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FIXED POINTS AND LINES IN 2-METRIC SPACES

ABDELKRIM ALIOUCHE AND CARLOS SIMPSON

ABSTRACT. We consider bounded 2-metric spaces satisfying an additional axiom, and show that a contractive mapping has either a fixed point or a fixed line.

1. INTRODUCTION

Gähler introduced in the 1960's the notion of 2-metric space [6] [7] [8], and the several authors have studied the question of fixed point theorems for mappings on such spaces. A 2-metric is a function $d(x, y, z)$ symmetric under permutations, satisfying the *tetrahedral inequality*

$$d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z) \text{ for all } x, y, z, a \in X.$$

as well as conditions (Z) and (N) which will be recalled below. In the prototypical example, $d(x, y, z)$ is the area of the triangle spanned by x, y, z .

This notion has been considered by several authors (see [5]), who have notably generalized Banach's principle to obtain fixed point theorems, for example White [22], Iseki [9], Rhoades [20], Khan [10], Singh, Tiwari and Gupta [21], Naidu and Prasad [18], Naidu [19] and Zhang [11], Abd El-Monsef, Abu-Donia, Abd-Rabou [2], Ahmed [3] and others.

The contractivity conditions used in these works are usually of the form

$$d(F(x), F(y), a) \leq \dots$$

for any $a \in X$. We may think of this as meaning that $d(x, y, a)$ is a family of distance-like functions of x and y , indexed by $a \in X$. This interpretation intervenes in our condition (I) below.

There have also been several different notions of a space together with a function of 3-variables. For example, Dhage [4] introduced the concept of D -metric space and proved the existence of a unique fixed point of a self-mapping satisfying a contractive condition. Dhage's definition uses the symmetry and tetrahedral axioms present in Gähler's

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definition, but includes the *coincidence* axiom that $d(x, y, z) = 0$ if and only if $x = y = z$.

A sequence $\{x_n\}$ in a D -metric space (X, d) is said by Dhage to be convergent to an element $x \in X$ (or d -convergent) [4] if given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_m, x_n, x) < \epsilon$ for all $m, n \geq N$. He calls a sequence $\{x_n\}$ in a D -metric space (X, d) Cauchy (or d -Cauchy) [4] if given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m, x_p) < \epsilon$ for all $n, m, p \geq N$.

These definitions, distinct from those used by Gähler et al, motivate the definition of the property $LIM(y, (x_i))$ in Definition 4.1 and studied in Theorem 4.6 below.

The question of fixed-point theorems on such spaces has proven to be somewhat delicate [15]. Mustafa and Sims introduced a notion of G -metric space [16] [17], in which the tetrahedral inequality is replaced by an inequality involving repetition of indices. In their point of view the function $d(x, y, z)$ is thought of as representing the perimeter of a triangle.

The question of fixed points for mappings on G -metric spaces has been considered by Abbas-Rhoades [1], Mustafa and co-authors [13], [14]. This is not an exhaustive description of the large literature on this subject.

We will consider a situation in which one can obtain either a fixed point or a fixed line, adding an additional axiom (I) to the definition of 2-metric.

In the present paper, after first considering an easy variant of the triangle inequality, we return to the case of 2-metric spaces, suppose that d is bounded (B), and add a quadratic axiom (I). The axiom (I) will be shown to hold in the example $X = S^2$ where $d(x, y, z)$ is given by a determinant (Section 5), which has appeared in [12]. This axiom allows us to consider a notion of *fixed line* of a mapping F which is contractive in the sense that

$$d(F(x), F(y), F(z)) \leq kd(x, y, z)$$

for $k < 1$. With these hypotheses on d and under appropriate compactness assumptions we prove that such a mapping has either a fixed point or a fixed line.

2. ASYMMETRIC TRIANGLE INEQUALITY

Suppose X is a set together with a function $\varphi(x, y)$ defined for $x, y \in X$ such that:

(R)— $\varphi(x, x) = 0$;

- (S)— $\varphi(x, y) = \varphi(y, x)$;
(AT)—for a constant $C \geq 1$ saying

$$\varphi(x, y) \leq \varphi(x, z) + C\varphi(z, y).$$

In this case we say that (X, φ) satisfies the *asymmetric triangle inequality*.

It follows that $\varphi(x, y) \geq 0$ for all $x, y \in X$. Furthermore, if we introduce a relation $x \sim y$ when $\varphi(x, y) = 0$, then the three axioms imply that this is an equivalence relation, and furthermore when $x \sim x'$ and $y \sim y'$ then $\varphi(x, y) = \varphi(x', y')$. Thus, φ descends to a function on the quotient X/\sim and on the quotient it has the property that $\varphi(\bar{x}, \bar{y}) = 0 \Leftrightarrow \bar{x} = \bar{y}$. In view of this discussion it is sometimes reasonable to add the *strict reflexivity* axiom

- (SR)—if $\varphi(x, y) = 0$ then $x = y$.

It is easy to see for (X, φ) satisfying the asymmetric triangle inequality, that the notion of limit for the distance function φ makes sense, similarly the notion of Cauchy sequence for φ makes sense, and we can say that (X, φ) is *complete* if every Cauchy sequence has a limit. If a sequence has a limit then it is Cauchy. The function φ is continuous, i.e., it transforms limits into limits. If furthermore the strictness axiom (SR) satisfied, then limit is unique.

A *point of accumulation* of a sequence $(x_i)_{i \in \mathbb{N}}$ is a limit of a subsequence, that is to say a point y such that there exists a subsequence $(x_{i(j)})_{j \in \mathbb{N}}$ with $i(j)$ increasing, such that $y = \lim_{j \rightarrow \infty} x_{i(j)}$. A set X provided with a distance function satisfying the asymmetric triangle inequality (i.e. (R), (S) and (AT)), is *compact* if every sequence has a point of accumulation. In other words, every sequence admits a convergent subsequence. This notion should perhaps be called “sequentially compact” but it is the only compactness notion which will be used in what follows.

Lemma 2.1. *Suppose (X, φ) satisfies the asymmetric triangle inequality with the constant C . Suppose $F : X \rightarrow X$ is a map such that $\varphi(Fx, Fy) \leq k\varphi(x, y)$ with $k < (1/C)$. Then for any $x \in X$ the sequence $\{F^i(x)\}$ is Cauchy. If (X, φ) is complete and strictly reflexive then its limit is the unique fixed point of F .*

Proof. Let x_0 be an arbitrary point in X and $\{x_n\}$ the sequence defined by $x_{n+1} = F(x_n) = F^n(x_0)$ for all positive integer n . We have

$$\varphi(x_{n+1}, x_n) = \varphi(Fx_n, Fx_{n-1}) \leq k\varphi(x_n, x_{n-1}).$$

By induction, we obtain

$$\varphi(x_{n+1}, x_n) \leq k^n \varphi(x_0, x_1)$$

Using the *asymmetric triangle inequality* several times we get for all positive integers n, m such that $m > n$

$$\begin{aligned} \varphi(x_n, x_m) &\leq \varphi(x_n, x_{n+1}) + C\varphi(x_{n+1}, x_{n+2}) + C^2\varphi(x_{n+2}, x_{n+3}) + \dots \\ &\quad \dots + C^{m-n-1}\varphi(x_{m-1}, x_m). \end{aligned}$$

Then

$$\begin{aligned} \varphi(x_n, x_m) &\leq k^n\varphi(x_0, x_1) + Ck^{n+1}\varphi(x_0, x_1) + C^2k^{n+2}\varphi(x_0, x_1) + \dots \\ &\quad \dots + C^{m-n-1}k^{m-1}\varphi(x_0, x_1). \end{aligned}$$

Therefore

$$\varphi(x_n, x_m) \leq (1 + Ck + C^2k^2 + \dots + C^{m-n-1}k^{m-n-1})k^n\varphi(x_0, x_1)$$

and so

$$\varphi(x_n, x_m) < \frac{k^n}{1 - Ck}\varphi(x_0, x_1)$$

Hence, the sequence $\{x_n\}$ is Cauchy. Since (X, φ) is complete, it converges to some $x \in X$. Now, we show that z is a fixed point of F . Suppose not. Then

$$\varphi(Fz, Fx_n) \leq k\varphi(z, x_{n-1})$$

As n tends to infinity we get $z = Fz$ using (SR). The uniqueness of z follows easily. \square

Corollary 2.2. *Suppose (X, φ) satisfies the asymmetric triangle inequality, is strictly reflexive and complete. If $F : X \rightarrow X$ is a map such that $\varphi(Fx, Fy) \leq k\varphi(x, y)$ with $k < 1$. Then F has a unique fixed point.*

Proof. Since $k < 1$ there exists $a_0 \geq 1$ such that $k^a < (1/C)$ for any $a \geq a_0$. Then the previous lemma applies to F^a whenever $a \geq a_0$, and F^a has a unique fixed point z_a . Choose $b \geq a_0$ and let z_b be the unique fixed point of F^b . Then

$$F^{ab}(z_b) = (F^b)^a(z_b) = z_b,$$

but also

$$F^{ab}(z_a) = (F^a)^b(z_a) = z_a.$$

Thus z_a and z_b are both fixed points of F^{ab} ; as $ab \geq a_0$ its fixed point is unique so $z_a = z_b$. Apply this with $b = a + 1$, so

$$F(z_a) = F(F^a(z_a)) = F^b(z_a) = F^b(z_b) = z_b = z_a.$$

Thus z_a is a fixed point of F . If z is another fixed point of F then it is also a fixed point of F^a so $z = z_a$; this proves uniqueness. \square

2.1. Triangle inequality with cost. If $d(x, y, z)$ is a function of three variables, the “triangle inequality with cost” is

$$\varphi(x, y) \leq \varphi(x, z) + \varphi(z, y) + d(x, y, z).$$

This enters into Lemma 3.2 below.

We mention in passing (but it will not be used elsewhere) a “triangle inequality with multiplicative cost”: suppose given a function $\varphi(x, y)$ plus a function of 3 variables $\psi(x, y, z)$ such that

$$\varphi(x, y) \leq (\varphi(x, z) + \varphi(z, y))e^{\psi(x, y, z)}.$$

Assume also that φ is invariant under transposition, with $\varphi(x, y) = 0 \Leftrightarrow x = y$ and that ψ is bounded above and below. We can define limits and Cauchy sequences, hence completeness and the function φ is continuous.

Proposition 2.3. *Suppose given φ, ψ as above. If F is a map such that $\varphi(F(x), F(y)) \leq k\varphi(x, y)$ and $\psi(F(x), F(y), F(z)) \leq k\psi(x, y, z)$, whenever both sides are positive, then we get a Cauchy sequence $F^k(x)$. If (X, φ) is complete then the limit of this Cauchy sequence is the unique fixed point of F .*

Proof. Let x_0 be an arbitrary point in X and $\{x_n\}$ the sequence defined by $x_{n+1} = F(x_n) = F^n(x_0)$ for all positive integer n . We have

$$\begin{aligned} \varphi(x_n, x_m) &\leq (\varphi(x_n, x_{n+1}) + \varphi(x_{n+1}, x_m))e^{\psi(x_n, x_m, x_{n+1})} \\ &\leq k^n \varphi(x_0, x_1) e^{\psi(x_n, x_m, x_{n+1})} + \varphi(x_{n+1}, x_m) e^{\psi(x_n, x_m, x_{n+1})} \\ &\leq k^n \varphi(x_0, x_1) e^{\psi(x_n, x_m, x_{n+1})} + \\ &(\varphi(x_{n+1}, x_{n+2}) + \varphi(x_{n+2}, x_m)) e^{\psi(x_{n+1}, x_m, x_{n+2}) + \psi(x_n, x_m, x_{n+1})} \\ &\leq k^n \varphi(x_0, x_1) e^{\psi(x_n, x_m, x_{n+1})} + \\ &k^{n+1} \varphi(x_0, x_1) e^{\psi(x_{n+1}, x_m, x_{n+2}) + \psi(x_n, x_m, x_{n+1})} + \\ &\varphi(x_{n+2}, x_m) e^{\psi(x_{n+1}, x_m, x_{n+2}) + \psi(x_n, x_m, x_{n+1})} \\ &\leq k^n \varphi(x_0, x_1) e^{\psi(x_n, x_m, x_{n+1})} + \\ &k^{n+1} \varphi(x_0, x_1) e^{\psi(x_{n+1}, x_m, x_{n+2}) + \psi(x_n, x_m, x_{n+1})} + \\ &k^{n+2} \varphi(x_0, x_1) e^{\psi(x_{n+2}, x_m, x_{n+3}) + \psi(x_{n+1}, x_m, x_{n+2}) + \psi(x_n, x_m, x_{n+1})} + \\ &\varphi(x_{n+3}, x_m) e^{\psi(x_{n+2}, x_m, x_{n+3}) + \psi(x_{n+1}, x_m, x_{n+2}) + \psi(x_n, x_m, x_{n+1})} \\ &\leq k^n \varphi(x_0, x_1) e^{\psi(x_n, x_m, x_{n+1})} + \\ &k^{n+1} \varphi(x_0, x_1) e^{\psi(x_{n+1}, x_m, x_{n+2}) + \psi(x_n, x_m, x_{n+1})} + \\ &k^{n+2} \varphi(x_0, x_1) e^{\psi(x_{n+2}, x_m, x_{n+3}) + \psi(x_{n+1}, x_m, x_{n+2}) + \psi(x_n, x_m, x_{n+1})} + \dots + \\ &k^{m-1} \varphi(x_0, x_1) e^{\psi(x_n, x_m, x_{n+1}) + \psi(x_{n+1}, x_m, x_{n+2}) + \psi(x_{n+2}, x_m, x_{n+3}) + \dots + \psi(x_{m-2}, x_m, x_{m-1})} \\ &\leq k^n \varphi(x_0, x_1) (e^{k^n \psi(x_0, x_p, x_1)} + k e^{k^n \psi(x_0, x_p, x_1) + k^{n+1} \psi(x_0, x_p, x_1)} + \end{aligned}$$

$$\begin{aligned}
 & k^2 e^{k^n \psi(x_0, x_p, x_1) + k^{n+1} \psi(x_0, x_p, x_1) + k^{n+2} \psi(x_0, x_p, x_1)} + \\
 & \dots\dots\dots + \\
 & k^{m-n-1} e^{k^n \psi(x_0, x_p, x_1) + k^{n+1} \psi(x_0, x_p, x_1) + k^{n+2} \psi(x_0, x_p, x_1) + \dots + k^{m-1} \psi(x_0, x_p, x_1)} \\
 & \leq k^n \varphi(x_0, x_1) (e^{Mk^n} + k e^{M(k^n + k^{n+1})} + k^2 e^{M(k^n + k^{n+1} + k^{n+2})} + \\
 & \dots\dots\dots + \\
 & k^{m-n-1} e^{M(k^n + k^{n+1} + k^{n+2} + \dots + k^{m-1})})
 \end{aligned}$$

since ψ is bounded. Hence, the sequence $\{x_n\}$ is Cauchy. Since (X, φ) is complete, it converges to some $x \in X$. The rest of the proof follows as in Lemma 2.1. □

3. BOUNDED 2-METRIC SPACES

Gähler defined the notion of 2-metric space to be a set X with function $d : X^3 \rightarrow \mathbb{R}$ denoted $(x, y, z) \mapsto d(x, y, z)$ satisfying the following axioms [6] [7] [8]:

(Sym)—that $d(x, y, z)$ is invariant under permutations of the variables x, y, z .

(Tetr)—for all a, b, c, x we have

$$d(a, b, c) \leq d(a, b, x) + d(b, c, x) + d(a, c, x).$$

(Z)—for all a, b we have $d(a, b, b) = 0$.

(N)—for all a, b there exists c such that $d(a, b, c) \neq 0$.

One can think of $d(x, y, z)$ as measuring how far are x, y, z from being “aligned” or “colinear”.

The 2-metric spaces (X, d) have been the subject of much study, see [1] and [2] for example. The prototypical example of a 2-metric space is obtained by setting $d(x, y, z)$ equal to the area of the triangle spanned by x, y, z .

Assume that the 2-metric is *bounded*, and by rescaling the bound can be supposed equal to 1:

(B)—the function is bounded by $d(x, y, z) \leq 1$ for all $x, y, z \in X$.

Define the *associated distance* by

$$\varphi(x, y) := \sup_{z \in X} d(x, y, z).$$

Lemma 3.1. *We have $d(x, y, z) \geq 0$ and hence $\varphi(x, y) \geq 0$. Also $\varphi(x, x) = 0$ and $\varphi(x, y) = \varphi(y, x)$.*

Proof. Applying the axiom (Tetr) with $b = c$, we get

$$d(a, b, b) \leq d(a, b, x) + d(b, a, x) + d(a, b, x).$$

By the axiom (Z) and the symmetry of d we obtain $d(a, b, x) \geq 0$ and so $d(x, y, z) \geq 0$. Then, $\varphi(x, y) \geq 0$. Symmetry of φ follows from invariance of d under permutations (Sym). \square

Lemma 3.2. *We have the triangle inequality with cost*

$$\varphi(x, y) \leq \varphi(x, z) + \varphi(z, y) + d(x, y, z).$$

Therefore

$$\varphi(x, y) \leq \varphi(x, z) + \varphi(z, y) + \min(\varphi(x, z), \varphi(z, y))$$

and hence the asymmetric triangle inequality (AT)

$$\varphi(x, y) \leq \varphi(x, z) + 2\varphi(z, y).$$

Proof. We have

$$\begin{aligned} d(x, y, z_0) &\leq d(x, y, z) + d(y, z_0, z) + d(x, z_0, z) \\ &\leq \varphi(x, z) + \varphi(z, y) + d(x, y, z). \end{aligned}$$

For the next statement, note that by definition

$$d(x, y, z) \leq \min(\varphi(x, z), \varphi(z, y)),$$

and for the last statement, $\min(\varphi(x, z), \varphi(z, y)) \leq \varphi(z, y)$. \square

In particular the distance φ satisfies the axioms (R), (S) and (AT) of Section 2. Note that axiom (N) for d is equivalent to strict reflexivity (SR) for φ ; if this is not assumed from the start, it can be fixed as follows.

3.1. Nondegeneracy. It is possible to start without supposing the nondegeneracy axiom (N), define an equivalence relation, and obtain a 2-metric on the quotient satisfying (N). For the next lemma and its corollary, we assume that d satisfies all of (Sym), (Tetr), (Z), (B), but not necessarily (N).

Lemma 3.3. *If a, b, x, y are any points then*

$$|d(a, b, x) - d(a, b, y)| \leq 2\varphi(x, y).$$

Proof. By condition (Tetr),

$$d(a, b, y) \leq d(a, b, x) + d(b, y, x) + d(a, y, x) \leq d(a, b, x) + 2\varphi(x, y).$$

The same in the other direction gives the required estimate. \square

Corollary 3.4. *If x, y are two points with $\varphi(x, y) = 0$ then for any a, b we have $d(a, b, x) = d(a, b, y)$. Therefore, if \sim is the equivalence relation considered in the second paragraph of Section 2, the function d descends to a function $(X/\sim)^3 \rightarrow \mathbb{R}$ satisfying the same properties but in addition its associated distance function is strictly reflexive and d satisfies (N).*

Proof. For the first statement, apply the previous lemma. This invariance applies in each of the three arguments since d is invariant under permutations, which in turn yields the descent of d to a function on $(X/\sim)^3$. The associated distance function is the descent of φ which is strictly reflexive. \square

In view of this lemma, we shall henceforth assume that φ satisfies (SR) or equivalently d satisfies (N) too. In particular the limit of a sequence is unique if it exists.

3.2. Surjective contractive mappings. If F is contractive on the function d , and also surjective, then it is contractive on φ too and usual arguments yield a fixed point.

Lemma 3.5. *Suppose $F : X \rightarrow X$ is a map such that*

$$d(F(x), F(y), F(z)) \leq kd(x, y, z)$$

for some constant $k > 0$. If F is surjective then $\varphi(F(x), F(y)) \leq k\varphi(x, y)$. If $k < 1$ and for any point x the sequence $\{F^i(x)\}$ is Cauchy; if (X, φ) is complete then this sequence has a limit which is a fixed point of F .

Proof. Suppose $x, y \in X$. For any $z \in X$, choose a preimage $w \in X$ such that $F(w) = z$ by surjectivity of F . Then

$$d(F(x), F(y), z) = d(F(x), F(y), F(w)) \leq kd(x, y, w) \leq k\varphi(x, y).$$

It follows that

$$\varphi(F(x), F(y)) = \sup_{z \in X} d(F(x), F(y), z) \leq k\varphi(x, y).$$

Apply 2.1 to obtain the rest of the lemma. \square

4. COLINEARITY

Consider a bounded 2-metric space (X, d) , that is to say satisfying axioms (Sym), (Tetr), (Z), (N) and (B), and require the following additional axiom:

(I)—for all a, b, c, x, y we have

$$d(a, b, x)d(c, x, y) \leq d(a, x, y) + d(b, x, y).$$

In Section 5 below we will see that a form of geodesic area function satisfies this additional axiom.

The term $d(c, x, y)$ may be replaced by its sup over c which is $\varphi(x, y)$. If we think of $d(a, b, x)$ as being a family of distance-like functions of a and b , indexed by $x \in X$, (I) can be rewritten

$$d(a, b, x) \leq (d(a, y, x) + d(y, b, x))\varphi(x, y)^{-1}$$

for $y \neq x$. This formulation may be related to the notion of “triangle inequality with multiplicative cost” discussed in Section 2.1.

Axiom (I) allows us to look at the question of fixed subsets of a contractive mapping F when F is not surjective.

Definition 4.1. *Say that (x, y, z) are colinear if $d(x, y, z) = 0$.*

Consider a sequence of points $x_i \in X$. The property $LIM(y, (x_i))$ is defined to mean:

$$\forall \epsilon > 0 \exists a_\epsilon, \forall i, j \geq a_\epsilon, d(y, x_i, x_j) < \epsilon.$$

Suppose $LIM(y, (x_i))$ and $LIM(y', (x_i))$. We would like to show that $d(y, y', x_i) \rightarrow 0$. However, this is not necessarily true: if (x_i) is Cauchy then the properties LIM are automatic (see Proposition 4.7 below). So, we need to include the hypothesis that our sequence is not Cauchy, in the following statements.

Lemma 4.2. *Suppose (x_i) is not Cauchy. If both $LIM(y, (x_i))$ and $LIM(y', (x_i))$ hold, then $d(y, y', x_i) \rightarrow 0$ as $i \rightarrow \infty$.*

Proof. The sequence (x_i) is supposed not to be Cauchy for φ , so there exists $\epsilon_0 > 0$ such that for any $m \geq 0$ there are $i, j \geq m$ with $\varphi(x_i, x_j) \geq \epsilon_0$. Therefore, in view of the definition of φ , for any m there exist $i(m), j(m) \geq m$ and a point $z(m) \in X$ such that $d(x_{i(m)}, x_{j(m)}, z(m)) \geq \epsilon_0/2$.

We now use condition (I) with $x = x_{i(m)}$ and $y = x_{j(m)}$ and $c = z(m)$, for $a = y$ and $b = y'$. This says

$$d(y, y', x_{i(m)})\epsilon_0/2 \leq d(y, x_{i(m)}, x_{j(m)}) + d(y', x_{i(m)}, x_{j(m)}).$$

If $LIM(y, (x_i))$, $LIM(y', (x_i))$, then for any ϵ we can assume m is big enough so that

$$d(y, x_{i(m)}, x_{j(m)}) \leq \epsilon\epsilon_0/4$$

and

$$d(y', x_{i(m)}, x_{j(m)}) \leq \epsilon\epsilon_0/4.$$

Putting these together gives $d(y, y', x_{i(m)}) \leq \epsilon$.

Choose m so that for all $j, k \geq m$ we have $d(y, x_j, x_k) \leq \epsilon$ and the same for y' . Then we have by (Tetr), for any $j \geq m$

$$d(y', y, x_j) \leq d(x_{i(m)}, y, x_j) + d(y', x_{i(m)}, x_j) + d(y', y, x_{i(m)}) \leq 3\epsilon.$$

Changing ϵ by a factor of three, we obtain the following statement: for any $\epsilon > 0$ there exists m such that for all $i \geq m$ we have $d(y', y, x_i) \leq \epsilon$. This is the required convergence. \square

Corollary 4.3. *If the sequence (x_i) is not Cauchy for the distance φ , then the following property holds:*

—if $LIM(y, (x_i))$, $LIM(y', (x_i))$, and $LIM(y'', (x_i))$ then (y, y', y'') are colinear.

Proof. We use the fact that

$$d(y, y', y'') \leq d(y, y', x_i) + d(y', y'', x_i) + d(y, y'', x_i).$$

By Lemma 4.2, all three terms on the right approach 0 as $i \rightarrow \infty$. This proves that $d(y, y', y'') = 0$. \square

Lemma 4.4. *Suppose that (x_i) is not Cauchy for the distance φ . If $LIM(y, (x_i))$, $LIM(y', (x_i))$, $\varphi(y, y') > 0$ and y'' is a point such that (y, y', y'') are colinear, then also $LIM(y'', (x_i))$.*

Proof. By Lemma 4.2, for any $\epsilon > 0$ there exists m such that for all $i \geq m$ we have $d(y', y, x_i) \leq \epsilon$.

Let u, v denote some x_i or x_j . By hypothesis $\varphi(y, y') > 0$ so there is a point z such that $d(y, y', z) = \epsilon_1 > 0$. Then condition (I) applied with $a = y'', b = u, x = y', c = z, y = y$ gives

$$d(y'', u, y')d(z, y', y) \leq d(y'', y', y) + d(u, y', y).$$

Hence

$$d(y'', u, y') \leq d(u, y', y)/\epsilon_1.$$

We can do the same for v , and also interchanging y and y' , to get

$$d(y'', v, y') \leq d(v, y', y)/\epsilon_1,$$

$$d(y'', u, y) \leq d(u, y, y')/\epsilon_1,$$

$$d(y'', v, y) \leq d(v, y, y')/\epsilon_1.$$

We have

$$\begin{aligned} d(y'', u, v) &\leq d(y, u, v) + d(y'', y, v) + d(y'', u, y) \\ &\leq d(y, u, v) + (d(u, y, y') + d(v, y, y'))/\epsilon_1. \end{aligned}$$

For $u = x_i$ and $v = x_j$ with $i, j \geq m$ as previously we get

$$d(y'', x_i, x_j) \leq d(y, x_i, x_j) + (d(x_i, y, y') + d(x_j, y, y'))/\epsilon_1.$$

By choosing m big enough this can be made arbitrarily small, thus giving the condition $LIM(y'', (x_i))$. \square

A *line* is a maximal subset $Y \subset X$ satisfying

$$(4.1) \quad \forall y, y', y'' \in Y, \quad d(y, y', y'') = 0.$$

A line is nonempty, by maximality since $d(y, y, y) = 0$ by (Z).

Lemma 4.5. *If $x \neq y$ are two points then there is a unique line Y containing x and y , and Y is the set of points a such that $d(a, x, y) = 0$.*

Proof. The set $\{x, y\}$ satisfies Condition (4.1), so there is at least one maximal such set Y containing x and y . If $a \in Y$ then automatically $d(a, x, y) = 0$.

Suppose $d(a, x, y) = 0$.

If $v \neq w$ then $\varphi(v, w) > 0$ which means that there exists $c \in X$ with $d(c, v, w) > 0$. Condition (I) therefore says that if $d(a, v, w) = 0$, $d(u, v, w) = 0$ and $v \neq w$ then $d(a, u, v) = 0$.

Setting $v = x, w = y$ this gives, for any $u \in Y$, that $d(a, u, x) = 0$.

Now suppose $u, v \in Y$. If $v = x$ then the preceding shows that $d(a, u, v) = 0$. If $v \neq x$ then apply the previous statement with $w := x$, noting that $d(u, v, w) = 0$ since $u, v, x \in Y$; and $d(a, v, w) = d(a, v, x) = 0$ by the preceding result. Thus we can conclude that $d(a, u, v) = 0$. This shows that a is colinear with any two points of Y . In particular, $Y \cup \{a\}$ also satisfies Condition (4.1) so by maximality, $a \in Y$. This shows that Y is the set of points a such that $d(a, x, y) = 0$, which characterizes it uniquely. \square

Theorem 4.6. *Suppose that (X, d) is a bounded 2-metric space satisfying axiom (I) as above. Suppose (x_i) is a sequence. Then there are the following possibilities (not necessarily exclusive):*

- there is no point y with $LIM(y, (x_i))$;
- there is exactly one point y with $LIM(y, (x_i))$;
- the sequence (x_i) is Cauchy for the distance φ ; or
- the subset $Y \subset X$ of points y such that $LIM(y, (x_i))$, is a line.

Proof. Consider the subset $Y \subset X$ of points y such that $LIM(y, (x_i))$ holds. We may assume that there are two distinct points $y_1 \neq y_2$ in Y , for otherwise one of the first two possibilities would hold. Suppose that (x_i) is not Cauchy for φ ; in particular, Lemmas 4.2, 4.4 and Corollary 4.3 apply.

If y, y', y'' are any three points in Y , then by Corollary 4.3, they are colinear. Thus Y is a subset satisfying Condition (4.1) in the definition of a line; to show that it is a line, we have to show that it is a maximal such subset.

Suppose $Y \subset Y_1$ and Y_1 also satisfies (4.1). Since $y_1 \neq y_2$, and we are assuming that φ satisfies strict reflexivity (SR), we have $\varphi(y_1, y_2) \neq 0$. By Lemma 3.1, $\varphi(y_1, y_2) > 0$. If $y \in Y_1$ then by (4.1), $d(y, y_1, y_2) = 0$. By Lemma 4.4, y must also satisfy $LIM(y, (x_i))$, thus $y \in Y$. This shows that $Y_1 \subset Y$, giving maximality of Y . Thus, Y is a line. \square

The following proposition shows that the case when (x_i) Cauchy has to be included in the statement of the theorem.

Proposition 4.7. *If (x_i) , (y_j) and (z_k) are Cauchy sequences, then the sequence $d(x_i, y_j, z_k)$ is Cauchy in the sense that for any $\epsilon > 0$ there exists M such that for $i, j, k, p, q, r \geq M$ then $|d(x_i, y_j, z_k) - d(x_p, y_q, z_r)| < \epsilon$. In particular d is continuous. If (x_i) is Cauchy then $LIM(y, (x_i))$ holds for any point $y \in X$.*

Proof. For given ϵ , by the Cauchy condition there is M such that for $i, j, k, p, q, r \geq M$ we have $\varphi(x_i, x_p) < \epsilon/6$, $\varphi(y_j, y_q) < \epsilon/6$, and $\varphi(z_k, z_r) < \epsilon/6$. Then by Lemma 3.3

$$|d(x_i, y_j, z_k) - d(x_i, y_j, z_r)| \leq \epsilon/3,$$

$$|d(x_i, y_j, z_r) - d(x_i, y_q, z_r)| \leq \epsilon/3,$$

and

$$|d(x_i, q, z_r) - d(x_p, y_q, z_r)| \leq \epsilon/3.$$

These give the Cauchy property

$$|d(x_i, y_j, z_k) - d(x_p, y_q, z_r)| \leq \epsilon.$$

This shows in particular that d is continuous. Suppose (x_i) is Cauchy and y is any point. Then the sequence $d(y, x_i, x_j)$ is Cauchy in the above sense in the two variables i, j , which gives exactly the condition $LIM(y, (x_i))$. \square

We say that a sequence (x_i) is *tri-Cauchy* if

$$\forall \epsilon > 0, \exists m_\epsilon, i, j, k \geq m_\epsilon \Rightarrow d(x_i, x_j, x_k) < \epsilon.$$

Lemma 4.8. *Suppose (x_i) is a tri-Cauchy sequence, and $y \in X$ is an accumulation point of the sequence with respect to the distance φ . Then $LIM(y, (x_i))$.*

Proof. The condition that y is an accumulation point means that there exists a subsequence $(x_{u(k)})$ such that $(x_{u(k)}) \rightarrow y$ with respect to the distance φ . We have by (Tetr)

$$d(y, x_i, x_j) \leq d(x_{u(k)}, x_i, x_j) + d(y, x_{u(k)}, x_j) + d(y, x_i, x_{u(k)}),$$

and all three terms on the right become small, for i, j big in the original sequence and k big in the subsequence. Hence $d(y, x_i, x_j) \rightarrow 0$ as $i, j \gg 0$, which is exactly the condition $LIM(y, (x_i))$. \square

We say that (X, d) is *tri-complete* if, for any tri-Cauchy sequence, the set Y of points satisfying $LIM(y, (x_i))$ is nonempty. By Theorem 4.6, Y is either a single point, a line, or (in case (x_i) is Cauchy) all of X .

Lemma 4.9. *Suppose (X, φ) is compact. Then it is tri-complete, and for any tri-Cauchy sequence (x_i) we have one of the following two possibilities:*

- (x_i) has a limit; or
- the subset Y of points y with $LIM(y, (x_i))$ is a line.

Proof. Suppose (x_i) is a tri-Cauchy sequence. By compactness there is at least one point of accumulation, so the set Y of points y with $LIM(y, (x_i))$ is nonempty by Lemma 4.8. This rules out the first possibility of Theorem 4.6.

Suppose Y consists of a single point y . We claim then that $x_i \rightarrow y$. Suppose not: then there is a subsequence which doesn't contain y in its closure, but since X is compact after going to a further subsequence we may assume that the subsequence has a limit point $y' \neq y$. But again by Lemma 4.8, we would have $LIM(y', (x_i))$, a contradiction. So in this case, the sequence (x_i) is Cauchy for φ and has y as its limit; thus we are also in the situation of the third possibility. Note however that, since (x_i) is Cauchy, the set of points Y consists of all of X by Proposition 4.7, so the second possibility doesn't occur unless X is a singleton.

From Theorem 4.6 the remaining cases are that (x_i) is Cauchy, in which case it has a limit by compactness; or that Y is a line. \square

5. AN EXAMPLE

Let $X := S^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3, x_1^2 + x_2^2 + x_3^2 = 1\}$. Define the function $d(x, y, z)$ by taking the absolute value of the determinant of the matrix containing x, y, z as column vectors:

$$d\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) := \left| \det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \right|.$$

This has appeared in Example 2.2 of [12].

Proposition 5.1. *This function satisfies axioms (Sym), (Tetr), (Z), (B), (I).*

Proof. Invariance under permutations (Sym) comes from the corresponding fact for determinants. Condition (Z) comes from vanishing of a determinant with two columns which are the same. Condition (B) comes from the fact that the determinant of a matrix whose columns have norm 1, is in $[-1, 1]$. We have to verify (Tetr) and (I).

For condition (Tetr), suppose given vectors $x, y, z \in S^2$ as above, and suppose that $d(x, y, z) > 0$ i.e. they are linearly independent. Suppose

given another vector $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ too. Then the determinants $d(a, y, z)$, $d(x, a, y)$ and $d(x, y, a)$ are the absolute values of the numerators appearing in Cramer's rule. This means that if we write

$$a = \alpha x + \beta y + \gamma z$$

then

$$\frac{d(a, y, z)}{d(x, y, z)} = |\alpha|,$$

$$\frac{d(x, a, z)}{d(x, y, z)} = |\beta|,$$

and

$$\frac{d(x, y, a)}{d(x, y, z)} = |\gamma|.$$

Now by the triangle inequality in \mathbb{R}^3 we have

$$1 = \|a\| \leq |\alpha| + |\beta| + |\gamma|$$

which gives exactly the relation (Tetr).

To prove (I), notice that it is invariant under orthogonal transformations of \mathbb{R}^3 so we may assume that

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}, \quad u^2 + v^2 = 1.$$

Notice that in this case, $\sup_{c \in S^2} d(c, x, y) = |v|$, so we are reduced to considering

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Now $d(a, x, y) = |a_3 v|$ and $d(b, x, y) = |b_3 v|$, so we have to show that

$$d(a, b, x)|v| \leq |a_3 v| + |b_3 v|.$$

But $d(a, b, x) = |a_2 b_3 - a_3 b_2|$ so this inequality is true (since $|a_2| \leq 1$ and $|b_2| \leq 1$). This completes the proof of (I). \square

Corollary 5.2. *If $X \subset S^2$ then with the same function $d(x, y, z)$, it still satisfies the axioms.*

\square

The "lines" for the function d defined above, are the great circles or geodesics on S^2 .

Remark 5.3. *The distance function φ is not strictly reflexive in this example, indeed the associated equivalence relation identifies antipodal points. The quotient S^2/\sim is the real projective plane. The corresponding function on the quotient is a bounded 2-metric satisfying (I).*

6. FIXED POINTS OF A MAP

Suppose that (X, φ) is compact meaning that every sequence has a convergent subsequence. Suppose $F : X \rightarrow X$ is a d -decreasing map i.e. one with $d(F(x), F(y), F(z)) \leq kd(x, y, z)$ for $0 < k < 1$.

Pick a point $x_0 \in X$ and define the sequence of iterates inductively by $x_{i+1} := F(x_i)$.

Corollary 6.1. *Suppose (X, φ) is compact and F is a d -decreasing map. Pick a point x_0 and define the sequence of iterates (x_i) with $x_{i+1} = F(x_i)$. This sequence is tri-Cauchy; hence either it is Cauchy with a unique limit point $y \in X$, or else the subset $Y \subset X$ of points y with $LIM(y, (x_i))$ is a line.*

Proof. Note first that the sequence (x_i) is tri-Cauchy. If $m \leq i, j, k$ then

$$x_i = F^m(x_{i-m}), \quad x_j = F^m(x_{j-m}), \quad x_k = F^m(x_{k-m}).$$

Hence using the global bound (B),

$$d(x_i, x_j, x_k) \leq k^m d(x_{i-m}, x_{j-m}, x_{k-m}) \leq k^m.$$

As $0 < k < 1$, for any ϵ there exists m such that $k^m < \epsilon$; this gives the tri-Cauchy property of the sequence of iterates.

Then by Lemma 4.9, either $(x_i) \rightarrow y$ or else the set Y of points y with $LIM(y, (x_i))$ is a line. \square

Theorem 6.2. *Suppose X is nonempty and d is a bounded 2-metric satisfying (I), such that (X, φ) is compact. Suppose F is a d -decreasing map for a constant $0 < k < 1$. Then, either F has a fixed point, or there is a line Y fixed in the sense that $F(Y) \subset Y$ and Y is the only line containing $F(Y)$.*

Proof. Pick $x_0 \in X$ and define the sequence of iterates (x_i) as above. By the previous corollary, either $x_i \rightarrow y$ or else the set Y of points y with $LIM(y, (x_i))$ is a line.

Suppose we are in the second possibility but not the first; thus (x_i) has more than one accumulation point. Recall that $LIM(y, (x_i))$ means $d(y, x_i, x_j) < \epsilon$ for $i, j \geq m_\epsilon$. Hence if $i, j \geq m_\epsilon + 1$ then

$$d(F(y), x_i, x_j) = d(F(y), F(x_{i-1}), F(x_{j-1})) < k\epsilon.$$

This shows the property $LIM(F(y), (x_i))$. Thus, $F(Y) \subset Y$.

Suppose Y_1 is a line with $F(Y) \subset Y_1$. If $F(Y)$ contains at least two distinct points then there is at most one line containing $F(Y)$ by Lemma 4.5 and we obtain the second conclusion of the theorem. Suppose $F(Y) = \{y_0\}$ consists of a single point. Then, $y_0 \in Y$ so $F(y_0) \in F(Y)$, which shows that $F(y_0) = y_0$; in this case F admits a fixed point.

We may now assume that we are in the first case of the first paragraph: $x_i \rightarrow y$. If $F(y) = y$ we have a fixed point, so assume¹ $F(y) \neq y$. Let Y be the unique line containing y and $F(y)$ by Lemma 4.5 which also says Y is the set of points colinear with y and $F(y)$.

We claim that $F(X) \subset Y$. If $z \in X$ then $LIM(z, (x_i))$ by the last part of Proposition 4.7. For a given ϵ there is m_ϵ such that $i, j \geq m_\epsilon \Rightarrow d(x_i, x_j, z) < \epsilon$. However, for i fixed we have $x_j \rightarrow y$ by hypothesis, and the continuity of d (Proposition 4.7) implies that $d(x_i, y, z) < \epsilon$. Apply F , giving

$$\forall i \geq m_\epsilon, \quad d(x_{i+1}, F(y), F(z)) < k\epsilon.$$

Again using continuity of d , we have

$$d(x_{i+1}, F(y), F(z)) \rightarrow d(y, F(y), F(z))$$

and the above then implies that $d(y, F(y), F(z)) < k\epsilon$ for any $\epsilon > 0$. Hence $d(y, F(y), F(z)) = 0$ which means $F(z) \in Y$. This proves that $F(X) \subset Y$ *a fortiori* $F(Y) \subset Y$. If $F(F(y))$ is distinct from $F(y)$ then Y is the unique line containing $F(Y)$, otherwise $F(y)$ is a fixed point of F . This completes the proof of the theorem. \square

Corollary 6.3. *Suppose $X \subset S^2$ is a closed subset. Define the function $d(x, y, z)$ as in Section 5. If $F : X \rightarrow X$ is a function such that $d(F(x), F(y), F(z)) \leq kd(x, y, z)$ for $0 < k < 1$ then either it has a fixed pair of antipodal points, or a fixed great circle.*

Proof. Recall that we should really be working with the image of X in the real projective space $\mathbb{RP}^2 = S^2 / \sim$ (Remark 5.3). On this quotient, the previous theorem applies. Note that by a ‘‘fixed great circle’’ we mean a subset $Y \subset X$ which is the intersection of X with a great circle, and such that Y is the intersection of X with the unique great circle containing $F(Y)$. \square

¹We need to consider this case: as F is not assumed to be surjective, it is not necessarily continuous for φ so the convergence of the sequence of iterates towards y doesn't directly imply that y is a fixed point.

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A. ALIOUCHE, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LARBI BEN M'HIDI, OUM-EL BOUAGHI, 04000, ALGERIA
E-mail address: alioumath@yahoo.fr

C. SIMPSON, CNRS, LABORATOIRE J. A. DIEUDONNÉ, UMR 6621, UNIVERSITÉ DE NICE-SOPHIA ANTIPOLIS, 06108 NICE, CEDEX 2, FRANCE
E-mail address: carlos@unice.fr