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# Twisting the Stern sequence

Roland Bacher

March 23, 2010

## 1 Main results

In this paper, we identify a complex sequence  $(a_n)_{n \in \mathbb{N}}$  with the corresponding function  $a : \mathbb{N} \mapsto \mathbb{C}$ . We write thus always  $a(n)$  instead of  $a_n$ .

The Stern sequence or Stern-Brocot sequence with first terms given by

0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, 3, ...

(cf. sequence A2487 of [8]) is the integral sequence  $s : \mathbb{N} \rightarrow \mathbb{N}$  recursively defined by  $s(0) = 0, s(1) = 1$  and  $s(2n) = s(n), s(2n + 1) = s(n) + s(n + 1)$  for  $n \geq 1$ . It is closely related to the Farey tree and induces a one-to-one map  $n \mapsto s(n)/s(n + 1)$  between  $\mathbb{N}$  and non-negative rational numbers, cf. [5] or Chapter 16 of [1]. It is also an example of a 2-regular sequence, see Chapter 16 of [2].

The following result gives a different, perhaps not very well-known, description of the Stern sequence.

**Proposition 1.1**  *$s(n)$  equals the number of distinct subsequences of the form  $1, 101, 10101, \dots = \{1(01)^*\}$  in the binary expansion of  $n$ .*

Proposition 1.1 is in fact a particular case of Proposition 2.2, an easy result concerning rational series in non-commuting variables.

**Example** The binary expansion of  $11 = 2^3 + 2^1 + 2^0$  is given by 1011. The string 1011 contains the following five subsequences

**1011, 1011, 1011, 1011, 1011**

printed with bold letters of the form  $1(01)^*$  and we have  $s(11) = s(5) + s(6) = s(2) + s(3) + s(3) = s(1) + 2(s(1) + s(2)) = 5s(1) = 5$ .

Proposition 1.1 allows to parametrize Stern sequences by counting a subsequence of the form  $1(01)^k$  with weight  $w^k$ , see Proposition 2.1 for formulae. For  $n = 11$  we get for instance  $3 + 2w$ .

In this paper we introduce a related sequence which will be called the *twisted Stern sequence* since it is obtained by twisting the recursive definition of the Stern sequence with a sign. More precisely, we define the twisted

Stern sequence  $t(0), t(1), \dots$  recursively by  $t(0) = 0, t(1) = 1$  and  $t(2n) = -t(n), t(2n+1) = -t(n) - t(n+1)$  for  $n \geq 1$ . It starts as

0, 1, -1, 0, 1, 1, 0, -1, -1, -2, -1, -1, 0, 1, 1, 2, 1, 3, 2, 3, 1, 2, 1, 1, 0, -1, \dots

An inspection of these first few terms shows already some striking similarities between the Stern sequence and its twisted relative. The aim of this paper is to describe a few properties of the Stern sequence and its twist.

The following result (the identity for  $s(n)$  in assertion (i) is probably well-known to the experts) is an illustration of the similarities between these two sequences:

**Theorem 1.2** (i) *We have*

$$\begin{aligned} s(2^e + n) &= s(2^e - n) + s(n) \\ t(2^e + n) &= (-1)^e s(2^e - n) + s(n) \end{aligned}$$

for all  $e \geq 0$  and for all  $n$  such that  $0 \leq n \leq 2^e$ .

(ii) *We have*

$$t(3 \cdot 2^e + n) = t(6 \cdot 2^e - n) = (-1)^e s(n)$$

for all  $e \geq 0$  and for all  $n$  such that  $0 \leq n \leq 2^{e+1}$ .

The failure for  $n > 2^{e+1}$  of the formula

$$t(3 \cdot 2^e + n) = (-1)^e s(n)$$

given by assertion (ii) can perhaps be mended by the following conjectural identity based on experimental observations.

**Conjecture 1.3** *There exists an integral sequence  $u(0), u(1), u(2), \dots$  such that we have*

$$\sum_{n=0}^{\infty} t(3 \cdot 2^e + n) z^n = (-1)^e \left( \sum_{n=0}^{\infty} u(n) z^{n \cdot 2^e} \right) \left( \sum_{m=0}^{\infty} s(m) z^m \right)$$

for all  $e \in \mathbb{N}$ .

If the conjecture holds, the ordinary generating function of the sequence  $u(0), u(1), \dots$  is given by

$$\sum_{n=0}^{\infty} u(n) z^n = \frac{\sum_{n=0}^{\infty} t(3+n) z^n}{\sum_{n=0}^{\infty} s(n) z^n}$$

and it starts as

$$1 - 2z^2 - 2z^5 + 4z^6 + 2z^7 - 6z^8 + 4z^9 + 2z^{10} - 6z^{11} + 8z^{12} + \dots$$

The first equality in assertion (ii) of Theorem 1.2 shows that the finite sequences  $(-1)^e t(3 \cdot 2^e), (-1)^e t(3 \cdot 2^e + 1), \dots, (-1)^e t(6 \cdot 2^e)$  of length  $3^e + 1$  are palindromic sequences of natural integers. The first few such sequences are

$$\begin{array}{cccccccccccccccc}
 0 & & & & & & & & \mathbf{1} & & & & & & & & \mathbf{1} & & & & & & & & & & 0 \\
 0 & & & & & 1 & & & \mathbf{1} & & & 2 & & & \mathbf{1} & & 1 & & & & & & & & & & & & 0 \\
 0 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & \mathbf{1} & 4 & 3 & 5 & 2 & 5 & 3 & 4 & \mathbf{1} & 3 & 2 & 3 & 1 & 2 & 1 & 1 & 1 & 0 \\
 0 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & \mathbf{1} & 4 & 3 & 5 & 2 & 5 & 3 & 4 & \mathbf{1} & 3 & 2 & 3 & 1 & 2 & 1 & 1 & 1 & 0
 \end{array}$$

with boldfaced 1's at one third and two thirds for highlighting the underlying partial self-similarity structure. The subsequence lying between the two boldfaced 1's appears also at the beginning of [7]. One notices that all sequences start and end with zero and that all existing central elements are equal to 2.

The polynomials defined by these palindromic sequences are described by the following result:

**Theorem 1.4** (i) *The polynomials*

$$\psi_e = (-1)^e \sum_{n=0}^{3 \cdot 2^e} t(3 \cdot 2^e + n) z^n$$

*have the factorisations*

$$\begin{aligned}
 \psi_e &= z(1 + z^{2^e})(1 + z + z^2)^e \prod_{n=0}^{e-2} (1 - z^{2^n} + z^{2^{n+1}})^{e-1-n} \\
 &= z(1 + z^{2^e}) \prod_{n=0}^{e-1} (1 + z^{2^n} + z^{2^{n+1}}).
 \end{aligned}$$

(ii) *We have*

$$\psi_{e+1} = \psi_e(1 + z^{3 \cdot 2^{e-1}}) + z^{2^e} \psi_{e-1}(1 + z^{2^e})$$

*for all*  $e \geq 1$ .

**Remark 1.5** *Theorem 1.4 implies the identity*

$$\sum_{n=0}^{\infty} t(n) z^n = z - z^2 + \sum_{e=0}^{\infty} (-1)^e z^{3 \cdot 2^e + 1} (1 + z^{2^e}) \prod_{n=0}^{e-1} (1 + z^{2^n} + z^{2^{n+1}}).$$

Assertion (ii) of Theorem 1.2 yields  $\lim_{e \rightarrow \infty} \psi_e = \sum_{n=0}^{\infty} s(n) t^n$ . The second factorisation contained in Theorem 1.4 gives a new proof of the following result due to Carlitz (see [6]):

**Corollary 1.6** *We have*  $\sum_{n=0}^{\infty} s(n) z^n = z \prod_{n=0}^{\infty} (1 + z^{2^n} + z^{2^{n+1}})$ .

(A direct proof of Corollary 1.6 is straightforward: The series  $U(z)$  defined by the right-hand-side starts as  $z + \dots = s(0) + s(1)z + \dots$  and its even, respectively odd, subseries are given by  $U(z^2)$ , respectively  $(\frac{1}{z} + z)U(z^2)$ . Its coefficients satisfy thus the same recursion relations as the elements of the Stern sequence.)

The Carlitz factorisation of Corollary 1.6 implies that  $\sum_{n=0}^{\infty} s(n)z^n$  has no non-zero roots in the open unit disc. This is not true for the ordinary generating series  $\sum_{n=0}^{\infty} t(n)z^n$  of the twisted Stern sequence which has (infinitely?) many non-zero roots in the open unit disc.

Given a natural integer  $k \geq 2$  and a natural integer  $i$ , we consider the endomorphism  $\rho(i)$  of the vector-space (or module) of formal power series defined by

$$\rho(i) \left( \sum_{n=0}^{\infty} a(n)z^n \right) = \sum_{n=0}^{\infty} a(i + nk)z^n .$$

The  $k$ -kernel of a formal power series  $A$  is the smallest vector space (or module when working over a ring)  $\mathcal{V}$  containing  $A$  such that  $\rho(0)\mathcal{V}, \dots, \rho(k-1)\mathcal{V} \subset \mathcal{V}$ . A formal power series is  $k$ -regular if its  $k$ -kernel is finitely generated. Easy examples of  $k$ -regular series are polynomials and ordinary generating series of periodic sequences.  $k$ -regular power series form a vector space (or module) which is preserved by many natural operations such as derivation, product, Hadamard product, shuffle product, ..., see [2] for details. The set of  $k$ -regular sequences with coefficients contained in a finite set (eg. in a finite field) coincides with the set of so-called  $k$ -automatic sequences, see Theorem 16.1.5 of [2].

A sequence  $a(0), a(1), \dots$  is called  $k$ -regular if its ordinary generating series  $\sum_{n=0}^{\infty} a(n)z^n$  is a  $k$ -regular formal power series.

The following result is also a consequence of the Carlitz factorisation:

**Theorem 1.7** *The logarithmic derivative*

$$H(z) = \frac{d}{dz} \log \left( \sum_{n=0}^{\infty} s(n+1)z^n \right) = \frac{\sum_{n=1}^{\infty} ns(n+1)z^{n-1}}{\sum_{n=0}^{\infty} s(n+1)z^n}$$

of  $\sum_{n=0}^{\infty} s(n+1)z^n$  is 2-regular. More precisely,  $H(z)$  is defined by the functional equation

$$H(z) = \frac{1+2z}{1+z+z^2} + 2zH(z^2) .$$

Coefficients of  $H(z)$  appear as sequence A163659 in [8].

2-regularity of the logarithmic derivation  $H(z)$  is a special case of the following result, perhaps already known to Schützenberger:

**Theorem 1.8** Given  $d$   $k$ -regular series  $A_1(z), \dots, A_d(z)$  over some commutative ring  $R$ ,  $d$  linear forms  $L_1(x_1, \dots, x_d), \dots, L_d(x_1, \dots, x_d)$  in  $d$  unknowns with coefficients in  $R[z]$  and  $d$  constants  $\alpha_1, \dots, \alpha_d \in R$  such that  $\alpha_i = A_i(0) + L_i(\alpha_1, \dots, \alpha_d) \pmod{z}$  for  $i = 1, \dots, d$ , the system of equations

$$\begin{aligned} U_1(z) &= A_1(z) + L_1(U_1(z^k), \dots, U_d(z^k)), \\ &\vdots \\ U_d(z) &= A_d(z) + L_d(U_1(z^k), \dots, U_d(z^k)) \end{aligned}$$

determines a unique set  $U_1(z), \dots, U_d(z)$  of  $d$   $k$ -regular sequences with constant coefficients  $\alpha_i = U_i(0)$  for  $i = 1, \dots, d$ .

**Remark 1.9** We have  $\alpha_i = A_i(0)$  if the linear form  $L_i$  has all its coefficients of in  $zR[z]$ .

The series

$$A(z) = \prod_{n=0}^{\infty} \frac{1}{1 - z^{2^n}}$$

satisfying  $A(z) = (\sum_{n=0}^{\infty} z^n) A(z^2)$  is not 2-regular (see Remark 1.11 below). This shows that Theorem 1.8 can not be extended to equations with linear forms having  $k$ -regular series as coefficients.

Theorem 1.7 can be generalised as follows:

**Theorem 1.10** Let  $P(z)$  be a polynomial with constant coefficient 1. Then the series

$$A = \prod_{n=0}^{\infty} P(z^{k^n})$$

is  $k$ -regular.

Moreover, if all roots of  $P(z)$  are complex roots of 1 having finite order, then the logarithmic derivative  $B = A'/A$  of  $A$  is also  $k$ -regular.

**Remark 1.11** Given a  $k$ -regular series  $A(z)$  with constant coefficient 1, the product  $\prod_{n=0}^{\infty} A(z^{k^n})$  is generally not  $k$ -regular. Indeed, starting with the 2-regular series  $A(z) = 1 + z + z^2 + \dots = \frac{1}{1-z}$ , the coefficient  $b(n)$  in the series

$$B = \sum_{n=0}^{\infty} b(n)z^n = \prod_{n=0}^{\infty} \frac{1}{1 - z^{2^n}}$$

counts the number of partitions of  $n$  into powers of 2, see sequence A123 in [8], and  $\log(b(2n))$  is asymptotically equal to  $\frac{1}{2 \log 2} \left( \log \frac{n}{\log n} \right)^2$  (see equation 1.3 in [4]) which is incompatible with  $k$ -regularity of  $B$  by Theorem 16.3.1 in [2].

**Remark 1.12** *Another famous sequence illustrating Theorem 1.10 is the sequence*

$$\prod_{n=0}^{\infty} (1 - z^{2^n}) = \sum_{n=0}^{\infty} (-1)^{tm(n)} z^n$$

*related to the Thue-Morse sequence  $n \mapsto tm(n)$  defined by digit-sums modulo 2 for binary expansions of natural integers.*

Another link between the two sequences  $s$  and  $t$  is given by determinants of  $2 \times 2$ -matrices. For  $n \geq 1$  we consider the matrix

$$M(n) = \begin{pmatrix} s(n) & s(n+1) \\ t(n) & t(n+1) \end{pmatrix}$$

with first row two consecutive terms of  $s$  and and second row the two corresponding consecutive terms of  $t$ . The first matrices are  $2 \times 2$ -submatrices defined by two consecutive rows of

$$\begin{array}{cccccccccccccccccccc} 1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 4 & 3 & 5 & 2 & 5 & 3 & 5 & 1 & 5 & 4 & 7 \\ 1 & -1 & 0 & 1 & 1 & 0 & -1 & -1 & -2 & -1 & -1 & 0 & 1 & 1 & 2 & 1 & 3 & 2 & 3 \end{array}$$

Injectivity of the map  $n \mapsto s(n+1)/s(n)$  shows that the matrices  $M(n)$  are all different. Their determinants are characterised by the following result.

**Theorem 1.13** *We have  $|\det(M(n))| = 2$  for all  $n \geq 1$ . More precisely,*

$$\det(M(n)) = -2(-1)^k \text{ if } 2^k \leq n < 2^{k+1} .$$

*In particular, all matrices  $M(n)$  are invertible for  $n \geq 1$ .*

Since the recursive definitions of the integral sequences  $s$  and  $t$  differ only by signs (and since they satisfy the same initial conditions) they have the same reduction modulo 2 characterised by the following (easy) result, already contained in [9]:

**Proposition 1.14** *The integers  $s(n)$  and  $t(n)$  are even if and only if  $n$  is divisible by 3.*

The determinants of the non-singular matrices  $M(n)$  are thus in some sense as small as possible: Indeed, since the sequences  $s$  and  $t$  coincide modulo 2, a matrix  $M(n)$  involves either a column consisting of even integers or all its four entries are odd integers and such matrices have even determinants.

We end this paper with a last result, going back to Stern (see [9]) for the first part of assertion (i):

**Theorem 1.15** (i) *The integer  $s(n)$  divides  $s(n-1) + s(n+1)$  for every  $n \geq 1$ . More precisely, we have*

$$\frac{s(n-1) + s(n+1)}{s(n)} = 1 + 2v_2(n)$$

where the 2-valuation  $v_2(n)$  is defined as the exponent of the highest power of 2 dividing  $n$ .

The function

$$C(z) = \sum_{n=1}^{\infty} \frac{s(n-1) + s(n+1)}{s(n)} z^n$$

satisfies  $C(0) = 0$  and

$$C(z) = z \frac{1+2z}{1-z^2} + C(z^2)$$

and is 2-regular.

(ii) *The integers  $t(n)$  and  $t(n-1) + t(n+1)$  are both zero if  $n = 3 \cdot 2^k$ . They are both non-zero otherwise and  $t(n)$  divides  $t(n-1) + t(n+1)$ . More precisely, we have*

$$\frac{t(n-1) + t(n+1)}{t(n)} = 1 + 2v_2(n)$$

if  $n \notin \{2^{\mathbb{N}}, 3 \cdot 2^{\mathbb{N}}\}$ ,  $(t(0) + t(2))/t(1) = (t(1) + t(3))/t(2) = -1$  and

$$\frac{t(2^e - 1) + t(2^e + 1)}{t(2^e)} = 1 + 2(e - 2)$$

for all  $e$  such that  $e \geq 2$ .

The rest of this paper is organized as follows: The next section contains a proof of Proposition 1.1 and a few complements.

Section 3 is devoted to the (easy) proofs of Theorem 1.2 and to a few more formulae and conjectures involving the Stern sequence and its twist.

Section 4 contains the proofs of Theorems 1.4, 1.7, 1.8, 1.10 and Corollary 1.6.

Section 5 contains the easy proof of Theorem 1.13 and a few related results.

Section 6 consists of the short proof of Theorem 1.15.

## 2 Proof of Proposition 1.1 and a few comments

We give first a bijective proof of Proposition 1.1. We describe then briefly a weighted version of the Stern sequence counting subsequences of the form



$1(01)^*$  with weights encoding their length. We give then a generalisation of Proposition 1.1 therefore providing a (sketch of a) second proof for Proposition 1.1.

**Proof of Proposition 1.1** We call a subsequence of a binary expansion  $B(n)$  *admissible* if it is of the form  $1(01)^*$ . For example,  $B(11) = 1011 = \epsilon_3\epsilon_2\epsilon_1\epsilon_0$  has five admissible subsequences given by the set

$$\{\epsilon_3, \epsilon_1, \epsilon_0, \epsilon_3\epsilon_2\epsilon_1, \epsilon_3\epsilon_2\epsilon_0\} .$$

Since the number  $b(n)$  of such subsequences satisfies clearly  $b(2n) = b(n)$ , the equality  $s(2n) = s(n)$  shows that we can restrict our attention to  $n$  odd. We consider the two cases  $n = 4n + 1$  and  $n = 4n - 1$ .

If  $w$  is an admissible subsequence of  $B(4n + 1)$ , then the digit  $\epsilon_1 = 0$  of the binary expansion of  $4n + 1$  is either contained in  $w$  or not. In the first case, admissibility of  $w$  shows that  $w$  contains also the last digit  $\epsilon_0$  of  $B(4n + 1)$  and removal of  $\epsilon_1\epsilon_0 = 01$  from  $w$  yields a bijection between such admissible subsequences and admissible subsequences of  $B(2n)$ . In the second case where  $\epsilon_1$  is not involved in  $w$ , we get a bijection between such admissible subsequences and admissible subsequences of the binary expansion  $B(2n + 1) = \dots\epsilon_3\epsilon_2\epsilon_0$  of the integer  $2n + 1$ . This shows the identity  $b(4n + 1) = b(2n) + b(2n + 1)$ .

We consider finally the case of an admissible subsequence  $w$  of  $B(4n - 1)$ . If  $\epsilon_0$  is not contained in  $w$ , then  $w$  can be associated with an admissible subsequence of  $B(4n - 2)$  or equivalently of  $B(2n - 1)$ . Denoting by  $l$  the least integer such that  $B(4n - 1) = \alpha 01^l$ , we consider now an admissible subsequence  $w$  of  $B(4n - 1)$  which contains  $\epsilon_0$ . If the admissible subsequence  $w$  is not of the form  $\beta\epsilon_l\epsilon_0$ , we transform it into the admissible subsequence  $\beta\tilde{\epsilon}_l = \beta 1$  of  $B(4n) = \alpha\tilde{\epsilon}_l\tilde{\epsilon}_{l-1}\dots\tilde{\epsilon}_0 = \alpha 10^l$  or equivalently of  $B(2n) = \alpha 10^{l-1}$  (obtained from  $B(4n)$  by erasing the last digit 0 never involved in an admissible subsequence). If  $w = \beta\epsilon_l\epsilon_0$  we transform it into the admissible subsequence  $\beta$  of  $B(4n)$  or equivalently of  $B(2n)$ . This shows  $b(4n - 1) = b(2n - 1) + b(2n)$  and ends the proof.  $\square$

## 2.1 A weighted variation of the Stern sequence

We denote by  $S(n) \in \mathbb{N}[w]$  the weighted number of subsequences of the form  $1(01)^*$  in the binary expansion  $B(n)$  of  $n$ , giving the weight  $w^k$  to a subsequence of the form  $1(01)^k$ . Similarly, we introduce  $S_e(n) \in \mathbb{N}[w]$  as the weighted number of subsequences of the form  $(10)^*$  in the binary expansion  $B(n)$  of  $n$ , with weight  $w^k$  for a subsequence of the form  $(10)^k$ .

**Proposition 2.1** (i) *Evaluating the polynomial  $S(n) \in \mathbb{N}[w]$  at  $w = 1$  yields the Stern sequence.*

(ii) The sequences  $S(n)$  and  $S_e(n)$  are uniquely determined by the initial conditions  $S(0) = 0, S(1) = S_e(0) = S_e(1) = 1$  and the recursive formulae

$$\begin{aligned} S(2n) &= S(n) \\ S(2n+1) &= S(n) + S_e(n) \\ S_e(2n) &= wS(n) + S_e(n) \\ S_e(2n+1) &= S_e(n) . \end{aligned}$$

(iii) The sequence  $S(n)$  is also uniquely determined by the initial conditions  $S(0) = 0, S(1) = 1$  and by the recursive formulae

$$\begin{aligned} S(2n) &= S(n) \\ S(4n+1) &= wS(2n) + S(2n+1) \\ S(4n-1) &= S(2n-1) + S(2m+1) + wS(2m) \end{aligned}$$

where  $n = 2^a(2m+1)$ .

**Proof of Proposition 2.1** Assertion (i) is obvious.

In the sequel, we use the notation introduced above during the proof of Proposition 1.1.

The initial values for  $S(n)$  and  $S_e(n)$  in assertion (ii) are easy to check. The identity  $S(2n) = S(n)$  is obvious since admissible subsequences of  $B(2n)$  never involve the last digit  $\epsilon_0 = 0$  in the binary expansion  $B(2n)$  of  $2n$ .

Admissible subsequences of  $B(2n+1)$  not containing the last digit  $\epsilon_0$  of  $B(2n+1)$  are in weight-preserving bijection with admissible subsequences of  $B(2n)$  or of  $B(n)$ . Removal of  $\epsilon_0$  induces a weight-preserving bijection between admissible subsequences of  $B(2n+1)$  involving the last digit  $\epsilon_0$  of  $B(2n+1)$  and monomial contributions to  $S_e(n)$ . This proves  $S(2n+1) = S(n) + S_e(n)$ .

Monomial contributions to  $S_e(2n)$  not involving the last digit  $\epsilon_0$  of  $B(2n)$  are in (weight-preserving) bijection with monomial contributions to  $S_e(n)$ . Removing the last digit of monomial contributions to  $S_e(2n)$  involving the last digit  $\epsilon_0$  of  $B(2n)$  yields admissible subsequences of  $B(n)$  with weight reduced by 1. This shows  $S_e(2n) = S_e(n) + wS_e(n)$ .

The identity  $S_e(2n+1) = S_e(n)$  is due to the fact that monomial contributions to  $S_e(2n+1)$  never involve the last digit  $\epsilon_0 = 1$  of  $B(2n+1)$ .

Assertion (iii) follows from the bijections used in the proof of Proposition 1.1. We leave the details to the reader.  $\square$

## 2.2 Counting weighted subsequences and subfactors

A famous result by Schützenberger implies essentially an identification of  $k$ -regular sequences with the set of rational formal power series in  $k$  non-commuting variables. (One has to be a little careful with leading zeros. A

way of dealing with them is to consider only formal power series involving no monomials starting with the variable  $x_0$  associated to the digit 0.)

Proposition 1.1 is then a particular case of the following well-known result which we give without proof. (A proof of stronger statements can be found in [3].)

**Proposition 2.2** *Let  $A$  be a rational formal power series in  $k$  non-commuting variables. Then the shuffle product of  $A$  with  $\frac{1}{1-(x_0+\dots+x_k)}$  and the ordinary non-commutative product  $\frac{1}{1-(x_0+\dots+x_k)}A\frac{1}{1-(x_0+\dots+x_k)}$  are both rational.*

The shuffle-product counts subsequences encoded and weighted by  $A$  in  $k$ -ary expansions of natural integers and the ordinary product in Proposition 2.2 counts subfactors (encoded by  $A$ ) in  $k$ -ary expansions.

Proposition 1.1 corresponds to the case where

$$A = x_1 \frac{1}{1 - x_0 x_1} = x_1 + x_1 x_0 x_1 + x_1 x_0 x_1 x_0 x_1 + x_1 x_0 x_1 x_0 x_1 + \dots$$

respectively

$$A = x_1 \frac{w}{1 - x_0 x_1} = x_1 + w x_1 x_0 x_1 + w^2 x_1 x_0 x_1 x_0 x_1 + \dots$$

in the weighted case with  $x_0, x_1$  non-commuting variables and  $w$  a central variable.

A famous example counting subsequences or subfactors (reduced to 1) is given by the Thue-Morse sequence corresponding to  $A = x_1$ .

Another famous example counting subfactors is given by the Rudin-Shapiro sequence associated to  $A = x_1^2$ .

### 3 Proof of Theorem 1.2 and more formulae

#### 3.1 Proof of Theorem 1.2

**Proof of assertion (i)** For  $n = 0$  we have

$$s(1 + 0) = s(1) + s(0) = 1 + 0 = 1, \quad s(1 + 1) = s(0) + s(1) = 0 + 1 = 1$$

and

$$t(1 + 0) = s(1) - s(0) = 1 - 0 = 1, \quad t(1 + 1) = s(0) - s(1) = 0 - 1 = -1.$$

The proof is now by induction on  $e$ . If  $n$  is even we have

$$s(2^e + n) = s(2^{e-1} + \frac{n}{2}) = s(2^{e-1} - \frac{n}{2}) + s(\frac{n}{2}) = s(2^e - n) + s(n)$$

and

$$\begin{aligned}
t(2^e + n) &= -t(2^{e-1} + \frac{n}{2}) \\
&= -(-1)^{e-1} \left( s(2^{e-1} - \frac{n}{2}) - s(\frac{n}{2}) \right) \\
&= (-1)^e (s(2^e - n) - s(n))
\end{aligned}$$

If  $n$  is odd, we have

$$\begin{aligned}
s(2^e + n) &= s(2^{e-1} + \frac{n-1}{2}) + s(2^{e-1} + \frac{n+1}{2}) \\
&= s(2^{e-1} - \frac{n-1}{2}) + s(\frac{n-1}{2}) + s(2^{e-1} - \frac{n+1}{2}) + s(\frac{n+1}{2}) \\
&= s(2^e - n) + s(n)
\end{aligned}$$

and

$$\begin{aligned}
t(2^e + n) &= -t(2^{e-1} + \frac{n-1}{2}) - t(2^{e-1} + \frac{n+1}{2}) \\
&= (-1)^e \left( s(2^{e-1} - \frac{n-1}{2}) - s(\frac{n-1}{2}) + s(2^{e-1} - \frac{n+1}{2}) - s(\frac{n+1}{2}) \right) \\
&= (-1)^e (s(2^e - n) - s(n))
\end{aligned}$$

**Proof of assertion (ii)** These formulae are easy to establish for  $e = 0$ .

For even  $n$  we have

$$t(3 \cdot 2^e + n) = -t(3 \cdot 2^{e-1} + \frac{n}{2}) = -(-1)^{e-1} s(\frac{n}{2}) = (-1)^e s(n)$$

and

$$t(6 \cdot 2^e - n) = -t(6 \cdot 2^{e-1} - \frac{n}{2}) = -(-1)^{e-1} s(\frac{n}{2}) = (-1)^e s(n)$$

and for odd  $n$  we get

$$\begin{aligned}
t(3 \cdot 2^e + n) &= -t(3 \cdot 2^{e-1} + \frac{n+1}{2}) - t(3 \cdot 2^{e-1} + \frac{n-1}{2}) \\
&= -(-1)^{e-1} (s(\frac{n+1}{2}) + s(\frac{n-1}{2})) = (-1)^e s(n)
\end{aligned}$$

and

$$\begin{aligned}
t(6 \cdot 2^e - n) &= -t(6 \cdot 2^{e-1} - \frac{n+1}{2}) - t(6 \cdot 2^{e-1} - \frac{n-1}{2}) \\
&= -(-1)^{e-1} (s(\frac{n+1}{2}) + s(\frac{n-1}{2})) = (-1)^e s(n) .
\end{aligned}$$

This completes the proof. □

### 3.2 A few other formulae

**Proposition 3.1** (i) *We have*

$$s(2^{e+1} + n) = s(2^e + n) + s(n)$$

for  $0 \leq n \leq 2^e$ , (see the remark by T. Tokita concerning the Stern-sequence A2487 in [8]).

(ii) *We have*

$$t(2^{e+1} + n) + t(2^e + n) = (-1)^{e+1} s(n)$$

for  $0 \leq n \leq 2^e$ .

The formulae of Proposition 3.1 have the following conjectural generalisation, analogous to Conjecture 1.3:

**Conjecture 3.2** (i) *The series*

$$A(z) = \frac{\sum_{n=0}^{\infty} (s(2+n) - s(1+n))z^n}{\sum_{n=0}^{\infty} s(n)z^n} = 1 - 2z + 2z^2 - 4z^4 + 4z^5 + 2z^6 + \dots$$

*satisfies*

$$\sum_{n=0}^{\infty} (s(2^{e+1} + n) - s(2^e + n))z^n = A(z^{2^e}) \sum_{n=0}^{\infty} s(n)z^n$$

for all  $e \in \mathbb{N}$ .

*Similarly, the series*

$$B(z) = -\frac{\sum_{n=0}^{\infty} (t(2+n) + t(1+n))z^n}{\sum_{n=0}^{\infty} s(n)z^n} = 1 - 2z - 2z^2 + 4z^3 + 6z^6 - 6z^7 + \dots$$

*satisfies*

$$(-1)^{e+1} \sum_{n=0}^{\infty} (t(2^{e+1} + n) + t(2^e + n))z^n = B(z^{2^e}) \sum_{n=0}^{\infty} s(n)z^n$$

for all  $e \in \mathbb{N}$ .

**Proof of Proposition 3.1** The formulae hold for  $n = 0$  and  $a \in \{0, 1\}$ . The induction step is an easy computation for odd  $a$  and obvious for even  $a$ .  $\square$

**Proposition 3.3** (i) *We have*

$$s(n) = -s(n - 2^e) + s(n - 2 \cdot 2^e) + 2s(n - 3 \cdot 2^e), \quad 2^{e+2} \leq n \leq 2^{e+3} - 2^e$$

for  $e \geq 0$ .

(ii) *We have*

$$t(n) = t(n - 2^e) - t(n - 2^{e+1})$$

for  $2^{e+2} \leq n \leq 2^{e+3}$ .

**Proof** The case  $e = 0$  implies  $n \in \{4, 5, 6, 7\}$  in assertion (i) and we have

$$\begin{aligned} s(4) &= 1 = -2 + 1 + 2 \cdot 1 = -s(3) + s(2) + 2s(1) \\ s(5) &= 3 = -1 + 2 + 2 \cdot 1 = -s(4) + s(3) + 2s(2) \\ s(6) &= 2 = -3 + 1 + 2 \cdot 2 = -s(5) + s(4) + 2s(3) \\ s(7) &= 3 = -2 + 3 + 2 \cdot 1 = -s(6) + s(5) + 2s(4) \end{aligned}$$

The induction step for  $e > 0$  is easy if  $n$  is even and involves the usual identity  $s(n) = s((n-1)/2) + s((n+1)/2)$  if  $n$  is odd.

The proof of assertion (ii) is similar.  $\square$

The following result gives a few partial sums associated to the Stern sequence and its twist:

**Proposition 3.4** *We have*

$$\begin{aligned} \sum_{n=1}^{2^e} s(n) &= \frac{3^e + 1}{2}, \quad e \geq 0 \\ \sum_{n=1}^{2^e} (-1)^n s(n) &= \frac{1 - 3^{e-1}}{2}, \quad e \geq 1 \\ \sum_{n=1}^{2^e} t(n) &= \frac{(-1)^e + 1}{2}, \quad e \geq 0 \\ \sum_{n=1}^{2^e} (-1)^n t(n) &= \frac{-3 + (-1)^e}{2}, \quad e \geq 0. \end{aligned}$$

**Proof** The first equality holds for  $e = 0$  and

$$\begin{aligned} \sum_{n=0}^{2^{e+1}} s(n) &= \sum_{n=0}^{2^e} s_{2n} + \sum_{n=1}^{2^e} s_{2n-1} \\ &= \sum_{n=0}^{2^e} s(n) + \sum_{n=1}^{2^e} (s(n-1) + s(n)) \\ &= -s(2^e) + 3 \sum_{n=0}^{2^e} s(n) = -1 + 3 \frac{3^e + 1}{2} = \frac{3^{e+1} + 1}{2} \end{aligned}$$

by induction.

For the next identity one finds similarly

$$\sum_{n=1}^{2^{e+1}} (-1)^n s(n) = 1 - \sum_{n=1}^{2^e} s(n) = 1 - \frac{3^e + 1}{2} = \frac{1 - 3^e}{2}.$$

The computations for the partial sums involving  $t(n)$  and  $(-1)^n t(n)$  are analogous.  $\square$

We end this section with a list of a few more identities.

**Proposition 3.5** *We have*

$$s(3 \cdot 2^e + n) = s(3 \cdot 2^e - n) \quad (1)$$

for all  $e, n$  such that  $0 \leq e \leq 2^n$ ,

$$s(3 \cdot 2^e + n) = s(3 \cdot 2^{e-1} + n) + 2s(n) \quad (2)$$

for all  $e, n$  such that  $0 \leq n \leq 2^{e-1}$ ,

$$t(2^e + n) = t(2^e + n - 2^{e-2}) - t(2^e + n - 2^{e-1}) \quad (3)$$

for all  $e, n$  such that  $e \geq 2$  and  $1 \leq n \leq 2^e$ ,

$$s(2^e + n) = (-1)^e t(2^e + n) + 2s(n) \quad (4)$$

for all  $e, n$  such that  $0 \leq n \leq 2^{e+1}$ ,

$$s(2^e + a) = (-1)^e t(2^e - n) - 3s(n) \quad (5)$$

for all  $e, n$  such that  $0 \leq n \leq 2^{e-1}$ ,

$$s(2^e - a) = (-1)^e t(2^e - n) + 2s(n) \quad (6)$$

for all  $e, n$  such that  $0 \leq n \leq 2^{e-1}$ ,

$$s(2^e - n) = (-1)^e t(2^e + n) + s(n) \quad (7)$$

for all  $e, n$  such that  $0 \leq n \leq 2^e$ .

Proofs are easy and left to the reader.

## 4 Proofs related to factorisations

**Proof of Theorem 1.4** We set  $\psi_e = z(1 + z^{2^e})(1 + z + z^2) \prod_{n=0}^{e-2} (1 - z^{2^n} + z^{2^{n+1}})^{e-1-n}$ . Iterating the trivial identity

$$(1 + z^n + z^{2n})(1 - z^n + z^{2n}) = (1 + z^{2n} + z^{4n})$$

we get the equivalent expression

$$\psi_e = z(1 + z^{2^e}) \prod_{n=0}^{e-1} (1 + z^{2^n} + z^{2^{n+1}}).$$

The proof of the identity  $\psi_e = (-1)^e \sum_{n=0}^{3 \cdot 2^e} t(3 \cdot 2^e + n) z^n$  is by induction on  $e$ . It holds for  $e = 0$ . The induction step follows from the recursive definition of the sequence  $t(0), t(1), \dots$  and from the equality  $\psi_{e+1}(z) = (\frac{1}{z} + 1 + z) \psi_e(z^2)$ .  $\square$

**Proof of Theorem 1.7:** Using the Carlitz factorisation

$$\tilde{S}(z) = \prod_{n=0}^{\infty} (1 + z^{2^n} + z^{2^{n+1}})$$

of  $\tilde{S}(z) = \sum_{n=0}^{\infty} s(n+1)z^n$  we have

$$H(z) = \frac{d}{dz} \log(\tilde{S}(z)) = \sum_{n=0}^{\infty} \frac{2^n z^{2^n-1} + 2^{n+1} z^{2^{n+1}-1}}{1 + z^{2^n} + z^{2^{n+1}}}.$$

The summand of index  $n = 0$  yields  $\frac{1+2z}{1+z+z^2}$  and the sum  $\sum_{n=1}^{\infty} \dots$  can be rewritten as  $2zH(z^2)$ .

The proof of 2-regularity of  $H(z)$  is an easy consequence of the functional equation for  $H$ , see Theorem 1.8 below.

Uniqueness of  $H$  defined by the functional equation  $H(z) = \frac{1+2z}{1+z+z^2} + 2zH(z^2)$  follows from the fact that the map

$$A(z) \mapsto \frac{1+2z}{1+z+z^2} + 2zA(z^2)$$

has a unique attracting fixpoint for formal power series (with respect to the obvious topology given by coefficient-wise convergency).  $\square$

**Proof of Theorem 1.8** We assume first that no linear form  $L_1, \dots, L_d$  involves coefficients of degree  $\geq k$ . For  $i = 0, \dots, k-1$ , we denote as before by  $\rho(i)$  the linear map

$$\rho(i) \left( \sum_{n=0}^{\infty} a(n)z^n \right) = \sum_{n=0}^{\infty} a(i+nk)z^n.$$

Given solutions  $U_1, \dots, U_d$ , we consider a finitely generated vector space or module  $\mathcal{V}$  containing  $U_1, \dots, U_d$  and the  $k$ -kernel of  $A_1, \dots, A_d$ . We have then

$$\rho(i)U_j = \rho(i)A_j + ([x^i]L_j)(U_1, \dots, U_d)$$

where  $[x^i]L_j \in R[x_1, \dots, x_d]$  is the linear form obtained from  $L_j$  by considering the coefficients of  $z^j$ . The power series  $\rho(i)U_j \in \mathcal{V}$  is thus a linear combination of  $U_1, \dots, U_d$  and of the  $k$ -kernel of  $A_j$ . The set  $\mathcal{V}$  contains thus the  $k$ -kernel of  $U_j$  and  $U_1, \dots, U_d$  are all  $k$ -regular.

If there are linear forms among  $L_1, \dots, L_d$  which are of degree  $\geq k$ , we introduce the  $k$ -regular series  $A_{d+1} = zA_1, \dots, A_{2d} = zA_d$ , the series  $U_{d+1} = zU_1, \dots, U_{2d} = zU_d$ , the linear forms  $L_{d+1} = zL_1, \dots, L_{2d} = zL_d \in zR[z][x_1, \dots, x_d]$  and set  $\alpha_{d+1} = \dots = \alpha_{2d} = 0$ . We have then the equations

$$U_i(z) = A_i(z) + zL_i(U_1(z^k), \dots, U_n(z^k))$$

and the identities  $\alpha_i = U_i(0)$  for  $i = 1, \dots, 2d$ . Modifying a linear form  $L_j$  of degree  $\geq k$  by substituting all occurrences of  $z^{k+i}U_j(z^k)$  with  $z^iU_{d+j}(z^k)$



we construct an equivalent system with strictly smaller maximal degree for the linear forms  $L_1, \dots, L_{2d}$ . Iteration of this construction leads eventually to a system containing only linear forms of degree strictly smaller than  $k$ .

Existence and unicity of the solution follow from unicity of the attracting fixpoint of the dynamical system defined by the map

$$U_i(z) \mapsto A_i(z) + L_i(U_1(z^k), \dots, U_d(z^k)), i = 1, \dots, d$$

starting from the point  $(\alpha_1, \dots, \alpha_d)$ .  $\square$

**Proof of Theorem 1.10** The first part follows Theorem 1.8 applied to the identity  $A(z) = P(z)A(z^k)$ . We present here however a second, independent proof.

Working over the field of complex numbers and using the fact that products of two  $k$ -regular series are  $k$ -regular (cf. Theorem 16.4.1 in [2]), it is enough to prove the result for polynomials of degree 1. We can thus assume that  $P(z) = 1 + \lambda z$ . The coefficient of  $z^n$  in  $A(z) = \prod_{m=0}^{\infty} (1 + \lambda z^{k^m})$  is then given by zero if the  $k$ -ary expansion of  $n$  involves digits greater than 1 and it is given by  $\lambda^\alpha$  otherwise, where  $\alpha$  equals the number of ones in the  $k$ -ary expansion of  $n$ . This implies  $k$ -regularity of  $A(z)$ .

We have

$$B(z) = \frac{d}{dz} \log(A(z)) = \sum_{n=0}^{\infty} \frac{P'(z^{k^n}) k^n z^{k^n-1}}{P(z^{k^n})}.$$

The summand of index  $n = 0$  yields  $\frac{P'(z)}{P(z)}$  and the remaining summation  $\sum_{n=1}^{\infty} \dots$  can be rewritten as  $kz^{k-1}B(z^k)$ . This shows that  $B(z)$  satisfies the functional equation

$$B(z) = \frac{P'(z)}{P(z)} + kz^{k-1}B(z^k).$$

Using Theorem 16.4.3 of [2] we see that the rational fraction  $\frac{P'(z)}{P(z)}$  is  $k$ -regular if and only if all zeroes of  $P(z)$  are roots of unity (ie. if  $P(z)$  divides  $(z^N - 1)^N$  for some integer  $N$ ). Theorem 1.8 implies then  $k$ -regularity of  $B(z)$ .  $\square$

## 5 Proof of Theorem 1.13 and other results involving matrices

**Proof of Theorem 1.13** The trivial identities

$$\begin{aligned} \det(M(2n)) &= \det \begin{pmatrix} s(2n) & s(2n+1) \\ t(2n) & t(2n+1) \end{pmatrix} \\ &= \det \begin{pmatrix} s(n) & s(n) + s(n+1) \\ -t(n) & -t(n) - t(n+1) \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= -\det \begin{pmatrix} s(n) & s(n+1) \\ t(n) & t(n+1) \end{pmatrix} \\
&= -\det(M(n))
\end{aligned}$$

and

$$\begin{aligned}
\det(M(2n-1)) &= \det \begin{pmatrix} s(2n-1) & s(2n) \\ t(2n-1) & t(2n) \end{pmatrix} \\
&= \det \begin{pmatrix} s(n-1) + s(n) & s(n) \\ -t(n-1) - t(n) & -t(n) \end{pmatrix} \\
&= -\det \begin{pmatrix} s(n-1) & s(n) \\ t(n-1) & t(n) \end{pmatrix} \\
&= -\det(M(n-1))
\end{aligned}$$

imply the result.  $\square$

**Proof of Proposition 1.14** The reduction modulo 2 of the Stern sequence  $s(0), s(1), \dots$  is the 3-periodic sequence  $0, 1, 1, 0, 1, 1, \dots$ . Indeed, this holds for  $s_0 = 0, s_1 = s_2 = 1$  and the recursive formulae

$$\begin{aligned}
s(6n) &= s(3n) \\
s(6n+1) &= s(3n) + s(3n+1) \\
s(6n+2) &= s(3n+1) \\
s(6n+3) &= s(3n+1) + s(3n+2) \\
s(6n+4) &= s(3n+2) \\
s(6n+5) &= s(3n+2) + s(3n+3)
\end{aligned}$$

imply the 3-periodicity of  $s(n) \pmod{2}$  by induction. The reduction modulo 2 of twisted Stern sequence  $t(0), t(1), \dots$  coincides with the reduction modulo 2 of the Stern sequence.  $\square$

## 5.1 Other results involving matrices

The proofs of the following results are easy and omitted.

**Proposition 5.1** (i) *The matrices*

$$\begin{pmatrix} s(n) & s(n+1) \\ s(2^e + n) & s(2^e + n + 1) \end{pmatrix}$$

have determinant  $-1$  for  $n$  such that  $0 \leq n < 2^e$  and determinant  $1$  for  $n$  such that  $2^e \leq n < 2^{e+1}$ .

(ii) *The matrices*

$$\begin{pmatrix} s(n) & s(n+1) \\ t(2^e + n) & t(2^e + n + 1) \end{pmatrix}$$

have determinant  $(-1)^{e+1}$  for  $n$  such that  $0 \leq n < 2^e$  and determinant  $(-1)^e$  for  $n$  such that  $2^e \leq n < 2^{e+2}$ .

(iii) The matrices

$$\begin{pmatrix} t(n) & t(n+1) \\ s(2^e + n) & s(2^e + n + 1) \end{pmatrix}$$

have determinant  $(-1)^{e+1}$  for  $n$  such that  $2^{e+1} < n < 5 \cdot 2^e$ .

(iv) The matrices

$$\begin{pmatrix} t(n) & t(n+1) \\ t(2^e + n) & t(2^e + n + 1) \end{pmatrix}$$

have determinant 1 for  $n$  such that  $2^{e-2} \leq n < 2^e$  or  $7 \cdot 2^e \leq n < 2^{e+3}$  and determinant  $-1$  for  $n$  such that  $2^e \leq n < 7 \cdot 2^e$ .

## 6 Proof for Theorem 1.15

For odd  $n$  we have

$$s(n) = s((n-1)/2) + s((n+1)/2) = s(n-1) + s(n+1) .$$

We have thus  $\frac{s(n-1)+s(n+1)}{s(n)} = 1 = 1 + 2v_2(n)$  since  $v_2(n) = 0$  for odd  $n$ .

For  $n$  even we have by induction

$$\begin{aligned} & \frac{s(n-1) + s(n+1)}{s(n)} \\ = & \frac{s((n-2)/2) + s(n/2) + s(n/2) + s((n+2)/2)}{s(n/2)} \\ = & \frac{s(n/2-1) + s(n/2+1)}{s(n/2)} + 2 \\ = & 1 + 2v_2(n/2) + 2 = 1 + 2v_2(n) . \end{aligned}$$

This ends the proof of assertion (i).

The twisted Stern sequence satisfies also the identities

$$\begin{aligned} t(n-1) + t(n+1) &= t(n), \quad n \text{ odd,} \\ t(n-1) + t(n+1) &= t(n/2-1) + t(n/2+1) + 2t(n), \quad n \text{ even} \end{aligned}$$

involved in the previous proof. This implies assertion (ii) by checking the initial cases and the case of  $n \in 3 \cdot 2^{\mathbb{N}}$ .  $\square$

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