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Abstract

We degenerate the finite gap solutions of the KdV equation from the general formulation in terms of abelian functions when the gaps tends to points, to recover solutions of KdV equations given a few years ago in terms of wronskians called solitons or positons. For this we establish a link between Fredholm determinants and Wronskians.

1 The KdV equation and solutions in terms of theta functions

We consider the Riemann surface Γ represented by $\cup_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1}$ of the algebraic curve defined by $\omega^2 = \prod_{j=1}^{2g+1}(z - E_j)$, with $E_j \neq E_k$, $j \neq k$. Let D be some divisor $D = \sum_{j=1}^{g} P_j$, $P_j \in \Gamma$. The so-called finite gap solution of the KdV equation

$$u_t = 6uu_x - u_{xxx} \tag{1}$$

can be expressed in the form

$$u(x,t) = -2\frac{d^2}{dx^2} \ln \theta(xg + tv + l) + C. \tag{2}$$

We recall briefly, the notations. In (2), θ is the Riemann function defined by

$$\theta(z) = \sum_{k \in \mathbf{Z}^g} \exp\{\pi i(Bk|k) + 2\pi i(k|z)\}, \tag{3}$$

constructed from the matrix of the B-periods of the surface Γ, and the vectors g, v, l are defined by

$$g_j = 2ic_{j1}, \tag{4}$$

$$v_j = 8i(\frac{c_{j1}}{2}\sum_{k=1}^{2g+1} E_k + c_{j2}), \tag{5}$$

$$l_j = -\sum_{k=1}^{g}\int_{\infty}^{P_k} dU_j + \frac{j}{2} - \frac{1}{2}\sum_{k=1}^{g} B_{kj}, \tag{6}$$

$$C = \sum_{k=1}^{2g+1} E_k - 2\sum_{k=1}^{g}\int_{a_k} zdU_k, \tag{7}$$

the coefficients c_{jk} being relating with abelian differential dU_j by

$$dU_j = \frac{\sum_{k=1}^{g} c_{jk}z^{g-k}}{\sqrt{\prod_{k=1}^{2g+1}(z-E_k)}}dz, \tag{8}$$

and coefficients c_{jk} can be obtained by solving the system of linear equations

$$\int_{a_k} dU_j = \delta_{jk}, \quad 1 \le j \le g, \quad 1 \le k \le g.$$

2 Degeneracy of solutions

We suppose that E_j are real, $E_m < E_j$ if $m < j$ and try to evaluate the limits of all objects in formula (2) when E_{2m}, E_{2m+1} tends to $-\alpha_m$, $-\alpha_m = \kappa_m^2$, $\kappa_m > 0$, for $1 \le m \le g$, and E_1 tends to 0.

2.0.1 Limit of $P(z) = \prod_{j=1}^{2g+1}(z - E_j)$

The limit of $P(z) = \prod_{j=1}^{2g+1}(z-E_j)$ is evidently equal to $\tilde{P}(z) = z\prod_{j=1}^{g}(z+\alpha_j)^2$

2.0.2 Limit of $dU_m = \frac{\sum_{k=1}^{g} c_{mk}z^{g-k}}{\sqrt{\prod_{k=1}^{2g+1}(z-E_k)}}dz$

The limit of dU_m is equal to $d\tilde{U}_m = \frac{\varphi_m(z)}{\sqrt{z}\prod_{j=1}^{g}(z+\alpha_j)}dz$, where $\varphi_m(z) = \sum_{k=1}^{g}\tilde{c}_{mk}z^{g-k}$. The normalization condition takes the form in the limit

$$\int_{a_k} dU_j \to \frac{2\pi i\varphi_j(-\alpha_k)}{\kappa_k\prod_{m\neq k}(\alpha_m - \alpha_k)} = \delta_{kj}, \tag{9}$$

which proves that the numbers $-\alpha_m$, $m \neq k$ are the zeros of the polynomials $\varphi_k(z)$, and so $\varphi_k(z)$ can be written as $\varphi_k(z) = \tilde{c}_{k1} \prod_{m \neq k}(z + \alpha_m)$. By (9), we get in the limit

$$\tilde{c}_{k1} = \frac{\kappa_k}{2\pi i}.$$

So

$$d\tilde{U}_k = \frac{\kappa_k}{2\pi i \sqrt{z}(z + \alpha_k)} dz$$

2.0.3 Limit of v_k and g_k

By identification of the powers of z^{g-2} in (10)

$$\tilde{\varphi_k} = \tilde{c_{k1}} \prod_{l \neq k}(z + \alpha_l) = \sum_{j=1}^{g} \tilde{c_{kj}} z^{g-j}, \tag{10}$$

we get in the limit

$$\tilde{c}_{k1} \sum_{l=1}^{g} -\alpha_l + \tilde{c}_{k2} = \frac{\kappa_k^3}{2\pi i}.$$

So we have the limit values of v_k and g_k :

$$\tilde{v}_k = \frac{4}{\pi} \kappa_k^3$$

and

$$\tilde{g}_k = \frac{1}{\pi} \kappa_k.$$

2.0.4 Limit of $U_j(P)$ and B_{mk}

For $\lambda_0 = -\alpha_m = -\kappa_m^2$, $I = \int_{\lambda_0}^{0} dU_k \to \frac{1}{2}\tilde{B}_{mk}$. The integral I can be easily evaluate along the real axis on the upper sheet of surface Γ and we get

$$I \to \frac{i}{2\pi} \ln \left| \frac{\kappa_m + \kappa_k}{\kappa_m - \kappa_k} \right|.$$

So we have the limit values of matrix B :

$$\tilde{B}_{mk} = \frac{i}{\pi} \ln \left| \frac{\kappa_m + \kappa_k}{\kappa_m - \kappa_k} \right|.$$

So iB_{kk} tends to $-\infty$. As previously, we have

$$\int_{\infty}^{P} dU_j \to -\frac{i}{2\pi} \ln \left| \frac{\kappa_j - \sqrt{z_P}}{\kappa_j + \sqrt{z_P}} \right|. \tag{11}$$

2.0.5 Limit of argument of exponential in $\theta(p)$

Let us denote A the argument of $\theta(p) = \sum_{k\in\mathbf{Z}^g} \exp\{\pi i(Bk|k) + 2\pi i(k|p)\}$.
A can be rewritten in the form

$$A = \pi i \sum_{j=1}^{g} B_{jj}k_j(k_j - 1) + 2\pi i \sum_{j>m} B_{mj}k_m k_j + \sum_{j=1}^{g} \pi i(2p_j + B_{jj})k_j. \quad (12)$$

Using the inequality $k_j(k_j - 1) \geq 0$ for all $k \in \mathbf{Z}^g$ and the fact that iB_{kk} tends to $-\infty$, we can reduce the limit $\tilde{\theta}$ of $\theta(p)$ to a finite sum taken over vectors $k \in \mathbf{Z}^g$ such that each k_j must be equal to 0 or 1.
So, A can be rewritten in the form

$$A = \pi i \sum_{j=1}^{g} B_{jj}k_j(k_j - 1) + 2\pi i \sum_{j>m} B_{mj}k_m k_j + \sum_{j=1}^{g} k_j[2\pi i(g_j x + v_j t)$$

$$-\pi i(-j + 2\sum_{k=1}^{g} \int_{\infty}^{P_k} dU_j + \sum_{m\neq j} B_{mj})].$$

In other words

$$A = \pi i \sum_{j=1}^{g} B_{jj}k_j(k_j - 1) + 2\pi i \sum_{j>m} B_{mj}k_m k_j + \sum_{j=1}^{g} k_j Q_j,$$

with

$$Q_j = 2\pi i(g_j x + v_j t) + \beta_j$$

and

$$\beta_j = -\pi i(-j + 2\sum_{k=1}^{g} \int_{\infty}^{P_k} dU_j + \sum_{m\neq j} B_{mj}).$$

The quantity β_j has a finite limit value $\tilde{\beta}_j$ independent from x and t.

2.0.6 Limit of $\theta(p)$

By means of the inequality $K_j(K_j - 1) \geq 0$ for all $K \in \mathbf{Z^g}$ and the previous relation iB_{kk} tends to $-\infty$, it turns out that the limit $\tilde{\theta}$ of $\theta(xg + tv + l)$ reduce to a finite sum taken over vectors $k \in \mathbf{Z^g}$ with the property that each k_j must be equal to 0 or 1.

$$\tilde{\theta} = \sum_{k\in\mathbf{Z}^g,\, k_j=0\, or\, 1} \exp\{\sum_{m>j} 2\ln\left|\frac{\kappa_m - \kappa_j}{\kappa_m + \kappa_j}\right| k_m k_j + i(\sum_{j=1}^{g} 2\kappa_j x + 8\kappa_j^3 t + 2\kappa_j x_j + \pi j$$

$$- \sum_{m \neq j} i \ln \left| \frac{\kappa_m + \kappa_j}{\kappa_m - \kappa_j} \right|) k_j \},$$

with

$$x_j = x_j(\kappa_j) = \frac{1}{2i\kappa_j} \sum_{k=1}^{g} \ln \left| \frac{\kappa_j - \sqrt{z_k}}{\kappa_j + \sqrt{z_k}} \right|$$

It can be reduced to

$$\tilde{\theta} = \sum_{J \in \{1,\ldots,g\}} \prod_{j \in J} (-1)^j \prod_{j \in J \ k \notin J} \left| \frac{\kappa_j + \kappa_k}{\kappa_j - \kappa_k} \right| \sum_{j \in J} \exp 2i(\kappa_j x + 4\kappa_j^3 t + \kappa_j x_j). \quad (13)$$

3 1-positon of order 1

A 1-positon of order 1 is given by (see [11]),

$$u = -2\partial_x^2 \log W(\phi, \partial_K \phi),$$

where W is the classical wronskian, and ϕ the function

$$\phi = \phi(x, K) = \sin(K(x + x_1(K) + 4K^2 t))$$

In this case

$$W = \frac{1}{2}(\sin 2\Theta - 2K\gamma),$$

where $\Theta = K(x + x_1(K) + 4K^2 t)$ and $\gamma = \partial_K \Theta$.
We consider here a Riemann surface of genus 2.
Using the previous section, when we take the limit as E_j tend to K_j^2 the function θ tends to $\tilde{\theta}$ which takes the form

$\tilde{\theta} = 1 - \frac{K_2+K_1}{K_2-K_1} \exp 2i(K_1 + 4K_1^3 t + K_1 x_1(K_1)) + \frac{K_2+K_1}{K_2-K_1} \exp 2i(K_2 + 4K_2^3 t + K_2 x_2(K_2))$
$- \exp 2i((K_1 + K_2) + 4(K_1^3 + K_2^3)t + K_1 x_1(K_1) + K_2 x_2(K_2)).$

It can be rewritten as

$$\begin{array}{l} \tilde{\theta} = 1 + (K_2 + K_1) \frac{\exp 2i(K_2 + 4K_2^3 t + K_2 x_2(K_2)) - \exp 2i(K_1 + 4K_1^3 t + K_1 x_1(K_1))}{K_2 - K_1} \\ - \exp 2i((K_1 + K_2) + 4(K_1^3 + K_2^3)t + K_1 x_1(K_1) + K_2 x_2(K_2)). \end{array} \quad (14)$$

Now, it is clear that when K_2 tends to $K_1 = K$, we get

$$\tilde{\theta} = 1 + 2Ki \exp(2i\Theta) 2i \partial_K \Theta - \exp(4i\Theta). \quad (15)$$

It can be reduced to

$$\tilde{\theta} = -2i\exp(2i\Theta)(\sin 2\Theta - 2K\gamma) = -4i\exp(2i\Theta)\times W. \tag{16}$$

Since Θ is a linear function of x, we recover exactly the 1-positon

$$u(x,t) = -2\partial_x^2 \ln\tilde{\theta} = -2\partial_x^2 W(\phi,,\partial_K\phi) = \frac{16K^2\sin\Theta(\sin\Theta - K\gamma\cos\Theta)}{(\sin 2\Theta - 2K\gamma)^2}.$$

4 1-positon of order 2

We define a 1-positon of order 2 by (see [11]),

$$u = -2\partial_x^2 \log W(\phi, \partial_K\phi, \partial_K^2\phi),$$

where W is the classical wronskian, and ϕ the function

$$\phi = \phi(x,K) = \sin(K(x + x_1(K) + 4K^2t))$$

In this case

$$W = 4\gamma^2K^2\cos\Theta - 2K\gamma\sin\Theta - 48K^3t\sin\Theta - \sin 2\Theta\sin\Theta,$$

where $\Theta = K(x + x_1(K) + 4K^2t)$ and $\gamma = \partial_K\Theta$.
Here we consider a Riemann surface of genus 3.
We can write the function θ using the previous section. When we take the limit as E_j tend to K_j^2 the function θ tends to $\tilde{\theta}$ which takes the form

$$\begin{array}{l}
\tilde{\theta} = 1 - \left|\frac{K_1+K_2}{K_1-K_2}\right|\left|\frac{K_1+K_3}{K_1-K_3}\right|\exp 2i(K_1 + 4K_1^3t + K_1x_1(K)) \\
+ \left|\frac{K_2+K_1}{K_2-K_1}\right|\left|\frac{K_2+K_3}{K_2-K_3}\right|\exp 2i(K_2 + 4K_2^3t + K_2x_2(K)) \\
- \left|\frac{K_3+K_1}{K_3-K_1}\right|\left|\frac{K_3+K_2}{K_3-K_2}\right|\exp 2i(K_3 + 4K_3^3t + K_3x_3(K)) \\
- \left|\frac{K_1+K_3}{K_1-K_3}\right|\left|\frac{K_2+K_3}{K_2-K_3}\right|\exp 2i(K_1 + 4K_1^3t + K_1x_1(K) + K_2 + 4K_2^3t + K_2x_2(K)) \\
+ \left|\frac{K_1+K_2}{K_1-K_2}\right|\left|\frac{K_3+K_2}{K_3-K_2}\right|\exp 2i(K_1 + 4K_1^3t + K_1x_1(K) + K_3 + 4K_3^3t + K_3x_3(K)) \\
- \left|\frac{K_2+K_1}{K_2-K_1}\right|\left|\frac{K_3+K_1}{K_3-K_1}\right|\exp 2i(K_2 + 4K_2^3t + K_2x_2(K) + K_3 + 4K_3^3t + K_3x_3(K)) \\
+ \exp 2i(K_1 + 4K_1^3t + K_1x_1(K) + K_2 + 4K_2^3t + K_2x_{21}(K) + K_3 + 4K_3^3t + K_3x_3(K))
\end{array}$$

Now, it is clear that when K_2 and K_3 tends to $K_1 = K$, we get

$$\begin{array}{l}
\tilde{\theta} = 1 - \exp(2i\Theta) - 2K^2(-4\gamma^2 + 48iK\gamma t)\exp(2i\Theta) - 4iK\gamma\exp(2i\Theta) + 16K^2\gamma^2\exp(4i\Theta) \\
- \exp(4i\Theta) + \exp(6i\Theta) + 4iK\gamma\exp(4i\Theta) + 2K^2(-4\gamma^2 + 48i\gamma Kt)\exp(4i\Theta).
\end{array}$$

It can be reduced to

$$\tilde{\theta} = -4\exp(3i\Theta)(4\gamma^2K^2\cos\Theta - 2K\gamma\sin\Theta - 48K^3t\sin\Theta - \sin 2\Theta\sin\Theta). \quad (17)$$

As Θ is linear in x, we recover exactly the 1-positon of order 2

$$u(x,t) = -2\partial_x^2 \ln\tilde{\theta} = -2\partial_x^2 W(\phi, \partial_K\phi, \partial_K^2\phi).$$

5 From Theta to Wronskian

5.1 From Theta to Fredholm

In a recent paper, Kirillov and Van Diejen [19] have given determinantal formulas for zonal spherical functions on hyperboloids. In particular, they compute $\det(I + A)$, where I is the unit matrix and $A = (a_{jk})_{1\leq j,k\leq m}$ the matrix defined as :

$$a_{jk} = \frac{2\epsilon_j K_j}{K_j + K_k}\prod_{l\neq j}\left|\frac{K_j + K_l}{K_j - K_l}\right|\exp(-2K_j x), \quad (18)$$

where $\epsilon_j \in \{-1; +1\}$.
Then $\det(I + A)$ has the following form

$$\det(I + A) = \sum_{J\subset\{1,\ldots,N\}}\prod_{i\in J}\epsilon_j \prod_{j\in J\, k\notin J}\left|\frac{K_j + K_k}{K_j - K_k}\right|\exp(-2x\sum_{j\in J}K_j). \quad (19)$$

We consider the following modified determinant $\det(I + A)$ where $A = (a_{jk})_{1\leq j,k\leq m}$ is the matrix defined as :

$$a_{jk} = \frac{2(-1)^j K_j}{K_j + K_k}\prod_{l\neq j}\left|\frac{K_j + K_l}{K_j - K_l}\right|\exp(i(2K_j x + 8K_j^3 t + 2K_j x_j), \quad (20)$$

where x_j in an arbitrary parameter.
Using the same strategy, we can compute $\det(I + A)$ which has the following form

$$\det(I + A) = \sum_{J\subset\{1,\ldots,N\}}\prod_{i\in J}(-1)^j \prod_{j\in J\, k\notin J}\left|\frac{K_j + K_k}{K_j - K_k}\right|\exp(i(2K_j x + 8K_j^3 t + 2K_j x_j). \quad (21)$$

If we compare this expression to (13), we have clearly the equality

$$\tilde{\theta} = \det(I + A). \quad (22)$$

It remains to find the link between this Fredholm determinant and a certain wronskian.

5.2 From Fredholm to Wronskians

In this section, we consider the following functions

$$\phi_j(x) = \sin(K_j x + 4K_j^3 t + K_j x_j), \tag{23}$$

where K_j are real numbers such that $K_1 \leq \ldots \leq K_N$, and x_j an arbitrary constant independent of x.
We use the following notations :
$\theta_j = K_j x + 4K_j^3 t + K_j x_j$.
$W = W(\phi_j, \ldots, \phi_N)$ is the classical Wronskian $W = \det[(\partial^j \phi_i)_{i,\, j\in[1,\ldots,N]}]$.
We consider the matrix $A = (a_{jk})_{j,\, k\in[1,\ldots,N]}$ defined by

$$a_{jk} = \frac{(-1)^j 2K_j}{K_j + K_k} \exp 2i(K_j x + 4K_j^3 t + K_j x_j). \tag{24}$$

Then we have the following statement

Theorem 5.1

$$\det(I + A) = \frac{2^N i^{\frac{N(N+5)}{2}} \exp(i\sum_{j=1}^N \theta_j)}{\prod_{j=2}^N \prod_{i=1}^{j-1}(K_j - K_i)} W(\phi_1, \ldots, \phi_N) \tag{25}$$

Proof : We start to remove the factor $(2i)^{-1}e^{i\theta_j}$ in each row j in the Wronskian W for $1 \leq \jmath \leq N$.
Then

$$W = \prod_{j=1}^N e^{i\theta_j}(2i)^{-N} \times W_1, \tag{26}$$

with

$$W_1 = \begin{vmatrix} (1-e^{-2i\theta_1}) & iK_1(1-e^{-2i\theta_1}) & \ldots & (iK_1)^{N-1}(1-e^{-2i\theta_1}) \\ (1-e^{-2i\theta_2}) & iK_2(1-e^{-2i\theta_2}) & \ldots & (iK_2)^{N-1}(1-e^{-2i\theta_2}) \\ \vdots & \vdots & \vdots & \vdots \\ (1-e^{-2i\theta_N}) & iK_N(1-e^{-2i\theta_N}) & \ldots & (iK_N)^{N-1}(1-e^{-2i\theta}) \end{vmatrix}$$

The determinant W_1 can be written as

$$W_1 = \det(\alpha_{jk} e_j + \beta_{jk}),$$

where $\alpha_{jk} = (-1)^k (iK_j)^{k-1}$, $e_j = e^{-2i\theta_j}$, and $\beta_{jk} = (iK_j)^{k-1}$.
The determinant $U = \det(\alpha_{jk}))$ is clearly equal to

$$U = (-1)^{\frac{N(N+1)}{2}} (i)^{\frac{N(N-1)}{2}} \prod_{N \geq l \geq m \geq 1} (K_l - K_m). \tag{27}$$

Then we use the following Lemma

Lemma 5.1 *Let* $A = (a_{ij})_{i,\, j \in [1,\ldots,N]}$, $B = (b_{ij})_{i,\, j \in [1,\ldots,N]}$, $(H_{ij})_{i,\, j \in [1,\ldots,N]}$, *the matrix formed by replacing the jth row of* A *by the ith row of* B
Then

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{a_{ij}}) \tag{28}$$

Proof : For $\tilde{A}$ the matrix of cofactors of A, we have the well known formula $A \times^t \tilde{A} = \det A I$.
So it is clear that $\det(\tilde{A}) = (\det(A))^{N-1}$.
The general term of the product $(c_{ij})_{i,j \in [1,..,N]} = (a_{ij}x_i + b_{ij})_{i,j \in [1,..,N]} \times (A_{ji})_{i,j \in [1,..,N]}$ can be written as
$c_{ij} = \sum_{s=1}^{N} (a_{is}x_i + b_{is}) \times A_{js}$
$= x_i \sum_{s=1}^{N} a_{is}A_{js} + \sum_{s=1}^{N} b_{is}A_{js}$
$= \delta_{ij} \det(A) x_i + \det(H_{ij})$.
We get
$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)})$.
Thus $\det(a_{ij}x_i + b_{ij}) = \det(A) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)})$.
□
Using the previous lemma (28), we get :

$$\det(\alpha_{ij}e_i + \beta_{ij}) = \det(\alpha_{ij}) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\alpha_{ij}}),$$

where $(H_{ij})_{i,\, j \in [1,\ldots,N]}$ is the matrix formed by replacing the jth row of U by the ith row of V defined previously.
We compute $\det(H_{ij})$ and we get

$$\det(H_{ij}) = (-1)^{\frac{N(N+1)}{2}+1} (i)^{\frac{N(N-1)}{2}} \prod_{N \geq l \geq m \geq 1\, l \neq j,\, m \neq j} (K_l - K_m) \prod_{l<j} (K_k - K_l) \prod_{l>j} (K_k - K_l). \tag{29}$$

So, we can simplify the quotient $q = \frac{\det(H_{ij})}{\det(\alpha_{ij})}$:

$$\begin{aligned}\det(H_{ij}) &= \frac{(-1)^j \prod_{l\neq j}(K_l + K_k)}{(-1)^{k-1}\prod_{l\neq k}(K_l - K_k)} \\ &= \frac{(-1)^{j-k+1}2K_k \prod_{l\neq k}(K_l + K_k)}{(K_j + K_k)\prod_{l\neq k}(K_l - K_k)} \\ &= \frac{(-1)^j 2K_k}{(K_j + K_k)} \prod_{l\neq k}\left|\frac{K_l + K_k}{K_l - K_k}\right|. \end{aligned} \tag{30}$$

So $\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\alpha_{jk}})$ can be expressed as

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\alpha_{jk}}) = \prod_{j=1}^{N} e^{-2i\theta_j} \det(\delta_{jk} + \frac{(-1)^j 2K_k}{(K_j + K_k)}) \prod_{l\neq k}\left|\frac{K_l + K_k}{K_l - K_k}\right| \prod_{j=1}^{N} e^{2i\theta_j}.$$

Then dividing each column k by K_k, $1 \leq k \leq N$ and multipling each row j by K_j, $1 \leq j \leq N$, we get

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\alpha_{jk}}) = \prod_{j=1}^{N} e^{-2i\theta_j} \det(\delta_{jk} + \frac{(-1)^j 2K_j}{(K_j + K_k)}) \prod_{l\neq k}\left|\frac{K_l + K_k}{K_l - K_k}\right| \prod_{j=1}^{N} e^{2i\theta_j},$$

and therefore

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\alpha_{jk}}) = \prod_{j=1}^{N} e^{-2i\theta_j} \det(I + A).$$

The Wronskian can be written as

$$W(\phi_1, \ldots, \phi_N) = \prod_{j=1}^{N} e^{i\theta_j}(2i)^{-N}(-1)^{\frac{N(N+1)}{2}}(i)^{\frac{N(N-1)}{2}} \prod_{j=2}^{N}\prod_{i=1}^{j-1}(K_j - K_i) \prod_{j=1}^{N} e^{-2i\theta_j} \det(I+A)$$

It follows that

$$\det(I + A) = \frac{e^{i\sum_{j=1}^{N}\theta_j}(2)^N(i)^{\frac{N(N+5)}{2}}}{\prod_{j=2}^{N}\prod_{i=1}^{j-1}(K_j - K_i)} W(\phi_1, \ldots, \phi_N) \tag{31}$$

□

6 Conclusion

- Now it is clear how to get multi-positons. It is the same strategy used for the preceding examples.
  If we want to get the following general positon

$$u = 2\partial_x^2 \ln W(\phi_1, \ldots, \phi_1^{(k_1)}, \phi_2, , \ldots, \phi_2^{(k_2)}, \ldots, \phi_l, , \ldots, \phi_l^{(k_l)}), \qquad (32)$$

  we consider a Riemann surface of genus $g = \sum_{j=1}^{l} k_j + l$.
  As defined in the first section, we suppose that E_j are real, $E_m \leq E_j$ if $m < j$ and we evaluate the limits of all objects in formula (2).
  We consider $K_i > 0$, for $1 \leq m \leq g$ such that $K_m \leq K_j$ if $m < j$. We choose the following limits :
  E_1 tends to 0.
  for $1 \leq i \leq k_1 + 1$, E_{2i}, E_{2i+1} tends to $-\alpha_1 = K_1^2$;
  for $1 \leq i \leq k_2 + 1$, $E_{2(k_1+1)+2i}$, $E_{2(k_1+1)+2i+1}$ tends to $-\alpha_2 = K_2^2$;
  we continue until;
  for $1 \leq i \leq k_l + 1$, $E_{2(k_1+\ldots k_{l-1}+l-1)+2i}$, $E_{2(k_1+\ldots k_{l-1}+l-1)+2i+1}$ tends to $-\alpha_l = K_l^2$.
  From the result (25), we get clearly in the limit the positon defined by (32).

- For the case of a soliton, it is more simple.
  If we want to get the following soliton

$$u = 2\partial_x^2 \ln W(\phi_1, \ldots, \phi_l), \qquad (33)$$

  we consider a Riemann surface of genus $g = l$.
  We choose E_j real such that $E_m < E_j$ if $m < j$.
  We consider $K_i > 0$, for $1 \leq m \leq g$. We choose the following limits :
  E_1 tends to 0.
  for $1 \leq i \leq l$, E_{2i}, E_{2i+1} tends to $-\alpha_i = K_i^2$;
  Then we get clearly in the limit the soliton defined by (33).

- We can also get the DPT potential.
  Let n and d be some non negative integers, $m = n + d$, If we choose the coefficients K_j defined by

$$K_p = p, \quad \text{if} \quad d \neq 0\,, \quad 1 \leq p \leq m - n,$$

$$K_j = n - m + 2j, \quad \text{if} \quad n \neq 0\,, \quad m - n + 1 \leq j \leq m,$$

Taking $t = 0$ and $x_j = 0$, then $u = 2\partial_x^2 \ln W(\phi_1, \ldots, \phi_l)$, is exactly the DPT potential

$$\frac{m(m+1)}{\sin^2(x)} + \frac{n(n+1)}{\cos^2(x)}.$$

We refer the reader to the paper [4] for the details or to an another forthcoming publication containing a different approach.

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