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Growth of Sobolev norms for solutions of time dependent Schrödinger operators with harmonic oscillator potential

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Abstract

It has been known for some time that solutions of linear Schrödinger operators on the torus, with bounded, smooth, time dependent potential, have Sobolev norms growing at most like t^ϵ when $t \rightarrow +\infty$ for any $\epsilon > 0$. This property is proved exploiting the fact that, on the circle, successive eigenvalues of the laplacian are separated by increasing gaps (and a more involved but similar property for clusters of eigenvalues in higher dimension). We study here the case of solutions of $(i\partial_t - \frac{\partial^2}{\partial x^2} + x^2 + V)u = 0$, where V is a time periodic order zero perturbation. In this case, the gap between successive eigenvalues of the stationary operator is constant. We show that there are order zero potentials V for which some solutions u have Sobolev norms of order s growing like $t^{s/2}$ when $t \rightarrow +\infty$.

0 Introduction

Let P_0 be an elliptic self-adjoint differential operator of positive order on some Riemannian manifold M . Consider $t \rightarrow V(t)$ a smooth family of self-adjoint operators on M , of order zero, and the Schrödinger equation

$$(1) \quad (i\partial_t + P_0 + V(t))u = 0.$$

Then $\|u(t, \cdot)\|_{L^2} = \|u(0, \cdot)\|_{L^2}$. Denote by \mathcal{H}^s the Sobolev space associated to P_0 , defined when s is an even integer $2k$ by $\|u\|_{\mathcal{H}^s} = \|u\|_{L^2} + \|P_0^k u\|_{L^2}$. We are interested in estimating $\|u(t, \cdot)\|_{\mathcal{H}^s}$ when t goes to infinity. If V is time independent, it is immediate that $\|u(t, \cdot)\|_{\mathcal{H}^s} = \|u(0, \cdot)\|_{\mathcal{H}^s}$. Over the last fifteen years, several results have been obtained by different authors for time dependent potentials V , under convenient spectral assumptions on P_0 and V . Our aim is to study this question when P_0 is the harmonic oscillator in one dimension, as this is the most important case which has not been studied by preceding authors. Before presenting our results, we discuss in a relatively lengthy way known results and the methods that have been developed to prove them.

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The problem of finding optimal bounds for $\|u(t, \cdot)\|_{\mathcal{H}^s}$ has been addressed by Nenciu [10] and Barbaroux and Joye [1], in the abstract framework of an operator P_0 and a perturbation $V(t)$ acting on elements of an Hilbert space, when the spectrum of P_0 is discrete and has increasing gaps. This means that the distinct eigenvalues $(\lambda_n)_n$ of P_0 satisfy conditions of type $|\lambda_{n+1} - \lambda_n| \geq c(\lambda_n + \lambda_{n+1})^{\rho/2}$ for some positive ρ . This assumption is satisfied for instance if $M = \mathbb{S}^d$ and $P_0 = -\Delta$ (with $\lambda_n \sim n^2$ and $\rho = 1/2$). If Π_n denotes the spectral projector associated to the eigenvalue λ_n , and if one assumes that $V(t)$ is an operator obeying conditions of type $\|\Pi_n \partial_t^\ell V(t) \Pi_{n'}\|_{\mathcal{L}(L^2)} \leq C_N \langle n - n' \rangle^{-N}$ for any N, ℓ , it follows from the results of [10, 1], that solutions of (1) verify $\|u(t, \cdot)\|_{\mathcal{H}^s} \leq C_\epsilon |t|^\epsilon$ when t goes to infinity, for any $\epsilon > 0$. Actually, because of the fact that L^2 norms are conserved, and since \mathcal{H}^s spaces are stable by interpolation, such a result follows from a seemingly weaker bound $\|u(t, \cdot)\|_{\mathcal{H}^s} \leq C(1 + |t|)^M$, where M is some positive number *independent* of s .

Let us recall the basic idea of the proof of such a result following the method of [10, 1], when for instance P_0 is of order 2: One looks for an operator Q such that

$$(2) \quad Q^*(i\partial_t + P_0 + V(t))Q = i\partial_t + P_0 + W(t) + \text{remainders},$$

where the remainders are of very negative order, and where $[W(t), P_0] = 0$ for any t . Clearly, any solution of $(i\partial_t + P_0 + W(t))w = 0$ will satisfy $\|P_0^k w(t, \cdot)\|_{L^2} = \|P_0^k w(0, \cdot)\|_{L^2}$ for any $t \in \mathbb{R}, k \in \mathbb{N}$. Taking the remainder in the right hand side of (2) into account, one easily deduces from that bounds $\|u(t, \cdot)\|_{\mathcal{H}^s} \leq C(1 + |t|)\|u(0, \cdot)\|_{\mathcal{H}^s}$ for solutions of (1). The operator Q is looked for as $Q = Q_0 + Q_1 + \dots$, with $Q_0 = Id$, $Q_1 = Q'_1 + Q''_1$, where Q'_1 is of order $-\rho$, Q''_1 of order -2ρ , $(Q'_1)^* = -Q'_1$, $(Q''_1)^* = Q''_1$. Decompose $V(t) = V_D(t) + V_{ND}(t)$, where $\Pi_n V_D \Pi_{n'} = 0$ if $n \neq n'$ and $\Pi_n V_{ND} \Pi_n = 0$ for any n . Then V_D will contribute to W in the right hand side of (2), and V_{ND} will be eliminated solving equation

$$(3) \quad -[Q'_1, P_0] + V_{ND} = 0.$$

Since this equation may be written, applying the spectral projectors Π_n (resp. $\Pi_{n'}$) on the left (resp. right), as

$$(4) \quad (\lambda_n - \lambda_{n'})\Pi_n Q'_1 \Pi_{n'} + \Pi_n V_{ND} \Pi_{n'} = 0 \quad (n \neq n'),$$

one sees that the gap condition on the λ_n 's implies that Q'_1 will be of order $-\rho$. In particular, terms in the right hand side of (2) of form $-[Q'_1, i\partial_t]$ are of order $-\rho$, so enter the construction of Q at subsequent orders.

The above approach uses in an essential way the gap condition on the eigenvalues of P_0 . In particular, it does not apply a priori to the case when $P_0 = -\Delta$ on the torus \mathbb{T}^d ($d \geq 2$). Nevertheless, Bourgain [2, 3] could prove that, if V is a smooth time dependent potential bounded as well as its derivatives, solutions of $(i\partial_t - \Delta + V)u = 0$ on \mathbb{T}^d obey bounds $\|u(t, \cdot)\|_{\mathcal{H}^s} \leq C_\epsilon(1 + |t|)^\epsilon$ for any $\epsilon > 0$. The method of proof is quite different from what we outlined above, and relies on reduction to a time periodic potential, and study of the eigenfunctions of the corresponding operator on $\mathbb{S}^1 \times \mathbb{T}^d$. The main technical ingredient is a partition of \mathbb{Z}^d (considered as the dual group of \mathbb{T}^d) in clusters, which are well separated for the distance $|n - n'| + ||n|^2 - |n'|^2|$ naturally associated to the Schrödinger operator. The same method has been applied in dimension one by Wei-Min Wang [12] to prove logarithmic bounds for the Sobolev

norm of solutions of (1), when V is analytic. We gave in [4] an alternative proof of the result of Bourgain, which works as well for perturbations V which are pseudo-differential operators of order zero, with smooth and bounded dependence on t , rediscovering the method used by Nenciu and Barbaroux-Joye, that we ignored at that time (We owe to discussions with P. Duclos and S. Kuksin references to these works and to related papers). The key point is a combination of the diagonalization method outlined above with the Bourgain lemma on separated clusters, which allows one to solve again (4) with a gain $-\rho$ on the order of Q versus the order of V . Moreover, we could extend such results to some other manifolds, like convenient surfaces of revolution. This method has been recently adapted to the case of analytic potentials by D. Fang and Q. Zhang [7] to recover and make more precise the logarithmic estimates of Wang. Remind that Bourgain [2] constructed examples of time periodic potentials for which $\|u(t, \cdot)\|_{H^s}$ is bounded from below by a power of $\log t$ when $s > 0$ and $t \rightarrow +\infty$.

A natural question is to determine whether estimates of type $\|u(t, \cdot)\|_{\mathcal{H}^s} = O(t^\epsilon)$, $t \rightarrow +\infty$ hold true when the assumption of increasing gaps $|\lambda_{n+1} - \lambda_n| \geq c(\lambda_n + \lambda_{n+1})^{\rho/2}$ is not satisfied with a positive ρ . Howland [9] obtained results on the structure of the spectrum of $i\partial_t + P_0 + V(t)$ on $\mathbb{S}^1 \times \mathbb{S}^1$, when V is time periodic and when the eigenvalues λ_n of P_0 satisfy a *shrinking* gap condition, like $\lambda_n \sim n^\alpha$ with $0 < \alpha < 1$. Duclos, Lev and Štoviček [5] have studied Sobolev bounds for solutions of such operators, under the same assumption of shrinking gap conditions. If one sets $\gamma = \frac{1-\alpha}{2}$, and if the perturbation $V(t)$ in (1) is periodic in time, small enough, and satisfies conditions of type

$$\|\Pi_n V(t) \Pi_{n'}\|_{\mathcal{L}(L^2)} = O((1 + |n| + |n'|)^{-2\gamma} \langle n - n' \rangle^{-N})$$

for any N , they could prove $O(|t|^\epsilon)$ -bounds for Sobolev norms. These assumptions on V are verified for instance when V is a pseudo-differential operator of order -2γ on \mathbb{S}^1 . The method of proof still relies on a conjugation of type (2), except that it is done in two steps. One first decomposes the potential $V(t)$ as $\bar{V} + \tilde{V}(t)$, where \bar{V} is the time average of V on a period. One eliminates $\tilde{V}(t)$ looking for an operator Q such that, instead of (3),

$$(5) \quad -[Q, i\partial_t + P_0 + \bar{V} + \tilde{V}(t)] + \tilde{V}(t) = \text{terms of lower order.}$$

This equation is solved taking Q such that $-[Q, i\partial_t] + \tilde{V}(t) = 0$. Since $\tilde{V}(t)$ is of order -2γ , Q will be of order -2γ . The shrinking gap condition on P_0 implies then that $[Q, P_0]$ is of order -4γ , so will contribute to the lower order terms in the right hand side. The same is true for the $[Q, \bar{V} + \tilde{V}(t)]$ contribution.

After the first step, one is morally reduced to to an operator of the form $i\partial_t + P_0 + \bar{V}$, where \bar{V} is time independent. The second step of the proof of the results of [5] is to find another operator Q such that

$$(6) \quad Q^*(i\partial_t + P_0 + \bar{V})Q = i\partial_t + P_0 + \bar{V}_D + \text{remainders,}$$

where \bar{V}_D commutes to P_0 . The method is similar to the one used in the case of increasing gaps, i.e. finding Q solving an equation of form (3). The difference is that under a shrinking gap condition, the operator Q that one finds is not of order smaller than the order of \bar{V} . Actually, since V is of order -2γ , where γ is related to the width of the gaps, one finds an operator Q of order zero. The key point is that Q is now time independent so that $[i\partial_t, Q] = 0$, and is small

since \bar{V} is small. Because of that, after finding the first approximation of Q to solve equation (6), one generates remainders which are of the same order as \bar{V} , but whose magnitude is ϵ^2 if the magnitude of \bar{V} is ϵ . Iterating the process, and showing that the resulting ϵ -series converges, allows one to make a reduction of (6) at infinite order i.e. to remove the remainders in the right hand side.

The results we describe so far use in an essential way that the gaps in the spectrum of P_0 are either increasing or shrinking. They do not apply to the case of the harmonic oscillator in one dimension $P_0 = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$, for which the eigenvalues are the integers (up to translation): this corresponds to the forbidden case $\alpha = 1$ or $\gamma = 0$ in the preceding results.

The goal of this paper is to study this problem, and to show that one may find order zero time dependent perturbations of P_0 such that there are solutions whose \mathcal{H}^s -norms grow like $t^{s/2}$, when t goes to infinity (This growth will be seen to be optimal). In other words, our results are in contrast with those of the cases of increasing or shrinking gaps, as one *cannot* obtain a bound for the Sobolev norms in terms of $|t|^M$ with a power M independent of s .

Let us describe the approach we follow. We consider an operator

$$(7) \quad i\partial_t + P_0 + \text{Op}^W(V),$$

where P_0 is the harmonic oscillator in one dimension, and $\text{Op}^W(V)$ the Weyl quantization of a time periodic order zero symbol V belonging to the symbolic classes naturally associated to P_0 . One first looks for an operator of order zero Q , such that

$$(8) \quad Q^*(i\partial_t + P_0 + \text{Op}^W(V))Q = i\partial_t + P_0 + \text{Op}^W(V^Z) + \text{remainders},$$

where V^Z is such that $[i\partial_t + P_0, V^Z] = 0$. The construction of Q is essentially reduced to solving an equation of the form

$$(9) \quad [i\partial_t + P_0, Q] + \text{Op}^W(V - V^Z)Q = \text{lower order operator}.$$

Notice that the difference with the case of increasing or shrinking gaps is that in the left hand side of (9), $[i\partial_t, Q]$ and $[P_0, Q]$ are of the same order (In the results described above, one of these commutators was of lower order, so that the equation to solve was either $[i\partial_t, Q] = B$ or $[P_0, Q] = B$ for a given B). Using symbolic calculus, we easily find V^Z with $[i\partial_t + P_0, V^Z] = 0$ and Q solving (9), so that, forgetting the remainders, the conjugation equation (8) reduces (7) to

$$(10) \quad i\partial_t + P_0 + \text{Op}^W(V^Z).$$

Conjugating the above operator by e^{itP_0} , one reduces it to

$$(11) \quad i\partial_t + \text{Op}^W(W),$$

where $\text{Op}^W(W) = e^{-itP_0}\text{Op}^W(V^Z)e^{itP_0}$. The commutator properties of $\text{Op}^W(V^Z)$ and $i\partial_t + P_0$ imply that W is independent of t . In that way, (7) is reduced to an autonomous ODE (11).

The second step of the proof is to construct a convenient W and an approximate solutions w_a of (11), whose \mathcal{H}^s -norm behaves like $t^{s/2}$ when t goes to infinity. We look for this function as

$w_a(t, x) = e^{i\psi(t, x)}b(t, x)$ where, for large t , $|\partial_x\psi(t, x)| \sim \sqrt{t}$, $|b(t, x)| \leq Ct^{-1/4}$, $x \rightarrow b(t, x)$ is supported for x/\sqrt{t} in a compact set K and $|b(t, x)| \geq ct^{-1/4}$ for x/\sqrt{t} in another compact set K' . In that way, $\|w_a(t, \cdot)\|_{L^2} \sim 1$ and $\|\partial_x^s w_a(t, \cdot)\|_{L^2} + \|x^s w_a(t, \cdot)\|_{L^2} \sim t^{s/2}$ i.e. $\|w_a(t, \cdot)\|_{\mathcal{H}^s} \sim t^{s/2}$ when t goes to infinity. To determine ψ , we construct the Lagrangian

$$\Lambda_t = \{(x, \partial_x\psi(t, x)); x \in \mathbb{R}\}.$$

At the initial time, we take a Lagrangian which is a graph above an interval $[A, 2A]$, and which stays inside an angular sector of angle $1/A$ around the positive x -axis of $\mathbb{R}^2 \simeq T^*\mathbb{R}$, where A is a large enough constant. We take an autonomous symbol W that may be written in polar coordinates as $W_\infty(\theta) +$ perturbation of order < -2 . The main contribution W_∞ is assumed to satisfy convenient non-degeneracy conditions along the positive x -axis $\theta = 0$. These conditions imply that the flow-out of the Hamiltonian vector field of W pulls the initial Lagrangian far away from zero, inside the sector of angle $1/A$. At large time t , the transported Lagrangian lies at a distance of order \sqrt{t} from the x -axis, and is the graph of a function $x \rightarrow \partial_x\psi(t, x)$, which has thus to verify $|\partial_x\psi(t, x)| \sim \sqrt{t}$. Once ψ has been constructed, one finds b so that $e^{i\psi}b$ be an approximate solution, solving transport equations. In that way, an approximate solution of $(i\partial_t + \text{Op}^W(W))w_a = 0$ has been constructed, for some convenient W . The last step of the proof is to show that one may find a time depending symbol V such that the reductions (8), (10), (11) bring to this W (up to remainders), and to deduce from the approximate solution w_a an approximate solution – and then an exact solution – of the initial operator (7), enjoying the wanted asymptotics at infinity.

1 Growth of Sobolev norms

1.1 Statement of the main theorem

The main result of this paper asserts that one may construct time dependent self-adjoint order zero perturbations of the harmonic oscillator $P_0 = \frac{1}{2}(-\frac{d^2}{dx^2} + x^2)$ in one space dimension, with solutions whose Sobolev norm grows at an optimal rate when time goes to infinity. Before stating the theorem, we need to introduce some notations.

Let us recall the definition of standard classes of pseudo-differential symbols adapted to the harmonic oscillator.

Definition 1.1.1 *Let $d \in \mathbb{N}^*$, $m \in \mathbb{R}$. One denotes by $\mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ the space of smooth functions $(t, x, \xi) \rightarrow a(t, x, \xi)$ defined on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, with values in \mathbb{C} , which are 2π -periodic in t , and such that for any $j, \alpha, \beta \in \mathbb{N}$, there is a constant $C > 0$ so that, for any (t, x, ξ) ,*

$$(1.1.1) \quad |\partial_t^j \partial_x^\alpha \partial_\xi^\beta a(t, x, \xi)| \leq C(1 + |x| + |\xi|)^{m - |\alpha| - |\beta|}.$$

We endow $\mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ with its natural semi-norms. We denote by $\mathcal{S}^m(\mathbb{R}^d \times \mathbb{R}^d)$ the subspace of time independent symbols.

We shall use Weyl quantization for the above symbols.

Definition 1.1.2 If $a \in \mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, we denote by $\text{Op}^W(a(t, \cdot))$ the operator defined on $\mathcal{S}(\mathbb{R}^d)$ by

$$(1.1.2) \quad \text{Op}^W(a(t, \cdot))u = \frac{1}{(2\pi)^d} \int e^{i(x-y)\xi} a\left(t, \frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

We refer to chapter 4 of the book of Shubin [11] for the main properties of these operators. We shall recall those which will be useful for us in subsection 1.2 below. Let us just mention at this point that if $a \in \mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, for fixed t , $\text{Op}^W(a(t, \cdot))$ extends as a bounded operator on $L^2(\mathbb{R}^d)$ and on $\mathcal{S}'(\mathbb{R}^d)$. Moreover, if one defines, for $s \in \mathbb{R}$, the Sobolev space \mathcal{H}^s associated to the harmonic oscillator, by

$$(1.1.3) \quad \mathcal{H}^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d); \text{Op}^W((1 + |x|^2 + |\xi|^2)^{s/2})u \in L^2(\mathbb{R}^d)\}$$

or equivalently, when $s \in \mathbb{N}$,

$$(1.1.4) \quad \mathcal{H}^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d); \text{ for any } \alpha, \beta \in \mathbb{N}^d \times \mathbb{N}^d \text{ with } |\alpha| + |\beta| \leq s, x^\alpha \partial^\beta u \in L^2(\mathbb{R}^d)\},$$

the operator $\text{Op}^W(a(t, \cdot))$ is bounded from $\mathcal{H}^s(\mathbb{R}^d)$ to $\mathcal{H}^{s-m}(\mathbb{R}^d)$, uniformly in t , for any s , when a is a symbol of order m .

Let state our main theorem.

Theorem 1.1.3 Let $d = 1$. There is a real valued symbol $V \in \mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$ and, for any $s > 0$, a constant $c > 0$, and a smooth solution $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ of the equation

$$(1.1.5) \quad (i\partial_t + P_0 + \text{Op}^W(V(t, \cdot)))u = 0,$$

such that for any $t \geq 0$, $u(t, \cdot) \in \mathcal{H}^s(\mathbb{R})$ and satisfies

$$(1.1.6) \quad \|u(t, \cdot)\|_{\mathcal{H}^s} \geq ct^{s/2}$$

for any large enough t .

Remarks: • We shall see, in the following subsection, that a solution of equation (1.1.5) with any V of order zero and with \mathcal{H}^s -Cauchy data, always satisfies a bound $\|u(t, \cdot)\|_{\mathcal{H}^s} \leq Ct^{s/2}$ when $t \rightarrow +\infty$. The lower bound (1.1.6) is thus optimal.

• The symbol V that we construct will be an element of $\mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$. In particular, it depends on x and on ξ . We do not know if a result of type (1.1.6) could be proved replacing in (1.1.5) the order zero perturbation $\text{Op}^W(V(t, \cdot))$ by a local potential $V(t, x)$. On the other hand, Grébert and Thomann [8] have studied recently equations of the form (1.1.5), where V is a small time quasi-periodic local potential satisfying convenient assumptions. They have been able to prove that, when the parameter of quasi-periodicity stays outside a subset of small measure, the equation may be reduced to a linear equation with constant coefficients. Consequently, the solutions of the Cauchy problem are time quasi-periodic, and so have uniformly bounded Sobolev norms. We refer to theorem 1.2 and corollary 1.3 of [8] for precise statements, and to corollary 1.4

of the same paper, as well as to the paper of Wang [13], for interpretation of such results in terms of Floquet spectrum. Notice also that similar results for the Schrödinger equation on the torus, with time quasi-periodic potential, had been previously proved by Eliasson and Kuksin [6].

- The conditions on the potential that we need to obtain (1.1.6) will be made more precise in subsection 3.1.

1.2 Symbolic calculus

In this subsection, we recall classical properties of operators of the form (1.1.2) and establish some related results that will be useful in the sequel.

Remind that if a is in $\mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, then

$$(1.2.1) \quad \text{Op}^W(a(t, \cdot))^* = \text{Op}^W(\bar{a}(t, \cdot)),$$

so that real valued symbols give rise to self-adjoint operators. If $a \in \mathcal{S}^{m_1}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, $b \in \mathcal{S}^{m_2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, there is a symbol $c \in \mathcal{S}^{m_1+m_2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ such that

$$(1.2.2) \quad \text{Op}^W(a(t, \cdot)) \circ \text{Op}^W(b(t, \cdot)) = \text{Op}^W(c(t, \cdot)).$$

Moreover, there is a sequence $(c_k)_k$ of symbols, $c_k \in \mathcal{S}^{m_1+m_2-2k}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, such that $c - \sum_{k=0}^{N-1} c_k \in \mathcal{S}^{m_1+m_2-2N}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, where

$$(1.2.3) \quad c_k(t, x, \xi) = 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha! \beta!} (\partial_\xi^\alpha D_x^\beta a) (\partial_\xi^\beta D_x^\alpha b).$$

The above expression is bilinear in (a, b) and symmetric (resp. antisymmetric) if k is even (resp. odd). This implies that

$$(1.2.4) \quad [\text{Op}^W(a(t, \cdot)), \text{Op}^W(b(t, \cdot))] = \text{Op}^W(e(t, \cdot)),$$

where $e \in \mathcal{S}^{m_1+m_2-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ is such that there is a sequence $(e_{2k+1})_k$ of symbols, with $e_{2k+1} \in \mathcal{S}^{m_1+m_2-4k-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, and

$$(1.2.5) \quad \begin{aligned} e(t, x, \xi) - \sum_{k=0}^{N-1} e_{2k+1}(t, x, \xi) &\in \mathcal{S}^{m_1+m_2-4N-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d) \\ e_1(t, x, \xi) = i\{a, b\} &= i \sum_{j=1}^d \left(-\frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} + \frac{\partial a}{\partial x_j} \frac{\partial b}{\partial \xi_j} \right). \end{aligned}$$

We set $p_0(x, \xi) = \frac{1}{2}(|x|^2 + |\xi|^2)$. If $a \in \mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ we get

$$(1.2.6) \quad \begin{aligned} P_0 = \text{Op}^W(p_0) &= -\frac{1}{2}\Delta + \frac{1}{2}|x|^2, \\ [P_0, \text{Op}^W(a(t, \cdot))] &= i\text{Op}^W(\{p_0, a\}(t, \cdot)). \end{aligned}$$

The above properties of symbolic calculus provide immediately a bound for the Sobolev norm of solutions of equations of form (1.1.5):

Lemma 1.2.1 *Let $V \in \mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ a real valued symbol. If $t, t' \in \mathbb{R}$, denote by $S(t, t')$ the operator solving*

$$(1.2.7) \quad \begin{aligned} [i\partial_t + P_0 + \text{Op}^W(V(t, \cdot))]S(t, t') &= 0 \\ S(t, t')|_{t=t'} &= \text{Id}. \end{aligned}$$

Then, for any $s \in \mathbb{R}_+$, there is a constant $C_s > 0$ such that, for any $t, t' \in \mathbb{R}$,

$$(1.2.8) \quad \|S(t, t')\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^s)} \leq C_s \langle t - t' \rangle^{s/2}.$$

In particular, any solution u of (1.1.5) with initial data in \mathcal{H}^s satisfies $\|u\|_{\mathcal{H}^s} = O(t^{s/2})$, $t \rightarrow +\infty$.

Proof: Consider w solution of

$$\begin{aligned} [i\partial_t + P_0 + \text{Op}^W(V(t, \cdot))]w(t, \cdot) &= 0 \\ w(t', x) &= w_0. \end{aligned}$$

Then, since $\text{Op}^W(V(t, \cdot))$ is self-adjoint on L^2 , $\|w(t, \cdot)\|_{L^2} = \|w_0\|_{L^2}$. Moreover, if $k \in \mathbb{N}$, it follows from (1.2.6) that

$$[i\partial_t + P_0 + \text{Op}^W(V(t, \cdot))](P_0^k w) = g,$$

where $\|g\|_{L^2} \leq C\|w(t, \cdot)\|_{\mathcal{H}^{2k-2}}$. It follows by induction that

$$\|w(t, \cdot)\|_{\mathcal{H}^{2k}} \leq C_k \langle t - t' \rangle^k \|w(t', \cdot)\|_{\mathcal{H}^{2k}}.$$

Inequality (1.2.8) follows for any positive s by interpolation. \square

In section 3, we shall study the action of operators of the form (1.1.2) (with $d = 1$) on some functions, given by the product of an oscillating term and a compactly supported profile. The classes of profiles we shall use is defined below.

Definition 1.2.2 *Let $j \in \mathbb{N}$, $t \rightarrow \mu_\ell(t)$, $\ell = 1, 2$ be two continuous functions from $[2, +\infty[$ to \mathbb{R}_+^* such that $\mu_1(t) < \mu_2(t)$ for any $t \geq 2$. Set $K(t) = [\mu_1(t), \mu_2(t)]$ and $K = (K(t))_{t \geq 2}$. One denotes by Σ_K^{-j} the space of smooth functions $(t, x) \rightarrow b(t, x)$, defined on $[2, +\infty[\times \mathbb{R}$, with values in \mathbb{C} , such that for any $t \geq 2$, $x \rightarrow b(t, x)$ is supported for $\frac{x}{\sqrt{t}} \in K(t)$ and satisfies, for any $\alpha, \beta \in \mathbb{N}$, estimates*

$$(1.2.9) \quad |\partial_t^\alpha \partial_x^\beta b(t, x)| \leq C_{\alpha, \beta} t^{-\frac{1}{4} - \alpha - \frac{\beta}{2} - j} (\log t)^{\alpha + \beta + 2j},$$

for any $t \geq 2, x \in \mathbb{R}$.

We assume from now on that we are given two positive real numbers $\mu'_1 < \mu'_2$ and $\nu > 0$ such that, for any $t \geq 2$, $\mu'_1 + \nu \leq \mu_1(t) < \mu_2(t) \leq \mu'_2 - \nu$. Let $(t, x) \rightarrow \psi(t, x)$ be a smooth real valued function, such that $Z(t, y) = \frac{1}{\sqrt{t}}(\partial_x \psi)(t, \sqrt{t}y)$ is defined on $[2, +\infty[\times [\mu'_1, \mu'_2]$ and satisfies for any (t, y)

$$(1.2.10) \quad |\partial_t^\alpha \partial_y^\beta Z(t, y)| \leq C_{\alpha, \beta} t^{-\alpha}.$$

Proposition 1.2.3 *Assume that there is $c > 0$ such that $|Z(t, y)| \geq c$ for any $t \geq 2$, any $y \in [\mu'_1, \mu'_2]$. Let $m \in \mathbb{R}, N \in \mathbb{N}, j \in \mathbb{N}, a \in \mathcal{S}^m(\mathbb{R} \times \mathbb{R})$. There is, for any $j' \in \mathbb{N}$, a linear map $L : \Sigma_K^{-j'} \rightarrow \Sigma_K^{-j'}$ and, for any $b \in \Sigma_K^{-j}$, there is a smooth function $r : [2, +\infty[\times \mathbb{R} \rightarrow \mathbb{C}$, satisfying for any $\alpha, \beta \in \mathbb{N}$ estimates*

$$(1.2.11) \quad |\partial_t^\alpha \partial_x^\beta r(t, x)| \leq C_{\alpha\beta} t^{-\frac{1}{4} - \alpha - \frac{\beta}{2} - N - 1} \langle x/\sqrt{t} \rangle^{-N} (\log t)^{\alpha+\beta},$$

such that

$$(1.2.12) \quad \begin{aligned} \text{Op}^W(a)[e^{i\psi(t,x)}b(t, x)] &= e^{i\psi(t,x)}[a(x, \partial_x\psi(t, x))b(t, x) \\ &\quad - i(\partial_\xi a)(x, \partial_x\psi(t, x))\partial_x b(t, x) \\ &\quad - \frac{i}{2}\left((\partial_x\partial_\xi a)(x, \partial_x\psi(t, x)) + (\partial_\xi^2 a)(x, \partial_x\psi(t, x))\partial_x^2\psi(t, x)\right)b(t, x)] \\ &\quad - ie^{i\psi(t,x)}\frac{(\log t)^2}{t^{2-\frac{m}{2}}}L(b)(t, x) + e^{i\psi(t,x)}r(t, x). \end{aligned}$$

Proof: Write the left hand side of (1.2.12)

$$(1.2.13) \quad T(x) = \frac{1}{2\pi} \int e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) e^{i\psi(t,y)} b(t, y) dy d\xi.$$

Define $B(t, y) = b(t, \sqrt{t}y)$ and compute

$$(1.2.14) \quad T(x\sqrt{t}) = \frac{t}{2\pi} \int e^{it(x-y)\xi + i\psi(t, y\sqrt{t})} a\left(\sqrt{t}\left(\frac{x+y}{2}\right), \sqrt{t}\xi\right) B(t, y) dy d\xi.$$

Notice first that for $|x|$ large enough, $\partial_x^\alpha [T(x\sqrt{t})] = O(t^{-\frac{M}{2}} |x|^{-M})$ for any α, M . Actually, since

$$\begin{aligned} \partial_x^\alpha \partial_\xi^\beta \left[a\left(\sqrt{t}\left(\frac{x+y}{2}\right), \sqrt{t}\xi\right) \right] &= O(t^{\frac{m+\alpha+\beta}{2}} (1 + |x| + |\xi|)^{m+}), \\ |t\xi - \sqrt{t}(\partial_x\psi)(t, y\sqrt{t})| &= t|\xi - Z(t, y)|, \end{aligned}$$

and since y stays in $[\mu'_1, \mu'_2]$, performing, for large $|\xi|$, ∂_y -integrations by parts, makes appear a $O((\sqrt{t}|\xi|)^{-M})$ decay inside the integral in (1.2.14). Moreover, ∂_ξ -integrations by parts provide a gain $O((\sqrt{t}|x|)^{-M})$ for large x . This shows that for x large enough, (1.2.14) and its derivatives are bounded by $C_M t^{-M/2} |x|^{-M}$ for any M . Returning to the original variables, we get a contribution to r in (1.2.12). We may therefore assume from now on that x in (1.2.14) satisfies $|x| \leq C$ for a fixed constant C . We perform the following change of variables

$$y = x - z, \xi = \frac{1}{\sqrt{t}} \left(\frac{\partial\psi}{\partial x} \right) \left(t, \sqrt{t} \left(x - \frac{z}{2} \right) \right) + \eta = Z \left(t, x - \frac{z}{2} \right) + \eta.$$

The phase in (1.2.14) may be expanded, using the expression of $\partial_x\psi$ in terms of Z , as

$$tz \left(\eta + Z \left(t, x - \frac{z}{2} \right) \right) + \psi \left(t, \sqrt{t} \left(x - \frac{z}{2} \right) \right) = \psi(t, \sqrt{t}x) + tz\eta + tz^3 g(t, x, z),$$

where g is smooth and satisfies estimates of type $\partial_t^\alpha \partial_x^\beta \partial_z^\gamma g = O(t^{-\alpha})$. We rewrite (1.2.14) as

$$(1.2.15) \quad T(x\sqrt{t}) = \frac{t}{2\pi} e^{i\psi(t, x\sqrt{t})} \int e^{it[z\eta + z^3 g(t, x, z)]} a\left(\sqrt{t}\left(x - \frac{z}{2}\right), \sqrt{t}\left(Z\left(t, x - \frac{z}{2}\right) + \eta\right)\right) B(t, x - z) dz d\eta.$$

Let us write

$$(1.2.16) \quad a\left(\sqrt{t}\left(x - \frac{z}{2}\right), \sqrt{t}\left(Z\left(t, x - \frac{z}{2}\right) + \eta\right)\right) = \sum_{k=0}^M (\sqrt{t}\eta)^k a_k\left(t, x - \frac{z}{2}, \eta\right),$$

where a_0, \dots, a_{M-1} are independent of η and

$$(1.2.17) \quad \begin{aligned} a_k\left(t, x - \frac{z}{2}\right) &= \frac{1}{k!} (\partial_\xi^k a)\left(\sqrt{t}\left(x - \frac{z}{2}\right), \sqrt{t}Z\left(t, x - \frac{z}{2}\right)\right), \quad 0 \leq k \leq M-1, \\ a_M\left(t, x - \frac{z}{2}, \eta\right) &= \frac{1}{(M-1)!} \int_0^1 (\partial_\xi^M a)\left(\sqrt{t}\left(x - \frac{z}{2}\right), \sqrt{t}Z\left(t, x - \frac{z}{2}\right) + \sqrt{t}\sigma\eta\right) (1-\sigma)^{M-1} d\sigma. \end{aligned}$$

Decompose from (1.2.16), $T(x\sqrt{t}) = \sum_{k=0}^M T_k(x\sqrt{t})$. For $0 \leq k \leq M-1$

$$(1.2.18) \quad T_k(x\sqrt{t}) = e^{i\psi(t, x\sqrt{t})} \left(\frac{i}{\sqrt{t}}\right)^k \partial_z^k \left[e^{itz^3 g} a_k\left(t, x - \frac{z}{2}\right) B(t, x - z) \right] \Big|_{z=0}.$$

The contributions coming from $k = 0$ or 1 give, returning to the original variables, the main contributions to the right hand side of (1.2.12). Let us show that for $k = 3, \dots, M-1$, we get expressions that may be incorporated to the $L(b)$ term in (1.2.12). Notice that every time a ∂_z -derivative acts on the exponential in (1.2.18), we have to make act at least two more derivatives on $\partial_z[itz^3 g]$ if we want to get a non-vanishing term. Consequently, (1.2.18) is a linear combination of quantities

$$(1.2.19) \quad e^{i\psi(t, x\sqrt{t})} \left(\frac{i}{\sqrt{t}}\right)^k t^\ell h_k(t, x),$$

where $3\ell \leq k$ and h_k is computed from derivatives of order smaller than $k - 3\ell$ of $a_k(t, x - z/2)B(t, x - z)$ taken at $z = 0$. The logarithmic losses coming, because of the derivatives, from $B(t, x) = b(t, \sqrt{t}x)$ will be at most $(\log t)^{k-3\ell}$. Moreover, the expression of a_k , estimates (1.1.1), and the fact that $\sqrt{t}x$ and $\sqrt{t}Z(t, x)$ are of size \sqrt{t} when x is in the support of $B(t, \cdot)$, show that

$$t^{\ell - \frac{k}{2}} |h_k(t, x)| \leq C t^{\ell - k} (\log t)^{k-3\ell} t^{-\frac{1}{4} - j + \frac{m}{2}} (\log t)^{2j}.$$

The conditions $3\ell \leq k, k \geq 2, k, \ell \in \mathbb{N}$, show that the prefactor $t^{\ell - k} (\log t)^{k-3\ell}$ is always smaller than $C t^{-2} (\log t)^2$. Since, moreover, ∂_x (resp. ∂_t) derivatives of h_k give a loss of $\log t$ (resp. a gain of t^{-1} and a loss of $\log t$), we conclude, returning to the original variables, that (1.2.18) gives to the right hand side of (1.2.12) a contribution of form $e^{i\psi(t, x)} \frac{(\log t)^2}{t^{2 - \frac{m}{2}}} c$, where c is a local linear expression in terms of b and its derivatives, belonging to Σ_K^{-j} when b is in Σ_K^{-j} .

We are left with studying T_M . Let $\chi \in C_0^\infty(\mathbb{R}^2)$ with small support, equal to one close to zero. Write

$$(1.2.20) \quad T_M(x\sqrt{t}) = \frac{t}{2\pi} e^{i\psi(t, \sqrt{t}x)} \int e^{it[z\eta + z^3 g(t, x, z)]} (\sqrt{t}\eta)^M \chi(z, \eta) a_M\left(t, x - \frac{z}{2}, \eta\right) B(t, x - z) dz d\eta + \tilde{r}(t, x),$$

where \tilde{r} is defined by a similar integral, with χ replaced by $1 - \chi$. In particular, the support of the integrand in the expression of \tilde{r} does not contain $(0, 0)$, which is the only critical point of the phase $z\eta + z^3g(t, x, z)$. Taking into account that a ∂_z or ∂_η integration by parts makes gain t^{-1} and lose at most \sqrt{t} , we conclude that $\tilde{r} = O(t^{-\infty})$ as well as its derivatives (Notice that for large $|\eta|$, ∂_z -integrations by parts provide a gain of $(\sqrt{t}|\eta|)^{-M'}$ for any M' , which makes converge the η -integral). Consequently, \tilde{r} contributes to r in (1.2.12), after returning to the original coordinates, since we have already reduced ourselves to the case of bounded x in (1.2.14).

To conclude the proof, let us show that the first term in the right hand side of (1.2.20) contributes to r in (1.2.12), up to the change of variables $x \rightarrow x/\sqrt{t}$. Rewrite this quantity as the product of $e^{i\psi(t, \sqrt{t}x)}$ and

$$(1.2.21) \quad I = \frac{1}{2\pi} \int e^{itz\eta} t^{1+\frac{M}{2}} (\eta - z^2g(t, x, z))^M c(t, x, z, \eta) dz d\eta$$

with

$$c(t, x, z, \eta) = \chi(z, \eta - z^2g(t, x, z)) a_M\left(t, x - \frac{z}{2}, \eta - z^2g\right) B(t, x - z).$$

If $\text{Supp } \chi$ is small enough, it follows from (1.1.1), (1.2.9) and (1.2.17), and from the fact that $x - \frac{z}{2}$ and $Z(t, x - \frac{z}{2})$ stay between two positive constants in the expression of a_M , that

$$(1.2.22) \quad |\partial_t^\alpha \partial_x^\beta \partial_z^\gamma \partial_\eta^\delta [c(t, x, z, \eta)]| \leq Ct^{-\frac{M}{2} - \alpha + \frac{m}{2} - \frac{1}{4} - j} (\log t)^{\alpha + \beta + \gamma + 2j}.$$

We may perform in (1.2.21) M integrations by parts in ∂_z or ∂_η to get

$$|\partial_t^\alpha \partial_x^\beta I| \leq Ct^{1-M-\alpha+\frac{m}{2}-\frac{1}{4}-j} (\log t)^{\alpha+\beta+2j+M}.$$

If we remember that we reduced ourselves to x remaining in a compact subset, we see, returning to the original variables, that we get estimates of form (1.2.11), if M has been chosen large enough relatively to N . This concludes the proof. \square

2 Reduction to an autonomous equation

The goal of this section is to reduce by conjugation equation (1.1.5) to an autonomous one. We study, in the first subsection, an homological equation on symbols, that will be the key of the arguments of the second subsection. Since our reduction works in any dimension, we present the arguments for a d -dimensional harmonic oscillator.

2.1 Homological equation on symbols

Let us define subspaces of the class of symbols $\mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ of definition 1.1.1.

Definition 2.1.1 Set for $m \in \mathbb{R}$

$$(2.1.1) \quad \begin{aligned} \mathcal{S}_Z^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d) &= \{a \in \mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d); \partial_t a + \{p_0, a\} = 0\}, \\ \mathcal{S}_B^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d) &= \{a \in \mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d); \text{there is } b \in \mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d) \\ &\quad \text{with } a = \partial_t b + \{p_0, b\}\}. \end{aligned}$$

For $t \in \mathbb{R}$, denote by $\rho(t)$ the rotation of angle t acting on each couple of variables (x_j, ξ_j) of $T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$. In complex coordinates

$$(2.1.2) \quad \rho(t) \cdot (x_1 + i\xi_1, \dots, x_d + i\xi_d) = e^{it}(x_1 + i\xi_1, \dots, x_d + i\xi_d).$$

Notice that $\dot{\rho}(t) = \rho(t)J$, where J is the $(2k) \times (2k)$ block diagonal matrix whose diagonal blocks are $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

If $a \in \mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, define

$$(2.1.3) \quad M_0(a)(t, x, \xi) = \frac{1}{2\pi} \int_0^{2\pi} a(s+t, \rho(s) \cdot (x, \xi)) ds.$$

Notice that if $a_0 = M_0(a)$,

$$\partial_t a_0 + \{p_0, a_0\} = \frac{1}{2\pi} \int_0^{2\pi} (\partial_t a)(s+t, \rho(s) \cdot (x, \xi)) ds + \frac{1}{2\pi} \int_0^{2\pi} (Da)(s+t, \rho(s) \cdot (x, \xi)) \cdot \rho(s) J \cdot (x, \xi) ds,$$

where Da is differentiation with respect to the variables of $\mathbb{R}^d \times \mathbb{R}^d$. Performing an integration by parts in the first integral, and using that $\dot{\rho} = \rho J$, we obtain $\partial_t a_0 + \{p_0, a_0\} = 0$ i.e. $M_0(a) \in \mathcal{S}_Z^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$.

Proposition 2.1.2 For any $a \in \mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, there is $b \in \mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ such that

$$(2.1.4) \quad \partial_t b + \{p_0, b\} = a - M_0(a)$$

(i.e. $a - M_0(a) \in \mathcal{S}_B^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$). Moreover, the semi-norms of b are estimated in terms of those of a , and b is real valued if a is real valued.

Proof: Set $\tilde{a} = a - M_0(a)$ and define

$$(2.1.5) \quad b(t, x, \xi) = \int_0^t \tilde{a}(s, \rho(s-t) \cdot (x, \xi)) ds.$$

Then b solves (2.1.4) and satisfies estimates (1.1.1). We just have to check that b is 2π -periodic in time. Since ρ is 2π -periodic, we need to check that

$$\int_0^{2\pi} \tilde{a}(s+t, \rho(s) \cdot (x, \xi)) ds = 0$$

for all t , i.e. coming back to the definition of \tilde{a} and $M_0(a)$,

$$\int_0^{2\pi} a(s+t, \rho(s) \cdot (x, \xi)) ds = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} a(s'+s+t, \rho(s') \rho(s) \cdot (x, \xi)) ds' ds$$

which follows from the group property of $\rho(\cdot)$. This concludes the proof. \square

2.2 Conjugation to an autonomous ODE

Definition 2.2.1 Let $\ell \in \mathbb{R}_+$. One denotes by $\mathcal{P}^{-\ell}$ the space of maps Ω defined on $\bigcup_m \mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, with values in the same space, satisfying the following: for any $N \in \mathbb{N}$, there is a family $(\Omega_j^N)_{1 \leq j \leq k_N}$ of j -linear maps

$$\Omega_j^N : \prod_{j'=1}^j \mathcal{S}^{m_{j'}}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d) \rightarrow \mathcal{S}^{-\ell + \sum_1^j m_{j'}}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d),$$

such that if $\Omega^N(a) = \sum_{j=1}^{k_N} \Omega_j^N(a, \dots, a)$, for any $a \in \mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$

$$(2.2.1) \quad \Omega(a) - \Omega_N(a) \in \mathcal{S}^{-2N-\ell}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d).$$

Remark: It follows from the definition and from (1.2.2), (1.2.3), that if $\Omega^{(1)} \in \mathcal{P}^{-\ell_1}$, $\Omega^{(2)} \in \mathcal{P}^{-\ell_2}$, then for any $a \in \mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, $\text{Op}^W(\Omega^{(1)}(a)) \circ \text{Op}^W(\Omega^{(2)}(a))$ may be written as $\text{Op}^W(\Omega(a))$ for some $\Omega \in \mathcal{P}^{-(\ell_1+\ell_2)}$.

Proposition 2.2.2 Let $N \in \mathbb{N}^*$. There is a map Ω belonging to the class \mathcal{P}^{-2} such that the following holds: for any real valued symbol V in $\mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, there are symbols U in $\mathcal{S}^{-2N}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, S in $\mathcal{S}^{-2N-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, m in $\mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ and a symbol of $\mathcal{S}^0(\mathbb{R}^d \times \mathbb{R}^d)$, independent of time,

$$(2.2.2) \quad W = [M_0(V) + \Omega(V)]|_{t=0},$$

such that

$$(2.2.3) \quad \begin{aligned} e^{-itP_0} \text{Op}^W(m)^* [i\partial_t + P_0 + \text{Op}^W(V)] \text{Op}^W(m) e^{itP_0} \\ = i\partial_t + \text{Op}^W(W) + e^{-itP_0} \text{Op}^W(U) e^{itP_0} \\ + e^{-itP_0} \text{Op}^W(S) e^{itP_0} \circ (i\partial_t) + (i\partial_t) \circ e^{-itP_0} \text{Op}^W(S) e^{itP_0}. \end{aligned}$$

Moreover W, U, S are real valued and

$$(2.2.4) \quad \text{Op}^W(m)^* \text{Op}^W(m) - \text{Id} = \text{Op}^W(r) \text{ with } r \in \mathcal{S}^{-4}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d).$$

We shall prove the proposition in several steps. We first construct a conjugation allowing one to replace the given symbol of order zero V , by the sum of an element of $\mathcal{S}_Z^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ and of a symbol of lower order.

Lemma 2.2.3 Let $V \in \mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, real valued. One may find a symbol $q \in \mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ satisfying

$$(2.2.5) \quad |q|^2 = 1, \quad \{q, \bar{q}\} = 0,$$

and real valued symbols $V_0^Z = M_0(V) \in \mathcal{S}_Z^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, $V_1 \in \mathcal{S}^{-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, $S_1 \in \mathcal{S}^{-4}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, such that

$$(2.2.6) \quad \begin{aligned} \text{Op}^W(q)^* (i\partial_t + P_0 + \text{Op}^W(V)) \text{Op}^W(q) = i\partial_t + P_0 + \text{Op}^W(V_0^Z) + \text{Op}^W(V_1) \\ + \text{Op}^W(S_1)(i\partial_t + P_0) + (i\partial_t + P_0) \text{Op}^W(S_1). \end{aligned}$$

Moreover V_1 (resp. S_1) may be written as $\Omega^{(-2)}(V)$ (resp. $\Omega^{(-4)}(V)$) for some $\Omega^{(-\ell)} \in \mathcal{P}^{-\ell}$.

Proof: Define $V_0^Z = M_0(V)$, $V_0^B = V - V_0^Z$ and take, according to proposition 2.1.2, an element $b \in \mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, real valued, solving the equation $\partial_t b + \{p_0, b\} = V_0^B$. Setting $Q = \text{Op}^W(q)$, the left hand side of (2.2.6) may be written

$$(2.2.7) \quad i\partial_t + P_0 + \frac{1}{2}[Q^*Q - I](i\partial_t + P_0) + \frac{1}{2}(i\partial_t + P_0)[Q^*Q - I] + \Sigma + Q^*\text{Op}^W(V_0^Z)Q,$$

where

$$(2.2.8) \quad \Sigma = \frac{1}{2}Q^*([i\partial_t + P_0, Q] + \text{Op}^W(V_0^B)Q) + \frac{1}{2}([Q^*, i\partial_t + P_0] + Q^*\text{Op}^W(V_0^B))Q.$$

We want to choose Q so that $[i\partial_t + P_0, Q] + \text{Op}^W(V_0^B)Q$ be of order -2 . By (1.2.6) and symbolic calculus, it is enough to solve

$$i\partial_t q + i\{p_0, q\} + V_0^B q = 0.$$

We take $q = e^{ib}$. Since b is of order zero and real valued, we get an element of $\mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ satisfying $|q| = 1$. Moreover, $\{q, \bar{q}\} = 0$. By (1.2.2) and (1.2.3), it follows that Σ may be written as $\text{Op}^W(\sigma)$ for some symbol σ in $\mathcal{S}^{-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, which is real valued since Σ is self-adjoint. Consequently, Σ contributes to $\text{Op}^W(V_1)$ in the right hand side of (2.2.6).

Since $Q^* = \text{Op}^W(\bar{q})$ and since $q\bar{q} = 1$, $\{q, \bar{q}\} = 0$, it follows from (1.2.2) and the fact that in (1.2.3) $c_1 = \frac{i}{2}\{a, b\}$, that $\frac{1}{2}(Q^*Q - I)$ may be written as $\text{Op}^W(S_1)$ for some real valued symbol S_1 in $\mathcal{S}^{-4}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$. Let us show that the last term in (2.2.7), $Q^*\text{Op}^W(V_0^Z)Q$, may be written as $\text{Op}^W(V_0^Z) + \text{Op}^W(\sigma)$, where σ will be again a real valued element of $\mathcal{S}^{-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ that will contribute to V_1 in (2.2.6). Actually, this follows from (1.2.4) and the fact that $Q^*Q - I$ is of order -4 .

To conclude the proof, we still have to show that V_1 (resp. S_1) may be written as $\Omega^{(-2)}(V)$ (resp. $\Omega^{(-4)}(V)$) for some $\Omega^{(-\ell)} \in \mathcal{P}^{-\ell}$. Notice first that b , given by (2.1.5) with a replaced by V , is linear in V . Moreover, it follows from (1.2.2), (1.2.3) that if a is a symbol, $\text{Op}^W(a) \circ \text{Op}^W(q)$ may be written as $\text{Op}^W(e^{ib}c)$, where c has an expansion $\sum c_k$, in which each c_k is a finite sum of expressions that are linear in a and multi-linear in b (and their derivatives). Applying this to $a = \bar{q}$, we see that $\frac{1}{2}[\text{Op}^W(q)^*\text{Op}^W(q) - I] = \text{Op}^W(S_1)$ is given by a symbol S_1 , such that each term in its expansion is a finite sum of multi-linear expressions in V and its derivatives. Since S_1 is of order -4 , this means that S_1 may be written as $\Omega^{(-4)}(V)$ for $\Omega^{(-4)} \in \mathcal{P}^{-4}$. In the same way, the contributions to V_1 coming from Σ , or from the symbol of $Q^*\text{Op}^W(V_0^Z)Q$, may be written as $\Omega^{(-2)}(V)$ for some $\Omega^{(-2)} \in \mathcal{P}^{-2}$. This concludes the proof. \square

The second step of the proof of proposition 2.2.2 is to conjugate the right hand side of (2.2.6) to an expression where V_1 and S_1 will be replaced by symbols of arbitrary negative order. We do that by induction. Next lemma is the general step.

Lemma 2.2.4 *Let j be a positive integer. Assume given real valued symbols V_{j-1}^Z in $\mathcal{S}_Z^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, V_j in $\mathcal{S}^{-2j}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, S_j in $\mathcal{S}^{-2j-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, a' in $\mathcal{S}^{-2j}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, a'' in $\mathcal{S}^{-2j-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$. One may find real valued symbols $\tilde{V}_{j+1} \in \mathcal{S}^{-2j-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, $\tilde{S}_{j+1} \in \mathcal{S}^{-2j-4}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ such that if $a = ia' + a''$, the operator*

$$(2.2.9) \quad (I + \text{Op}^W(a))^*[i\partial_t + P_0 + \text{Op}^W(V_{j-1}^Z) + \text{Op}^W(V_j) + \text{Op}^W(S_j)(i\partial_t + P_0) + (i\partial_t + P_0)\text{Op}^W(S_j)](I + \text{Op}^W(a))$$

may be written as

$$\begin{aligned}
(2.2.10) \quad & i\partial_t + P_0 + \text{Op}^W(V_{j-1}^Z) + \text{Op}^W(V_j) - i[\text{Op}^W(a'), i\partial_t + P_0] \\
& + [\text{Op}^W(a'') + \text{Op}^W(S_j) + \frac{1}{2}\text{Op}^W(a')^*\text{Op}^W(a)'](i\partial_t + P_0) \\
& + (i\partial_t + P_0)[\text{Op}^W(a'') + \text{Op}^W(S_j) + \frac{1}{2}\text{Op}^W(a')^*\text{Op}^W(a)'] \\
& + \text{Op}^W(\tilde{V}_{j+1}) + \text{Op}^W(\tilde{S}_{j+1})(i\partial_t + P_0) + (i\partial_t + P_0)\text{Op}^W(\tilde{S}_{j+1}).
\end{aligned}$$

Moreover, if we assume that V_{j-1}^Z (resp. V_j , resp. S_j , resp. a' , resp. a'') may be written in terms of a given symbol of order zero V as $\Omega^{(0)}(V)$ (resp. $\Omega^{(-2j)}(V)$, resp. $\Omega^{(-2j-2)}(V)$, resp. $\Omega^{(-2j)}(V)$, resp. $\Omega^{(-2j-2)}(V)$) for elements $\Omega^{(-\ell)} \in \mathcal{P}^{-\ell}$, then \tilde{V}_{j+1} (resp. \tilde{S}_{j+1}) is of the form $\Omega^{(-2j-2)}(V)$ (resp. $\Omega^{(-2j-4)}(V)$) for other elements $\Omega^{(-\ell)}$ of $\mathcal{P}^{-\ell}$.

Proof: Since $\text{Op}^W(a')$, $\text{Op}^W(a'')$, $\text{Op}^W(S_j)$ are self-adjoint, we may write (2.2.9) as

$$\begin{aligned}
(2.2.11) \quad & i\partial_t + P_0 + \text{Op}^W(V_{j-1}^Z) + \text{Op}^W(V_j) - i[\text{Op}^W(a'), i\partial_t + P_0] \\
& + (\text{Op}^W(a'') + \text{Op}^W(S_j))(i\partial_t + P_0) + (i\partial_t + P_0)(\text{Op}^W(a'') + \text{Op}^W(S_j)) \\
& - i[\text{Op}^W(a'), \text{Op}^W(V_{j-1}^Z)] + \text{Op}^W(a'')\text{Op}^W(V_{j-1}^Z) + \text{Op}^W(V_{j-1}^Z)\text{Op}^W(a'') \\
& + \text{Op}^W(a)^*\text{Op}^W(V_j) + \text{Op}^W(V_j)\text{Op}^W(a) \\
& + \text{Op}^W(a)^*\text{Op}^W(S_j)^*(i\partial_t + P_0) + (i\partial_t + P_0)\text{Op}^W(S_j)\text{Op}^W(a) \\
& + \text{Op}^W(a)^*(i\partial_t + P_0)\text{Op}^W(a) \\
& + \text{Op}^W(a)^*\text{Op}^W(V_{j-1}^Z + V_j)\text{Op}^W(a) \\
& + \text{Op}^W(a)^*\text{Op}^W(S_j)^*(i\partial_t + P_0)\text{Op}^W(a) + \text{Op}^W(a)^*(i\partial_t + P_0)\text{Op}^W(S_j)\text{Op}^W(a).
\end{aligned}$$

The first two lines of (2.2.11) give contributions to the first three lines of (2.2.10). Consider the sixth line in (2.2.11). Write

$$\begin{aligned}
\text{Op}^W(a)^*(i\partial_t + P_0)\text{Op}^W(a) &= \frac{1}{2}[\text{Op}^W(a)^*\text{Op}^W(a)(i\partial_t + P_0) + (i\partial_t + P_0)\text{Op}^W(a)^*\text{Op}^W(a)] \\
&+ \frac{1}{2}[\text{Op}^W(a)^*, i\partial_t + P_0]\text{Op}^W(a) + \frac{1}{2}\text{Op}^W(a)^*[i\partial_t + P_0, \text{Op}^W(a)].
\end{aligned}$$

Since a is of order $-2j$, (1.2.2) and (1.2.4) show that the last two terms are of order $-4j \leq -2j - 2$, so contribute to the \tilde{V}_{j+1} term in (2.2.10). On the other hand

$$\text{Op}^W(a)^*\text{Op}^W(a)(i\partial_t + P_0) = \text{Op}^W(a')^*\text{Op}^W(a')(i\partial_t + P_0) + \text{Op}^W(b)(i\partial_t + P_0)$$

for some symbol b of order $-4j - 2 \leq -2j - 2$, again by (1.2.2). The first term in the right hand side gives the missing contribution in the second line of (2.2.10). The second term contributes to the S_{j+1} term in (2.2.10). These contributions are real valued by self-adjointness.

By (1.2.2) and (1.2.4), the third and fourth lines of (2.2.11) are made of self-adjoint operators of order $-2j - 2$, so will contribute to the $\text{Op}^W(\tilde{V}_{j+1})$ term in (2.2.10). In the fifth line, write

$$\text{Op}^W(S_j)\text{Op}^W(a) = \text{Op}^W(c_1) + \text{Op}^W(c_2)$$

where c_1, c_2 are of order $-4j-2 \leq -2j-4$, c_1 real, c_2 purely imaginary. The fifth line of (2.2.11) is thus

$$\text{Op}^W(c_1)(i\partial_t + P_0) + (i\partial_t + P_0)\text{Op}^W(c_1) + [i\partial_t + P_0, \text{Op}^W(c_2)].$$

The last term contributes to $\text{Op}^W(\tilde{V}_{j+1})$ and the first two ones to the last two expressions in (2.2.10).

The last but one line in (2.2.11) is of order $-4j \leq -2j-2$ and self-adjoint, so contributes to $\text{Op}^W(\tilde{V}_{j+1})$ in (2.2.10).

Consider finally the first term in the last line of (2.2.11) and write it as

$$\text{Op}^W(a)^*\text{Op}^W(S_j)^*\text{Op}^W(a)(i\partial_t + P_0) + \text{Op}^W(a)^*\text{Op}^W(S_j)^*[i\partial_t + P_0, \text{Op}^W(a)].$$

Again, by symbolic calculus, the last term is of order $-6j-2 \leq -2j-2$, so contributes to $\text{Op}^W(\tilde{V}_{j+1})$ in (2.2.10). The coefficient of $(i\partial_t + P_0)$ in the first term is also of order $-6j-2 \leq -2j-4$, so gives rise to a contribution to the last but one term in (2.2.10). These contributions are real by self-adjointness. The assertions of the end of the statement of the lemma, concerning the structure of $\tilde{V}_{j+1}, \tilde{S}_{j+1}$ as expressions of form $\Omega^{(-2j-2)}(V), \Omega^{(-2j-4)}(V)$, follow from the fact that the corresponding operators are obtained by composition from operators of the same form, using the remark following definition 2.2.1. \square

The last but one step in the proof of proposition 2.2.2 is the following:

Lemma 2.2.5 *Let V be a real valued symbol in $\mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$ and N a positive integer. One may find a real symbol V^Z in $\mathcal{S}_Z^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, a real symbol V_{N+1} in $\mathcal{S}^{-2N-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, a real symbol S_{N+1} in $\mathcal{S}^{-2N-4}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, a symbol $m \in \mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, such that*

$$(2.2.12) \quad \text{Op}^W(m)\text{Op}^W(m)^* - I = \text{Op}^W(r) \text{ with } r \in \mathcal{S}^{-4}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d),$$

and

$$(2.2.13) \quad \begin{aligned} \text{Op}^W(m)^*(i\partial_t + P_0 + \text{Op}^W(V))\text{Op}^W(m) &= i\partial_t + P_0 + \text{Op}^W(V^Z) + \text{Op}^W(V_{N+1}) \\ &\quad + \text{Op}^W(S_{N+1})(i\partial_t + P_0) + (i\partial_t + P_0)\text{Op}^W(S_{N+1}). \end{aligned}$$

Moreover we may write

$$(2.2.14) \quad V^Z = M_0(V) + \Omega^{(-2)}(V),$$

where M_0 is given by (2.1.3) and $\Omega^{(-2)}$ is an element of \mathcal{P}^{-2} .

Proof: We proceed by induction on N . By lemma 2.2.3, formula (2.2.13) holds with $N = 0, m = q$. Assume that (2.2.13) has been proved at some order $N = j-1$, with V^Z in the right hand side replaced by some V_{j-1}^Z , and with a symbol V_j (resp. S_j) that may be written as $\Omega^{(-2j)}(V)$ (resp. $\Omega^{(-2j-2)}(V)$) for some $\Omega^{(-2j)}$ in \mathcal{P}^{-2j} (resp. $\Omega^{(-2j-2)}$ in $\mathcal{P}^{-(2j+2)}$). We compose this equality on the left with $(I + \text{Op}^W(a))$, for some symbol $a = ia' + a''$ to be chosen, with a' in $\mathcal{S}^{-2j}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, a'' in $\mathcal{S}^{-2j-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, a' and a'' real valued. We apply

lemma 2.2.4 to get expressions (2.2.10), in which we decompose $V_j = M_0(V_j) + V_j^B$, where $M_0(V_j) \in \mathcal{S}_Z^{-2j}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, $V_j^B \in \mathcal{S}_B^{-2j}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$. We define $V_j^Z = V_{j-1}^Z + M_0(V_j)$. Since V_j may be written $\Omega^{(-2j)}(V)$, we obtain from assumption (2.2.14) for V_{j-1}^Z that the same type of equality will be satisfied by V_j^Z . We choose next a' so that

$$(2.2.15) \quad -i[\text{Op}^W(a'), i\partial_t + P_0] + \text{Op}^W(V_j^B)$$

is an operator of order $-2j - 2$. At the level of symbols, we have to solve $\partial_t a' + \{p_0, a'\} = V_j^B$, which is possible by proposition 2.1.2. We get a real symbol a' of order $-2j$ given by a linear expression in V_j^B , so that may be written as $\Omega^{(-2j)}(V)$ for some $\Omega^{(-2j)}$ in \mathcal{P}^{-2j} . In that way, (2.2.15) is an operator of order $-2j - 2$, self-adjoint, which contributes to V_{j+1} in (2.2.13) at rank $N = j$. The \tilde{V}_{j+1} contribution to (2.2.10) may be as well incorporated inside V_{j+1} . We choose next

$$a'' = -S_j - \frac{1}{2}|a'|^2 \in \mathcal{S}^{-2j-2}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d).$$

By the induction hypothesis, a'' may be written $\Omega^{(-2j-2)}(V)$ for some $\Omega^{(-2j-2)}$ in $\mathcal{P}^{-(2j+2)}$, and this definition of a'' implies that the coefficient of $(i\partial_t + P_0)$, in the second and third lines of (2.2.10), may be written $\text{Op}^W(c)$ for a real symbol c in $\mathcal{S}^{-2j-4}(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$. Moreover, c may be written as $\Omega^{(-2j-4)}(V)$ for some $\Omega^{(-2j-4)}$ in $\mathcal{P}^{-2(j+2)}$. This symbol, together with \tilde{S}_{j+1} in (2.2.10), contributes to S_{j+1} in (2.2.13) at rank $N = j$. Since (2.2.12), with $\text{Op}^W(m)$ replaced by $\text{Op}^W(m)(I + \text{Op}^W(a))$, holds as

$$(I + \text{Op}^W(a))^*(I + \text{Op}^W(a)) - I = 2\text{Op}^W(a'') + \text{Op}^W(a)^*\text{Op}^W(a)$$

is of order $-2j - 2 \leq -4$, we have obtained (2.2.12) and (2.2.13) at rank j . This concludes the proof of the lemma. \square

Proof of proposition 2.2.2: We multiply (2.2.13) with N replaced by $N - 1$ by e^{-itP_0} on the left and e^{itP_0} on the right. The main contribution of the right hand side of (2.2.13) is

$$(2.2.16) \quad e^{-itP_0}(i\partial_t + P_0 + \text{Op}^W(V^Z))e^{itP_0} = i\partial_t + e^{-itP_0}\text{Op}^W(V^Z)e^{itP_0}.$$

Notice that

$$\frac{d}{dt} \left[e^{-itP_0}\text{Op}^W(V^Z)e^{itP_0} \right] = ie^{-itP_0}[\text{Op}^W(V^Z), P_0]e^{itP_0}.$$

By the second formula (1.2.6) and the fact that V^Z is in $\mathcal{S}_Z^0(\mathbb{S}^1 \times \mathbb{R}^d \times \mathbb{R}^d)$, i.e. Poisson commutes to p_0 , we see that this quantity vanishes. Consequently, (2.2.16) is equal to $i\partial_t + \text{Op}^W(V^Z)|_{t=0}$. We define $W(x, \xi) = V^Z(0, x, \xi)$. This is a symbol of order 0, independent of time, and (2.2.14) gives (2.2.2). The contributions coming by conjugation from the V_N and S_N terms of the right hand side of (2.2.13) give the last three terms of (2.2.3). This concludes the proof. \square

3 Construction of the solution

This section is devoted to the construction of a potential V and of a solution u to (1.1.5) for which (1.1.6) holds. The main step will be to show that, if W is a convenient autonomous

symbol belonging to $\mathcal{S}^0(\mathbb{R} \times \mathbb{R})$, we may construct an approximate solution of the equation $(i\partial_t + \text{Op}^W(W))u = 0$ satisfying (1.1.6). Then, inverting (2.2.2), we shall obtain a time dependent potential V , such that the corresponding Schrödinger equation $i\partial_t + P_0 + \text{Op}^W(V)$ may be reduced by conjugation to the right hand side of (2.2.3). This right hand side being equal to $(i\partial_t + \text{Op}^W(W))$ modulo operators of very negative order, we shall deduce from the approximate solution already constructed an approximate solution of $(i\partial_t + P_0 + \text{Op}^W(V))u = 0$. Modifying it to get an exact solution, we shall obtain the conclusion of theorem 1.1.3.

We first study the Hamiltonian flow of an autonomous symbol of order zero satisfying convenient assumptions. From now on, the space dimension is $d = 1$.

3.1 The Hamiltonian flow associated to a potential

We use polar coordinates (ρ, θ) on $T^*\mathbb{R}$ ($x = \rho \cos \theta, \xi = \rho \sin \theta$). A function W on $T^*\mathbb{R}$ will be denoted indifferently by $W(x, \xi)$ or $W(\rho, \theta)$. Let $W \in \mathcal{S}^0(\mathbb{R} \times \mathbb{R})$ satisfy the following assumptions: There is $m > 2$, a symbol $\widetilde{W} \in \mathcal{S}^{-m}(\mathbb{R} \times \mathbb{R})$ and a symbol $W_\infty \in \mathcal{S}^0(\mathbb{R} \times \mathbb{R})$, with $W_\infty(\lambda(x, \xi)) = W_\infty(x, \xi)$ for any $\lambda \geq 1$, any (x, ξ) with $x^2 + \xi^2 \geq 1$, such that $W = W_\infty + \widetilde{W}$. In polar coordinates, we may write for $\rho \geq 1$

$$(3.1.1) \quad W(\rho, \theta) = W_\infty(\theta) + \widetilde{W}(\rho, \theta),$$

with uniform estimates for $\rho \geq 1$

$$(3.1.2) \quad \partial_\theta^\alpha W_\infty(\theta) = O(1), \quad (\rho \partial_\rho)^\alpha \partial_\theta^\beta \widetilde{W}(\rho, \theta) = O(\rho^{-m}).$$

For A a large positive constant to be chosen, we consider the Lagrangian submanifold of $T^*\mathbb{R}$ given in cartesian and polar coordinates by

$$(3.1.3) \quad \begin{aligned} \Lambda_0^A &= \{(As \cos(s/A), As \sin(s/A)); s \in [1, 2]\} \\ &= \{(\rho_0(s), \theta_0(s)); \rho_0(s) = As, \theta_0(s) = \frac{s}{A}; s \in [1, 2]\}. \end{aligned}$$

Our objective is to show that, under convenient assumptions on W , and for large enough A , the Hamiltonian flow of the symbol, given in cartesian coordinates by

$$(3.1.4) \quad \dot{x} = -\frac{\partial W}{\partial \xi}, \quad \dot{\xi} = \frac{\partial W}{\partial x},$$

and in polar coordinates by

$$(3.1.5) \quad \dot{\rho} = -\frac{1}{\rho} \frac{\partial W}{\partial \theta}, \quad \dot{\theta} = \frac{1}{\rho} \frac{\partial W}{\partial \rho},$$

is globally well defined for initial data in Λ_0^A , and transports at large time $t > 0$ the Lagrangian Λ_0^A onto a Lagrangian Λ_t^A , which is the graph of a function.

Proposition 3.1.1 *Assume*

$$(3.1.6) \quad \frac{\partial W_\infty}{\partial \theta}(0) < 0, \quad \frac{\partial^2 W_\infty}{\partial \theta^2}(0) \neq 0.$$

Denote by $(\rho(t, s), \theta(t, s))$ the flow of (3.1.5) at time t , issued from the point $(\rho_0(s), \theta_0(s))$ with parameter s on Λ_0^A at time $t = 0$. There are $A_0 > 0, C > 0$, such that for any $A \geq A_0$, any $s \in [1, 2]$, the flow is defined for any $t \geq 0$ and satisfies

$$(3.1.7) \quad C^{-1}(A^2 + t)^{1/2} \leq \rho(t, s) \leq C(A^2 + t)^{1/2}.$$

Moreover, there are smooth functions $s \rightarrow \lambda(s)$, $s \rightarrow \theta_\infty(s)$, defined on $[1, 2]$, satisfying for any $s \in [1, 2]$

$$(3.1.8) \quad C^{-1} \leq \lambda(s) \leq C, \quad C^{-1}A^{-1} \leq \left| \frac{\partial \lambda}{\partial s}(s) \right| \leq CA^{-1}, \quad \left| \theta_\infty(s) - \frac{s}{A} \right| \leq \frac{C}{Am},$$

and smooth functions $\sigma_j : [0, +\infty[\times [1, 2] \rightarrow \mathbb{R}$, satisfying

$$(3.1.9) \quad \begin{aligned} |\partial_t^\alpha \partial_s^\beta \sigma_1(t, s)| &\leq C_{\alpha\beta} (A^2 + t)^{-1-\alpha} \\ |\partial_t^\alpha \partial_s^\beta \sigma_2(t, s)| &\leq C_{\alpha\beta} (A^2 + t)^{-\frac{m}{2}-\alpha}, \end{aligned}$$

such that

$$(3.1.10) \quad \begin{aligned} \rho(t, s) &= (t\lambda(s) + (As)^2)^{1/2}(1 + \sigma_1(t, s)) \\ \theta(t, s) &= \theta_\infty(s) + \sigma_2(t, s). \end{aligned}$$

Before starting the proof, we introduce some notation. Let us write equation (3.1.5) under an integral form

$$(3.1.11) \quad \begin{aligned} \rho(t, s)^2 &= (As)^2 - 2 \int_0^t \frac{\partial W}{\partial \theta}(\rho(t', s), \theta(t', s)) dt' \\ \theta(t, s) &= \frac{s}{A} + \int_0^t \frac{1}{\rho(t', s)} \frac{\partial \widetilde{W}}{\partial \rho}(\rho(t', s), \theta(t', s)) dt' \end{aligned}$$

so that, in the second equation, the right hand side depends only on \widetilde{W} , which is of order $-m$. One sees by induction that there are, for any $j \in \mathbb{N}^*$, and for any $(a, b) \in \mathbb{N} \times \mathbb{N}$ with $2 \leq a + b \leq j$,

- symbols W'_{ab} (resp. W''_{ab}) of degree $-m - 2a$ when $a > 0$ and 0 when $a = 0$ (resp. of degree $-m - 2 - 2a$),
- polynomials $P'_{ab}(X_1, \dots, X_{j+1-a-b}, Y_1, \dots, Y_{j+1-a-b})$, which are homogeneous of degree a in $X = (X_1, \dots, X_{j+1-a-b})$ and b in $Y = (Y_1, \dots, Y_{j+1-a-b})$, such that

$$(3.1.12) \quad \begin{aligned} \partial_s^j [\rho(t, s)^2] &= \partial_s^j [(As)^2] - \int_0^t \frac{1}{\rho(t', s)} \frac{\partial^2 \widetilde{W}}{\partial \rho \partial \theta}(\rho(t', s), \theta(t', s)) \partial_s^j [\rho(t', s)^2] dt' \\ &\quad - 2 \int_0^t \frac{\partial^2 W}{\partial \theta^2}(\rho(t', s), \theta(t', s)) \partial_s^j \theta(t', s) dt' \\ &\quad + \sum_{2 \leq a+b \leq j} \int_0^t W'_{ab}(\rho(t', s), \theta(t', s)) P'_{ab}((\partial_s^\alpha \rho^2)_{\alpha>0}, (\partial_s^\beta \theta)_{\beta>0}) dt' \end{aligned}$$

and

$$\begin{aligned}
(3.1.13) \quad \partial_s^j[\theta(t, s)] &= \partial_s^j\left[\frac{s}{A}\right] + \frac{1}{2} \int_0^t \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\frac{1}{\rho} \frac{\partial \widetilde{W}}{\partial \rho}\right]\right)(\rho(t', s), \theta(t', s)) \partial_s^j[\rho(t', s)^2] dt' \\
&+ \int_0^t \frac{\partial}{\partial \theta} \left[\frac{1}{\rho} \frac{\partial \widetilde{W}}{\partial \rho}\right](\rho(t', s), \theta(t', s)) \partial_s^j \theta(t', s) dt' \\
&+ \sum_{2 \leq a+b \leq j} \int_0^t W''_{ab}(\rho(t', s), \theta(t', s)) P''_{ab}((\partial_s^\alpha \rho^2)_{\alpha>0}, (\partial_s^\beta \theta)_{\beta>0}) dt'.
\end{aligned}$$

In (3.1.12), a (resp. b) represents the number of times $\frac{\partial W}{\partial \theta}$ has been differentiated by $\frac{1}{\rho} \frac{\partial}{\partial \rho}$ (resp. $\frac{\partial}{\partial \theta}$) to compute W'_{ab} . In the same way, in (3.1.13), a (resp. b) is the number of $\frac{1}{\rho} \frac{\partial}{\partial \rho}$ (resp. $\frac{\partial}{\partial \theta}$) derivatives acting on $\frac{1}{\rho} \frac{\partial \widetilde{W}}{\partial \rho}$ to give W''_{ab} . The polynomial expressions P'_{ab}, P''_{ab} give the multi-linear expressions, homogeneous of degree a (resp. b) in the derivatives of ρ^2 (resp. θ), that multiply these symbols. The conditions on the orders of W'_{ab}, W''_{ab} come from the fact that $\frac{1}{\rho} \frac{\partial}{\partial \rho}$ (resp. $\frac{\partial}{\partial \theta}$) acting on a symbol gives a gain of two powers of ρ in decay (resp. does not ameliorate the decay). The order of the symbol W'_{ab} is better when $a > 0$ since, as soon as one takes at least one ρ -derivative of (3.1.1), the W_∞ contribution disappears.

Proof of proposition 3.1.1: Let us check that the flow is global and that (3.1.7) holds. In particular, this will imply that, if A is large enough, $\rho(t, s)$ stays larger than 1, so that the decomposition (3.1.1) holds. Let $c > 0, \delta_0 > 0$, be two small constants so that by (3.1.6)

$$(3.1.14) \quad -\frac{\partial W_\infty}{\partial \theta}(\theta) \geq c \text{ for any } \theta \text{ with } |\theta| \leq \delta_0.$$

Denote by T the supremum of those $t \in [0, +\infty[$ such that the solution of (3.1.5) exists on $[0, t[$, and satisfies for any $t' \in [0, t[$, any $s \in [1, 2]$,

$$(3.1.15) \quad \rho(t', s)^2 \geq \frac{1}{2}(As)^2 + ct' \text{ and } |\theta(t', s)| \leq \delta_0.$$

To check that $T = +\infty$ if $A \geq A_0$ for a large enough A_0 , we just need to see that there is $C_0 > 0$, such that (3.1.15) implies

$$(3.1.16) \quad C_0^2(A^2 + t') \geq \rho(t', s)^2 \geq \frac{3}{4}(As)^2 + 2ct' \text{ and } |\theta(t', s)| \leq \frac{\delta_0}{2},$$

for any $t' \in [0, t[$, any $s \in [1, 2]$. Since, in the right hand side of the second equation (3.1.5),

$$\rho(t', s)^{-1} \left| \frac{\partial \widetilde{W}}{\partial \rho}(\rho(t', s), \theta(t', s)) \right| \leq C \left(\frac{1}{2}(As)^2 + ct' \right)^{-\frac{m}{2}-1}$$

by (3.1.15), we get for $s \in [1, 2]$, $0 < t < T$,

$$|\theta(t, s)| \leq \frac{s}{A} + \frac{C'}{A^m} \leq \frac{\delta_0}{2}$$

for large enough A 's. It follows from (3.1.14), (3.1.1), (3.1.15) and the first equation in (3.1.5) that

$$\frac{d}{dt} \rho(t, s)^2 \geq 2c - C \left(\frac{1}{2}(As)^2 + ct \right)^{-m/2},$$

for any $s \in [1, 2]$, $t \in [0, T]$. Since $m > 2$, this implies

$$\rho(t, s)^2 \geq (As)^2 + 2ct - C'(As)^{-m+2},$$

which gives the lower bound in the first estimate (3.1.16) for large enough A . The upper bound follows from (3.1.11). We have thus established global existence and (3.1.7).

To prove (3.1.8), we notice that (3.1.5) and (3.1.7) imply for $t \leq t'$

$$(3.1.17) \quad |\theta(t, s) - \theta(t', s)| \leq C(A^2 + t)^{-\frac{m}{2}}.$$

Consequently, $\theta_\infty(s) = \lim_{t \rightarrow +\infty} \theta(t, s)$ exists and satisfies the last estimate in (3.1.8). We set

$$(3.1.18) \quad \lambda(s) = -2 \frac{\partial W_\infty}{\partial \theta}(\theta_\infty(s)).$$

Since, for A large enough, $|\theta_\infty(s)| \leq \delta_0$ for any $s \in [1, 2]$, we deduce from (3.1.14) the first formula in (3.1.8). Let us prove that there are numbers B_j such that, for any $j \in \mathbb{N}$, any $t \geq 0$, any $s \in [1, 2]$,

$$(3.1.19) \quad \left| \frac{\partial^j}{\partial s^j} [\rho(t, s)^2] \right| \leq B_j(A^2 + t), \quad \left| \frac{\partial^j}{\partial s^j} \theta(t, s) \right| \leq \frac{B_j}{A}.$$

Notice that if such estimates hold for derivatives of order up to $j - 1$, (3.1.12), (3.1.13) and (3.1.7) give for

$$w_1(t, s) = |\partial_s^j [\rho(t, s)^2]|, \quad w_2(t, s) = |\partial_s^j \theta(t, s)|,$$

estimates

$$(3.1.20) \quad w_1(t, s) \leq w_1(0, s) + C \int_0^t (A^2 + t')^{-\frac{m}{2}-1} w_1(t', s) dt' + C \int_0^t w_2(t', s) dt' + \tilde{B}_{j-1}(t+1)$$

and

$$(3.1.21) \quad \begin{aligned} w_2(t, s) &\leq w_2(0, s) + C \int_0^t (A^2 + t')^{-\frac{m}{2}-2} w_1(t', s) dt' \\ &\quad + C \int_0^t (A^2 + t')^{-\frac{m}{2}-1} w_2(t', s) dt' + \tilde{B}_{j-1} \int_0^t (A^2 + t')^{-\frac{m}{2}-1} dt', \end{aligned}$$

where \tilde{B}_{j-1} is a constant depending only on B_1, \dots, B_{j-1} , and where we have used that in the last terms of (3.1.12), (3.1.13), the decay of W'_{ab}, W''_{ab} compensates the growth of $\partial_s^\alpha \rho^2$. If B_j is large enough relatively to \tilde{B}_{j-1} , and if A is large enough relatively to the constant C in the right hand side of (3.1.20), (3.1.21), one deduces by bootstrap from these inequalities that (3.1.19) holds at rank j (using that $m > 2$).

It follows from (3.1.5), (3.1.7) and estimates (3.1.19) that, for any s , any $t \leq t'$,

$$(3.1.22) \quad |\partial_s^j [\theta(t, s) - \theta(t', s)]| \leq C_j(A^2 + t)^{-\frac{m}{2}}.$$

We conclude that θ_∞ is smooth and that $\partial_s^j [\theta_\infty(s) - \frac{s}{A}] = O(A^{-m})$. Plugging this information in (3.1.18) and using the second condition in (3.1.6), we get the estimate of $\partial \lambda / \partial s$ in (3.1.8), taking A large enough. Moreover, (3.1.22) shows that $\partial_s^j [\theta(t, s) - \theta_\infty(s)] = O((A^2 + t)^{-m/2})$.

This, together with time derivative estimates for ρ, θ coming from (3.1.5), shows that the second formula in (3.1.10) holds. To get the first one, compute from (3.1.11), $\rho(t, s)^2 = (As)^2 + \lambda(s)t + \sigma(t, s)$ with

$$(3.1.23) \quad \sigma(t, s) = -2 \int_0^t \left[\frac{\partial W_\infty}{\partial \theta}(\theta(t', s)) - \frac{\partial W_\infty}{\partial \theta}(\theta_\infty(s)) \right] dt' - 2 \int_0^t \frac{\partial \widetilde{W}}{\partial \theta}(\rho(t', s), \theta(t', s)) dt'.$$

Since $m > 2$, it follows from the estimates of σ_2 in (3.1.9), (3.1.19) and (3.1.5) that

$$|\partial_t^\alpha \partial_s^\beta \sigma(t, s)| \leq C_{\alpha, \beta} (A^2 + t)^{-\alpha}.$$

This implies the first formula in (3.1.9) and concludes the proof. \square

We denote by $\widetilde{\Lambda}_0^A$ the submanifold of the characteristic variety of $i\partial_t + \text{Op}^W(W)$ given by

$$\widetilde{\Lambda}_0^A = \{(t, x; \tau, \xi); t = 0, \tau = W(x, \xi), (x, \xi) \in \Lambda_0^A\} \subset T^*\mathbb{R}^2.$$

The flow-out of $\widetilde{\Lambda}_0^A$ by the Hamiltonian vector field of the symbol $-\tau + W(x, \xi)$ is given by

$$(3.1.24) \quad \widetilde{\Lambda}_t^A = \{(t, x; \tau, \xi); \tau = W(x, \xi), (x, \xi) \in \Lambda_t^A, t \geq 0\},$$

where $\Lambda_t^A \subset T^*\mathbb{R}$ is the image of Λ_0^A by the flow of (3.1.4) at time t . If we denote by (3.1.10) this flow expressed in polar coordinates, we get

$$(3.1.25) \quad \Lambda_t^A = \{(\rho(t, s) \cos \theta(t, s), \rho(t, s) \sin \theta(t, s)); s \in [1, 2]\}.$$

Our next task will be to show that for large enough A, t , (3.1.25) may be written as the graph of a function.

Define $Y(+\infty, s) = \lambda(s)^{1/2} \cos \theta_\infty(s)$. To fix ideas, we suppose from now on that $\partial\lambda/\partial s$, which has fixed sign on $[1, 2]$ by (3.1.8), is positive (this corresponds, according to (3.1.18), to the case when $\frac{\partial^2 W_\infty}{\partial \theta^2}(0) < 0$ in (3.1.6)). Consequently,

$$C^{-1}A^{-1} \leq \frac{\partial \lambda}{\partial s}(s) \leq CA^{-1}, \quad s \in [1, 2].$$

By these inequalities and (3.1.19),

$$(3.1.26) \quad \begin{aligned} \frac{\partial}{\partial s} Y(+\infty, s) &= \frac{\lambda'(s)}{2\sqrt{\lambda(s)}} \cos \theta_\infty(s) - \sqrt{\lambda(s)} \theta'_\infty(s) \sin \theta_\infty(s), \\ &\geq \frac{1}{2} C^{-3/2} A^{-1} + O(A^{-2}), \end{aligned}$$

uniformly for $s \in [1, 2]$. We fix from now on A larger than the constant A_0 of proposition 3.1.1, and large enough so that the above lower bound is strictly positive and so that $\frac{3}{A} \geq \theta_\infty(s) \geq \frac{1}{2A}$ for any $s \in [1, 2]$ (using (3.1.8)). Then $s \rightarrow Y(+\infty, s)$ is a strictly increasing diffeomorphism from $[1, 2]$ onto $[Y(+\infty, 1), Y(+\infty, 2)]$. Fix a small positive number δ and define

$$(3.1.27) \quad \begin{aligned} \mu'_1 &= Y(+\infty, 1 + \delta) < Y(+\infty, 1 + 2\delta) < \mu_1 = Y(+\infty, 1 + 3\delta) \\ &< \mu_2 = Y(+\infty, 2 - 3\delta) < Y(+\infty, 2 - 2\delta) < \mu'_2 = Y(+\infty, 2 - \delta). \end{aligned}$$

By (3.1.10), there are symbols $\hat{\sigma}_1, \hat{\sigma}_2$ of order -1 in t such that the flow of (3.1.4) may be written

$$(3.1.28) \quad \begin{aligned} X(t, s) &\stackrel{\text{def}}{=} \rho(t, s) \cos \theta(t, s) = \sqrt{t} \lambda(s)^{1/2} \cos \theta_\infty(s) (1 + \hat{\sigma}_1(t, s)) \\ \Xi(t, s) &\stackrel{\text{def}}{=} \rho(t, s) \sin \theta(t, s) = \sqrt{t} \lambda(s)^{1/2} \sin \theta_\infty(s) (1 + \hat{\sigma}_2(t, s)). \end{aligned}$$

If we set $Y(t, s) = \frac{1}{\sqrt{t}} X(t, s) = Y(+\infty, s) (1 + \hat{\sigma}_1(t, s))$, for $t \geq t_0$ with t_0 large enough, $s \rightarrow Y(t, s)$ is a diffeomorphism from $[1, 2]$ onto its image. Set $\mu_1(t) = Y(t, 1 + 2\delta)$, $\mu_2(t) = Y(t, 2 - 2\delta)$. Then $\mu_1(t) = Y(+\infty, 1 + 2\delta) + O(1/t)$, $\mu_2(t) = Y(+\infty, 2 - 2\delta) + O(1/t)$, so that (3.1.27) implies that for $t \geq t_0$ large enough and some $\nu > 0$,

$$(3.1.29) \quad [\mu_1, \mu_2] \subset [\mu_1(t), \mu_2(t)] \subset [\mu'_1 + \nu, \mu'_2 - \nu] \subset [\mu'_1, \mu'_2] \subset]Y(t, 1), Y(t, 2)[.$$

Proposition 3.1.2 *Assume that A has been fixed as above. There is $t_0 \geq 2$ and a smooth function $(t, y) \rightarrow Z(t, y)$, defined on $[t_0, +\infty[\times]\mu'_1, \mu'_2]$, with values in a compact subset of $]0, +\infty[$, such that $\partial_t^\alpha \partial_y^\beta Z(t, y) = O(t^{-\alpha})$ and, for any $t \geq t_0$,*

$$(3.1.30) \quad \Lambda_t^A \cap \pi^{-1}([\mu'_1 \sqrt{t}, \mu'_2 \sqrt{t}]) = \left\{ \left(x, \sqrt{t} Z \left(t, \frac{x}{\sqrt{t}} \right) \right); \mu'_1 \sqrt{t} \leq x \leq \mu'_2 \sqrt{t} \right\},$$

where $\pi : (x, \xi) \rightarrow x$.

Proof: By (3.1.29), for $\frac{x}{\sqrt{t}} \in]\mu'_1, \mu'_2]$, we may solve the equation $\frac{x}{\sqrt{t}} = Y(t, s)$ in s . The solution may be written $s = g(t, x/\sqrt{t})$ where $g : [t_0, +\infty[\times]\mu'_1, \mu'_2] \rightarrow [1, 2]$ is a smooth function with $\partial_t^\alpha \partial_y^\beta g(t, y) = O(t^{-\alpha})$. Plugging this expression in the second formula (3.1.28), we get $\xi = \sqrt{t} Z(t, x/\sqrt{t})$ for a function Z satisfying the conclusions of the statement. Notice that $Z(t, y)$ stays larger than a fixed positive constant, for $t \geq t_0$ large enough, $y \in]\mu'_1, \mu'_2]$, since in the second formula (3.1.28), $\theta_\infty(s) \sim A^{-1}$. \square

Proposition 3.1.2 shows that for $t \geq t_0$, Λ_t^A is the graph of a function above $[\mu'_1 \sqrt{t}, \mu'_2 \sqrt{t}]$, so that

$$\tilde{\Lambda}^A \cap \pi^{-1}(\{(t, x); t \geq t_0, \mu'_1 \sqrt{t} \leq \frac{x}{\sqrt{t}} \leq \mu'_2\})$$

is also the graph of a function of (t, x) . Since $\tilde{\Lambda}^A$ is Lagrangian, there is a smooth function

$$(3.1.31) \quad \psi : \{(t, x); t \geq t_0, \mu'_1 \sqrt{t} \leq x \leq \mu'_2 \sqrt{t}\} \rightarrow \mathbb{R}$$

such that for any (t, x) in the domain of definition

$$(3.1.32) \quad \partial_x \psi(t, x) = \sqrt{t} Z \left(t, \frac{x}{\sqrt{t}} \right), \quad \partial_t \psi(t, x) = W \left(x, \sqrt{t} Z \left(t, \frac{x}{\sqrt{t}} \right) \right).$$

3.2 Construction of an approximate solution

We assume from now on that the constants A and t_0 have been fixed large enough, as in the statement of proposition 3.1.2. Our objective is to prove the following:

Proposition 3.2.1 *Let $\mu_1(t), \mu_2(t)$ be defined as in (3.1.29) and set $K(t) = [\mu_1(t), \mu_2(t)]$, $K = (K(t))_{t \geq t_0}$. Let N be a fixed integer. There is a sequence b_j of elements of the class Σ_K^{-j} of definition 1.2.2, $0 \leq j \leq N-1$, and a smooth function $(t, x) \rightarrow r_N(t, x)$ from $[t_0, +\infty[\times \mathbb{R}$ to \mathbb{C} , satisfying for any α, β*

$$(3.2.1) \quad |\partial_t^\alpha \partial_x^\beta r_N(t, x)| \leq C_{\alpha\beta} t^{-\frac{1}{4}-N-\alpha-\frac{\beta}{2}-1} (\log t)^{\alpha+\beta+2N} \left(1 + \frac{|x|}{\sqrt{t}}\right)^{-N},$$

such that if $b = \sum_{j=0}^{N-1} b_j$,

$$(3.2.2) \quad (i\partial_t + \text{Op}^W(W))[e^{i\psi(t,x)}b(t, x)] = e^{i\psi(t,x)}r_N(t, x)$$

and $\|b_0(t, \cdot)\|_{L^2}$ is a positive constant independent of t .

We shall prove the proposition solving transport equations. Define for $x \in [\mu'_1\sqrt{t}, \mu'_2\sqrt{t}]$

$$(3.2.3) \quad U(t, x) = -(\partial_\xi W)(x, \partial_x \psi(t, x))$$

and consider transport equations of the form

$$(3.2.4) \quad \frac{\partial b}{\partial t} + U(t, x) \frac{\partial b}{\partial x} + \frac{1}{2}(\partial_x U(t, x))b = \frac{1}{t^2}f,$$

where f is a given source term and $t \geq t_0$. Since the Hamiltonian flow of $\tau - W(x, \xi)$ foliates $\tilde{\Lambda}^A$, we may write for any $s \in [1, 2]$ that the integral curves $t \rightarrow (X(t, s), \Xi(t, s))$, given by (3.1.28), satisfy for $t \geq t_0$,

$$(3.2.5) \quad \begin{aligned} \Xi(t, s) &= \partial_x \psi(t, X(t, s)) \\ \frac{d}{dt}X(t, s) &= U(t, X(t, s)). \end{aligned}$$

We denote by $X^{-1}(t, x)$ the inverse of the diffeomorphism $X(t, \cdot)$. By definition of $\mu_1(t), \mu_2(t)$, $X^{-1}(t, \cdot)$ sends the interval $[\sqrt{t}\mu_1(t), \sqrt{t}\mu_2(t)]$ onto $[1+2\delta, 2-2\delta]$, and (3.1.29) shows that $X^{-1}(t, \cdot)$ is defined on $[\sqrt{t}\mu_1, \sqrt{t}\mu_2]$ for any $t \geq t_0$.

Lemma 3.2.2 (i) *There is a constant $C > 0$ such that for any $t \geq t' \geq t_0$, any $s \in [1+2\delta, 2-2\delta]$,*

$$(3.2.6) \quad C^{-1} \left(\frac{t'}{t}\right)^{1/2} \leq \exp \left[- \int_{t'}^t (\partial_x U)(\tau, X(\tau, s)) d\tau \right] \leq C \left(\frac{t'}{t}\right)^{1/2}.$$

(ii) *For any $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$, there is $C_{\alpha\beta} > 0$ such that, for any t, τ in $[t_0, +\infty[$, any $x \in [\sqrt{t}\mu_1(t), \sqrt{t}\mu_2(t)]$,*

$$(3.2.7) \quad \begin{aligned} |\partial_t^\alpha \partial_x^\beta U(t, x)| &\leq C_{\alpha\beta} t^{-\frac{1}{2}-\alpha-\frac{\beta}{2}} \\ |\partial_t^\alpha \partial_x^\beta [(\partial_x U)(\tau, X(\tau, X^{-1}(t, x)))]| &\leq C_{\alpha\beta} \tau^{-1} t^{-\alpha-\frac{\beta}{2}}. \end{aligned}$$

Proof: (i) Let γ_0 be a smooth given function, compactly supported in $]\sqrt{t_0}\mu_1(t_0), \sqrt{t_0}\mu_2(t_0)[$ and consider the solution of

$$(3.2.8) \quad \begin{aligned} \frac{\partial \gamma}{\partial t} + U(t, x) \frac{\partial \gamma}{\partial x} + \frac{1}{2}(\partial_x U(t, x))\gamma &= 0 \\ \gamma(t_0, x) &= \gamma_0(x). \end{aligned}$$

Since

$$(3.2.9) \quad \frac{d}{dt} \gamma(t, X(t, s)) = -\frac{1}{2}(\partial_x U)(t, X(t, s))\gamma(t, X(t, s))$$

for any $t \geq t_0$, $x \rightarrow \gamma(t, x)$ is compactly supported in $]\sqrt{t}\mu_1(t), \sqrt{t}\mu_2(t)[$ and, since the equation is on divergence form,

$$(3.2.10) \quad \int \gamma(t_0, x)^2 dx = \int \gamma(t, x)^2 dx$$

for any $t \geq t_0$. By (3.2.9) we may write

$$\gamma(t, X(t, s)) \exp \left[\frac{1}{2} \int_{t_0}^t (\partial_x U)(\tau, X(\tau, s)) d\tau \right] = \gamma(t_0, X(t_0, s)),$$

so that

$$\int \gamma(t, x)^2 dx = \int \gamma(t_0, X(t_0, X^{-1}(t, x)))^2 \exp \left[- \int_{t_0}^t (\partial_x U)(\tau, X(\tau, X^{-1}(t, x))) d\tau \right] dx.$$

Using (3.2.10) to express the left hand side and changing variables in the right hand one, we get

$$\int \gamma_0(x)^2 dx = \int \gamma_0(X(t_0, s))^2 \left| \frac{\partial X}{\partial s}(t, s) \right| \exp \left[- \int_{t_0}^t (\partial_x U)(\tau, X(\tau, s)) d\tau \right] ds.$$

Taking $\gamma_0(x) = \frac{1}{\sqrt{\epsilon}} \chi\left(\frac{x - X(t_0, a)}{\epsilon}\right)$, where $\int \chi(z)^2 dz = 1$, $a \in [1, 2]$, $\epsilon \in]0, 1]$, we obtain when $\epsilon \rightarrow 0$

$$\Gamma(a) = \left| \frac{\partial X}{\partial s}(t, a) \right| \exp \left[- \int_{t_0}^t (\partial_x U)(\tau, X(\tau, a)) d\tau \right],$$

where $\Gamma(a)$ is some positive continuous function of a . Since by (3.1.26), (3.1.28), $\left| \frac{\partial X}{\partial s}(t, a) \right| \sim \sqrt{t}$, we get (3.1.27) in the special case $t' = t_0$. The general case follows by division.

(ii) The first inequality in (3.2.7) follows from (3.2.3), (3.1.32), the fact that W is a symbol of order zero, and the bounds of Z and its derivatives given in proposition 3.1.2.

The left hand side of the second inequality in (3.2.7) has modulus bounded in terms of expressions

$$|(\partial_x^{p+1} U)(\tau, X(\tau, X^{-1}(t, x)))| |\partial_t^{\alpha_1} \partial_x^{\beta_1} X(\tau, X^{-1}(t, x))| \cdots |\partial_t^{\alpha_p} \partial_x^{\beta_p} X(\tau, X^{-1}(t, x))|,$$

where $\alpha_1 + \cdots + \alpha_p = \alpha$, $\beta_1 + \cdots + \beta_p = \beta$, $\alpha_j + \beta_j \geq 1$. By (3.2.3) and (3.1.32), for any $q \in \mathbb{N}$,

$$|\partial_x^q U(\tau, x)| = \left| \frac{\partial^q}{\partial x^q} \left[(\partial_\xi W) \left(x, \sqrt{\tau} Z \left(\tau, \frac{x}{\sqrt{\tau}} \right) \right) \right] \right| = O(\tau^{-\frac{1}{2} - \frac{q}{2}}),$$

when x is replaced by $X(\tau, s)$, $s \in [1 + 2\delta, 2 - 2\delta]$, since Z stays between two positive constants as well as $\frac{1}{\sqrt{\tau}}X(\tau, s)$. Moreover, by (3.1.28),

$$(3.2.11) \quad |\partial_t^{\alpha_j} \partial_x^{\beta_j} X(\tau, X^{-1}(t, x))| = O(\sqrt{\tau} t^{-\alpha_j - \frac{\beta_j}{2}}).$$

This gives the second upper bound in (3.2.7) and concludes the proof. \square

Lemma 3.2.3 (i) Let $c :]\sqrt{t_0}\mu_1(t_0), \sqrt{t_0}\mu_2(t_0)[\rightarrow \mathbb{C}$ be a smooth compactly supported function. The solution b_0 , defined for $t \geq t_0$, of the equation

$$(3.2.12) \quad \begin{aligned} \frac{\partial b_0}{\partial t}(t, x) + U(t, x) \frac{\partial b_0}{\partial x}(t, x) + \frac{1}{2}(\partial_x U(t, x))b_0(t, x) &= 0 \\ b_0(t, x) &= c(x), \end{aligned}$$

is an element of Σ_K^0 . Moreover, for any $t \geq t_0$, $\|b_0(t, \cdot)\|_{L^2} = \|c\|_{L^2}$.

(ii) Let $j \in \mathbb{N}^*$, c_{j-1} a given element of $\Sigma_K^{-(j-1)}$, defined for $t \geq t_0$. The equation

$$(3.2.13) \quad \frac{\partial b_j}{\partial t}(t, x) + U(t, x) \frac{\partial b_j}{\partial x}(t, x) + \frac{1}{2}(\partial_x U(t, x))b_j(t, x) = \frac{(\log t)^2}{t^2} c_{j-1}(t, x)$$

has a solution $b_j \in \Sigma_K^{-j}$ defined for $t \geq t_0$.

Proof: (i) Writing the equation under form (3.2.9), we get for $t \geq t_0$ and $\frac{x}{\sqrt{t}} \in [\mu_1(t), \mu_2(t)]$

$$(3.2.14) \quad b_0(t, x) = c(X(t_0, X^{-1}(t, x)))E(t_0, t, x)$$

with

$$(3.2.15) \quad E(t', t, x) = \exp\left[-\frac{1}{2} \int_{t'}^t (\partial_x U)(\tau, X(\tau, X^{-1}(t, x))) d\tau\right].$$

It follows from the assumption on c and from the definition of $\mu_1(t)$, $\mu_2(t)$, that $b_0(t, x)$ is supported for x/\sqrt{t} in $K(t) = [\mu_1(t), \mu_2(t)]$. By (3.2.6), we get that $|b_0(t, x)| \leq Ct^{-1/4}$. We may write $\partial_x^\beta E(t', t, x)$ in terms of expressions

$$\prod_{j=1}^q \left(-\frac{1}{2} \int_{t'}^t \partial_x^{\beta_j} [(\partial_x U)(\tau, X(\tau, X^{-1}(t, x)))] d\tau \right) E(t', t, x)$$

with $\beta_1 + \dots + \beta_q = \beta$, $\beta_j \geq 1$. By (3.2.7) the absolute value of each integral is bounded from above by $C|\log \frac{t}{t'}| t^{-\beta_j/2}$ so that, taking (3.2.6) into account,

$$(3.2.16) \quad |\partial_x^\beta E(t', t, x)| \leq C \left(\frac{t}{t'}\right)^{-1/4} C \left\langle \log \frac{t}{t'} \right\rangle^\beta t^{-\beta/2}.$$

We deduce from this and (3.2.11) that $b_0(t, x)$ satisfies estimates (1.2.9) with $j = \alpha = 0$. Using (3.2.12) and (3.2.7) we obtain that estimates (1.2.9) with $\alpha > 0$ are also valid, so that b_0 belongs to Σ_K^0 . The fact that the L^2 -norm of b_0 is conserved has been already checked in (3.2.10).

(ii) We write equation (3.2.13) under the form

$$\frac{\partial}{\partial t} b_j(t, X(t, s)) + \frac{1}{2} (\partial_x U)(t, X(t, s)) b_j(t, X(t, s)) = \frac{(\log t)^2}{t^2} c_{j-1}(t, X(t, s)).$$

We solve this equation writing

$$(3.2.17) \quad b_j(t, x) = \tilde{b}_j(t, X^{-1}(t, x))$$

with

$$(3.2.18) \quad \tilde{b}_j(t, s) = - \int_t^{+\infty} \frac{(\log \tau)^2}{\tau^2} c_{j-1}(\tau, X(\tau, s)) \exp\left[\frac{1}{2} \int_t^\tau (\partial_x U)(\tau', X(\tau', s)) d\tau'\right] d\tau.$$

By (3.2.6), the exponential inside the integral is $O((t/\tau)^{-1/4})$ and by (1.2.9), $c_{j-1}(\tau, X(\tau, s)) = O(\tau^{-\frac{1}{4}-(j-1)}(\log \tau)^{2(j-1)})$. This shows that the integral in (3.2.18) converges and that

$$|\tilde{b}_j(t, s)| \leq Ct^{-j-\frac{1}{4}}(\log t)^{2j}.$$

We get a solution vanishing when $t \rightarrow +\infty$, supported at time t in the same interval as $c_{j-1}(t, \cdot)$. Let us write $\partial_x^{\beta_1} b_j(t, x)$ as a linear combination of expressions

$$(3.2.19) \quad \int_t^{+\infty} \frac{(\log \tau)^2}{\tau^2} \partial_x^{\beta_1} [c_{j-1}(\tau, X(\tau, X^{-1}(t, x)))] \partial_x^{\beta_2} E(\tau, t, x) d\tau$$

with $\beta_1 + \beta_2 = \beta$. By (3.2.11) and (1.2.9)

$$|\partial_x^{\beta_1} [c_{j-1}(\tau, X(\tau, X^{-1}(t, x)))]| \leq C\tau^{-\frac{1}{4}-(j-1)} t^{-\beta_1/2} (\log \tau)^{\beta_1+2(j-1)}.$$

Using (3.2.16), we conclude that (3.2.19) is bounded from above by $Ct^{-\frac{1}{4}-\frac{\beta}{2}-j}(\log t)^{\beta+2j}$ for any $j \geq 1$. This gives (1.2.9) with $\alpha = 0$. One recovers the estimates for $\alpha > 0$ using equation (3.2.13) and (3.2.7). This concludes the proof. \square

Proof of proposition 3.2.1: If b is an element of Σ_K^0 with $K(t) = [\mu_1(t), \mu_2(t)]$, it follows from proposition 1.2.3 applied with $m = 0, j = 0$ that, using notation (3.2.3),

$$(3.2.20) \quad \begin{aligned} (i\partial_t + \text{Op}^W(W))(e^{i\psi(t,x)} b(t, x)) &= e^{i\psi(t,x)} [-\partial_t \psi(t, x) + W(x, \partial_x \psi(t, x))] b(t, x) \\ &+ i e^{i\psi(t,x)} [\partial_t b(t, x) + U(t, x) \partial_x b(t, x) + \frac{1}{2} \partial_x U(t, x) b(t, x) - \frac{(\log t)^2}{t^2} L(b)(t, x)] \\ &+ e^{i\psi(t,x)} r(t, x), \end{aligned}$$

where L is a linear operator acting from Σ_K^{-j} to itself for any j , and r satisfies (1.2.11). By (3.1.32) the first term in the right hand side vanishes. We look for b as a sum $b = \sum_{j=0}^{N-1} b_j$ with $b_j \in \Sigma^{-j}$. We take for b_0 the solution of (3.2.12) with some non trivial initial data c at time $t = t_0$, compactly supported inside $] \mu_1(t_0)\sqrt{t_0}, \mu_2(t_0)\sqrt{t_0} [$. We next solve (3.2.13) with $j = 1$ and $c_0 = L(b_0) \in \Sigma_K^0$. We obtain by lemma 3.2.3 an element $b_1 \in \Sigma_K^{-1}$. Iterating the process, we construct recursively b_2, \dots, b_{N-1} . The term $\frac{(\log t)^2}{t^2} L(b_{N-1})$ satisfies estimates of type (3.2.1) as well as r , because of (1.2.11). This concludes the proof. \square

3.3 Proof of the theorem

This subsection will be devoted to the proof of theorem 1.1.3. Let $W \in \mathcal{S}^0(\mathbb{R} \times \mathbb{R})$ be a time independent symbol satisfying conditions (3.1.1), (3.1.2), (3.1.6). Let us construct a time dependent symbol V in $\mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$ satisfying equation (2.2.2) up to some remainder. The right hand side of that equation is

$$(3.3.1) \quad M_0(V)|_{t=0} + \Omega(V)|_{t=0},$$

where M_0 is given by (2.1.3) and, for a given N and a family Ω_j , $1 \leq j \leq k_{N-1}$, of j -linear maps sending $\mathcal{S}^{m_1}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}) \times \dots \times \mathcal{S}^{m_j}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$ to $\mathcal{S}^{m_1+\dots+m_j-2}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$,

$$(3.3.2) \quad \Omega(V) - \sum_{j=1}^{k_{N-1}} \Omega_j(V, \dots, V) \in \mathcal{S}^{-2N}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}).$$

Lemma 3.3.1 *Let $N \in \mathbb{N}$, $W \in \mathcal{S}^0(\mathbb{R} \times \mathbb{R})$. One may find for $j = 0, \dots, N-1$ symbols $V_j \in \mathcal{S}^{-2j}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$ such that if $V = \sum_{j=0}^{N-1} V_j$*

$$(3.3.3) \quad W - M_0(V)|_{t=0} - \Omega(V)|_{t=0} \in \mathcal{S}^{-2N}(\mathbb{R} \times \mathbb{R}).$$

Proof: Define for $U \in \mathcal{S}^m(\mathbb{R} \times \mathbb{R})$ an element $EU \in \mathcal{S}^m(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$ by $(EU)(t, x, \xi) = U(\rho(-t) \cdot (x, \xi))$. Then $(M_0EU)|_{t=0} = U$. Using (3.3.2), we decompose

$$\Omega(V) = \sum_{j=1}^{N-1} \tilde{\Omega}_j((V_k)_{0 \leq k \leq j-1}) + R(V_0, \dots, V_{N-1})$$

where $\tilde{\Omega}_j$ belongs to $\mathcal{S}^{-2j}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$ and depends only on V_k for $k < j$, and R is in $\mathcal{S}^{-2N}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$. We define recursively

$$V_0 = EW, \quad V_j = -E[\tilde{\Omega}_j(V_0, \dots, V_{j-1})|_{t=0}], \quad 1 \leq j \leq N-1.$$

We obtain equation (3.3.3). □

Let us apply proposition 2.2.2 with the symbol V associated to W by lemma 3.3.1. We obtain a symbol $m \in \mathcal{S}^0(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$ such that

$$(3.3.4) \quad \begin{aligned} & e^{-itP_0} \text{Op}^W(m)^* [i\partial_t + P_0 + \text{Op}^W(V)] \text{Op}^W(m) e^{itP_0} \\ &= i\partial_t + \text{Op}^W(W) + \text{Op}^W(U_2) + e^{-itP_0} \text{Op}^W(U_1) e^{itP_0} \\ & \quad + e^{-itP_0} \text{Op}^W(S) e^{itP_0} \circ (i\partial_t) + (i\partial_t) \circ e^{-itP_0} \text{Op}^W(S) e^{itP_0}, \end{aligned}$$

where U_1 and U_2 are in $\mathcal{S}^{-2N}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$ and S belongs to $\mathcal{S}^{-2N-2}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R})$.

The main remaining step in the proof of theorem 1.1.3 is given by the following:

Proposition 3.3.2 *Let $s \in \mathbb{R}_+$, $N \in \mathbb{N}$ with $N > s + 2$. There is a smooth function $(t, x) \rightarrow w_a(t, x)$ defined for $t \geq t_0$, $x \in \mathbb{R}$, compactly supported in x at fixed t , and a constant $C > 0$ such that, for $t \geq t_0$,*

$$(3.3.5) \quad \begin{aligned} \|w_a(t, \cdot)\|_{\mathcal{H}^s} &\geq C^{-1}t^{s/2}, \\ \|(i\partial_t + P_0 + \text{Op}^W(V))w_a(t, \cdot)\|_{\mathcal{H}^s} &\leq Ct^{\frac{s}{2}-N+1}. \end{aligned}$$

Proof: Let b be the function given by proposition 3.2.1 and set

$$(3.3.6) \quad v_a(t, x) = e^{i\psi(t, x)}b(t, x).$$

By (3.2.2)

$$(i\partial_t + \text{Op}^W(W))v_a = e^{i\psi(t, x)}r_N(t, x)$$

with r_N satisfying estimate (3.2.1). We set

$$(3.3.7) \quad w_a(t, x) = \text{Op}^W(m(t, \cdot))e^{itP_0}v_a(t, x).$$

We apply the operator (3.3.4) to v_a . We get

$$(3.3.8) \quad \begin{aligned} \text{Op}^W(m)^*[i\partial_t + P_0 + \text{Op}^W(V)]w_a &= e^{itP_0}[e^{i\psi(t, x)}r_N] + e^{itP_0}\text{Op}^W(U_2)v_a \\ &+ \text{Op}^W(U_1)e^{itP_0}v_a + \text{Op}^W(S)e^{itP_0}(i\partial_tv_a) \\ &+ e^{itP_0}i\partial_t[e^{-itP_0}\text{Op}^W(S)e^{itP_0}v_a]. \end{aligned}$$

Proposition 1.2.3 applied to a symbol a of order $s - 2N$ shows that $\text{Op}^W(a)v_a = e^{i\psi(t, x)}(c(t, x) + r(t, x))$, where c is supported for x/\sqrt{t} staying in a fixed compact set and satisfies $|c(t, x)| \leq Ct^{-\frac{1}{4} + \frac{s}{2} - N}$, and where r satisfies (1.2.11) with an arbitrary large N . We conclude that

$$(3.3.9) \quad \|v_a(t, \cdot)\|_{\mathcal{H}^{s-2N}} \leq Ct^{\frac{s}{2}-N}, \quad \|\partial_tv_a(t, \cdot)\|_{\mathcal{H}^{s-2N-2}} \leq Ct^{\frac{s}{2}-N}.$$

Using these inequalities, together with the fact that e^{itP_0} is isometric, and that U_1, U_2 are of order $-2N$, and S of order $-2N - 2$, we rewrite (3.3.8) as

$$\text{Op}^W(m)^*[i\partial_t + P_0 + \text{Op}^W(V)]w_a = g$$

with $\|g(t, \cdot)\|_{\mathcal{H}^s} \leq Ct^{\frac{s}{2}-N}(\log t)^{2N}$. Using (2.2.4), we may construct a symbol q of order zero such that $\text{Op}^W(q)\text{Op}^W(m)^* = Id - \text{Op}^W(r)$, where r is of order $-2N - 2$. We get

$$(3.3.10) \quad (i\partial_t + P_0 + \text{Op}^W(V))w_a = \tilde{g}$$

with

$$\tilde{g} = \text{Op}^W(q)g(t, \cdot) + \text{Op}^W(r)(i\partial_t + P_0 + \text{Op}^W(V))w_a.$$

Since by (3.3.7)

$$\|\partial_tv_a\|_{\mathcal{H}^{s-2N-2}} \leq C[\|\partial_tv_a\|_{\mathcal{H}^{s-2N-2}} + \|v_a\|_{\mathcal{H}^{s-2N}}]$$

we get $\|\tilde{g}(t, \cdot)\|_{\mathcal{H}^s} \leq Ct^{\frac{s}{2}-N}(\log t)^{2N}$. This implies the second estimate in (3.3.5).

We are left with showing that if b is conveniently chosen in (3.3.6), the lower bound in (3.3.5) holds. Denoting now by q a symbol of order 0 such that $\text{Op}^W(q)\text{Op}^W(m) = Id + \text{Op}^W(r)$ with r of order $-2N$, we may write

$$\begin{aligned} \|v_a(t, \cdot)\|_{\mathcal{H}^s} &= \|e^{itP_0}v_a(t, \cdot)\|_{\mathcal{H}^s} \\ &\leq \|\text{Op}^W(q)\text{Op}^W(m)e^{itP_0}v_a(t, \cdot)\|_{\mathcal{H}^s} + \|\text{Op}^W(r)e^{itP_0}v_a(t, \cdot)\|_{\mathcal{H}^s} \\ &\leq C\|w_a(t, \cdot)\|_{\mathcal{H}^s} + C\|v_a(t, \cdot)\|_{\mathcal{H}^{s-2N}}. \end{aligned}$$

Because of (3.3.9), we see that we just need to get a lower bound for

$$\|v_a(t, \cdot)\|_{\mathcal{H}^s} \sim \|\text{Op}^W((x^2 + \xi^2 + 1)^{s/2})v_a(t, \cdot)\|_{L^2}.$$

By (3.3.6) and proposition 1.2.3,

$$\text{Op}^W((x^2 + \xi^2 + 1)^{s/2})v_a(t, x) = e^{i\psi(t, x)}[\tilde{b}(t, x) + \tilde{c}(t, x)]$$

where

$$\begin{aligned} \tilde{b}(t, x) &= (x^2 + (\partial_x \psi(t, x))^2 + 1)^{s/2}b(t, x) \\ |c(t, x)| &\leq Ct^{-\frac{1}{4} + \frac{s}{2}} \frac{(\log t)}{t} \left\langle \frac{x}{\sqrt{t}} \right\rangle^{-M}, \end{aligned}$$

where $M > 1$ and where we have used that on the support of $b(t, \cdot)$, $|x| \sim \sqrt{t}$ and $|\partial_x \psi(t, x)| \sim \sqrt{t}$. These properties show that

$$\|v_a(t, \cdot)\|_{\mathcal{H}^s} \geq ct^{s/2}\|b(t, \cdot)\|_{L^2} + o(t^{s/2}), \quad t \rightarrow +\infty.$$

We just need to check that $\|b(t, \cdot)\|_{L^2} \geq c > 0$ for large enough t . This follows from proposition 3.2.1, as $b = \sum_{j=0}^{N-1} b_j$ with $\|b_0(t, \cdot)\|_{L^2}$ equal to a positive constant and $\|b_j(t, \cdot)\|_{L^2} = O(t^{-j}(\log t)^{2j})$, $t \rightarrow +\infty$ when $j \geq 1$. This concludes the proof of the proposition. \square

Proof of theorem 1.1.3: Let w_a be the function of proposition 3.3.2, that we extend smoothly for all $t \geq 0$. We look for an exact solution u of (1.1.5) as $u = w_a + \tilde{u}$, where \tilde{u} satisfies

$$(i\partial_t + P_0 + \text{Op}^W(V))\tilde{u} = -\tilde{g}$$

with \tilde{g} given by (3.3.10). By (3.3.5), $\|\tilde{g}\|_{\mathcal{H}^s} \leq Ct^{\frac{s}{2} - N + 1}$. Using the notation introduced in the statement of lemma 1.2.1 we may take

$$\tilde{u}(t, \cdot) = \int_t^{+\infty} S(t, t')\tilde{g}(t', \cdot) dt'$$

and by (1.2.8)

$$\|\tilde{u}(t, \cdot)\|_{\mathcal{H}^s} \leq C \int_t^{+\infty} \langle t - t' \rangle^{s/2} \langle t' \rangle^{-N + \frac{s}{2} + 1} dt' \leq C \langle t \rangle^{-N + s + 2}$$

if N has been taken strictly larger than $s + 2$. Consequently

$$\|u(t, \cdot)\|_{\mathcal{H}^s} \geq \|w_a(t, \cdot)\|_{\mathcal{H}^s} - \|\tilde{u}(t, \cdot)\|_{\mathcal{H}^s} \geq \frac{1}{2}C^{-1}t^{s/2}$$

when $t \rightarrow +\infty$, using (3.3.5). This concludes the proof. \square

References

- [1] J.-M. Barbaroux and A. Joye: *Expectation values of observables in time dependent quantum mechanics*, J. Statist. Phys. 90 (1998), no. 5-6, 1225–1249.
- [2] J. Bourgain: *Growth of Sobolev norms in linear Schrödinger equations with quasi-periodic potential*, Comm. Math. Phys. 204 (1999), no. 1, 207–247.
- [3] J. Bourgain: *On growth of Sobolev norms in linear Schrödinger equations with smooth time dependent potential*, J. Anal. Math. 77 (1999), 315–348.
- [4] J.-M. Delort: *Growth of Sobolev norms of solutions of linear Schrödinger equations on some compact manifolds*, to appear, International Mathematics Research Notices, doi:10.1093/imrn/rnp213.
- [5] P. Duclos, O. Lev and P. Štoviček: *On the energy growth of some periodically driven quantum systems with shrinking gaps in the spectrum*, J. Stat. Phys. 130 (2008), no. 1, 169–193.
- [6] H. Eliasson and S. Kuksin: *On reducibility of Schrödinger equations with quasiperiodic potentials*, Comm. Math. Phys. 286 (2009), no. 1, 125–135.
- [7] D. Fang and Q. Zhang: *On the logarithmic growth of Sobolev norms for solutions of linear Schrödinger equations with analytic potentials on the circle*, preprint, (2010), 30pp.
- [8] B. Grébert and L. Thomann: *KAM for the quantum harmonic oscillator*, preprint, (2010), 52 pp.
- [9] J. S. Howland: *Floquet operators with singular spectrum III*, Ann. Inst. H. Poincaré Phys. Théor. 69 (1998), no. 2, 265–273.
- [10] G. Nenciu: *Adiabatic theory: Stability of systems with increasing gaps*, Ann. Inst. H. Poincaré Phys. Théor. 67 (1997), no. 4, 411–424.
- [11] M. A. Shubin: *Pseudodifferential operators and spectral theory*, Second edition. Springer-Verlag, Berlin, 2001. xii+288 pp.
- [12] W.-M. Wang: *Logarithmic bounds on Sobolev norms for time dependent linear Schrödinger equations*, Comm. Partial Differential Equations 33 (2008), no. 10-12, 2164–2179.
- [13] W.-M. Wang: *Pure point spectrum of the Floquet Hamiltonian for the quantum harmonic oscillator under time quasi-periodic perturbations*, Comm. Math. Phys. 277 (2008), no. 2, 459–496.