $M$ between $C_{i+1}$ and $C_{i+2}$. We may consider that $0 \leq i<k-1$.
For $j \in\{i, i+2, i+4, \ldots\}$ we denote by $e_{j}$ the unique edge of $G \backslash M$ having
one end vertex in $C_{j-1}$ and the other in $C_{j}$. Let us denote by $A$ the set $\left\{e_{i}, e_{i+2}, e_{i+4}, \ldots\right\}$. We note that $|A|=\frac{k}{2}$.
Assume without loss of generality that the two edges of $M$ between $C_{i-1}$ and $C_{i}$ have end vertices in $C_{i}$ which are $x_{i}$ and $y_{i}$ (then $z_{i}$ is the end vertex of $e_{i}$ in $C_{i}$ ). Two cases may now occur.

Case 1: The end vertices in $C_{i+1}$ of the two edges of $M$ between $C_{i+1}$ and $C_{i+2}$ are $x_{i+1}$ and $y_{i+1}$ (then $z_{i+1}$ is the end vertex of $e_{i+2}$ in $C_{i+1}$ ).
In that case the 2 - factor $G \backslash M$ contains the cycle of length $6 x_{i} x_{i+1} t_{i+1} y_{i+1} y_{i} t_{i}$ while the edge $z_{i} z_{i+1}$ of $G \backslash M$ relies $e_{i}$ and $e_{i+2}$.

Case 2: The end vertices in $C_{i+1}$ of the two edges of $M$ between $C_{i+1}$ and $C_{i+2}$ are $y_{i+1}$ and $z_{i+1}$ (respectively $x_{i+1}$ and $z_{i+1}$ ). Then $x_{i+1}$ (respectively $y_{i+1}$ ) is the end vertex of $e_{i+2}$ in $C_{i+1}$.
In that case the edges $e_{i}$ and $e_{i+2}$ are connected in $G \backslash M$ by the path $z_{i} z_{i+1} t_{i+1} y_{i+1} y_{i} t_{i} x_{i} x_{i+1}$ (respectively $z_{i} z_{i+1} t_{i+1} x_{i+1} x_{i} t_{i} y_{i} y_{i+1}$ ).

The same reasoning can be done for $\left\{e_{i+2}, e_{i+4}\right\},\left\{e_{i+4}, e_{i+6}\right\}$, and so on. Then, we see that the set $A$ is contained in a unique cycle $\Gamma$ of $G \backslash M$ which crosses each claw. Thus, the length $l$ of $\Gamma$ is at least $k$. More precisely, each $e_{j}$ in $A$ contributes for 1 in $l$, in Case 1 the edge $z_{i} z_{i+1}$ contributes for 1 in $l$ and in Case 2 the path $z_{i} z_{i+1} t_{i+1} y_{i+1} y_{i} t_{i} x_{i} x_{i+1}$ contributes for 7 in $l$. Let us suppose that Case 1 appears $p$ times $\left(0 \leq p \leq \frac{k}{2}\right)$, that is to say $G \backslash M$ contains $p$ cycles of length 6 . Since Case 2 appears $\frac{k}{2}-p$ times, the length of $\Gamma$ is $l=\frac{k}{2}+p+7\left(\frac{k}{2}-p\right)=4 k-6 p$.

Remark 14 If $k$ is even then by Lemmas 因, 因 and $13 F S(j, k)$ has an even 2 -factor. That is to say $F S(j, k)$ is a cubic 3-edge colourable graph .

## 4 Perfect matchings and hamiltonian cycles of $F(j, k)$

### 4.1 Perfect matchings of type 1 and hamiltonicity

Theorem 15 Let $M$ be a perfect matching of type 1 of $G=F S(j, k)$. Then the 2 -factor $G \backslash M$ is a hamiltonian cycle except for $k$ odd and $j=2$, and for $k$ even and $j=1$ or 3 .

Proof Suppose that there exists a perfect matching $M$ of type 1 of $G$ such that $G \backslash M$ is not a hamiltonian cycle. By Lemma 6 and Lemma 8 the 2factor $G \backslash M$ is made of exactly two cycles $\Gamma_{1}$ and $\Gamma_{2}$ whose lengths have the same parity as $k$. Without loss of generality we suppose that $l\left(\Gamma_{1}\right) \leq l\left(\Gamma_{2}\right)$.

Assume moreover that among the perfect matchings of type 1 of $G$ such that the 2 -factor $G \backslash M$ is composed of two cycles, $M$ has been chosen in such a way that the length of the longest cycle $\Gamma_{2}$ is as great as possible. By Lemma 12 there exists at most one $\Gamma_{1}$-major claw.

Case 1: There exists one $\Gamma_{1}$-major claw.
Without loss of generality, suppose that $C_{0}$ is intersected by $\Gamma_{1}$ in $\left\{y_{0}, t_{0}, x_{0}\right\}$ and that $y_{k-1}^{\prime} y_{0}$ belongs to $\Gamma_{1}$. Since for every $i \neq 0$ the claw $C_{i}$ is $\Gamma_{2}$-major, $\Gamma_{1}$ contains the vertices $y_{0}, t_{0}, x_{0}, x_{1}, x_{2}, \ldots, x_{k-1}$.

- If $k=2 r+1$ with $r \geq 1$ then $\Gamma_{2}$ contains the path

$$
z_{0} z_{1} t_{1} y_{1} y_{2} t_{2} z_{2} \ldots z_{2 r-1} t_{2 r-1} y_{2 r-1} y_{2 r} t_{2 r} z_{2 r}
$$

Thus, $y_{0} x_{k-1}, x_{0} y_{k-1}, z_{0} z_{k-1}$ are edges of $G$. This means that $\cup_{i=0}^{i=k-1}\left\{C_{i} \backslash\left\{t_{i}\right\}\right\}$ induces two cycles, that is to say $j=2$ and $G=F S(2, k)$.

- If $k=2 r+2$ with $r \geq 1$ then $\Gamma_{2}$ contains the path

$$
z_{0} z_{1} t_{1} y_{1} y_{2} t_{2} z_{2} \ldots z_{2 r-1} t_{2 r-1} y_{2 r-1} y_{2 r} t_{2 r} z_{2 r} z_{2 r+1} t_{2 r+1} y_{2 r+1}
$$

Thus, $x_{0} z_{k-1}, y_{0} x_{k-1}$ and $z_{0} y_{k-1}$ are edges. This means that $\cup_{i=0}^{i=k-1}\left\{C_{i} \backslash\left\{t_{i}\right\}\right\}$ induces one cycle, that is to say $j=1$ and $G=F S(1, k)$.

Case 2: There is no $\Gamma_{1}$-major claw.
Suppose that $x_{0}$ belongs to $\Gamma_{1}$. Then, $\Gamma_{1}$ contains $x_{0}, x_{1}, \ldots, x_{k-1}$.

- If $k=2 r+1$ with $r \geq 1$ then $\Gamma_{2}$ contains the path

$$
y_{0} t_{0} z_{0} z_{1} t_{1} y_{1} y_{2} \ldots z_{2 r-1} t_{2 r-1} y_{2 r-1} y_{2 r} t_{2 r} z_{2 r}
$$

Thus, $x_{0} x_{k-1}, y_{0} z_{k-1}$ and $z_{0} y_{k-1}$ are edges of $G$ and the set $\cup_{i=0}^{i=k-1}\left\{C_{i} \backslash\left\{t_{i}\right\}\right\}$ induces two cycles, that is to say $j=2$ and $G=F S(2, k)$.

- If $k=2 r+2$ with $r \geq 1$ then $\Gamma_{2}$ contains the path

$$
y_{0} t_{0} z_{0} z_{1} t_{1} y_{1} y_{2} \ldots y_{2 r} t_{2 r} z_{2 r} z_{2 r+1} t_{2 r+1} y_{2 r+1} .
$$

Thus, $x_{0} x_{k-1}, y_{0} y_{k-1}$ and $z_{0} z_{k-1}$ are edges. This means that $\cup_{i=0}^{i=k-1}\left\{C_{i} \backslash\left\{t_{i}\right\}\right\}$ induces three cycles, that is to say $j=3$ and $G=F S(3, k)$.

Definition 16 A cubic graph $G$ is said to be 2-factor hamiltonian [6] if every 2 -factor of $G$ is a hamiltonian cycle (or equivalently, if for every perfect
matching $M$ of $G$ the 2-factor $G \backslash M$ is a hamiltonian cycle).

By Theorem 15 for any odd $k \geq 3$ and $j \in\{1,3\}$ or for any even $k$ and $j=2$, and for every perfect matching $M$ of type 1 in $F S(j, k)$ the 2 -factor $F S(j, k) \backslash M$ is a hamiltonian cycle. By Lemma $13 F S(2, k)(k \geq 4)$ may have a perfect matching $M$ of type 2 such that the 2-factor $F S(2, k) \backslash M$ is not a hamiltonian cycle (it may contains cycles of length 6).

Then we have the following.
Corollary 17 A graph $G=F S(j, k)$ is 2-factor hamiltonian if and only if $k$ is odd and $j=1$ or 3 .

We note that $F S(1,3)$ is the "Triplex Graph" of Robertson, Seymour and Thomas [15]. We shall examine others known results about 2-factor hamiltonian cubic graphs in Section 5 .

Corollary 18 The chromatic index of a graph $G=F S(j, k)$ is 4 if and only if $j=2$ and $k$ is odd.

Proof When $j=2$ and $k$ is odd, any 2 -factor must have at least two cycles, by Theorem 15. Then Lemma 8 implies that any 2 -factor is composed of two odd cycles. Hence $G$ has chromatic index 4 .

When $j=1$ or 3 and $k$ is odd by Theorem $15 F S(j, k)$ is hamiltonian. If $k$ is even then by Lemmas 6, 8 and $13 F S(j, k)$ has an even 2-factor.

### 4.2 Perfect matchings of type 2 and hamiltonicity

At this point of the discourse one may ask what happens for perfect matchings of type 2 in $F S(j, k)$ ( $k$ even). Can we characterize and count perfect matchings of type 2 , complementary 2 -factor of which is a hamiltonian cycle ? An affirmative answer shall be given.

Let us consider a perfect matching $M$ of type 2 in $F S(j, 2 p)$ with $p \geq 2$. Suppose that there are no edges of $M$ between $C_{2 i-1}$ and $C_{2 i}$ (for any $i \geq 1$ ), that is $M$ is a matching of type 2.0 (see Definition (2). Consider two consecutive claws $C_{2 i}$ and $C_{2 i+1}(0 \leq i \leq p-1)$. There are three cases:

Case (x): $\left\{y_{2 i} y_{2 i+1}, z_{2 i} z_{2 i+1}\right\} \subset M$ (then, $\left.M \cap\left(C_{2 i} \cup C_{2 i+1}\right)=\left\{x_{2 i} t_{2 i}, x_{2 i+1} t_{2 i+1}\right\}\right)$.
Case (y): $\left\{x_{2 i} x_{2 i+1}, z_{2 i} z_{2 i+1}\right\} \subset M\left(\right.$ then, $\left.M \cap\left(C_{2 i} \cup C_{2 i+1}\right)=\left\{y_{2 i} t_{2 i}, y_{2 i+1} t_{2 i+1}\right\}\right)$.

Case (z): $\left\{x_{2 i} x_{2 i+1}, y_{2 i} y_{2 i+1}\right\} \subset M$ (then, $\left.M \cap\left(C_{2 i} \cup C_{2 i+1}\right)=\left\{z_{2 i} t_{2 i}, z_{2 i+1} t_{2 i+1}\right\}\right)$.
The subgraph induced on $C_{2 i} \cup C_{2 i+1}$ is called a block. In Case (x) (respectively Case (y), Case (z)) a block is called a block of type $X$ (respectively block of type $Y$, block of type $Z$ ). Then $F S(j, 2 p)$ with a perfect matchings $M$ of type 2.0 can be seen as a sequence of $p$ blocks properly relied. In other words, a perfect matchings $M$ of type 2 in $F S(j, 2 p)$ is entirely described by a word of length $p$ on the alphabet of three letters $\{X, Y, Z\}$. The block $C_{0} \cup C_{1}$ is called initial block and the block $C_{2 p-1} \cup C_{2 p}$ is called terminal block. These extremal blocks are not considered here as consecutive blocks.

By Lemma [13, $F S(j, 2 p) \backslash M$ has no 6 -cycles if and only if $F S(j, 2 p) \backslash M$ is a unique even cycle. It is an easy matter to prove that two consecutive blocks do not induce a 6 -cycle if and only if they are not of the same type. Then the possible configurations for two consecutive blocks are $X Y, X Z, Y X$, $Y Z, Z X$ and $Z Y$. To eliminate a possible 6 -cycle in $C_{0} \cup C_{2 p-1}$ we have to determine for every $j \in\{1,2,3\}$ the forbidden extremal configurations. An extremal configuration shall be denoted by a word on two letters in $\{X, Y, Z\}$ such that the left letter denotes the type of the initial block $C_{0} \cup C_{1}$ and the right letter denotes the type of the terminal block $C_{2 p-1} \cup C_{2 p}$. We suppose that the extremal blocks are connected for $j=1$ by the edges $x_{2 p-1} z_{0}, y_{2 p-1} x_{0}$ and $z_{2 p-1} y_{0}$, for $j=2$ by the edges $x_{2 p-1} x_{0}, y_{2 p-1} z_{0}$ and $z_{2 p-1} y_{0}$ and for $j=3$ by the edges $x_{2 p-1} x_{0}, y_{2 p-1} y_{0}$ and $z_{2 p-1} z_{0}$. Then, it is easy to verify that we have the following result.

Lemma 19 Let $M$ be a perfect matching of type 2.0 of $G=F S(j, 2 p)$ (with $p \geq 2$ ) such that the $2-$ factor $G \backslash M$ is a hamiltonian cycle. Then the forbidden extremal configurations are
$X Y, Y Z$ and $Z X$ for $F S(1,2 p)$,
$X X, Y Z$ and $Z Y$ for $F S(2,2 p)$,
and $X X, Y Y$ and $Z Z$ for $F S(3,2 p)$.

Thus, any perfect matching $M$ of type 2.0 of $F S(j, 2 p)$ such that the 2 -factor $G \backslash M$ is a hamiltonian cycle is totally characterized by a word of length $p$ on the alphabet $\{X, Y, Z\}$ having no two identical consecutive letters and such that the sub-word [initial letter][terminal letter] is not a forbidden configuration. Then, we are in position to obtain the number of such perfect matchings in $F S(j, 2 p)$. Let us denote by $\mu_{2.0}^{\prime}(j, 2 p)$ (respectively $\mu_{2.1}^{\prime}(j, 2 p)$, $\left.\mu_{2}^{\prime}(j, 2 p)\right)$ the number of perfect matchings of type 2.0 (respectively type 2.1 , type 2) complementary to a hamiltonian cycle in $F S(j, 2 p)$. Clearly $\mu_{2}^{\prime}(j, 2 p)=$ $\mu_{2.0}^{\prime}(j, 2 p)+\mu_{2.1}^{\prime}(j, 2 p)$ and $\mu_{2.0}^{\prime}(j, 2 p)=\mu_{2.1}^{\prime}(j, 2 p)$.

Theorem 20 The numbers $\mu_{2}^{\prime}(j, 2 p)$ of perfect matchings of type 2 complementary to hamiltonian cycles in $F S(j, 2 p)(j \in\{1,2,3\})$ are given by:

$$
\begin{aligned}
\mu_{2}^{\prime}(1,2 p) & =2^{p+1}+(-1)^{p+1} 2, \\
\mu_{2}^{\prime}(2,2 p) & =2^{p+1}, \\
\text { and } \quad \mu_{2}^{\prime}(3,2 p) & =2^{p+1}+(-1)^{p} 4 .
\end{aligned}
$$

Proof Consider, as previously, perfect matchings of type 2.0. Let $\alpha$ and $\beta$ be two letters in $\{X, Y, Z\}$ (not necessarily distinct). Let $A_{\alpha \beta}^{p}$ be the set of words of length $p$ on $\{X, Y, Z\}$ having no two consecutive identical letters, beginning by $\alpha$ and ending by a letter distinct from $\beta$. Denote the number of words in $A_{\alpha \beta}^{p}$ by $a_{\alpha \beta}^{p}$. Let $B_{\alpha \beta}^{p}$ be the set of words of length $p$ on $\{X, Y, Z\}$ having no two consecutive identical letters, beginning by $\alpha$ and ending by $\beta$. Denote by $b_{\alpha \beta}^{p}$ the number of words in $B_{\alpha \beta}^{p}$.

Clearly, the number of words of length $p$ having no two consecutive identical letters and beginning by $\alpha$ is $2^{p-1}$. Then $a_{\alpha \beta}^{p}+b_{\alpha \beta}^{p}=2^{p-1}$. The deletion of the last $\beta$ of a word in $B_{\alpha \beta}^{p}$ gives a word in $A_{\alpha \beta}^{p-1}$ and the addition of $\beta$ to the right of a word in $A_{\alpha \beta}^{p-1}$ gives a word in $B_{\alpha \beta}^{p}$.

Thus $b_{\alpha \beta}^{p}=a_{\alpha \beta}^{p-1}$ and for every $p \geq 3 a_{\alpha \beta}^{p}=2^{p-1}-a_{\alpha \beta}^{p-1}$. We note that $a_{\alpha \beta}^{2}=2$ if $\alpha=\beta$, and $a_{\alpha \beta}^{2}=1$ if $\alpha \neq \beta$. If $\alpha=\beta$ we have to solve the recurrent sequence : $u_{2}=2$ and $u_{p}=2^{p-1}-u_{p-1}$ for $p \geq 3$. If $\alpha \neq \beta$ we have to solve the recurrent sequence : $v_{2}=1$ and $v_{p}=2^{p-1}-v_{p-1}$ for $p \geq 3$. Then we obtain $u_{p}=\frac{2}{3}\left(2^{p-1}+(-1)^{p}\right)$ and $v_{p}=\frac{1}{3}\left(2^{p}+(-1)^{p+1}\right)$ for $p \geq 2$.

By Lemma 19

$$
\begin{aligned}
\mu_{2.0}^{\prime}(1,2 p) & =a_{X Y}^{p}+a_{Y Z}^{p}+a_{Z X}^{p}=3 v_{p}=2^{p}+(-1)^{p+1} \\
\mu_{2.0}^{\prime}(2,2 p) & =a_{X X}^{p}+a_{Y Z}^{p}+a_{Z Y}^{p}=u_{p}+2 v_{p}=2^{p} \\
\text { and } \quad \mu_{2.0}^{\prime}(3,2 p) & =a_{X X}^{p}+a_{Y Y}^{p}+a_{Z Z}^{p}=3 u_{p}=2^{p}+(-1)^{p} 2 .
\end{aligned}
$$

Since $\mu_{2}^{\prime}(j, 2 p)=\mu_{2.0}^{\prime}(j, 2 p)+\mu_{2.1}^{\prime}(j, 2 p)$ and $\mu_{2.0}^{\prime}(j, 2 p)=\mu_{2.1}^{\prime}(j, 2 p)$ we obtain the announced results.

Remark 21 We see that $\mu_{2}^{\prime}(j, 2 p) \simeq 2^{p+1}$ and this is to compare with the number $\mu_{2}(j, 2 p)=2 \times 3^{p}$ of perfect matchings of type 2 in $F S(j, 2 p)$ (see backward in Section (2).

### 4.3 Strong matchings and Jaeger's graphs

For a given graph $G=(V, E)$ a strong matching (or induced matching) is a matching $S$ such that no two edges of $S$ are joined by an edge of $G$. That is, $S$ is the set of edges of the subgraph of $G$ induced by the set $V(S)$. We consider cubic graphs having a perfect matching which is the union of two strong matchings that we call Jaeger's graph (in his thesis [g] Jaeger called these cubic graphs equitable). We call Jaeger's matching a perfect matching $M$ of a cubic graph $G$ which is the union of two strong matchings $M_{B}$ and $M_{R}$. Set $B=V\left(M_{B}\right)$ (the blue vertices) and $R=V\left(M_{R}\right)$ (the red vertices). An edge of $G$ is said mixed if its end vertices have distinct colours. Since the set of mixed edges is $E(G) \backslash M$, the 2-factor $G \backslash M$ is even and $|B|=|M|$. Thus, every Jaeger's graph $G$ is a cubic 3-edge colourable graph and for any Jaeger's matching $M=M_{B} \cup M_{R},\left|M_{B}\right|=\left|M_{R}\right|$. See, for instance, [3] and [4] for some properties of these graphs.

In this subsection we determine the values of $j$ and $k$ for which a graph $F S(j, k)$ is a Jaeger's graph.

Lemma 22 If $G=F S(j, k)$ is a Jaeger's graph (with $k \geq 3$ ) and $M=$ $M_{B} \cup M_{R}$ is a Jaeger's matching of $G$ then $M$ is a perfect matching of type 1 .

Proof Suppose that $M$ is of type 2 and suppose without loss of generality that there are two edges of $M$ between $C_{0}$ and $C_{1}$, for instance $x_{0} x_{1}$ and $y_{0} y_{1}$. Then $C_{0} \cap M=\left\{t_{0} z_{0}\right\}$ and $C_{1} \cap M=\left\{t_{1} z_{1}\right\}$. Suppose that $x_{0} x_{1}$ and $y_{0} y_{1}$ belong to $M_{B}$. Since $M_{B}$ is a strong matching, $t_{0} z_{0}$ and $t_{1} z_{1}$ belong to $M \backslash M_{B}=M_{R}$. This is impossible because $M_{R}$ is also a strong matching. By symmetry there are no two edges of $M_{R}$ between $C_{0}$ and $C_{1}$. Then there is one edge of $M_{B}$ between $C_{0}$ and $C_{1}, x_{0} x_{1}$ for instance, and one edge of $M_{R}$ between $C_{0}$ and $C_{1}, y_{0} y_{1}$ for instance. Since $M_{B}$ and $M_{R}$ are strong matchings, there is no edge of $M$ in $C_{0} \cup C_{1}$, a contradiction. Thus, $M$ is a perfect matching of type 1 .

Lemma 23 If $G=F S(j, k)$ is a Jaeger's graph (with $k \geq 3$ ) then either $(j=1$ and $k \equiv 1$ or $2(\bmod 3))$ or $(j=3$ and $k \equiv 0(\bmod 3))$.

Proof Let $M=M_{B} \cup M_{R}$ be a Jaeger's matching of $G$. By Lemma 22 $M$ is a perfect matching of type 1 . Suppose without loss of generality that $M_{B} \cap E\left(C_{0}\right)=\left\{x_{0} t_{0}\right\}$. Since $M_{B}$ is a strong matching there is no edge of $M_{B}$ between $C_{0}$ and $C_{1}$. Suppose, without loss of generality, that the edge in $M_{R}$ joining $C_{0}$ to $C_{1}$ is $y_{0} y_{1}$. Consider the claws $C_{0}, C_{1}$ and $C_{2}$. Since $M_{B}$ and $M_{R}$ are strong matchings, we can see that the choices of $x_{0} t_{0} \in M_{B}$ and $y_{0} y_{1} \in M_{R}$ fixes the positions of the other edges of $M_{B}$ and $M_{R}$. More
precisely, $\left\{t_{1} z_{1}, y_{2} t_{2}\right\} \subset M_{B}$ and $\left\{x_{1} x_{2}, z_{2} z_{3}^{\prime}\right\} \subset M_{R}$. This unique configuration is depicted in Figure 5 .


Fig. 5. Strong matchings $M_{B}$ (bold edges) and $M_{R}$ (dashed edges)
If $k \geq 4$ then we see that $z_{2} z_{3} \in M_{R}, x_{3} t_{3} \in M_{B}$, and $y_{3} y_{4}^{\prime} \in M_{R}$. So, the local situation in $C_{3}$ is similar to that in $C_{0}$, and we can see that there is a unique Jaeger's matching $M=M_{B} \cup M_{R}$ such that $x_{0} t_{0} \in M_{B}$ and $y_{0} y_{1} \in M_{R}$ in the graph $F S(j, k)$. We have to verify the coherence of the connections between the claws $C_{k-1}$ and $C_{0}$. We note that $M_{B}=M \cap\left(\cup_{i=0}^{i=k-1} E\left(C_{i}\right)\right)$ and $M_{R}$ is a strong matching included in the 2-factor induced by $\cup_{i=0}^{i=k-1}\left\{V\left(C_{i}\right) \backslash\left\{t_{i}\right\}\right\}$.

Case 1: $k=3 p$ with $p \geq 1$.
We have $x_{0} t_{0} \in M_{B}, y_{k-1} t_{k-1} \in M_{B}, x_{k-2} x_{k-1} \in M_{R}$ and $z_{k-1}^{\prime} z_{0}=z_{k-1} z_{0}^{\prime} \in$ $M_{R}$ (that is, $z_{k-1} z_{0} \in M_{R}$ ). Thus, $z_{k-1} z_{0}, y_{k-1} y_{0}$ and $x_{k-1} x_{0}$ are edges of $F S(j, 3 p)$ and we must have $j=3$.

Case 2: $k=3 p+1$ with $p \geq 1$.
We have $x_{0} t_{0} \in M_{B}, x_{k-1} t_{k-1} \in M_{B}$ (that is, $\left.x_{k-1} x_{0} \notin E(G)\right), z_{k-2} z_{k-1} \in M_{R}$ and $z_{k-1}^{\prime} z_{0}=y_{k-1} y_{0}^{\prime} \in M_{R}$ (that is, $y_{k-1} z_{0} \in M_{R}$ ). Thus, $y_{k-1} z_{0}, x_{k-1} y_{0}$ and $z_{k-1} x_{0}$ are edges of $F S(j, 3 p+1)$ and we must have $j=1$.

Case 3: $k=3 p+2$ with $p \geq 1$.
We have $x_{0} t_{0} \in M_{B}, z_{k-1} t_{k-1} \in M_{B}, y_{k-2} y_{k-1} \in M_{R}$ and $z_{k-1}^{\prime} z_{0}=x_{k-1} x_{0}^{\prime} \in$ $M_{R}$ (that is $x_{k-1} z_{0} \in M_{R}$ ). Thus, $x_{k-1} z_{0}, y_{k-1} x_{0}$ and $z_{k-1} y_{0}$ are edges of $F S(j, 3 p+2)$ and we must have $j=1$.

Remark 24 It follows from Lemma 23 that for every $k \geq 3$ the graph $F S(2, k)$ is not a Jaeger's graph. This is obvious when $k$ is odd, since the flower snarks have chromatic index 4.

Then, we obtain the following.
Theorem 25 For $j \in\{1,2,3\}$ and $k \geq 2$, the graph $G=F S(j, k)$ is a Jaeger's graph if and only if

$$
\begin{aligned}
& \text { either } k \equiv 1 \text { or } 2(\bmod 3) \text { and } j=1 \text {, } \\
& \text { or } k \equiv 0(\bmod 3) \text { and } j=3 .
\end{aligned}
$$

Moreover, FS(1,2) has 3 Jaeger's matchings and for $k \geq 3$ a Jaeger's graph $G=F S(j, k)$ has exactly 6 Jaeger's matchings.

Proof For $k=2$ we remark that $F S(1,2)$ (that is the cube) has exactly three distinct Jaeger's matchings $M_{1}, M_{2}$ and $M_{3}$. Following our notations: $M_{1}=\left\{x_{0} t_{0}, t_{1} z_{1}\right\} \cup\left\{y_{0} y_{1}, z_{0} x_{1}\right\}, M_{2}=\left\{z_{0} t_{0}, t_{1} y_{1}\right\} \cup\left\{y_{0} z_{1}, x_{0} x_{1}\right\}$ and $M_{3}=$ $\left\{y_{0} t_{0}, t_{1} x_{1}\right\} \cup\left\{z_{0} z_{1}, x_{0} y_{1}\right\}$.

For $k \geq 3$, by Lemma 23, condition

$$
(*)(j=1 \text { and } k \equiv 1 \text { or } 2(\bmod 3)) \text { or }(j=3 \text { and } k \equiv 0(\bmod 3))
$$

is a necessary condition for $F S(j, k)$ to be a Jaeger's graph.
Consider the function $\Phi_{X, Y}: V(G) \rightarrow V(G)$ such that for every $i$ in $\mathbf{Z}_{k}$, $\Phi_{X, Y}\left(t_{i}\right)=t_{i}, \Phi_{X, Y}\left(z_{i}\right)=z_{i}, \Phi_{X, Y}\left(x_{i}\right)=y_{i}$ and $\Phi_{X, Y}\left(y_{i}\right)=x_{i}$. Define similarly $\Phi_{X, Z}$ and $\Phi_{Y, Z}$. For $j=1$ or 3 these functions are automorphisms of $F S(j, k)$. Thus, the process described in the proof of Lemma 23 is a constructive process of all Jaeger's matchings in a graph $F S(j, k)$ (with $k \geq 3$ ) verifying condition $\left.{ }^{*}\right)$.

We remark that for any choice of an edge $e$ of $C_{0}$ to be in $M_{B}$ there are two distinct possible choices for an edge $f$ between $C_{0}$ and $C_{1}$ to be in $M_{R}$, and such a pair $\{e, f\}$ corresponds exactly to one Jaeger's matching. Then, a Jaeger's graph $F S(j, k)$ (with $k \geq 3$ ) has exactly 6 Jaeger's matchings.

Remark 26 The Berge-Fulkerson Conjecture states that if $G$ is a bridgeless cubic graph, then there exist six perfect matchings $M_{1}, \ldots, M_{6}$ of $G$ (not necessarily distinct) with the property that every edge of $G$ is contained in exactly two of $M_{1}, \ldots, M_{6}$ (this conjecture is attributed to Berge in [16] but appears in [5]). Using each colour of a cubic 3-edge colourable graph twice, we see that such a graph verifies the Berge-Fulkerson Conjecture. Very few is known about this conjecture except that it holds for the Petersen graph and for cubic 3 -edge colourable graphs. So, Berge-Fulkerson Conjecture holds for Jaeger's graphs, but generally we do not know if we can find six distinct perfect matchings. We remark that if $F S(j, k)$, with $k \geq 3$, is a Jaeger's graph then its six Jaeger's matchings are such that every edge is contained in exactly two of them.

## 5 2-factor hamiltonian cubic graphs

Recall that a simple graph of maximum degree $d>1$ with edge chromatic number equal to $d$ is said to be a Class 1 graph. For any $d$-regular simple graph (with $d>1$ ) of even order and of Class 1, for any minimum edgecolouring of such a graph, the set of edges having a given colour is a perfect matching (or 1-factor). Such a regular graph is also called a 1-factorable graph. A Class $1 d$-regular graph of even order is strongly hamiltonian or perfectly 1-factorable (or is a Hamilton graph in the Kotzig's terminology [11]) if it has an edge colouring such that the union of any two colours is a hamiltonian cycle. Such an edge colouring is said to be a Hamilton decomposition in the Kotzig's terminology. In (10) by using two operations $\rho$ and $\pi$ (described also in [1]) and starting from the $\theta$-graph (two vertices joined by three parallel edges) he obtains all strongly hamiltonian cubic graphs, but these operations does not always preserve planarity. In his paper [11] he describes a method for constructing planar strongly hamiltonian cubic graphs and he deals with the relation between strongly hamiltonian cubic graphs and 4-regular graphs which can be decomposed into two hamiltonian cycles. See also [12] and a recent work on strongly hamiltonian cubic graphs [2] in which the authors give a new construction of strongly hamiltonian graphs.

A Class 1 regular graph such that every edge colouring is a Hamilton decomposition is called a pure Hamilton graph by Kotzig [11]. Note that $K_{4}$ is a pure Hamilton graph and every cubic graph obtained from $K_{4}$ by a sequence of triangular extensions is also a pure Hamilton cubic graph. In the paper [11] of Kotzig, a consequence of his Theorem 9 (p.77) concerning pure Hamilton graphs is that the family of pure Hamilton graphs that he exhibits is precisely the family obtained from $K_{4}$ by triangular extensions. Are there others pure Hamilton cubic graphs? The answer is "yes".

We remark that 2-factor hamiltonian cubic graphs defined above (see Definition (16) are pure Hamilton graphs (in the Kotzig's sense) but the converse is false because $K_{4}$ is 2-factor hamiltonian and the pure Hamilton cubic graph on 6 vertices obtained from $K_{4}$ by a triangular extension (denoted by $P R_{3}$ ) is not 2-factor hamiltonian. Observe that the operation of triangular extension preserves the property "pure Hamilton", but does not preserve the property "2-factor hamiltonian". The Heawood graph $H_{0}$ (on 14 vertices) is pure Hamiltonian, more precisely it is 2-factor hamiltonian (see [7] Proposition 1.1 and Remark 2.7). Then, the graphs obtained from the Heawood graph $H_{0}$ by triangular extensions are also pure Hamilton graphs.

A minimally 1-factorable graph $G$ is defined by Labbate and Funk (7) as a Class 1 regular graph of even order such that every perfect matching of $G$ is contained in exactly one 1 -factorization of $G$. In their article they study
bipartite minimally 1 -factorable graphs and prove that such a graph $G$ has necessarily a degree $d \leq 3$. If $G$ is a minimally 1 -factorable cubic graph then the complementary 2 -factor of any perfect matching has a unique decomposition into two perfect matchings, therefore this 2 -factor is a hamiltonian cycle of $G$, that is $G$ is 2 -factor hamiltonian. Conversely it is easy to see that any 2 -factor hamiltonian cubic graph is minimally 1 -factorable. The complete bipartite graph $K_{3,3}$ and the Heawood graph $H_{0}$ are examples of 2-factor hamiltonian bipartite graph given by Labbate and Funk. Starting from $H_{0}$, from $K_{1,3}$ and from three copies of any tree of maximum degree 3 and using three operations called amalgamations the authors exhibit an infinite family of bipartite 2-factor hamiltonian cubic graphs, namely the poly $-H B-R-R^{2}$ graphs (see $\sqrt{7}$ for more details). Except $H_{0}$, these graphs are exactly cyclically 3 -edge connected. Others structural results about 2-factor hamiltonian bipartite cubic graph are obtained in [13], [14. These results have been completed and a simple method to generate 2-factor hamiltonian bipartite cubic graphs was given in [6].

Proposition 27 (Lemma 3.3, [6]) Let $G$ be a 2-factor hamiltonian bipartite cubic graph. Then $G$ is 3 -connected and $|V(G)| \equiv 2(\bmod 4)$.

Let $G_{1}$ and $G_{2}$ be disjoint cubic graphs, $x \in v\left(G_{1}\right), y \in v\left(G_{2}\right)$. Let $x_{1}, x_{2}, x_{3}$ (respectively $y_{1}, y_{2}, y_{3}$ ) be the neighbours of $x$ in $G_{1}$ (respectively, of $y$ in $G_{2}$ ). The cubic graph $G$ such that $V(G)=\left(V\left(G_{1}\right) \backslash\{x\}\right) \cup\left(V\left(G_{2}\right) \backslash\{y\}\right)$ and $E(G)=$ $\left(E\left(G_{1}\right) \backslash\left\{x_{1} x, x_{2} x, x_{3} x\right\}\right) \cup\left(E\left(G_{2}\right) \backslash\left\{y_{1} y, y_{2} y, y_{3} y\right\}\right) \cup\left\{x_{1} y_{1}, x_{2} y_{2}, x_{3} y_{3}\right\}$ is said to be a star product and $G$ is denoted by $\left(G_{1}, x\right) *\left(G_{2}, y\right)$. Since $\left\{x_{1} y_{1}, x_{2} y_{2}, x_{3} y_{3}\right\}$ is a cyclic edge-cut of $G$, a star product of two 3 -connected cubic graphs has cyclic edge-connectivity 3 .

Proposition 28 (Proposition 3.1, [6]) If a bipartite cubic graph $G$ can be represented as a star product $G=\left(G_{1}, x\right) *\left(G_{2}, y\right)$, then $G$ is 2-factor hamiltonian if and only if $G_{1}$ and $G_{2}$ are 2-factor hamiltonian.

Then, taking iterated star products of $K_{3,3}$ and the Heawood graph $H_{0}$ an infinite family of 2 -factor hamiltonian cubic graphs is obtained. These graphs (except $K_{3,3}$ and $H_{0}$ ) are exactly cyclically 3 -edge connected. In [6] the authors conjecture that the process is complete.

Conjecture 29 (Funk, Jackson, Labbate, Sheehan (2003)[G]) Let G be a bipartite 2-factor hamiltonian cubic graph. Then $G$ can be obtained from $K_{3,3}$ and the Heawood graph $H_{0}$ by repeated star products.

The authors precise that a smallest counterexample to Conjecture 29 is a cyclically 4 -edge connected cubic graph of girth at least 6 , and that to show this result it would suffice to prove that $H_{0}$ is the only 2 -factor hamiltonian
cyclically 4－edge connected bipartite cubic graph of girth at least 6 ．Note that some results have been generalized in［⿴囗⿰丨丨⿱一口𧘇$]$ ．

To conclude，we may ask what happens for non bipartite 2－factor hamiltonian cubic graphs．Recall that $K_{4}$ and $F S(1,3)$（the＂Triplex Graph＂of Robertson， Seymour and Thomas（15）are 2－factor hamiltonian cubic graphs．By Corol－ lary 17 the graphs $F S(j, k)$ with $k$ odd and $j=1$ or 3 introduced in this paper form a new infinite family of non bipartite 2 －factor hamiltonian cubic graphs． We remark that they are cyclically 6 －edge connected．Can we generate others families of non bipartite 2 －factor hamiltonian cubic graphs？Since $P R_{3}$（the cubic graph on 6 vertices obtained from $K_{4}$ by a triangular extension）is not 2－factor hamiltonian and $P R_{3}=K_{4} * K_{4}$ ，the star product operation is surely not a possible tool．

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