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New examples for the KLS conjecture

Nolwen Huet¹

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Abstract

We show that the conjecture of Kannan, Lovász and Simonovits is true for log-concave measures of the form $\rho(|x|_B)dx$ on \mathbb{R}^n and $\rho(t, |x|_B)dx$ on \mathbb{R}^{1+n} , where $|x|_B$ is the norm associated to any convex body B already satisfying the conjecture. In particular, the conjecture holds for convex bodies of revolution.

1 Introduction

Let $K \subset \mathbb{R}^n$ be a convex body. We denote the uniform probability measure on K by μ_K . We say that K is isotropic if

- its barycenter $\int x d\mu_K(x)$ is 0,
- $\int \langle x, \theta \rangle^2 d\mu_K(x)$ is constant over $\theta \in \mathbb{S}^{n-1}$.

This means that the covariance of μ_K is a multiple of the identity. In this case,

$$\forall \theta \in \mathbb{S}^{n-1}, \quad \int \langle x, \theta \rangle^2 d\mu_K(x) = \frac{\mathbb{E}_{\mu_K} |X|^2}{n} = \mathbb{E}_{\mu_K} (X_1^2).$$

Here $\langle \cdot, \cdot \rangle$ stands for the euclidean scalar product of \mathbb{R}^n and $|\cdot|$ the associated euclidean norm; \mathbb{S}^{n-1} is the euclidean sphere of radius 1, \mathbb{E}_μ is the expectation under the measure μ , and X_1 the first coordinate of X . Let us note that others authors require more, namely $\text{Vol}(K) = 1$ in [16] or $\int \langle x, \theta \rangle^2 d\mu_K = 1$ in [10], but we do not. The conjecture of Kannan, Lovász and Simonovits (KLS conjecture for short) from [10] can be stated as follows:

KLS conjecture for convex bodies. *There exists a universal constant C such that for every $n \geq 1$ and every isotropic convex body $K \subset \mathbb{R}^n$, the Poincaré constant $C_P(\mu_K)$ of μ_K is bounded from above by $C \mathbb{E}_{\mu_K} (X_1^2) = C \mathbb{E}_{\mu_K} |X|^2/n$.*

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Let us recall that the Poincaré constant $C_P(\mu)$ of a measure μ is the best constant C such that, for every smooth function f ,

$$\text{Var}_\mu f \leq C \int |\nabla f|^2 d\mu,$$

where ∇f is the gradient of f and $\text{Var}_\mu f = \int (f - \int f d\mu)^2 d\mu$ is its variance under μ . So the conjecture tells us that the Poincaré inequality for convex bodies is tight for linear functions, up to a universal constant.

Up to now, this conjecture is known to be true for ℓ^p -balls with $p \geq 1$ (see [8] for the euclidean case $p = 2$, [20] for $p \in [1, 2]$ and [13] for $p \geq 2$), for the hypercube (see [9] or [6]), and for the regular simplex (see [1]). Moreover, as the Poincaré inequality is stable under tensorization ([5]), every product of convex sets satisfying the KLS conjecture, satisfies also it. Quite obviously, it is stable under dilation too.

One can also extend the definition of isotropy and the conjecture to any log-concave measure μ on \mathbb{R}^n , by just replacing μ_K by μ in the statements. Recall that a measure μ on \mathbb{R}^n is log-concave if it satisfies the following Brunn-Minkowski inequality for all compact sets A, B and real $\lambda \in [0, 1]$:

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

Equivalently μ is absolutely continuous with respect to the Lebesgue measure on an affine space of dimension $m \leq n$ and its density is log-concave (see [7]). Then the KLS conjecture becomes

KLS conjecture for log-concave measures. *Let μ be a log-concave probability measure on \mathbb{R}^n . If μ is isotropic then $C_P(\mu) \leq C \mathbb{E}_\mu(X_1^2) = C \mathbb{E}_\mu|X|^2/n$.*

This conjecture seems rather difficult to tackle. It has been only checked in cases where some additional structure is involved. For instance, the conjecture holds for any log-concave product measure or more generally for any product of log-concave measures satisfying the KLS conjecture [5], which generalizes the case of the hypercube. Bobkov proved also the conjecture for spherically symmetric log-concave measures in [3] which generalizes the case of the euclidean ball. Let us note also that the conjecture is also true for small dimensions, since

$$C_P(\mu) \leq C \mathbb{E}_\mu|X|^2 \leq Cn \mathbb{E}_\mu(X_1^2) \tag{1}$$

whenever μ is an isotropic log-concave measure on \mathbb{R}^n . This follows from a theorem proved by Kannan, Lovász and Simonovits thanks to their localization lemma, and also deduced by Bobkov from its isoperimetric inequality for log-concave measures.

Theorem 1 (Kannan–Lovász–Simonovits [10, 2]). *Let $X = (X_1, \dots, X_n)$ be a log-concave random vector in \mathbb{R}^n of law μ . Then, there exists a universal constant C such that*

$$C_P(\mu) \leq C \mathbb{E}_\mu|X - \mathbb{E}(X)|^2 = C \sum_{i=1}^n \text{Var}_\mu(X_i).$$

This was improved by Bobkov [4] who proved that

$$C_P(\mu) \leq C (\text{Var}|X|^2)^{1/2}.$$

This enables Klartag [11] to show the following improvement of the bound (1), when proving power-law estimates for the central limit theorem for convex sets: there exists $\varepsilon > 0$ such that

$$C_P(\mu) \leq Cn^{1-\varepsilon} \text{E}(X_1^2).$$

If moreover μ is uniform on an unconditional convex set, i.e. invariant under coordinate reflexions, he shows [12] that

$$C_P(\mu) \leq C(\log n)^2 \text{E}(X_1^2). \quad (2)$$

In Section 2, we deal with the case of log-concave measures symmetric with respect to the norm associated to a convex body B satisfying the KLS conjecture. It encompasses the result of Bobkov on spherically symmetric log-concave measures. Its proof relied on the tensorization of the radial measure with the uniform measure on the sphere. He used the following property of the law of the radius:

Theorem 2 (Bobkov [3]). *Let ν be a probability measure on \mathbb{R}_+ defined by*

$$\nu(dr) = r^{n-1} \rho(r) \mathbb{1}_{\mathbb{R}_+}(r) dr$$

with ρ log-concave. Then

$$\text{Var}_\nu(r) \leq \frac{\text{E}_\nu(r^2)}{n}.$$

Here, a natural idea would be to use the polar representation $X = R\theta$, where $R = |X|_B$ is the norm of X , and the distribution of θ is the cone measure on ∂B . However, in the non-euclidean case, the differentiation on ∂B is more difficult to handle. So we choose to decompose X into SU where $S \in \mathbb{R}_+$ and U is uniform on B . We study as well the convex bodies of revolution in Section 3 by the same method, and more general log-concave measures with a finite number of symmetries. To simplify the statements, we use the following definition.

Definition. An isotropic convex body B (respectively an isotropic log-concave probability μ) is said to satisfy KLS with constant C if $C_P(\mu_B) \leq C\text{E}_{\mu_B}(|X|^2)/n$ (respectively $C_P(\mu) \leq C\text{E}_\mu(|X|^2)/n$).

If X, Y are random variables and μ is a probability measure, we will note $X \sim Y$ when the two random variables have the same distribution, and $X \sim \mu$ when μ is the law of X . We close this introduction by stating a very useful theorem due to E. Milman (see [15, Theorem 2.4] where a stronger result is stated).

Theorem 3 (E. Milman [15]). *Let μ be a log-concave probability measure. If there exists $C > 0$ such that for every smooth function f ,*

$$\text{Var}_\mu(f) \leq C \|\nabla f\|_\infty^2,$$

then $C_P(\mu) \leq cC$, where $c > 0$ is a universal constant.

2 B -symmetric log-concave measures

Let B be a convex body on \mathbb{R}^n , whose interior contains 0. We associate to B the (non-necessary symmetric) norm $|\cdot|_B$ defined by

$$|x|_B = \inf \left\{ \lambda > 0, \frac{x}{\lambda} \in B \right\}.$$

Then B and its boundary ∂B correspond respectively to the unit ball and the unit sphere for this norm. Let us note respectively μ_B and σ_B the uniform measure on B and the cone measure on ∂B normalized so as to be probability measures. Recall that the cone measure is characterized by the following decomposition formula for all measurable $f : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\int f d\mu_B = \int f(r\theta) nr^{n-1} \mathbb{1}_{[0,1]}(r) dr d\sigma_B(\theta).$$

Let μ be a probability measure on \mathbb{R}^n with density $\rho(|x|_B)$ with respect to the Lebesgue measure. If ρ is log-concave and non-increasing then μ is log-concave. In that case, we say that μ is a B -symmetric log-concave probability measure. Note that the isotropy of μ amounts to the isotropy of B .

Proposition 4. *Let B be an isotropic convex body of \mathbb{R}^n satisfying KLS with constant C . Then any probability measure $\mu(dx) = \rho(|x|_B)dx$ with ρ log-concave and non-increasing on \mathbb{R}^+ , satisfies KLS with a constant depending only on C .*

To prove the latter, we use the following decomposition of μ .

Lemma 5. *Let X be a random variable on \mathbb{R}^n of law $\mu(dx) = \rho(|x|_B)dx$ with ρ log-concave and non-increasing on \mathbb{R}^+ . Then $X \sim SU$ where U and S are independent random variables respectively on B and \mathbb{R}^+ , of law $U \sim \mu_B$ and $S \sim -\text{Vol}(B)s^n \rho'(s) \mathbb{1}_{\mathbb{R}^+}(s)ds$.*

Note that, as ρ is log-concave, ρ is locally Lipschitz, and thereby is differentiable almost everywhere and satisfies the fundamental theorem of calculus (see for instance [19]).

Proof. Let us recall the classical polar decomposition: if θ and R are independent random variables respectively on ∂B and \mathbb{R}^+ , of law $\theta \sim \sigma_B$ and $R \sim n\text{Vol}(B)r^{n-1}\rho(r)\mathbb{1}_{\mathbb{R}^+}(r)dr$, then $R\theta \sim \mu$. Let $T \sim nt^{n-1}\mathbb{1}_{[0,1]}(t)dt$ be independent of $\theta \sim \sigma_B$. By the same polar decomposition, $U = T\theta$ is uniform on B . So, if $S \sim -\text{Vol}(B)s^n \rho'(s) \mathbb{1}_{\mathbb{R}^+}(s)ds$ is independent of T and θ , and $X = SU = (ST)\theta$, it remains only to show that $ST \sim R$ to prove that $X \sim \mu$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function, then

$$\begin{aligned} \mathbb{E}(f(ST)) &= \int f(st) n\text{Vol}(B) t^{n-1} s^n (-\rho'(s)) \mathbb{1}_{[0,1]}(t) \mathbb{1}_{\mathbb{R}^+}(s) dt ds \\ &= \int f(r) n\text{Vol}(B) r^{n-1} \mathbb{1}_{\mathbb{R}^+}(r) \left(\int_r^{+\infty} -\rho'(s) ds \right) dr \\ &= \int f(r) n\text{Vol}(B) r^{n-1} \rho(r) \mathbb{1}_{\mathbb{R}^+}(r) dr. \end{aligned}$$

In the last equality, we use that $-\rho'(x) \geq 0$ and $\rho(x) \rightarrow 0$ when $x \rightarrow +\infty$. Actually, as ρ is log-concave, non-increasing, and non-constant as a density, there exists $c_1 > 0$ such that $\rho(x) \leq e^{-c_1 x}$ for x large enough. \square

We need then a kind of Poincaré inequality for S in the above representation.

Lemma 6. *Let S be a random variable on \mathbb{R}^+ of law $\eta(ds) = -s^n \rho'(s) ds$ where $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a log-concave non-increasing function. Then, for every smooth function f on \mathbb{R}^+*

$$\text{Var}_\eta(f) \leq c \frac{\mathbb{E}(S^2)}{n} \|f'\|_\infty^2,$$

where $c > 0$ is a universal constant.

Proof. By integration by parts, we see that $-\int_0^{+\infty} r^n \rho'(r) dr = \int_0^{+\infty} n r^{n-1} \rho(r) dr = 1$. So $\nu(dr) = n r^{n-1} \rho(r) dr$ is a probability measure, and by Theorems 1 and 2, if $R \sim \nu$, then there exists a universal constant $c > 0$ such that

$$C_P(\nu) \leq c \text{Var}(R) \leq c \frac{\mathbb{E}(R^2)}{n}.$$

Let f be a smooth function on \mathbb{R}^+ such that $\mathbb{E}(f(R)) = 0$ and $\|f'\|_\infty < \infty$. We perform again an integration by parts:

$$\begin{aligned} \mathbb{E}(f^2(S)) &= - \int_0^{+\infty} f^2(r) r^n \rho'(r) dr \\ &= \int_0^{+\infty} (f^2(r) n r^{n-1} + 2f(r) f'(r) r^n) \rho(r) dr \\ &= \mathbb{E}(f^2(R)) + \frac{2}{n} \mathbb{E}(R f(R) f'(R)). \end{aligned}$$

In the second equality, we use that $f^2(r) r^n \rho(r) \rightarrow 0$ when $r \rightarrow +\infty$. This comes from the fact that f is Lipschitz, so there exist a and b such that $|f(r)| \leq ar + b$. The Poincaré inequality for R leads to

$$\mathbb{E}(f^2(R)) \leq c \frac{\mathbb{E}(R^2)}{n} \|f'\|_\infty^2.$$

Moreover,

$$\begin{aligned} \mathbb{E}(R f(R) f'(R)) &\leq \|f'\|_\infty \sqrt{\mathbb{E}(R^2)} \sqrt{\mathbb{E}(f^2(R))} \\ &\leq \|f'\|_\infty \sqrt{\mathbb{E}(R^2)} \sqrt{c \frac{\mathbb{E}(R^2)}{n} \|f'\|_\infty^2} \\ &= \sqrt{\frac{c}{n}} \mathbb{E}(R^2) \|f'\|_\infty^2. \end{aligned}$$

Thus,

$$\mathbb{E}(f^2(S)) \leq \left(c + 2\sqrt{c/n}\right) \frac{\mathbb{E}(R^2)}{n} \|f'\|_\infty^2.$$

Now, for any smooth function g on \mathbb{R}^+ , we set $f = g - \mathbb{E}(g(R))$. As $\text{Var}_\eta(g) \leq \mathbb{E}(g(S) - a)^2$ for every $a \in \mathbb{R}$, it holds

$$\text{Var}_\eta(g) \leq \left(c + 2\sqrt{c/n}\right) \frac{\mathbb{E}(R^2)}{n} \|g'\|_\infty^2.$$

To conclude, let us remark that

$$\mathbb{E}(R^2) = \int_0^{+\infty} nr^{n+1}\rho(r)dr = -\frac{n}{n+2} \int_0^{+\infty} r^{n+2}\rho'(r)dr = \frac{n}{n+2}\mathbb{E}(S^2).$$

□

We can now prove Proposition 4 by tensorization.

Proof of Proposition 4. Let $\mu(dx) = \rho(|x|_B)dx$ be a probability measure with ρ log-concave non-increasing. Let S and U be independent random variables respectively on \mathbb{R}^+ and uniform on B , such that $X = SU \sim \mu$ as in Lemma 5. Let f be a smooth function on \mathbb{R}^n . By Lemma 6, there exists a universal constant $c > 0$ such that

$$\begin{aligned} \mathbb{E}_S(f^2(SU)) &\leq [\mathbb{E}_S(f(SU))]^2 + c\frac{\mathbb{E}(S^2)}{n} \max_{s \geq 0} (\langle \nabla f(sU), U \rangle^2) \\ &\leq [\mathbb{E}_S(f(SU))]^2 + c\frac{\mathbb{E}(S^2)}{n} |U|^2 \|\nabla f\|_\infty^2. \end{aligned} \quad (3)$$

As B satisfies KLS with constant C , we can also apply the Poincaré inequality to $u \mapsto \mathbb{E}_S(f(Su))$:

$$\begin{aligned} \mathbb{E}_U [\mathbb{E}_S(f(SU))]^2 &\leq [\mathbb{E}_U \mathbb{E}_S(f(SU))]^2 + C\frac{\mathbb{E}(|U|^2)}{n} \mathbb{E}_U |\mathbb{E}_S(S\nabla f(SU))|^2 \\ &\leq [\mathbb{E}_U \mathbb{E}_S(f(SU))]^2 + C\frac{\mathbb{E}(|U|^2)}{n} \mathbb{E}(S^2) \|\nabla f\|_\infty^2. \end{aligned}$$

So, if we take the expectation with respect to U in (3), we obtain:

$$\begin{aligned} \text{Var}_\mu f &\leq (c + C)\frac{\mathbb{E}(S^2)\mathbb{E}(|U|^2)}{n} \|\nabla f\|_\infty^2 \\ &= (c + C)\frac{\mathbb{E}(|X|^2)}{n} \|\nabla f\|_\infty^2, \end{aligned}$$

since S and U are independent, and $X = SU$. We conclude with Theorem 3. □

3 Convex bodies of revolution

The same method works with convex bodies of revolution or more generally for convex bodies $K \subset \mathbb{R}^{n+1}$ defined by

$$K = \{(t, x) \in I \times \mathbb{R}^n; |x|_B \leq R(t)\},$$

where I is a bounded interval of \mathbb{R} and $R : I \rightarrow \mathbb{R}^+$ is a concave function. Actually we show the KLS conjecture for the corresponding measures $\rho(t, |x|_B)dt dx$ and more generally for log-concave measures with a finite number of symmetries and whose first variable takes value in a finite-dimensional space.

Proposition 7. *Let n_0, \dots, n_k be positive integers and $N = \sum n_i$. Let B_1, \dots, B_k be isotropic convex bodies respectively of $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_k}$ satisfying KLS with constant C . Then every isotropic probability measure $\mu(dx) = \rho(x_0, |x_1|_{B_1}, \dots, |x_k|_{B_k})dx$ on \mathbb{R}^N with ρ log-concave and non-increasing with respect to its k last variables, satisfies KLS with a constant depending only on C, n_0 , and k .*

When $n_0 = k = 1$ and B_1 is an euclidean ball, we deduce the following.

Corollary 8. *The convex bodies of revolution satisfy the KLS conjecture.*

Proof of Proposition 7. The same method as for B -symmetric measures applies. In the same way as in Lemma 5, let $(X_0, S) = (X_0, S_1, \dots, S_k)$ and U_1, \dots, U_k be independent variables respectively on $\mathbb{R} \times (\mathbb{R}^+)^k$ and B_1, \dots, B_k , with

$$(X_0, S) \sim (-1)^k \text{Vol}(B_1) \cdots \text{Vol}(B_k) s_1^{n_1} \cdots s_k^{n_k} \partial_{1, \dots, k}^k \rho(x_0, s) \mathbf{1}_{(\mathbb{R}^+)^k}(s) dx_0 ds,$$

$$\forall i, \quad U_i \sim \mu_{B_i}.$$

Be careful that $\partial_i \rho$ denotes here the partial derivative of $s_i \mapsto \rho(x_0, s_1, \dots, s_k)$. Then $X = (X_0, S_1 U_1, \dots, S_k U_k)$ is of law μ . Moreover, we can suppose without loss of generality that that for all i ,

$$\mathbb{E}|U_i|^2 = 1.$$

Else, we choose $\lambda_i^{-2} = \mathbb{E}(|U_i|^2)$ such that $V_i \sim \mu_{\lambda_i B}$ satisfies $\mathbb{E}(|V_i|^2) = 1$, and replace ρ by $\rho_\lambda : (x_0, s_1, \dots, s_k) \mapsto \rho(x_0, \lambda_1 s_1, \dots, \lambda_k s_k)$ which is still log-concave and non-increasing with respect to its k last variables. In this case, $\mu(dx) = \rho_\lambda(x_0, |x_1|_{\lambda_1 B_1}, \dots, |x_k|_{\lambda_k B_k})dx$.

Recall now that μ is isotropic, i.e. that

$$\mathbb{E}(X) = 0$$

and

$$\forall y \in \mathbb{R}^N, \quad \mathbb{E}(\langle X, y \rangle^2) = \frac{\mathbb{E}(|X|^2)}{N} |y|^2.$$

This implies that each B_i had to be itself an isotropic convex body, X_0 is an isotropic log-concave variable, and

$$\forall i, \quad \frac{\mathbb{E}(S_i^2)}{n_i} = \frac{\mathbb{E}(|X_0|^2)}{n_0} = \frac{\mathbb{E}(|X|^2)}{N}, \quad (4)$$

since $\mathbb{E}(|U_i|^2) = 1$. In the same spirit as Lemma 6, one can show

Lemma 9. *For any smooth function f on $\mathbb{R}^{n_0} \times (\mathbb{R}^+)^k$,*

$$\text{Var}_{(X_0, S)}(f) \leq c(n_0 + k^2) \frac{\mathbb{E}|X|^2}{N} \|\|\nabla f\|\|_\infty^2,$$

where $c > 0$ is a universal constant.

Let us postpone the proof of this lemma and show how to deduce the claim of the proposition. Let f be a smooth function on \mathbb{R}^N . Let us note $\nabla_i f$ the derivative of

$$x_i \in \mathbb{R}^{n_i} \mapsto f(x_0, x_1, \dots, x_k),$$

and $su = (s_1u_1, \dots, s_ku_k)$ whenever $s = (s_1, \dots, s_k) \in \mathbb{R}^k$ and $u = (u_1, \dots, u_k)$ with $u_i \in \mathbb{R}^{n_i}$. As in the proof of Proposition 4, we apply Poincaré inequality for (X_0, S) and then for each U_i . First we consider $(x_0, s) \mapsto f(x_0, sU)$:

$$\begin{aligned} \mathbb{E}_{(X_0, S)}(f^2(X_0, SU)) &\leq [\mathbb{E}_{(X_0, S)}(f(X_0, SU))]^2 \\ &\quad + c(n_0 + k^2) \frac{\mathbb{E}(|X|^2)}{N} \max_{(x_0, s) \in \mathbb{R} \times (\mathbb{R}^+)^k} \left(|\nabla_0 f(x_0, sU)|^2 + \sum_i \langle \nabla_i f(x_0, sU), U_i \rangle^2 \right) \\ &\leq [\mathbb{E}_{(T, S)}(f(X_0, SU))]^2 + c(n_0 + k^2) \frac{\mathbb{E}(|X|^2)}{N} (1 + \sum_i |U_i|^2) \|\nabla f\|_\infty^2. \end{aligned}$$

Then, as each B_i satisfies KLS with constant C ,

$$\begin{aligned} \mathbb{E}_U [\mathbb{E}_{(X_0, S)}(f(X_0, SU))]^2 &\leq [\mathbb{E}(f(X))]^2 + \sum_i C \frac{\mathbb{E}(|U_i|^2)}{n_i} \mathbb{E}_{U_i} |\mathbb{E}_{(X_0, S)}(S_i \nabla_i f(X_0, SU))|^2 \\ &\leq [\mathbb{E}(f(X))]^2 + \sum_i C \frac{\mathbb{E}(|U_i|^2)}{n_i} \mathbb{E}(S_i^2) \mathbb{E}(|\nabla_i f(X)|^2) \\ &\leq [\mathbb{E}(f(X))]^2 + C \frac{\mathbb{E}(|X|^2)}{N} \mathbb{E}(|\nabla f(X)|^2). \end{aligned}$$

The second inequality comes from Cauchy–Schwarz inequality and the independence of S and U . It follows

$$\begin{aligned} \text{Var}_\mu(f) &\leq [c(n_0 + k^2)(1 + \sum_i \mathbb{E}(|U_i|^2)) + C] \frac{\mathbb{E}(|X|^2)}{N} \|\nabla f\|_\infty^2 \\ &= [2c(n_0 + k^2)(1 + k) + C] \frac{\mathbb{E}(|X|^2)}{N} \|\nabla f\|_\infty^2. \end{aligned}$$

We are done, thanks to Theorem 3. □

Proof of Lemma 9. Let $R = (R_1, \dots, R_k)$ be a random variable on $(\mathbb{R}^+)^k$ such that

$$(X_0, R) \sim \nu(dx_0 dr) = \left(\prod_{i=1}^k \text{Vol}(B_i) n_i \mathbb{1}_{\mathbb{R}^+}(r_i) r_i^{n_i-1} \right) \rho(x_0, r) dt dr.$$

Then one can see thanks to the classical polar decomposition of μ that

$$\forall i, \quad R_i \sim S_i |U_i|_{B_i}.$$

In particular,

$$\forall i, \quad \frac{\mathbb{E}(R_i^2)}{n_i} = \frac{n_i}{n_i + 2} \frac{\mathbb{E}(S_i^2)}{n_i} = \frac{n_i}{n_i + 2} \frac{\mathbb{E}(|X|^2)}{N} \leq \frac{\mathbb{E}(|X|^2)}{N}.$$

Moreover ν is a log-concave (non-isotropic) measure and by Theorem 1, there exists a universal constant $c > 0$ such that

$$C_P(\nu) \leq c \left(\mathbb{E}(|X_0|^2) + \sum_i \text{Var}(R_i) \right).$$

Now the density of R_i is proportional to

$$r_i^{n_i-1} \int \rho(x_0, r) dx_0 dr_1 \cdots dr_{i-1} dr_{i+1} \cdots dr_k$$

and $r_i \mapsto \int \rho(x_0, r) dx_0 dr_1 \cdots dr_{i-1} dr_{i+1} \cdots dr_k$ is a non-increasing function which is also log-concave by Prékopa–Leindler’s theorem [17, 14, 18]. According to Theorem 2,

$$\text{Var}(R_i) \leq \frac{\mathbb{E}(R_i^2)}{n_i} \leq \frac{\mathbb{E}(|X|^2)}{N}.$$

Eventually,

$$C_P(\nu) \leq c(n_0 + k) \frac{\mathbb{E}(|X|^2)}{N}.$$

As for Lemma 6, to prove the claim, it is enough to consider smooth functions f on $\mathbb{R} \times (\mathbb{R}^+)^k$ such that $\mathbb{E}(f(R)) = 0$ and $\|\|\nabla f\|\|_\infty < \infty$. We apply k successive integrations by parts and bound the derivatives of f by $\|\|\nabla f\|\|_\infty$ each times it appears:

$$\begin{aligned} & \mathbb{E}(f^2(X_0, S)) \\ &= Z \int f^2(x_0, r) (-1)^k r_1^{n_1} \cdots r_k^{n_k} \partial_{1, \dots, k}^k \rho(x_0, r) \mathbf{1}_{(\mathbb{R}^+)^k}(r) dx_0 dr \\ &= \int \left[f^2 + 2 \frac{r_1}{n_1} f \partial_1 f \right] (-1)^{k-1} n_1 r_1^{n_1-1} r_2^{n_2} \cdots r_k^{n_k} \partial_{2, \dots, k}^k \rho(x_0, r) \mathbf{1}_{(\mathbb{R}^+)^k}(r) dx_0 dr \\ &\leq \int \left[f^2 + 2 \frac{r_1}{n_1} f \|\|\nabla f\|\|_\infty \right] (-1)^{k-1} n_1 r_1^{n_1-1} r_2^{n_2} \cdots r_k^{n_k} \partial_{2, \dots, k}^k \rho(x_0, r) \mathbf{1}_{(\mathbb{R}^+)^k}(r) dx_0 dr \\ &= \int \left[f^2 + 2 \frac{r_1}{n_1} f \|\|\nabla f\|\|_\infty + 2 \frac{r_2}{n_2} f \partial_2 f + 2 \frac{r_1 r_2}{n_1 n_2} \partial_2 f \|\|\nabla f\|\|_\infty \right] \\ &\quad (-1)^{k-2} n_1 n_2 r_1^{n_1-1} r_2^{n_2-1} r_3^{n_3} \cdots r_k^{n_k} \partial_{2, \dots, k}^k \rho(x_0, r) \mathbf{1}_{(\mathbb{R}^+)^k}(r) dx_0 dr \\ &\leq \int \left[f^2 + 2 \left(\frac{r_1}{n_1} + \frac{r_2}{n_2} \right) f \|\|\nabla f\|\|_\infty + 2 \frac{r_1 r_2}{n_1 n_2} \|\|\nabla f\|\|_\infty^2 \right] \\ &\quad (-1)^{k-2} n_1 n_2 r_1^{n_1-1} r_2^{n_2-1} r_3^{n_3} \cdots r_k^{n_k} \partial_{2, \dots, k}^k \rho(x_0, r) \mathbf{1}_{(\mathbb{R}^+)^k}(r) dx_0 dr \\ &\vdots \\ &\leq \mathbb{E}(f^2(R)) + 2 \|\|\nabla f\|\|_\infty \mathbb{E} \left(f(R) \sum_i \frac{R_i}{n_i} \right) + 2 \|\|\nabla f\|\|_\infty^2 \mathbb{E} \left(\sum_{i < j} \frac{R_i R_j}{n_i n_j} \right). \end{aligned} \quad (5)$$

By Cauchy–Schwarz inequality,

$$\mathbb{E} \left(f(R) \sum_i \frac{R_i}{n_i} \right) \leq \sqrt{\mathbb{E}(f^2(R))} \sqrt{\mathbb{E} \left(\sum_i \frac{R_i}{n_i} \right)^2}, \quad (6)$$

and

$$\mathbb{E} \left(\sum_i \frac{R_i}{n_i} \right)^2 \leq \left(\sum_i \frac{1}{n_i} \right) \mathbb{E} \left(\sum_i \frac{R_i^2}{n_i} \right) \leq k^2 \frac{\mathbb{E}|X|^2}{N}. \quad (7)$$

The Poincaré inequality leads to

$$\mathbb{E}(f^2(R)) \leq c(n_0 + k) \frac{\mathbb{E}(|X|^2)}{N} \|\nabla f\|_\infty^2. \quad (8)$$

Noting that $\sum_{i < j} \frac{R_i R_j}{n_i n_j} \leq \left(\sum_i \frac{R_i}{n_i}\right)^2$ and plugging the estimates (6), (7), and (8) in the inequality (5), it follows

$$\begin{aligned} \mathbb{E}(f^2(S)) &\leq \left(c(n_0 + k) + 2\sqrt{c(n_0 + k)k} + 2k^2\right) \frac{\mathbb{E}|X|^2}{N} \|\nabla f\|_\infty^2 \\ &\leq (2c(n_0 + k) + 3k^2) \frac{\mathbb{E}|X|^2}{N} \|\nabla f\|_\infty^2. \end{aligned}$$

□

Remark. In the special case of the euclidean ball, one can improve the above estimate using the classical polar decomposition $X = (R_1\theta_1, \dots, R_k\theta_k)$ with $\theta_i \sim \sigma_{\mathbb{S}^{n_i-1}}$:

Proposition 10. *Let $\mu(dx) = \rho(x_0, |x_1|, \dots, |x_k|)dx$ be an isotropic log-concave probability measure on \mathbb{R}^N , with $x_i \in \mathbb{R}^{n_i}$. Then there exists a universal constant c such that*

$$C_P(\mu) \leq c(n_0 + k) \frac{\mathbb{E}_\mu |X|^2}{N}.$$

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