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# On a family of cubic graphs containing the flower snarks

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## Abstract

We consider cubic graphs formed with  $k \geq 2$  disjoint claws  $C_i \sim K_{1,3}$  ( $0 \leq i \leq k-1$ ) such that for every integer  $i$  modulo  $k$  the three vertices of degree 1 of  $C_i$  are joined to the three vertices of degree 1 of  $C_{i-1}$  and joined to the three vertices of degree 1 of  $C_{i+1}$ . Denote by  $t_i$  the vertex of degree 3 of  $C_i$  and by  $T$  the set  $\{t_1, t_2, \dots, t_{k-1}\}$ . In such a way we construct three distinct graphs, namely  $FS(1, k)$ ,  $FS(2, k)$  and  $FS(3, k)$ . The graph  $FS(j, k)$  ( $j \in \{1, 2, 3\}$ ) is the graph where the set of vertices  $\cup_{i=0}^{i=k-1} V(C_i) \setminus T$  induce  $j$  cycles (note that the graphs  $FS(2, 2p+1)$ ,  $p \geq 2$ , are the flower snarks defined by Isaacs [8]). We determine the number of perfect matchings of every  $FS(j, k)$ . A cubic graph  $G$  is said to be *2-factor hamiltonian* if every 2-factor of  $G$  is a hamiltonian cycle. We characterize the graphs  $FS(j, k)$  that are 2-factor hamiltonian (note that  $FS(1, 3)$  is the "Triplex Graph" of Robertson, Seymour and Thomas [15]). A *strong matching*  $M$  in a graph  $G$  is a matching  $M$  such that there is no edge of  $E(G)$  connecting any two edges of  $M$ . A cubic graph having a perfect matching union of two strong matchings is said to be a *Jaeger's graph*. We characterize the graphs  $FS(j, k)$  that are Jaeger's graphs.

*Key words:* cubic graph; perfect matching; strong matching; counting; hamiltonian cycle; 2-factor hamiltonian

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## 1 Introduction

The complete bipartite graph  $K_{1,3}$  is called, as usually, a *claw*. Let  $k$  be an integer  $\geq 2$  and let  $G$  be a cubic graph on  $4k$  vertices formed with  $k$  disjoint claws  $C_i = \{x_i, y_i, z_i, t_i\}$  ( $0 \leq i \leq k-1$ ) where  $t_i$  (the *center* of  $C_i$ ) is joined to the three independent vertices  $x_i, y_i$  and  $z_i$  (the *external* vertices of  $C_i$ ). For every integer  $i$  modulo  $k$   $C_i$  has three neighbours in  $C_{i-1}$  and three neighbours in  $C_{i+1}$ . For any integer  $k \geq 2$  we shall denote the set of integers modulo  $k$  as  $\mathbf{Z}_k$ . In the sequel of this paper indices  $i$  of claws  $C_i$  belong to  $\mathbf{Z}_k$ .

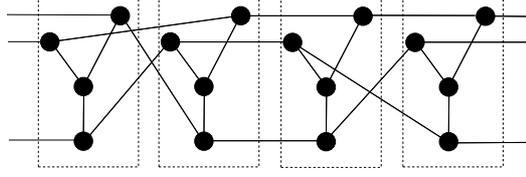


Fig. 1. Four consecutive claws

By renaming some external vertices of claws we can suppose, without loss of generality, that  $\{x_i x_{i+1}, y_i y_{i+1}, z_i z_{i+1}\}$  are edges for any  $i$  distinct from  $k-1$ . That is to say the subgraph induced on  $X = \{x_0, x_1, \dots, x_{k-1}\}$  (respectively  $Y = \{y_0, y_1, \dots, y_{k-1}\}$ ,  $Z = \{z_0, z_1, \dots, z_{k-1}\}$ ) is a path or a cycle (as induced subgraph of  $G$ ). Denote by  $T$  the set of the internal vertices  $\{t_0, t_1, \dots, t_{k-1}\}$ .

Up to isomorphism, the matching joining the external vertices of  $C_{k-1}$  to those of  $C_0$  (also called, for  $k \geq 3$ , *edges between  $C_{k-1}$  and  $C_0$* ) determines the graph  $G$ . In this way we construct essentially three distinct graphs, namely  $FS(1, k)$ ,  $FS(2, k)$  and  $FS(3, k)$ . The graph  $FS(j, k)$  ( $j \in \{1, 2, 3\}$ ) is the graph where the set of vertices  $\cup_{i=0}^{i=k-1} \{C_i \setminus \{t_i\}\}$  induces  $j$  cycles. For  $k \geq 3$  and any  $j \in \{1, 2, 3\}$  the graph  $FS(j, k)$  is a simple cubic graph. When  $k$  is odd, the  $FS(2, k)$  are the graphs known as the flower snarks [8]. We note that  $FS(3, 2)$  and  $FS(2, 2)$  are multigraphs, and that  $FS(1, 2)$  is isomorphic to the cube. For  $k = 2$  the notion of "edge between  $C_{k-1}$  and  $C_0$ " is ambiguous, so we must define it precisely. For two parallel edges having one end in  $C_0$  and the other in  $C_1$ , for instance two parallel edges having  $x_0$  and  $x_1$  as endvertices, we denote one edge by  $x_0 x_1$  and the other by  $x_1 x_0$ . An edge in  $\{x_1 x_0, x_1 y_0, x_1 z_0, y_1 x_0, y_1 y_0, y_1 z_0, z_1 x_0, z_1 y_0, z_1 z_0\}$ , if it exists, is an *edge between  $C_1$  and  $C_0$* . We will say that  $x_0 x_1$ ,  $y_0 y_1$  and  $z_0 z_1$  are *edges between  $C_0$  and  $C_1$* .

By using an ad hoc translation of the indices of claws (and of their vertices) and renaming some external vertices of claws, we see that for any reasoning about a sequence of  $h \geq 3$  consecutive claws ( $C_i, C_{i+1}, C_{i+2}, \dots, C_{i+h-1}$ ) there is no loss of generality to suppose that  $0 \leq i < i+h-1 \leq k-1$ . For a sequence of claws ( $C_p, \dots, C_r$ ) with  $0 \leq p < r \leq k-1$ , since 0 is a possible value for subscript  $p$  and since  $k-1$  is a possible value for subscript  $r$ , it will be useful from time to time to denote by  $x'_{p-1}$  the neighbour in  $C_{p-1}$  of the vertex  $x_p$  of  $C_p$  (recall that  $x'_{p-1} \in \{x_{k-1}, y_{k-1}, z_{k-1}\}$  if  $p = 0$ ), and to denote by  $x'_{r+1}$  the neighbour in  $C_{r+1}$  of the vertex  $x_r$  of  $C_r$  (recall that  $x'_{r+1} \in \{x_0, y_0, z_0\}$  if  $r = k-1$ ). We shall make use of analogous notations for neighbours of  $y_p$ ,  $z_p$ ,  $y_r$  and  $z_r$ .

We shall prove in the following lemma that there are essentially two types of perfect matchings in  $FS(j, k)$ .

**Lemma 1** *Let  $G \in \{FS(j, k), j \in \{1, 2, 3\}, k \geq 2$  and let  $M$  be a perfect matching of  $G$ . Then the 2-factor  $G \setminus M$  induces a path of length 2 and an isolated vertex in each claw  $C_i$  ( $i \in \mathbf{Z}_k$ ) and  $M$  fulfils one (and only one) of*

the three following properties :

- i) For every  $i$  in  $\mathbf{Z}_k$   $M$  contains exactly one edge joining the claw  $C_i$  to the claw  $C_{i+1}$ ,
- ii) For every even  $i$  in  $\mathbf{Z}_k$   $M$  contains exactly two edges between  $C_i$  and  $C_{i+1}$  and none between  $C_{i-1}$  and  $C_i$ ,
- iii) For every odd  $i$  in  $\mathbf{Z}_k$   $M$  contains exactly two edges between  $C_i$  and  $C_{i+1}$  and none between  $C_{i-1}$  and  $C_i$ .

Moreover, when  $k$  is odd  $M$  satisfies only item i).

**Proof** Let  $M$  be a perfect matching of  $G = FS(j, k)$  for some  $j \in \{1, 2, 3\}$ . Since  $M$  contains exactly one edge of each claw, it is obvious that  $G \setminus M$  induces a path of length 2 and an isolated vertex in each claw  $C_i$ .

For each claw  $C_i$  of  $G$  the vertex  $t_i$  must be saturated by an edge of  $M$  whose end (distinct from  $t_i$ ) is in  $\{x_i, y_i, z_i\}$ . Hence there are exactly two edges of  $M$  having one end in  $C_i$  and the other in  $C_{i-1} \cup C_{i+1}$ .

If there are two edges of  $M$  between  $C_i$  and  $C_{i+1}$  then there is no edge of  $M$  between  $C_{i-1}$  and  $C_i$ . If there are two edges of  $M$  between  $C_{i-1}$  and  $C_i$  then there is no edge of  $M$  between  $C_i$  and  $C_{i+1}$ . Hence, we get ii) or iii) and we must have an even number  $k$  of claws in  $G$ .

Assume now that there is only one edge of  $M$  between  $C_{i-1}$  and  $C_i$ . Then there exists exactly one edge between  $C_i$  and  $C_{i+1}$  and, extending this trick to each claw of  $G$ , we get i) when  $k$  is even or odd.  $\square$

**Definition 2** We say that a perfect matching  $M$  of  $FS(j, k)$  is of *type 1* in Case i) of Lemma 1 and of *type 2* in Cases ii) and iii). If necessary, to distinguish Case ii) from Case iii) we shall say *type 2.0* in Case ii) and *type 2.1* in Case iii). We note that the numbers of perfect matchings of type 2.0 and of type 2.1 are equal.

**Notation :** The length of a path  $P$  (respectively a cycle  $\Gamma$ ) is denoted by  $l(P)$  (respectively  $l(\Gamma)$ ).

## 2 Counting perfect matchings of $FS(j, k)$

We shall say that a vertex  $v$  of a cubic graph  $G$  is *inflated* into a triangle when we construct a new cubic graph  $G'$  by deleting  $v$  and adding three new vertices inducing a triangle and joining each vertex of the neighbourhood  $N(v)$  of  $v$  to

a single vertex of this new triangle. We say also that  $G'$  is obtained from  $G$  by a *triangular extension*. The converse operation is the *contraction* or *reduction* of the triangle. The number of perfect matchings of  $G$  is denoted by  $\mu(G)$ .

**Lemma 3** *Let  $G$  be a bipartite cubic graph and let  $\{V_1, V_2\}$  be the bipartition of its vertex set. Assume that each vertex in some subset  $W_1 \subseteq V_1$  is inflated into a triangle and let  $G'$  be the graph obtained in that way. Then  $\mu(G) = \mu(G')$ .*

**Proof** Note that  $\{V_1, V_2\}$  is a balanced bipartition and, by Knig's Theorem, the graph  $G$  is a cubic 3-edge colourable graph. So,  $G'$  is also a cubic 3-edge colourable graph (hence,  $G$  and  $G'$  have perfect matchings). Let  $M$  be a perfect matching of  $G'$ . Each vertex of  $V_1 \setminus W_1$  is saturated by an edge whose second end vertex is in  $V_2$ . Let  $A \subseteq V_2$  be the set of vertices so saturated in  $V_2$ . Assume that some triangle of  $G'$  is such that the three vertices are saturated by three edges having one end in the triangle and the second one in  $V_2$ . Then we need to have at least  $|W_1| + 2$  vertices in  $V_2 \setminus A$ , a contradiction. Hence,  $M$  must have exactly one edge in each triangle and the contraction of each triangle in order to get back  $G$  transforms  $M$  in a perfect matching of  $G$ . Conversely, each perfect matching of  $G$  leads to a unique perfect matching of  $G'$  and we obtain the result.  $\square$

Let us denote by  $\mu(j, k)$  the number of perfect matchings of  $FS(j, k)$ ,  $\mu_1(j, k)$  its number of perfect matchings of type 1 and  $\mu_2(j, k)$  its number of perfect matchings of type 2.

**Lemma 4** *We have*

- $\mu(1, 3) = \mu_1(1, 3) = 9$
- $\mu(2, 3) = \mu_1(2, 3) = 8$
- $\mu(3, 3) = \mu_1(3, 3) = 6$
- $\mu(1, 2) = 9, \mu_1(1, 2) = 3$
- $\mu(2, 2) = 10, \mu_1(2, 2) = 4$
- $\mu(3, 2) = 12, \mu_1(3, 2) = 6$

**Proof** The cycle containing the external vertices of the claws of the graph  $FS(1, 3)$  is  $x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, x_0$ . Consider a perfect matching  $M$  containing the edge  $t_0x_0$ . There are two cases: *i*)  $x_1x_2 \in M$  and *ii*)  $x_1t_1 \in M$ . In Case *i*) we must have  $y_0y_1, t_1z_1, t_2z_2, z_0y_2 \in M$ . In Case *ii*) there are two sub-cases: *ii).a*  $x_2y_0 \in M$  and *ii).b*  $x_2t_2 \in M$ . In Case *ii).a* we must have  $y_1y_2, t_2z_2, z_0z_1 \in M$  and in Case *ii).b* we must have  $y_0y_1, y_2z_0, z_1z_2 \in M$ . Thus, there are exactly 3 distinct perfect matching containing  $t_0x_0$ . By symmetry, there are 3 distinct perfect matchings containing  $t_0y_0$ , and 3 distinct matchings containing  $t_0z_0$ , therefore  $\mu(1, 3) = 9$ .

It is well known that the Petersen graph has exactly 6 perfect matchings. Since  $FS(2, 3)$  is obtained from the Petersen graph by inflating a vertex into a triangle these 6 perfect matchings lead to 6 perfect matchings of  $FS(2, 3)$ . We have two new perfect matchings when considering the three edges connected to this triangle (we have two ways to include these edges into a perfect matching). Hence  $\mu(2, 3) = 8$ .

$FS(3, 3)$  is obtained from  $K_{3,3}$  by inflating three vertices in the same colour of the bipartition. Since  $K_{3,3}$  has six perfect matchings, applying Lemma 3 we get immediately the result for  $\mu(3, 3)$ .

It is a routine matter to obtain the values for  $FS(j, 2)$  ( $j \in \{1, 2, 3\}$ ).  $\square$

**Theorem 5** *The numbers  $\mu(i, k)$  of perfect matchings of  $FS(i, k)$  ( $i \in \{1, 2, 3\}$ ) are given by:*

- $\mu(2, k) = 2^k$

When  $k$  is odd •  $\mu(1, k) = 2^k + 1$

- $\mu(3, k) = 2^k - 2$

- $\mu(2, k) = 2 \times 3^{\frac{k}{2}} + 2^k$

When  $k$  is even •  $\mu(1, k) = 2 \times 3^{\frac{k}{2}} + 2^k - 1$

- $\mu(3, k) = 2 \times 3^{\frac{k}{2}} + 2^k + 2$

**Proof** We shall prove this result by induction on  $k$  and we distinguish the case " $k$  odd" and the case " $k$  even".

The following trick will be helpful. Let  $i \neq 0$  and let  $C_{i-2}$ ,  $C_{i-1}$ ,  $C_i$  and  $C_{i+1}$  be four consecutive claws of  $FS(j, k)$  ( $j \in \{1, 2, 3\}$ ). We can delete  $C_{i-1}$  and  $C_i$  and join the three external vertices of  $C_{i-2}$  to the three external vertices of  $C_{i+1}$  by a matching in such a way that the resulting graph is  $FS(j', k - 2)$ . We have three distinct ways to reduce  $FS(j, k)$  into  $FS(j', k - 2)$  when deleting  $C_{i-1}$  and  $C_i$ .

**Case 1:** We add the edges  $\{x_{i-2}x_{i+1}, y_{i-2}y_{i+1}, z_{i-2}z_{i+1}\}$  and we get  $G_1 = FS(j_1, k - 2)$

**Case 2:** We add the edges  $\{x_{i-2}y_{i+1}, y_{i-2}z_{i+1}, z_{i-2}x_{i+1}\}$  and we get  $G_2 = FS(j_2, k - 2)$ .

**Case 3:** We add the edges  $\{x_{i-2}z_{i+1}, y_{i-2}x_{i+1}, z_{i-2}y_{i+1}\}$  and we get  $G_3 = FS(j_3, k - 2)$ .

Following the cases, we shall precise the values of  $j_1, j_2$  and  $j_3$ .

It is an easy task to see that each perfect matching of type 1 of  $FS(j, k)$  leads to a perfect matching of either  $G_1$  or  $G_2$  or  $G_3$  and, conversely, each perfect matching of type 1 of  $G_1$  allows us to construct 2 distinct perfect matchings of type 1 of  $FS(j, k)$ , while each perfect matching of type 1 of  $G_2$  and  $G_3$  allows us to construct 1 perfect matching of type 1 of  $FS(j, k)$ .

We have

$$\mu_1(j, k) = 2\mu_1(G_1) + \mu_1(G_2) + \mu_1(G_3) \quad (1)$$

CLAIM 1  $\mu_1(2, k) = 2^k$

**Proof** Since the result holds for  $FS(2, 3)$  and  $FS(2, 2)$  by Lemma 4, in order to prove the result by induction on the number  $k$  of claws, we assume that the property holds for  $FS(2, k - 2)$  with  $k - 2 \geq 2$ .

In that case  $G_1, G_2$  and  $G_3$  are isomorphic to  $FS(2, k - 2)$ . Using Equation 1 we have, as claimed

$$\mu_1(2, k) = 4\mu_1(2, k - 2) = 2^k$$

□

CLAIM 2  $\mu_1(1, k) = 2^k - (-1)^k$  and  $\mu_1(3, k) = 2^k + 2(-1)^k$

**Proof** Since the result holds for  $FS(1, 3)$ ,  $FS(1, 2)$ ,  $FS(3, 3)$ , and  $FS(3, 2)$ , by Lemma 4, in order to prove the result by induction on the number  $k$  of claws, we assume that the property holds for  $FS(1, k - 2)$ , and  $FS(3, k - 2)$  with  $k - 2 \geq 2$ .

When considering  $FS(1, k)$ ,  $G_1$  is isomorphic to  $FS(1, k - 2)$ , and among  $G_2$  and  $G_3$  one of them is isomorphic to  $FS(3, k - 2)$  and the other to  $FS(1, k - 2)$ . In the same way, when considering  $FS(3, k)$ ,  $G_1$  is isomorphic to  $FS(3, k - 2)$ , and  $G_2$  and  $G_3$  are isomorphic to  $FS(1, k - 2)$ .

Using Equation 1 we have,

$$\mu_1(1, k) = 2\mu_1(1, k - 2) + \mu_1(1, k - 2) + \mu_1(3, k - 2)$$

and

$$\mu_1(1, k) = 2(2^{k-2} + 1) + 2^{k-2} + 1 + 2^{k-2} - 2 = 2^k + 1$$

$$\mu_1(3, k) = 2(2^{k-2} - 2) + 2^{k-2} + 1 + 2^{k-2} + 1 = 2^k - 2$$

□

When  $k$  is odd, we have  $\mu_2(j, k) = 0$  by Lemma 1 and hence  $\mu(j, k) = \mu_1(j, k)$

When  $k$  is even it remains to count the number of perfect matchings of type 2. From Lemma 1, for every two consecutive claws  $C_i$  and  $C_{i+1}$ , we have either two edges of  $M$  joining the external vertices of  $C_i$  to those of  $C_{i+1}$  or none. We have 3 ways to choose 2 edges between  $C_i$  and  $C_{i+1}$ , each choice of these two edges can be completed in a unique way in a perfect matching of the subgraph  $C_i \cup C_{i+1}$ . Hence we get easily that the number of perfect matchings of type 2 in  $FS(j, k)$  ( $j \in \{1, 2, 3\}$ ) is

$$\mu_2(j, k) = 2 \times 3^{\frac{k}{2}} \quad (2)$$

Using Claims 1 and 2 and Equation 2 we get the results for  $\mu(j, k)$  when  $k$  is even.

□

### 3 Some structural results about perfect matchings of FS(j, k)

#### 3.1 Perfect matchings of type 1

**Lemma 6** *Let  $M$  be a perfect matching of type 1 of  $G = FS(j, k)$ . Then the 2-factor  $G \setminus M$  has exactly one or two cycles and each cycle of  $G \setminus M$  has at least one vertex in each claw  $C_i$  ( $i \in \mathbf{Z}_k$ ).*

**Proof** Let  $M$  be a perfect matching of type 1 in  $G$ . Let us consider the claw  $C_i$  for some  $i$  in  $\mathbf{Z}_k$ . Assume without loss of generality that the edge of  $M$  contained in  $C_i$  is  $t_i x_i$ . The cycle of  $G \setminus M$  visiting  $x_i$  comes from  $C_{i-1}$ , crosses  $C_i$  by using the vertex  $x_i$  and goes to  $C_{i+1}$ . By Lemma 1, the path  $y_i t_i z_i$  is contained in a cycle of  $G \setminus M$ . The two edges incident to  $y_i$  and  $z_i$  joining  $C_i$  to  $C_{i-1}$  (as well as those joining  $C_i$  to  $C_{i+1}$ ) are not contained both in  $M$  (since  $M$  has type 1). Thus, the cycle of  $G \setminus M$  containing  $y_i t_i z_i$  comes from  $C_{i-1}$ , crosses  $C_i$  and goes to  $C_{i+1}$ . Thus, we have at most two cycles in  $G \setminus M$ , as claimed, and we can note that each claw must be visited by these cycles. □

**Definition 7** Let us suppose that  $M$  is a perfect matching of type 1 in  $G = FS(j, k)$  such that the 2-factor  $G \setminus M$  has exactly two cycles  $\Gamma_1$  and  $\Gamma_2$ . A claw  $C_i$  intersected by three vertices of  $\Gamma_1$  (respectively  $\Gamma_2$ ) is said to be  $\Gamma_1$ -major (respectively  $\Gamma_2$ -major).

**Lemma 8** Let  $M$  be a perfect matching of type 1 of  $G = FS(j, k)$  such that the 2-factor  $G \setminus M$  has exactly two cycles. Then, the lengths of these two cycles have the same parity as  $k$ , and those lengths are distinct when  $k$  is odd.

**Proof** Let  $\Gamma_1$  and  $\Gamma_2$  be the two cycles of  $G \setminus M$ . By Lemma 6, for each  $i$  in  $\mathbf{Z}_k$  these two cycles must cross the claw  $C_i$ . Let  $k_1$  be the number of  $\Gamma_1$ -major claws and let  $k_2$  be the number of  $\Gamma_2$ -major claws. We have  $k_1 + k_2 = k$ ,  $l(\Gamma_1) = 3k_1 + k_2$  and  $l(\Gamma_2) = 3k_2 + k_1$ . When  $k$  is odd, we must have either  $k_1$  odd and  $k_2$  even, or  $k_1$  even and  $k_2$  odd. Then  $\Gamma_1$  and  $\Gamma_2$  have distinct odd lengths. When  $k$  is even, we must have either  $k_1$  and  $k_2$  even, or  $k_2$  and  $k_1$  odd. Then  $\Gamma_1$  and  $\Gamma_2$  have even lengths.  $\square$

**Lemma 9** Let  $M$  be a perfect matching of type 1 of  $G = FS(j, k)$  such that the 2-factor  $G \setminus M$  has exactly two cycles  $\Gamma_1$  and  $\Gamma_2$ . Suppose that there are two consecutive  $\Gamma_1$ -major claws  $C_j$  and  $C_{j+1}$  with  $j \in \mathbf{Z}_k \setminus \{k-1\}$ . Then there is a perfect matching  $M'$  of type 1 such that the 2-factor  $G \setminus M'$  has exactly two cycles  $\Gamma'_1$  and  $\Gamma'_2$  having the following properties:

- a) for  $i \in \mathbf{Z}_k \setminus \{j, j+1\}$   $C_i$  is  $\Gamma'_2$ -major if and only if  $C_i$  is  $\Gamma_2$ -major,
- b)  $C_j$  and  $C_{j+1}$  are  $\Gamma'_2$ -major,
- c)  $l(\Gamma'_1) = l(\Gamma_1) - 4$  and  $l(\Gamma'_2) = l(\Gamma_2) + 4$ .

**Proof** Consider the claws  $C_j$  and  $C_{j+1}$ . Since  $C_j$  is a  $\Gamma_1$ -major claw suppose without loss of generality that  $t_j z_j$  belongs to  $M$  and that  $\Gamma_1$  contains the path  $x'_{j-1} x_j t_j y_j y_{j+1}$  where  $x'_{j-1}$  denotes the neighbour of  $x_j$  in  $C_{j-1}$  (then  $x_j x_{j+1}$  belongs to  $M$ ). Since  $C_{j+1}$  is  $\Gamma_1$ -major and  $\Gamma_2$  goes through  $C_j$  and  $C_{j+1}$ , the cycle  $\Gamma_1$  must contain the path  $y_{j+1} t_{j+1} x_{j+1} x'_{j+2}$  where  $x'_{j+2}$  denotes the neighbour of  $x_{j+1}$  in  $C_{j+2}$  (then  $M$  contains  $t_{j+1} z_{j+1}$  and  $y_{j+1} y'_{j+2}$ ). Denote by  $P_1$  the path  $x'_{j-1} x_j t_j y_j y_{j+1} t_{j+1} x_{j+1} x'_{j+2}$ . Note that  $\Gamma_2$  contains the path  $P_2 = z'_{j-1} z_j z_{j+1} z'_{j+1}$  where  $z'_{j-1}$  and  $z'_{j+1}$  are defined similarly. See to the left part of Figure 2.

Let us perform the following local transformation: delete  $x_j x_{j+1}$ ,  $t_j z_j$  and  $t_{j+1} z_{j+1}$  from  $M$  and add  $z_j z_{j+1}$ ,  $t_j x_j$  and  $t_{j+1} x_{j+1}$ . Let  $M'$  be the resulting perfect matching. Then the subpath  $P_1$  of  $\Gamma_1$  is replaced by  $P'_1 = x'_{j-1} x_j x_{j+1} x'_{j+2}$  and the subpath  $P_2$  of  $\Gamma_2$  is replaced by  $P'_2 = z'_{j-1} z_j t_j y_j y_{j+1} t_{j+1} z_{j+1} z'_{j+2}$  (see Figure 2). We obtain a new 2-factor containing two new cycles  $\Gamma'_1$  and  $\Gamma'_2$ . Note that  $C_j$  and  $C_{j+1}$  are  $\Gamma'_2$ -major claws and for  $i$  in  $\mathbf{Z}_k \setminus \{j, j+1\}$   $C_i$  is  $\Gamma'_2$ -major (respectively  $\Gamma'_1$ -major) if and only if  $C_i$  is  $\Gamma_2$ -major (respectively

$\Gamma_1$ -major). The length of  $\Gamma_1$  (now  $\Gamma'_1$ ) decreases of 4 units while the length of  $\Gamma_2$  (now  $\Gamma'_2$ ) increases of 4 units.  $\square$

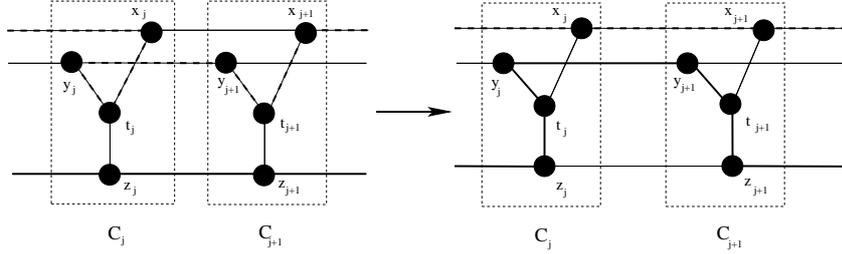


Fig. 2. Local transformation of type 1

The operation depicted in Lemma 9 above will be called a *local transformation of type 1*.

**Lemma 10** *Let  $M$  be a perfect matching of type 1 of  $G = FS(j, k)$  such that the 2-factor  $G \setminus M$  has exactly two cycles  $\Gamma_1$  and  $\Gamma_2$ . Suppose that there are three consecutive claws  $C_j, C_{j+1}$  and  $C_{j+2}$  with  $j$  in  $\mathbf{Z}_k \setminus \{k-1, k-2\}$  such that  $C_j$  and  $C_{j+2}$  are  $\Gamma_1$ -major and  $C_{j+1}$  is  $\Gamma_2$ -major. Then there is a perfect matching  $M'$  of type 1 such that the 2-factor  $G \setminus M'$  has exactly two cycles  $\Gamma'_1$  and  $\Gamma'_2$  having the following properties:*

- a) for  $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$   $C_i$  is  $\Gamma'_2$ -major if and only if  $C_i$  is  $\Gamma_2$ -major,
- b)  $C_j$  and  $C_{j+2}$  are  $\Gamma'_2$ -major and  $C_{j+1}$  is  $\Gamma'_1$ -major,
- c)  $l(\Gamma'_1) = l(\Gamma_1) - 2$  and  $l(\Gamma'_2) = l(\Gamma_2) + 2$ .

**Proof** Since  $C_j$  is  $\Gamma_1$ -major, as in the proof of Lemma 9 suppose that  $\Gamma_1$  contains the path  $x'_{j-1}x_jt_jy_jy_{j+1}$  (that is edges  $t_jz_j$  and  $x_jx_{j+1}$  belong to  $M$ ). Since  $C_{j+1}$  is  $\Gamma_2$ -major the cycle  $\Gamma_1$  contains the edge  $y_{j+1}y_{j+2}$ . Then we see that  $\Gamma_1$  contains the path  $Q_1 = x'_{j-1}x_jt_jy_jy_{j+1}y_{j+2}t_{j+2}z_{j+2}z'_{j+3}$  and that  $\Gamma_2$  contains the path  $Q_2 = z'_{j-1}z_jz_{j+1}t_{j+1}x_{j+1}x_{j+2}x'_{j+3}$ . Note that  $y_{j+1}t_{j+1}$ ,  $z_{j+1}z_{j+2}$  and  $t_{j+2}x_{j+2}$  belong to  $M$ .

Let us perform the following local transformation: delete  $t_jz_j$ ,  $x_jx_{j+1}$ ,  $z_{j+1}z_{j+2}$  and  $x_{j+2}t_{j+2}$  from  $M$  and add  $x_jt_j$ ,  $z_jz_{j+1}$ ,  $x_{j+1}x_{j+2}$  and  $z_{j+2}t_{j+2}$  to  $M$ . Let  $M'$  be the resulting perfect matching. Then the subpath  $Q_1$  of  $\Gamma_1$  is replaced by  $Q'_1 = x'_{j-1}x_jx_{j+1}t_{j+1}z_{j+1}z_{j+2}z'_{j+3}$  and the subpath  $Q_2$  of  $\Gamma_2$  is replaced by  $Q'_2 = z'_{j-1}z_jt_jy_jy_{j+1}y_{j+2}t_{j+2}x_{j+2}x'_{j+3}$  (see Figure 3). We obtain a new 2-factor containing two new cycles named  $\Gamma'_1$  and  $\Gamma'_2$ . Note that  $C_j$  and  $C_{j+2}$  are now  $\Gamma'_2$ -major claws and  $C_{j+1}$  is  $\Gamma'_1$ -major. The length of  $\Gamma_1$  decreases of 2 units while the length of  $\Gamma_2$  increases of 2 units. It is clear that for  $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$   $C_i$  is  $\Gamma'_2$ -major (respectively  $\Gamma'_1$ -major) if and only if  $C_i$  is  $\Gamma_2$ -major (respectively  $\Gamma_1$ -major).  $\square$

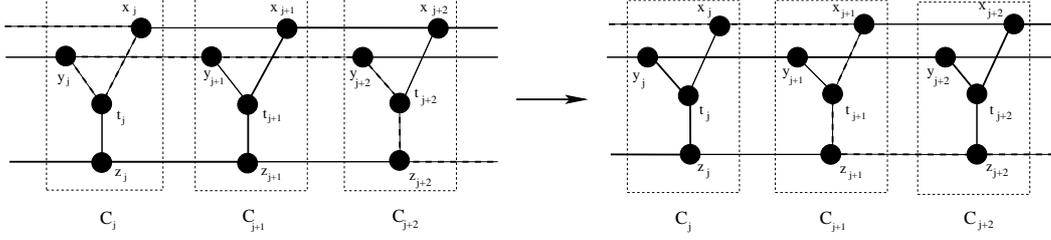


Fig. 3. Local transformation of type 2

The operation depicted in Lemma 10 above will be called a *local transformation of type 2*.

**Lemma 11** *Let  $M$  be a perfect matching of type 1 of  $G = FS(j, k)$  such that the 2-factor  $G \setminus M$  has exactly two cycles  $\Gamma_1$  and  $\Gamma_2$ . Suppose that there are three consecutive claws  $C_j, C_{j+1}$  and  $C_{j+2}$  with  $j$  in  $\mathbf{Z}_k \setminus \{k-1, k-2\}$  such that  $C_{j+1}$  and  $C_{j+2}$  are  $\Gamma_2$ -major and  $C_j$  is  $\Gamma_1$ -major. Then there is a perfect matching  $M'$  of type 1 such that the 2-factor  $G \setminus M'$  has exactly two cycles  $\Gamma'_1$  and  $\Gamma'_2$  having the following properties:*

- for  $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$   $C_i$  is  $\Gamma'_2$ -major if and only if  $C_i$  is  $\Gamma_2$ -major,
- $C_j$  and  $C_{j+1}$  are  $\Gamma'_2$ -major and  $C_{j+2}$  is  $\Gamma'_1$ -major,
- $l(\Gamma'_1) = l(\Gamma_1)$  and  $l(\Gamma'_2) = l(\Gamma_2)$ .

**Proof** Since  $C_j$  is  $\Gamma_1$ -major, as in the proof of Lemma 9 suppose that  $\Gamma_1$  contains the path  $x'_{j-1}x_jt_jy_jy_{j+1}$  (that is edges  $t_jz_j$  and  $x_jx_{j+1}$  belong to  $M$ ). Since  $C_{j+1}$  and  $C_{j+2}$  are  $\Gamma_2$ -major, the unique vertex of  $C_{j+1}$  (respectively  $C_{j+2}$ ) contained in  $\Gamma_1$  is  $y_{j+1}$  (respectively  $y_{j+2}$ ). Note that the perfect matching  $M$  contains the edges  $t_jz_j, x_jx_{j+1}, t_{j+1}y_{j+1}, z_{j+1}z_{j+2}$  and  $t_{j+2}y_{j+2}$ . Then the path  $R_1 = x'_{j-1}x_jt_jy_jy_{j+1}y_{j+2}y'_{j+3}$  is a subpath of  $\Gamma_1$  and the path  $R_2 = z'_{j-1}z_jz_{j+1}t_{j+1}x_{j+1}x_{j+2}t_{j+2}z_{j+2}z'_{j+3}$  is a subpath of  $\Gamma_2$ . See to the left part of Figure 4.

Let us perform the following local transformation: delete  $t_jz_j, x_jx_{j+1}, t_{j+1}y_{j+1}, z_{j+1}z_{j+2}$  and  $t_{j+2}y_{j+2}$  from  $M$  and add  $x_jt_j, z_jz_{j+1}, t_{j+1}x_{j+1}, y_{j+1}y_{j+2}$  and  $t_{j+2}z_{j+2}$ . Let  $M'$  be the resulting perfect matching. Then the subpath  $R_1$  of  $\Gamma_1$  is replaced by  $R'_1 = x'_{j-1}x_jx_{j+1}x_{j+2}t_{j+2}y_{j+2}y'_{j+3}$  and the subpath  $R_2$  of  $\Gamma_2$  is replaced by  $R'_2 = z'_{j-1}z_jt_jy_jy_{j+1}t_{j+1}z_{j+1}z_{j+2}z'_{j+3}$ . We obtain a new 2-factor containing two new cycles named  $\Gamma'_1$  and  $\Gamma'_2$  such that  $l(\Gamma'_1) = l(\Gamma_1)$  and  $l(\Gamma'_2) = l(\Gamma_2)$  (see Figure 4). It is clear that for  $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$   $C_i$  is  $\Gamma'_2$ -major (respectively  $\Gamma'_1$ -major) if and only if  $C_i$  is  $\Gamma_2$ -major (respectively  $\Gamma_1$ -major). Note that  $C_j$  and  $C_{j+1}$  are  $\Gamma'_2$ -major and  $C_{j+2}$  is  $\Gamma'_1$ -major.  $\square$

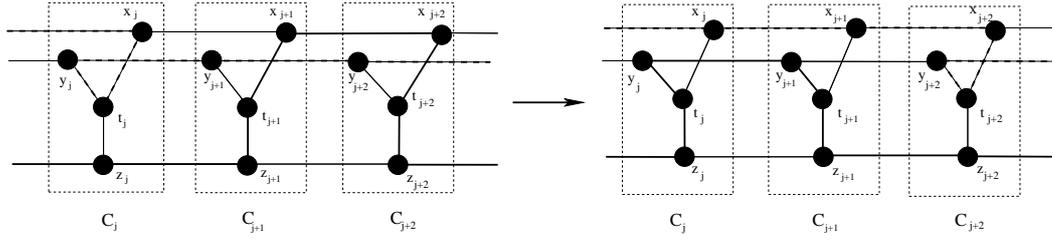


Fig. 4. Local transformation of type 3

The operation depicted in Lemma 11 above will be called a *local transformation of type 3*.

**Lemma 12** *Let  $M$  be a perfect matching of type 1 of  $G = FS(j, k)$  such that the 2-factor  $G \setminus M$  has exactly two cycles  $\Gamma_1$  and  $\Gamma_2$  such that  $l(\Gamma_1) \leq l(\Gamma_2)$  and  $l(\Gamma_2)$  is as great as possible. Then there exists at most one  $\Gamma_1$ -major claw.*

**Proof** Suppose, for the sake of contradiction, that there exist at least two  $\Gamma_1$ -major claws. Since  $l(\Gamma_2)$  is maximum, by Lemma 9 these claws are not consecutive. Then consider two  $\Gamma_1$ -major claws  $C_i$  and  $C_{i+h+1}$  (with  $h \geq 1$ ) such that the  $h$  consecutive claws  $(C_{i+1}, \dots, C_{i+h})$  are  $\Gamma_2$ -major. Since  $l(\Gamma_2)$  is maximum, by Lemma 10 the number  $h$  is at least 2. Then by applying  $r = \lfloor \frac{h}{2} \rfloor$  consecutive local transformations of type 3 (Lemma 11) we obtain a perfect matching  $M^{(r)}$  such that the 2-factor  $G \setminus M^{(r)}$  has exactly two cycles  $\Gamma_1^{(r)}$  and  $\Gamma_2^{(r)}$  with  $l(\Gamma_1^{(r)}) = l(\Gamma_1)$  and  $l(\Gamma_2^{(r)}) = l(\Gamma_2)$  and such that  $C_{i+2\lfloor \frac{h}{2} \rfloor}$  and  $C_{i+h+1}$  are  $\Gamma_1^{(r)}$ -major. Since  $l(\Gamma_2^{(r)})$  is maximum, we can conclude by Lemma 9 and by Lemma 10 that  $h$  is neither even nor odd, a contradiction.  $\square$

### 3.2 Perfect matchings of type 2

We give here a structural result about perfect matchings of type 2 in  $G = FS(j, k)$ .

**Lemma 13** *Let  $M$  be a perfect matching of type 2 of  $G = FS(j, k)$  (with  $k \geq 4$ ). Then the 2-factor  $G \setminus M$  has exactly one cycle of even length  $l \geq k$  and a set of  $p$  cycles of length 6 where  $l + 6p = 4k$  (with  $0 \leq p \leq \frac{k}{2}$ ).*

**Proof** Let  $M$  be a perfect matching of type 2 in  $G$ . By Lemma 1 the number  $k$  of claws is even. Let  $i$  in  $\mathbf{Z}_k$  such that there are two edges of  $M$  between  $C_{i-1}$  and  $C_i$ . There are no edges of  $M$  between  $C_i$  and  $C_{i+1}$  and two edges of  $M$  between  $C_{i+1}$  and  $C_{i+2}$ . We may consider that  $0 \leq i < k - 1$ . For  $j \in \{i, i + 2, i + 4, \dots\}$  we denote by  $e_j$  the unique edge of  $G \setminus M$  having

one end vertex in  $C_{j-1}$  and the other in  $C_j$ . Let us denote by  $A$  the set  $\{e_i, e_{i+2}, e_{i+4}, \dots\}$ . We note that  $|A| = \frac{k}{2}$ .

Assume without loss of generality that the two edges of  $M$  between  $C_{i-1}$  and  $C_i$  have end vertices in  $C_i$  which are  $x_i$  and  $y_i$  (then  $z_i$  is the end vertex of  $e_i$  in  $C_i$ ). Two cases may now occur.

**Case 1:** *The end vertices in  $C_{i+1}$  of the two edges of  $M$  between  $C_{i+1}$  and  $C_{i+2}$  are  $x_{i+1}$  and  $y_{i+1}$  (then  $z_{i+1}$  is the end vertex of  $e_{i+2}$  in  $C_{i+1}$ ).*

In that case the 2-factor  $G \setminus M$  contains the cycle of length 6  $x_i x_{i+1} t_{i+1} y_{i+1} y_i t_i$  while the edge  $z_i z_{i+1}$  of  $G \setminus M$  relies  $e_i$  and  $e_{i+2}$ .

**Case 2:** *The end vertices in  $C_{i+1}$  of the two edges of  $M$  between  $C_{i+1}$  and  $C_{i+2}$  are  $y_{i+1}$  and  $z_{i+1}$  (respectively  $x_{i+1}$  and  $z_{i+1}$ ). Then  $x_{i+1}$  (respectively  $y_{i+1}$ ) is the end vertex of  $e_{i+2}$  in  $C_{i+1}$ .*

In that case the edges  $e_i$  and  $e_{i+2}$  are connected in  $G \setminus M$  by the path  $z_i z_{i+1} t_{i+1} y_{i+1} y_i t_i x_i x_{i+1}$  (respectively  $z_i z_{i+1} t_{i+1} x_{i+1} x_i t_i y_i y_{i+1}$ ).

The same reasoning can be done for  $\{e_{i+2}, e_{i+4}\}$ ,  $\{e_{i+4}, e_{i+6}\}$ , and so on. Then, we see that the set  $A$  is contained in a unique cycle  $\Gamma$  of  $G \setminus M$  which crosses each claw. Thus, the length  $l$  of  $\Gamma$  is at least  $k$ . More precisely, each  $e_j$  in  $A$  contributes for 1 in  $l$ , in Case 1 the edge  $z_i z_{i+1}$  contributes for 1 in  $l$  and in Case 2 the path  $z_i z_{i+1} t_{i+1} y_{i+1} y_i t_i x_i x_{i+1}$  contributes for 7 in  $l$ . Let us suppose that Case 1 appears  $p$  times ( $0 \leq p \leq \frac{k}{2}$ ), that is to say  $G \setminus M$  contains  $p$  cycles of length 6. Since Case 2 appears  $\frac{k}{2} - p$  times, the length of  $\Gamma$  is  $l = \frac{k}{2} + p + 7(\frac{k}{2} - p) = 4k - 6p$ .  $\square$

**Remark 14** If  $k$  is even then by Lemmas 6, 8 and 13  $FS(j, k)$  has an even 2-factor. That is to say  $FS(j, k)$  is a cubic 3-edge colourable graph .

## 4 Perfect matchings and hamiltonian cycles of $F(j, k)$

### 4.1 Perfect matchings of type 1 and hamiltonicity

**Theorem 15** *Let  $M$  be a perfect matching of type 1 of  $G = FS(j, k)$ . Then the 2-factor  $G \setminus M$  is a hamiltonian cycle except for  $k$  odd and  $j = 2$ , and for  $k$  even and  $j = 1$  or 3.*

**Proof** Suppose that there exists a perfect matching  $M$  of type 1 of  $G$  such that  $G \setminus M$  is not a hamiltonian cycle. By Lemma 6 and Lemma 8 the 2-factor  $G \setminus M$  is made of exactly two cycles  $\Gamma_1$  and  $\Gamma_2$  whose lengths have the same parity as  $k$ . Without loss of generality we suppose that  $l(\Gamma_1) \leq l(\Gamma_2)$ .

Assume moreover that among the perfect matchings of type 1 of  $G$  such that the 2-factor  $G \setminus M$  is composed of two cycles,  $M$  has been chosen in such a way that the length of the longest cycle  $\Gamma_2$  is as great as possible. By Lemma 12 there exists at most one  $\Gamma_1$ -major claw.

**Case 1:** There exists one  $\Gamma_1$ -major claw.

Without loss of generality, suppose that  $C_0$  is intersected by  $\Gamma_1$  in  $\{y_0, t_0, x_0\}$  and that  $y'_{k-1}y_0$  belongs to  $\Gamma_1$ . Since for every  $i \neq 0$  the claw  $C_i$  is  $\Gamma_2$ -major,  $\Gamma_1$  contains the vertices  $y_0, t_0, x_0, x_1, x_2, \dots, x_{k-1}$ .

- If  $k = 2r + 1$  with  $r \geq 1$  then  $\Gamma_2$  contains the path

$$z_0 z_1 t_1 y_1 y_2 t_2 z_2 \dots z_{2r-1} t_{2r-1} y_{2r-1} y_{2r} t_{2r} z_{2r}.$$

Thus,  $y_0 x_{k-1}, x_0 y_{k-1}, z_0 z_{k-1}$  are edges of  $G$ . This means that  $\cup_{i=0}^{i=k-1} \{C_i \setminus \{t_i\}\}$  induces two cycles, that is to say  $j = 2$  and  $G = FS(2, k)$ .

- If  $k = 2r + 2$  with  $r \geq 1$  then  $\Gamma_2$  contains the path

$$z_0 z_1 t_1 y_1 y_2 t_2 z_2 \dots z_{2r-1} t_{2r-1} y_{2r-1} y_{2r} t_{2r} z_{2r} z_{2r+1} t_{2r+1} y_{2r+1}.$$

Thus,  $x_0 z_{k-1}, y_0 x_{k-1}$  and  $z_0 y_{k-1}$  are edges. This means that  $\cup_{i=0}^{i=k-1} \{C_i \setminus \{t_i\}\}$  induces one cycle, that is to say  $j = 1$  and  $G = FS(1, k)$ .

**Case 2:** There is no  $\Gamma_1$ -major claw.

Suppose that  $x_0$  belongs to  $\Gamma_1$ . Then,  $\Gamma_1$  contains  $x_0, x_1, \dots, x_{k-1}$ .

- If  $k = 2r + 1$  with  $r \geq 1$  then  $\Gamma_2$  contains the path

$$y_0 t_0 z_0 z_1 t_1 y_1 y_2 \dots z_{2r-1} t_{2r-1} y_{2r-1} y_{2r} t_{2r} z_{2r}.$$

Thus,  $x_0 x_{k-1}, y_0 z_{k-1}$  and  $z_0 y_{k-1}$  are edges of  $G$  and the set  $\cup_{i=0}^{i=k-1} \{C_i \setminus \{t_i\}\}$  induces two cycles, that is to say  $j = 2$  and  $G = FS(2, k)$ .

- If  $k = 2r + 2$  with  $r \geq 1$  then  $\Gamma_2$  contains the path

$$y_0 t_0 z_0 z_1 t_1 y_1 y_2 \dots y_{2r} t_{2r} z_{2r} z_{2r+1} t_{2r+1} y_{2r+1}.$$

Thus,  $x_0 x_{k-1}, y_0 y_{k-1}$  and  $z_0 z_{k-1}$  are edges. This means that  $\cup_{i=0}^{i=k-1} \{C_i \setminus \{t_i\}\}$  induces three cycles, that is to say  $j = 3$  and  $G = FS(3, k)$ .

□

**Definition 16** A cubic graph  $G$  is said to be *2-factor hamiltonian* [6] if every 2-factor of  $G$  is a hamiltonian cycle (or equivalently, if for every perfect

matching  $M$  of  $G$  the 2-factor  $G \setminus M$  is a hamiltonian cycle).

By Theorem 15 for any odd  $k \geq 3$  and  $j \in \{1, 3\}$  or for any even  $k$  and  $j = 2$ , and for every perfect matching  $M$  of type 1 in  $FS(j, k)$  the 2-factor  $FS(j, k) \setminus M$  is a hamiltonian cycle. By Lemma 13  $FS(2, k)$  ( $k \geq 4$ ) may have a perfect matching  $M$  of type 2 such that the 2-factor  $FS(2, k) \setminus M$  is not a hamiltonian cycle (it may contain cycles of length 6).

Then we have the following.

**Corollary 17** *A graph  $G = FS(j, k)$  is 2-factor hamiltonian if and only if  $k$  is odd and  $j = 1$  or 3.*

We note that  $FS(1, 3)$  is the "Triplex Graph" of Robertson, Seymour and Thomas [15]. We shall examine other known results about 2-factor hamiltonian cubic graphs in Section 5.

**Corollary 18** *The chromatic index of a graph  $G = FS(j, k)$  is 4 if and only if  $j = 2$  and  $k$  is odd.*

**Proof** When  $j = 2$  and  $k$  is odd, any 2-factor must have at least two cycles, by Theorem 15. Then Lemma 8 implies that any 2-factor is composed of two odd cycles. Hence  $G$  has chromatic index 4.

When  $j = 1$  or 3 and  $k$  is odd by Theorem 15  $FS(j, k)$  is hamiltonian. If  $k$  is even then by Lemmas 6, 8 and 13  $FS(j, k)$  has an even 2-factor.  $\square$

#### 4.2 Perfect matchings of type 2 and hamiltonicity

At this point of the discourse one may ask what happens for perfect matchings of type 2 in  $FS(j, k)$  ( $k$  even). Can we characterize and count perfect matchings of type 2, complementary 2-factor of which is a hamiltonian cycle? An affirmative answer shall be given.

Let us consider a perfect matching  $M$  of type 2 in  $FS(j, 2p)$  with  $p \geq 2$ . Suppose that there are no edges of  $M$  between  $C_{2i-1}$  and  $C_{2i}$  (for any  $i \geq 1$ ), that is  $M$  is a matching of type 2.0 (see Definition 2). Consider two consecutive claws  $C_{2i}$  and  $C_{2i+1}$  ( $0 \leq i \leq p-1$ ). There are three cases:

Case (x):  $\{y_{2i}y_{2i+1}, z_{2i}z_{2i+1}\} \subset M$  (then,  $M \cap (C_{2i} \cup C_{2i+1}) = \{x_{2i}t_{2i}, x_{2i+1}t_{2i+1}\}$ ).

Case (y):  $\{x_{2i}x_{2i+1}, z_{2i}z_{2i+1}\} \subset M$  (then,  $M \cap (C_{2i} \cup C_{2i+1}) = \{y_{2i}t_{2i}, y_{2i+1}t_{2i+1}\}$ ).

Case (z):  $\{x_{2i}x_{2i+1}, y_{2i}y_{2i+1}\} \subset M$  (then,  $M \cap (C_{2i} \cup C_{2i+1}) = \{z_{2i}t_{2i}, z_{2i+1}t_{2i+1}\}$ ).

The subgraph induced on  $C_{2i} \cup C_{2i+1}$  is called a *block*. In Case (x) (respectively Case (y), Case (z)) a block is called a *block of type X* (respectively *block of type Y*, *block of type Z*). Then  $FS(j, 2p)$  with a perfect matchings  $M$  of type 2.0 can be seen as a sequence of  $p$  blocks properly relied. In other words, a perfect matchings  $M$  of type 2 in  $FS(j, 2p)$  is entirely described by a word of length  $p$  on the alphabet of three letters  $\{X, Y, Z\}$ . The block  $C_0 \cup C_1$  is called *initial block* and the block  $C_{2p-1} \cup C_{2p}$  is called *terminal block*. These extremal blocks are not considered here as consecutive blocks.

By Lemma 13,  $FS(j, 2p) \setminus M$  has no 6-cycles if and only if  $FS(j, 2p) \setminus M$  is a unique even cycle. It is an easy matter to prove that two consecutive blocks do not induce a 6-cycle if and only if they are not of the same type. Then the possible configurations for two consecutive blocks are  $XY$ ,  $XZ$ ,  $YX$ ,  $YZ$ ,  $ZX$  and  $ZY$ . To eliminate a possible 6-cycle in  $C_0 \cup C_{2p-1}$  we have to determine for every  $j \in \{1, 2, 3\}$  the forbidden extremal configurations. An extremal configuration shall be denoted by a word on two letters in  $\{X, Y, Z\}$  such that the left letter denotes the type of the initial block  $C_0 \cup C_1$  and the right letter denotes the type of the terminal block  $C_{2p-1} \cup C_{2p}$ . We suppose that the extremal blocks are connected for  $j = 1$  by the edges  $x_{2p-1}z_0$ ,  $y_{2p-1}x_0$  and  $z_{2p-1}y_0$ , for  $j = 2$  by the edges  $x_{2p-1}x_0$ ,  $y_{2p-1}z_0$  and  $z_{2p-1}y_0$  and for  $j = 3$  by the edges  $x_{2p-1}x_0$ ,  $y_{2p-1}y_0$  and  $z_{2p-1}z_0$ . Then, it is easy to verify that we have the following result.

**Lemma 19** *Let  $M$  be a perfect matching of type 2.0 of  $G = FS(j, 2p)$  (with  $p \geq 2$ ) such that the 2-factor  $G \setminus M$  is a hamiltonian cycle. Then the forbidden extremal configurations are*

$XY$ ,  $YZ$  and  $ZX$  for  $FS(1, 2p)$ ,

$XX$ ,  $YZ$  and  $ZY$  for  $FS(2, 2p)$ ,

and  $XX$ ,  $YY$  and  $ZZ$  for  $FS(3, 2p)$ .

Thus, any perfect matching  $M$  of type 2.0 of  $FS(j, 2p)$  such that the 2-factor  $G \setminus M$  is a hamiltonian cycle is totally characterized by a word of length  $p$  on the alphabet  $\{X, Y, Z\}$  having no two identical consecutive letters and such that the sub-word [initial letter][terminal letter] is not a forbidden configuration. Then, we are in position to obtain the number of such perfect matchings in  $FS(j, 2p)$ . Let us denote by  $\mu'_{2,0}(j, 2p)$  (respectively  $\mu'_{2,1}(j, 2p)$ ,  $\mu'_2(j, 2p)$ ) the number of perfect matchings of type 2.0 (respectively type 2.1, type 2) complementary to a hamiltonian cycle in  $FS(j, 2p)$ . Clearly  $\mu'_2(j, 2p) = \mu'_{2,0}(j, 2p) + \mu'_{2,1}(j, 2p)$  and  $\mu'_{2,0}(j, 2p) = \mu'_{2,1}(j, 2p)$ .

**Theorem 20** *The numbers  $\mu'_2(j, 2p)$  of perfect matchings of type 2 complementary to hamiltonian cycles in  $FS(j, 2p)$  ( $j \in \{1, 2, 3\}$ ) are given by:*

$$\mu'_2(1, 2p) = 2^{p+1} + (-1)^{p+1}2,$$

$$\mu'_2(2, 2p) = 2^{p+1} ,$$

and  $\mu'_2(3, 2p) = 2^{p+1} + (-1)^p4$ .

**Proof** Consider, as previously, perfect matchings of type 2.0. Let  $\alpha$  and  $\beta$  be two letters in  $\{X, Y, Z\}$  (not necessarily distinct). Let  $A_{\alpha\beta}^p$  be the set of words of length  $p$  on  $\{X, Y, Z\}$  having no two consecutive identical letters, beginning by  $\alpha$  and ending by a letter distinct from  $\beta$ . Denote the number of words in  $A_{\alpha\beta}^p$  by  $a_{\alpha\beta}^p$ . Let  $B_{\alpha\beta}^p$  be the set of words of length  $p$  on  $\{X, Y, Z\}$  having no two consecutive identical letters, beginning by  $\alpha$  and ending by  $\beta$ . Denote by  $b_{\alpha\beta}^p$  the number of words in  $B_{\alpha\beta}^p$ .

Clearly, the number of words of length  $p$  having no two consecutive identical letters and beginning by  $\alpha$  is  $2^{p-1}$ . Then  $a_{\alpha\beta}^p + b_{\alpha\beta}^p = 2^{p-1}$ . The deletion of the last  $\beta$  of a word in  $B_{\alpha\beta}^p$  gives a word in  $A_{\alpha\beta}^{p-1}$  and the addition of  $\beta$  to the right of a word in  $A_{\alpha\beta}^{p-1}$  gives a word in  $B_{\alpha\beta}^p$ .

Thus  $b_{\alpha\beta}^p = a_{\alpha\beta}^{p-1}$  and for every  $p \geq 3$   $a_{\alpha\beta}^p = 2^{p-1} - a_{\alpha\beta}^{p-1}$ . We note that  $a_{\alpha\beta}^2 = 2$  if  $\alpha = \beta$ , and  $a_{\alpha\beta}^2 = 1$  if  $\alpha \neq \beta$ . If  $\alpha = \beta$  we have to solve the recurrent sequence :  $u_2 = 2$  and  $u_p = 2^{p-1} - u_{p-1}$  for  $p \geq 3$ . If  $\alpha \neq \beta$  we have to solve the recurrent sequence :  $v_2 = 1$  and  $v_p = 2^{p-1} - v_{p-1}$  for  $p \geq 3$ . Then we obtain  $u_p = \frac{2}{3}(2^{p-1} + (-1)^p)$  and  $v_p = \frac{1}{3}(2^p + (-1)^{p+1})$  for  $p \geq 2$ .

By Lemma 19

$$\mu'_{2,0}(1, 2p) = a_{XY}^p + a_{YZ}^p + a_{ZX}^p = 3v_p = 2^p + (-1)^{p+1},$$

$$\mu'_{2,0}(2, 2p) = a_{XX}^p + a_{YZ}^p + a_{ZY}^p = u_p + 2v_p = 2^p ,$$

and  $\mu'_{2,0}(3, 2p) = a_{XX}^p + a_{YY}^p + a_{ZZ}^p = 3u_p = 2^p + (-1)^p2$ .

Since  $\mu'_2(j, 2p) = \mu'_{2,0}(j, 2p) + \mu'_{2,1}(j, 2p)$  and  $\mu'_{2,0}(j, 2p) = \mu'_{2,1}(j, 2p)$  we obtain the announced results.  $\square$

**Remark 21** We see that  $\mu'_2(j, 2p) \simeq 2^{p+1}$  and this is to compare with the number  $\mu_2(j, 2p) = 2 \times 3^p$  of perfect matchings of type 2 in  $FS(j, 2p)$  (see backward in Section 2).

### 4.3 Strong matchings and Jaeger's graphs

For a given graph  $G = (V, E)$  a *strong matching* (or *induced matching*) is a matching  $S$  such that no two edges of  $S$  are joined by an edge of  $G$ . That is,  $S$  is the set of edges of the subgraph of  $G$  induced by the set  $V(S)$ . We consider cubic graphs having a perfect matching which is the union of two strong matchings that we call *Jaeger's graph* (in his thesis [9] Jaeger called these cubic graphs *equitable*). We call *Jaeger's matching* a perfect matching  $M$  of a cubic graph  $G$  which is the union of two strong matchings  $M_B$  and  $M_R$ . Set  $B = V(M_B)$  (the blue vertices) and  $R = V(M_R)$  (the red vertices). An edge of  $G$  is said *mixed* if its end vertices have distinct colours. Since the set of mixed edges is  $E(G) \setminus M$ , the 2-factor  $G \setminus M$  is even and  $|B| = |M|$ . Thus, every Jaeger's graph  $G$  is a cubic 3-edge colourable graph and for any Jaeger's matching  $M = M_B \cup M_R$ ,  $|M_B| = |M_R|$ . See, for instance, [3] and [4] for some properties of these graphs.

In this subsection we determine the values of  $j$  and  $k$  for which a graph  $FS(j, k)$  is a Jaeger's graph.

**Lemma 22** *If  $G = FS(j, k)$  is a Jaeger's graph (with  $k \geq 3$ ) and  $M = M_B \cup M_R$  is a Jaeger's matching of  $G$  then  $M$  is a perfect matching of type 1.*

**Proof** Suppose that  $M$  is of type 2 and suppose without loss of generality that there are two edges of  $M$  between  $C_0$  and  $C_1$ , for instance  $x_0x_1$  and  $y_0y_1$ . Then  $C_0 \cap M = \{t_0z_0\}$  and  $C_1 \cap M = \{t_1z_1\}$ . Suppose that  $x_0x_1$  and  $y_0y_1$  belong to  $M_B$ . Since  $M_B$  is a strong matching,  $t_0z_0$  and  $t_1z_1$  belong to  $M \setminus M_B = M_R$ . This is impossible because  $M_R$  is also a strong matching. By symmetry there are no two edges of  $M_R$  between  $C_0$  and  $C_1$ . Then there is one edge of  $M_B$  between  $C_0$  and  $C_1$ ,  $x_0x_1$  for instance, and one edge of  $M_R$  between  $C_0$  and  $C_1$ ,  $y_0y_1$  for instance. Since  $M_B$  and  $M_R$  are strong matchings, there is no edge of  $M$  in  $C_0 \cup C_1$ , a contradiction. Thus,  $M$  is a perfect matching of type 1.  $\square$

**Lemma 23** *If  $G = FS(j, k)$  is a Jaeger's graph (with  $k \geq 3$ ) then either ( $j = 1$  and  $k \equiv 1$  or  $2 \pmod{3}$ ) or ( $j = 3$  and  $k \equiv 0 \pmod{3}$ ).*

**Proof** Let  $M = M_B \cup M_R$  be a Jaeger's matching of  $G$ . By Lemma 22  $M$  is a perfect matching of type 1. Suppose without loss of generality that  $M_B \cap E(C_0) = \{x_0t_0\}$ . Since  $M_B$  is a strong matching there is no edge of  $M_B$  between  $C_0$  and  $C_1$ . Suppose, without loss of generality, that the edge in  $M_R$  joining  $C_0$  to  $C_1$  is  $y_0y_1$ . Consider the claws  $C_0$ ,  $C_1$  and  $C_2$ . Since  $M_B$  and  $M_R$  are strong matchings, we can see that the choices of  $x_0t_0 \in M_B$  and  $y_0y_1 \in M_R$  fixes the positions of the other edges of  $M_B$  and  $M_R$ . More

precisely,  $\{t_1z_1, y_2t_2\} \subset M_B$  and  $\{x_1x_2, z_2z'_3\} \subset M_R$ . This unique configuration is depicted in Figure 5.

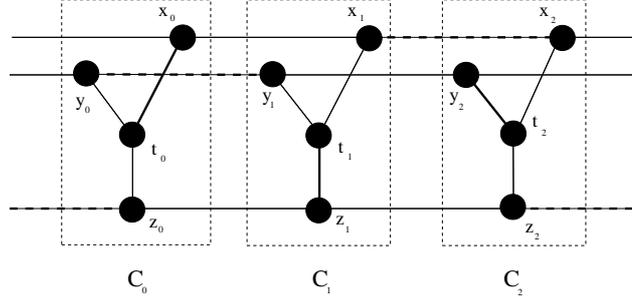


Fig. 5. Strong matchings  $M_B$  (bold edges) and  $M_R$  (dashed edges)

If  $k \geq 4$  then we see that  $z_2z_3 \in M_R$ ,  $x_3t_3 \in M_B$ , and  $y_3y'_4 \in M_R$ . So, the local situation in  $C_3$  is similar to that in  $C_0$ , and we can see that there is a unique Jaeger's matching  $M = M_B \cup M_R$  such that  $x_0t_0 \in M_B$  and  $y_0y_1 \in M_R$  in the graph  $FS(j, k)$ . We have to verify the coherence of the connections between the claws  $C_{k-1}$  and  $C_0$ . We note that  $M_B = M \cap (\cup_{i=0}^{i=k-1} E(C_i))$  and  $M_R$  is a strong matching included in the 2-factor induced by  $\cup_{i=0}^{i=k-1} \{V(C_i) \setminus \{t_i\}\}$ .

**Case 1:**  $k = 3p$  with  $p \geq 1$ .

We have  $x_0t_0 \in M_B$ ,  $y_{k-1}t_{k-1} \in M_B$ ,  $x_{k-2}x_{k-1} \in M_R$  and  $z'_{k-1}z_0 = z_{k-1}z'_0 \in M_R$  (that is,  $z_{k-1}z_0 \in M_R$ ). Thus,  $z_{k-1}z_0$ ,  $y_{k-1}y_0$  and  $x_{k-1}x_0$  are edges of  $FS(j, 3p)$  and we must have  $j = 3$ .

**Case 2:**  $k = 3p + 1$  with  $p \geq 1$ .

We have  $x_0t_0 \in M_B$ ,  $x_{k-1}t_{k-1} \in M_B$  (that is,  $x_{k-1}x_0 \notin E(G)$ ),  $z_{k-2}z_{k-1} \in M_R$  and  $z'_{k-1}z_0 = y_{k-1}y'_0 \in M_R$  (that is,  $y_{k-1}z_0 \in M_R$ ). Thus,  $y_{k-1}z_0$ ,  $x_{k-1}y_0$  and  $z_{k-1}x_0$  are edges of  $FS(j, 3p + 1)$  and we must have  $j = 1$ .

**Case 3:**  $k = 3p + 2$  with  $p \geq 1$ .

We have  $x_0t_0 \in M_B$ ,  $z_{k-1}t_{k-1} \in M_B$ ,  $y_{k-2}y_{k-1} \in M_R$  and  $z'_{k-1}z_0 = x_{k-1}x'_0 \in M_R$  (that is  $x_{k-1}z_0 \in M_R$ ). Thus,  $x_{k-1}z_0$ ,  $y_{k-1}x_0$  and  $z_{k-1}y_0$  are edges of  $FS(j, 3p + 2)$  and we must have  $j = 1$ .  $\square$

**Remark 24** It follows from Lemma 23 that for every  $k \geq 3$  the graph  $FS(2, k)$  is not a Jaeger's graph. This is obvious when  $k$  is odd, since the flower snarks have chromatic index 4.

Then, we obtain the following.

**Theorem 25** For  $j \in \{1, 2, 3\}$  and  $k \geq 2$ , the graph  $G = FS(j, k)$  is a Jaeger's graph if and only if

either  $k \equiv 1$  or  $2 \pmod{3}$  and  $j = 1$ ,

or  $k \equiv 0 \pmod{3}$  and  $j = 3$ .

Moreover,  $FS(1,2)$  has 3 Jaeger's matchings and for  $k \geq 3$  a Jaeger's graph  $G = FS(j,k)$  has exactly 6 Jaeger's matchings.

**Proof** For  $k = 2$  we remark that  $FS(1,2)$  (that is the cube) has exactly three distinct Jaeger's matchings  $M_1$ ,  $M_2$  and  $M_3$ . Following our notations:  $M_1 = \{x_0t_0, t_1z_1\} \cup \{y_0y_1, z_0x_1\}$ ,  $M_2 = \{z_0t_0, t_1y_1\} \cup \{y_0z_1, x_0x_1\}$  and  $M_3 = \{y_0t_0, t_1x_1\} \cup \{z_0z_1, x_0y_1\}$ .

For  $k \geq 3$ , by Lemma 23, condition

(\*) ( $j = 1$  and  $k \equiv 1$  or  $2 \pmod{3}$ ) or ( $j = 3$  and  $k \equiv 0 \pmod{3}$ )

is a necessary condition for  $FS(j,k)$  to be a Jaeger's graph.

Consider the function  $\Phi_{X,Y} : V(G) \rightarrow V(G)$  such that for every  $i$  in  $\mathbf{Z}_k$ ,  $\Phi_{X,Y}(t_i) = t_i$ ,  $\Phi_{X,Y}(z_i) = z_i$ ,  $\Phi_{X,Y}(x_i) = y_i$  and  $\Phi_{X,Y}(y_i) = x_i$ . Define similarly  $\Phi_{X,Z}$  and  $\Phi_{Y,Z}$ . For  $j = 1$  or  $3$  these functions are automorphisms of  $FS(j,k)$ . Thus, the process described in the proof of Lemma 23 is a constructive process of all Jaeger's matchings in a graph  $FS(j,k)$  (with  $k \geq 3$ ) verifying condition (\*).

We remark that for any choice of an edge  $e$  of  $C_0$  to be in  $M_B$  there are two distinct possible choices for an edge  $f$  between  $C_0$  and  $C_1$  to be in  $M_R$ , and such a pair  $\{e, f\}$  corresponds exactly to one Jaeger's matching. Then, a Jaeger's graph  $FS(j,k)$  (with  $k \geq 3$ ) has exactly 6 Jaeger's matchings.  $\square$

**Remark 26** The *Berge-Fulkerson Conjecture* states that if  $G$  is a bridgeless cubic graph, then there exist six perfect matchings  $M_1, \dots, M_6$  of  $G$  (not necessarily distinct) with the property that every edge of  $G$  is contained in exactly two of  $M_1, \dots, M_6$  (this conjecture is attributed to Berge in [16] but appears in [5]). Using each colour of a cubic 3-edge colourable graph twice, we see that such a graph verifies the Berge-Fulkerson Conjecture. Very few is known about this conjecture except that it holds for the Petersen graph and for cubic 3-edge colourable graphs. So, Berge-Fulkerson Conjecture holds for Jaeger's graphs, but generally we do not know if we can find six distinct perfect matchings. We remark that if  $FS(j,k)$ , with  $k \geq 3$ , is a Jaeger's graph then its six Jaeger's matchings are such that every edge is contained in exactly two of them.

## 5 2-factor hamiltonian cubic graphs

Recall that a simple graph of maximum degree  $d > 1$  with edge chromatic number equal to  $d$  is said to be a *Class 1 graph*. For any  $d$ -regular simple graph (with  $d > 1$ ) of even order and of Class 1, for any minimum edge-colouring of such a graph, the set of edges having a given colour is a perfect matching (or 1-factor). Such a regular graph is also called a *1-factorable graph*. A Class 1  $d$ -regular graph of even order is *strongly hamiltonian* or *perfectly 1-factorable* (or is a *Hamilton graph* in the Kotzig's terminology [11]) if it has an edge colouring such that the union of any two colours is a hamiltonian cycle. Such an edge colouring is said to be a *Hamilton decomposition* in the Kotzig's terminology. In [10] by using two operations  $\rho$  and  $\pi$  (described also in [11]) and starting from the  $\theta$ -graph (two vertices joined by three parallel edges) he obtains all strongly hamiltonian cubic graphs, but these operations does not always preserve planarity. In his paper [11] he describes a method for constructing planar strongly hamiltonian cubic graphs and he deals with the relation between strongly hamiltonian cubic graphs and 4-regular graphs which can be decomposed into two hamiltonian cycles. See also [12] and a recent work on strongly hamiltonian cubic graphs [2] in which the authors give a new construction of strongly hamiltonian graphs.

A Class 1 regular graph such that every edge colouring is a Hamilton decomposition is called a *pure Hamilton graph* by Kotzig [11]. Note that  $K_4$  is a pure Hamilton graph and every cubic graph obtained from  $K_4$  by a sequence of triangular extensions is also a pure Hamilton cubic graph. In the paper [11] of Kotzig, a consequence of his Theorem 9 (p.77) concerning pure Hamilton graphs is that the family of pure Hamilton graphs that he exhibits is precisely the family obtained from  $K_4$  by triangular extensions. Are there others pure Hamilton cubic graphs ? The answer is "yes".

We remark that 2-factor hamiltonian cubic graphs defined above (see Definition 16) are pure Hamilton graphs (in the Kotzig's sense) but the converse is false because  $K_4$  is 2-factor hamiltonian and the pure Hamilton cubic graph on 6 vertices obtained from  $K_4$  by a triangular extension (denoted by  $PR_3$ ) is not 2-factor hamiltonian. Observe that the operation of triangular extension preserves the property "pure Hamilton", but does not preserve the property "2-factor hamiltonian". The Heawood graph  $H_0$  (on 14 vertices) is pure Hamiltonian, more precisely it is 2-factor hamiltonian (see [7] Proposition 1.1 and Remark 2.7). Then, the graphs obtained from the Heawood graph  $H_0$  by triangular extensions are also pure Hamilton graphs.

A *minimally 1-factorable* graph  $G$  is defined by Labbate and Funk [7] as a Class 1 regular graph of even order such that every perfect matching of  $G$  is contained in exactly one 1-factorization of  $G$ . In their article they study

bipartite minimally 1-factorable graphs and prove that such a graph  $G$  has necessarily a degree  $d \leq 3$ . If  $G$  is a minimally 1-factorable cubic graph then the complementary 2-factor of any perfect matching has a unique decomposition into two perfect matchings, therefore this 2-factor is a hamiltonian cycle of  $G$ , that is  $G$  is 2-factor hamiltonian. Conversely it is easy to see that any 2-factor hamiltonian cubic graph is minimally 1-factorable. The complete bipartite graph  $K_{3,3}$  and the Heawood graph  $H_0$  are examples of 2-factor hamiltonian bipartite graph given by Labbate and Funk. Starting from  $H_0$ , from  $K_{1,3}$  and from three copies of any tree of maximum degree 3 and using three operations called *amalgamations* the authors exhibit an infinite family of bipartite 2-factor hamiltonian cubic graphs, namely the *poly - HB - R - R<sup>2</sup>* graphs (see [7] for more details). Except  $H_0$ , these graphs are exactly cyclically 3-edge connected. Others structural results about 2-factor hamiltonian bipartite cubic graph are obtained in [13], [14]. These results have been completed and a simple method to generate 2-factor hamiltonian bipartite cubic graphs was given in [6].

**Proposition 27** (Lemma 3.3, [6]) *Let  $G$  be a 2-factor hamiltonian bipartite cubic graph. Then  $G$  is 3-connected and  $|V(G)| \equiv 2 \pmod{4}$ .*

Let  $G_1$  and  $G_2$  be disjoint cubic graphs,  $x \in v(G_1)$ ,  $y \in v(G_2)$ . Let  $x_1, x_2, x_3$  (respectively  $y_1, y_2, y_3$ ) be the neighbours of  $x$  in  $G_1$  (respectively, of  $y$  in  $G_2$ ). The cubic graph  $G$  such that  $V(G) = (V(G_1) \setminus \{x\}) \cup (V(G_2) \setminus \{y\})$  and  $E(G) = (E(G_1) \setminus \{x_1x, x_2x, x_3x\}) \cup (E(G_2) \setminus \{y_1y, y_2y, y_3y\}) \cup \{x_1y_1, x_2y_2, x_3y_3\}$  is said to be a *star product* and  $G$  is denoted by  $(G_1, x) * (G_2, y)$ . Since  $\{x_1y_1, x_2y_2, x_3y_3\}$  is a cyclic edge-cut of  $G$ , a star product of two 3-connected cubic graphs has cyclic edge-connectivity 3.

**Proposition 28** (Proposition 3.1, [6]) *If a bipartite cubic graph  $G$  can be represented as a star product  $G = (G_1, x) * (G_2, y)$ , then  $G$  is 2-factor hamiltonian if and only if  $G_1$  and  $G_2$  are 2-factor hamiltonian.*

Then, taking iterated star products of  $K_{3,3}$  and the Heawood graph  $H_0$  an infinite family of 2-factor hamiltonian cubic graphs is obtained. These graphs (except  $K_{3,3}$  and  $H_0$ ) are exactly cyclically 3-edge connected. In [6] the authors conjecture that the process is complete.

**Conjecture 29** (Funk, Jackson, Labbate, Sheehan (2003)[6]) *Let  $G$  be a bipartite 2-factor hamiltonian cubic graph. Then  $G$  can be obtained from  $K_{3,3}$  and the Heawood graph  $H_0$  by repeated star products.*

The authors precise that a smallest counterexample to Conjecture 29 is a cyclically 4-edge connected cubic graph of girth at least 6, and that to show this result it would suffice to prove that  $H_0$  is the only 2-factor hamiltonian

cyclically 4-edge connected bipartite cubic graph of girth at least 6. Note that some results have been generalized in [1].

To conclude, we may ask what happens for non bipartite 2-factor hamiltonian cubic graphs. Recall that  $K_4$  and  $FS(1, 3)$  (the "Triplex Graph" of Robertson, Seymour and Thomas [15]) are 2-factor hamiltonian cubic graphs. By Corollary 17 the graphs  $FS(j, k)$  with  $k$  odd and  $j = 1$  or  $3$  introduced in this paper form a new infinite family of non bipartite 2-factor hamiltonian cubic graphs. We remark that they are cyclically 6-edge connected. Can we generate others families of non bipartite 2-factor hamiltonian cubic graphs ? Since  $PR_3$  (the cubic graph on 6 vertices obtained from  $K_4$  by a triangular extension) is not 2-factor hamiltonian and  $PR_3 = K_4 * K_4$ , the star product operation is surely not a possible tool.

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