# On the Hierarchy of Functioning Rules in Distributed Computing

Alain Bui, Marc Bui, Christian Lavault

RAIRO Rech. Opér.
(vol. 33, n° 1, 1999, pp. 15-27)

# ON THE HIERARCHY OF FUNCTIONING RULES IN DISTRIBUTED COMPUTING (*)

by A. Bui ([1, 4]), M. Bui ([2, 4]) and C. Lavault ([3])

Communicated by Catherine Roucairol

Abstract. – In previous papers, we used a Markovian model to determine the *optimal functioning rules* of a distributed system in various settings. Searching optimal functioning rules amounts to solve an optimization problem under constraints. The hierarchy of solutions arising from the above problem is called the "*first order hierarchy*", and may possibly yield equivalent solutions. The present paper emphasizes a specific technique for deciding between two equivalent solutions, which establishes the "*second order hierarchy*".

Keywords: Distributed Systems, Performance evaluation, Markov Chains, Optimization.

Résumé. – Dans des travaux précédents, nous avons déterminé grâce à un modèle Markovien, *les règles de fonctionnement optimal* d'un système distribué pour divers problèmes. La recherche des règles de fonctionnement optimal revient en fait à résoudre un problème d'optimisation sous contrainte. La hiérarchie des solutions obtenues, que nous appelons "*la hiérarchisation du premier odre*", peut générer des solutions équivalentes. Dans le présent article, nous développons une technique spéciale pour départager deux solutions équivalentes : "*la hiérarchisation du second ordre*".

Mots clés : Systèmes Distribués, Évaluation de performance, Chaînes de Markov, Optimisation.

## 1. INTRODUCTION

In our previous papers [2-5], we presented a stochastic model which allows a behavioral study of distributed computing, and we showed its usefulness. For example, thanks to our model, we solved the dining philosophers problem (cf. [9, 11]) without taking left-handing and right-handing into consideration

(*) Received November 1995.

([1]) LaRIA, Univ. Picardie Jules-Verne, 5, rue du Moulin-Neuf, 80000 Amiens, France, e-mail: bui@laria.u-picardie.fr

([2]) Département de Génie Informatique, Univ. de Technologie de Compiègne, URA CNRS 817, Centre de Recherche de Royallieu, BP 649, Compiègne Cedex, France, e-mail: Marc.Bui@utc.fr

([3]) LIPN, Univ. Paris-Nord, Institut Galilée, URA CNRS 1507, av. J.-B.-Clément, 94430 Villetaneuse, France, e-mail: Christian.Lavault@ura1507.univ-paris13.fr

([4]) Corresponding authors.

Recherche opérationnelle/Operations Research, 0399-0559/99/01/$ 7.00

(cf. [4]). Similarly, we settled in [5], the Multiway-Rendez Vous problem raised in [10]. Thanks to it, we also proposed in [2], an "identikit" of the configurations of sites to which corresponds a degree of efficiency for the functioning of some fault-tolerant distributed routing algorithms (e.g. [12, 13]).

On the other hand, our model is based on the interconnection of $N$ finite Markov chains (each one representing a distributed process having only one acyclic ergodic class of states and possibly transient states), and it differs from the other models (see [1, 6, 8]) since it handles a formal specification of distributed systems through local consideration. It makes it possible to determine the optimal functioning rules of a distributed system. Searching optimal functioning rules amounts to optimize a "guide function" $F$ under constraints: we use a function $F$ involving the mean recurrence times of ergodic states and the mean sojourn times within transient states starting from another transient state. The hierarchy of solutions arising from the above problem is called the "*first order hierarchy*" (abbreviation for "first order conditional moment hierarchy"), which may possibly yield equivalent solutions. The aim of our present paper is to emphasize a special technique for deciding between two equivalent solutions, which establishes the "*second order hierarchy*" (abbreviation for "second order central conditional moment hierarchy").

Consider $N$ processors, represented by $N$ random function $({}^kX_t)_{t\in\mathbb{N}}$ evolving as $N$ finite homogeneous Markov Chains, with $r$ similar states; their transition matrices are denoted ${}^kP = ({}^kp_{ij})$, $(k \in \{1,\ldots,N\},\ i,j \in \{1,\ldots,r\})$, respectively. These Markov Chains are assumed to have one acyclic ergodic class (the same class, whatever $k \in \{1,\ldots,N\}$), and possibly transient states (even in the form of several transient classes). The above notation ${}^kP$ actually expresses the fact that each transition matrix depends on a multi-dimensional parameter $\rho_k$, which characterizes the matrix, *e.g.* $\rho_k = ({}^kp_{11},\ldots,{}^kp_{ij},\ldots,{}^kp_{rr})$. The distributed system is made up of a network of processes logically represented by the interconnection of the $N$ Markov chains. This interconnection defines a set of relations between the parameters $\rho_1,\ldots,\rho_N$, which characterizes the network,

$$\mathcal{R}_j(\rho_1,\ldots,\rho_N) \geq 0, \quad j \in \mathcal{I},$$

where $\mathcal{I}$ is a set of indices. When there exists a $\rho = (\rho_1,\ldots,\rho_N)$ which verifies the previous relations, we will write $\rho \in \mathcal{R}$ for short; we will also

call it a functioning rule or solution to the choice problem of functioning rules for the network.

Now, the problem of choosing functioning rules inevitably raises the following question: which criterion can we decide on to provide a functioning rule preference over any other? We propose the following answer: according to the context, we define a "*guide function*" $F$ mapping the real $N$-tuple $\rho = (\rho_1, \ldots, \rho_N)$ into $\mathbb{R}$; the role of $F$ is to "guide" the working of the system. Searching optimal functioning rules amounts to an optimization process (maximization or minimization) of the guide function $F$ under constraints.

A functioning rule is said to be optimal if and only if the corresponding $\rho$ maximizes (resp. minimizes) $F$ when the optimality criterion is maximization (resp. minimization). In such a case, a functioning rule $\rho$ is said to be better than a functioning rule $\rho'$ if and only if $F(\rho) > F(\rho')$ (resp. $-F(\rho) > -F(\rho')$). Subsequently, an optimal functioning rule (if any) is obviously better than a functioning rule which is not optimal.

Two solutions $\rho$ and $\rho'$ are said to be equivalent if and only if $F(\rho) = F(\rho')$.

Any functioning rule such that $\rho$ maximizes (resp. minimizes) $F$ when the optimality criterion is minimization (resp. maximization) is a bad rule. Obviously, every functioning rule which is not bad (it is then said advisable) is better than a bad functioning rule.

## 2. FIRST ORDER HIERARCHY

In order to be more concrete in the choice of the guide function, we consider the following mathematical objects (where the left upper index $k$ still indicates the $k$-th process):

- On the assumption that we deal with ergodic states, let ${}^kT_{ii'}$, denote the mean time to reach the state $i'$, starting from the state $i$. This mean time may be regarded as the conditional expectation of the random number ${}^kf_{i'}$, of transitions before entering $i'$ for the first when starting from the initial state $i$, *viz.*

$$ {}^kT_{ii'} = E({}^kf_{i'} | {}^kX_0 = i). $$

  In the particular case when $i = i'$, ${}^kT_{ii}$ (denoted by ${}^kT_i$ is the mean recurrence time of the state $i$.

- On the assumption that there are transient states, let ${}^kS_{jj'}$ denote the mean sojourn time within the transient state $j'$ starting from the transient state $j$. This mean sojourn time may be regarded as the conditional expectation, starting from the state $j$, of the random number ${}^kn_{j'}$ of times that the process is in state $j'$, *viz*

$$ {}^kS_{jj'} = E({}^kn_{j'}|{}^kX_0 = j). $$

Since we assumed that there is only one acyclic ergodic class (with or without transient state), we know that $\lim_{n\to\infty}({}^kP)^n = {}^kQ$ exists and that it is a matrix with identical rows $({}^kq_1, \ldots, {}^kq_r)$. Let us consider two particular cases.

If there is no transient state, the matrix ${}^k\mathcal{T}$ with components $E({}^kf_{i'}|{}^kX_0 = i)$ is given by

$$ {}^k\mathcal{T} = (I - {}^kZ + J \cdot {}^kZ_{dg}){}^k\Delta, \tag{2} $$

where

$$ {}^kZ = (I - {}^kP + {}^kQ)^{-1}, $$

where ${}^kZ_{dg}$ results from ${}^kZ$ by setting off-diagonal entries equal to 0, where ${}^k\Delta$ is the diagonal matrix with $i$-th entry $\frac{1}{{}^kq_i}$, and where $J$ is the matrix will all entries 1. The diagonal of the matrix ${}^k\mathcal{T}$ provides the ${}^kT_i$. If there are transient states and if we assume the absorbing case, then denote ${}^k\Lambda$ the matrix with components ${}^kS_{jj'}$, $j$ and $j'$ being the transient states, ${}^k\Lambda$ is such that

$$ {}^k\Lambda = (I - {}^kW)^{-1}, \tag{3} $$

where ${}^kW$ is the restriction of ${}^kP$ to the transient states.

Thus, the ${}^kT_{ii'}$'s and the ${}^kS_{jj'}$ are depending on the parameter $\rho_k = ({}^kp_{11}, \ldots, {}^kp_{ij}, \ldots, {}^kp_{rr})$. According to the context, the guide function $F$ is defined either through the ${}^kS_{jj'}$'s, or through the ${}^kT_{ii'}$'s.

Since it involves conditional first order moments (namely the conditional expectations of the ${}^kf_i$'s and the ${}^kn_j$'s), we call *first order hierarchy*, the hierarchy which arises from the ordering induced by the above function $F$.

## 3. SECOND ORDER HIERARCHY

Consider the function

$$ F : \rho = (\rho_1, \ldots, \rho_k, \ldots, \rho_N) \mapsto F(\rho). $$

How can we decide between two equivalent solutions $\rho$ and $\rho'$?

The above question has already been answered in [3] by studying a particular problem (performance evaluation of distributed routing algorithms and construction of a fuzzy set of solutions). Therefore, following the work in [3], we use conditional variances, *i.e.* second order central conditional moments.

The idea is as follows: the variance expresses the dispersion of values around the mean, thus, between two equivalent solutions $\rho$ and $\rho'$ according to the first order hierarchy, we decide and prefer the solution with "globally" smaller conditional variances. The meaning of the world "global" is highly dependable on the context. Yet, we usually deal with sums of conditional variances; thus, the criterion of preference establishes a second hierarchy, which will be called the *second order hierarchy*, because it arises from conditional variances, *i.e.* from second order central conditional moments. The computation of variances is different according to whether states are transient or ergodic. According to the two cases:

1. When the computations deal with ergodic states, the conditional variances $\sigma^2({}^k f_{i'} | {}^k X_0 = i)$ are given by the matrix ${}^k V$

$$ {}^k V = {}^k \mathcal{T} \cdot [(2 \cdot {}^k Z_{dg} \cdot {}^k \Delta) - I] + 2 \cdot [{}^k Z \cdot {}^k \mathcal{T} - J \cdot ({}^k Z \cdot {}^k \mathcal{T})_{dg}] - {}^k \mathcal{T}_{sq} \quad (4) $$

   where ${}^k \mathcal{T}$ is given by (2), ${}^k \mathcal{T}_{sq}$ results from ${}^k \mathcal{T}$ by squaring each entry, and where $({}^k Z \cdot {}^k \mathcal{T})_{dg}$ results from ${}^k Z \cdot {}^k \mathcal{T}$ by setting off-diagonal entries equal to 0.

2. When the computation is concerning the transient states, the conditional variances $\sigma^2({}^k n_{j'} | {}^k X_0 = j)$ are given by the matrix ${}^k Y$

$$ {}^k Y = {}^k \Lambda \cdot (2 \cdot {}^k \Lambda_{dg} - I) - {}^k \Lambda_{sq} \quad (5) $$

   where ${}^k \Lambda$ is given by (3), where ${}^k \Lambda_{dg}$ results from ${}^k \Lambda$ by setting off-diagonal entries equal to 0, and where ${}^k \Lambda_{sq}$ results from ${}^k \Lambda$ by squaring each entry.

   In particular, when the sum $\sum_{j'} {}^k n_{j'}$ is involved, (the sum being taken for all the transient states), one may use more subtle considerations. The conditional expectations $E(\sum_{j'} {}^k n_{j'} | {}^k X_0 = j)$, (namely the mean sojourn times in the set of all transient states) are given by the vector $({}^k \Lambda \xi)$, where all the values of the column vector $\xi$ are 1's. Hence, the conditional variances $\sigma^2(\sum_{j'} {}^k n_{j'} | {}^k X_0 = j)$ are given by the column vector ${}^k V^*$,

$$ {}^k V^* = (2 \cdot {}^k \Lambda - I)({}^k \Lambda \xi) - ({}^k \Lambda \xi)_{sq} \quad (6) $$

where $({}^k\Lambda\xi)_{sq}$ denotes the column vector which results from ${}^k\Lambda\xi$ by squaring each entry.

At this point, an important remark is imperative, so that the issue should not be confused with regard to the interpretation of $F$.

In the first order hierarchy, we considered $F$ as a function of the ${}^kT_{ii'} = E({}^kf_{i'}|{}^kX_0 = i)$ and of the ${}^kS_{jj'} = E({}^kn_{j'}|{}^kX_0 = j)$, which are conditional expectations. This is the reason why we advocate the criterion of the smallest conditional variances of the $f_{i'}$'s and $n_{j'}$'s in the second order hierarchy. Therefore, the latter $F$ must not be mistaken for a function with random variables ${}^kf_{i'}$ and ${}^kn_{j'}$: in that case we should have to compute the expectation $E[F(\ldots,{}^kf_{i'},{}^kn_{j'},\ldots)]$ and consider the variance $\sigma^2[(F(\ldots,{}^kf_{i'},{}^kn_{j'},\ldots))]$; such is not the case in the present settings.

## 4. APPLICATIONS

Here are two examples of applications: the first exemple uses ergodic states and the second one uses transient states.

### 4.1. Example 1

In order to propose a kind of "identikit" of the configurations of sites providing a good functioning of a type of distributed algorithms in [3], we use our stochastic model as follows. We consider a network of $N$ Markov processes with five states (where state 2 is the waiting state and state 3 is the updating state); the transition matrix of the $k$-th processor is

$${}^kP = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \Gamma - b_k & b_k & c & 1-c-\Gamma \\ 0 & d_k & 1-d_k & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

where

$$\begin{cases} 0 < c < 1 - \Gamma & \\ 0 < b_k < \Gamma & \text{and} \quad 0 < d_k < 1 \\ \sum_{k=1}^{N} b_k - B = 0 & \text{and} \quad \sum_{k=1}^{N} d_k - D = 0. \end{cases}$$

It is easily seen, that each of these Markov processes has only one acyclic ergodic class {1,2,3,4,5} and has no transient state.

In the present problem, $\rho_k = (b_k, d_k)$, $k \in \{1, \ldots, N\}$. We know that $\lim_{n\to\infty} ({}^kP)^n$ exists and that the above identical rows matrix satisfies the relation ${}^kQ \cdot {}^kP = {}^kQ$. Solving this equation, we obtain the expression of ${}^kQ$, one row of which is $\frac{1}{G}((1-\Gamma)d_k, d_k, b_k, cd_k, \alpha d_k)$, where $G = b_k + (3-2\Gamma)d_k$ and $\alpha = 1 - c - \Gamma$. Then we have

$$ {}^kT_1 = \frac{1}{(1-\Gamma)d_k}, {}^kT_2 = \frac{G}{d_k}, {}^kT_3 = \frac{G}{b_k}, {}^kT_4 = \frac{G}{cd_k}, {}^kT_5 = \frac{G}{\alpha d_k}. $$

Our criterion of choice (detailed in [2]) is that the updating state should appear the more frequent possible, and the waiting state, the less frequent possible. This defines the guide function $F$:

$$ F(b_1, d_1, \ldots, b_N, d_N) = \sum_{k=1}^{N} \frac{{}^kT_3}{{}^kT_2} = \sum_{k=1}^{N} \frac{d_k}{b_k} $$

to be minimized under the constraints $\sum_{k=1}^N b_k = B$ and $\sum_{k=1}^N d_k = D$.

In order to solve the problem, we can start either with $(b_1, \ldots, b_N)$, or with $(d_1, \ldots, d_N)$. We thus propose a bounding of $(d_1, \ldots, d_N)$, that is to perform a closure of $(]0,1[)^N$, which is the domain of $(d_1, \ldots, d_N)$.

A bounding for $d = (d_1, \ldots, d_N)$ is a pair $(\eta, m)$ where $\eta$ is a number of $]0,1[$ such that $\eta \leq d_k \leq 1 - \eta$ for $k \in \{1, \ldots, N\}$ and where $m$ is an integer $> 0$ such that $d_1 = \ldots = d_m = \eta, d_{m+1} = \ldots = d_{N-1} = 1 - \eta$, $d_N = \rho(\eta)$; $\rho(\eta) = D - m\eta - (N - m - 1)(1 - \eta)$. Owing to the above constraints, $m$ necessarily depends on $\eta$ : $m = m(\eta)$. In [3], we give a necessary and sufficient condition for the existence of such a bounding, and we present a catalogue of possible boundings $(\eta, m(\eta))$. We also show that the $N$-tuples $(b_1, \ldots, b_N)$ associated with those $(d_1, \ldots, d_N)$ which make $F$ minimum, have the following properties. A number $m(\eta)$ among the $b_k$'s is equal to $b_*^{(1)}(\eta) = \frac{B\sqrt{\eta}}{\delta(\eta)}$, a number $N - m(\eta) - 1$ among the $b_k$'s is equal to $b_*^{(2)}(\eta) = \frac{B\sqrt{1-\eta}}{\delta(\eta)}$, and one of the $b_k$'s is equal to $b_*^{(3)}(\eta) = \frac{B\sqrt{\rho(\eta)}}{\delta(\eta)}$, where

$$ \delta(\eta) = m(\eta)\sqrt{\eta} + (N - m(\eta) - 1)\sqrt{1-\eta} + \sqrt{\rho(\eta)}. $$

Since $b_*^{(i)}(\eta)$, $i = 1, 2, 3$, must belong to the range $]0, \Gamma[$, we give in [2] necessary and sufficient conditions for that:

$$B < \left(1 + (N-1)\sqrt{\frac{D-1}{(N-1)-D}}\right)\Gamma, \quad \text{if} \quad 2 \leq D < \frac{N}{2};$$

$$B < \left((N-1) + \sqrt{\frac{(N-1)-D}{D-1}}\right)\Gamma, \quad \text{if} \quad \frac{N}{2} < D < N-1.$$

Note that if $D = \frac{N}{2}$, the existence of such a bounding is always insured. Such a bounding is called "a bounding for $(d_1, \ldots, d_N)$ valid for $(b_1, \ldots, b_N) \in (]0, \Gamma[)^N$", and the following result is proved in [2].

PROPOSITION 1: *Let $(\eta, m(\eta))$ be a bounding for $(d_1, \ldots, d_N)$ valid for $(b_1, \ldots, b_N) \in (]0, \Gamma[)^N$.*
*The best solution $(b_1, d_1, \ldots, b_N, d_N) \in (]0, \Gamma[\times[\eta, 1-\eta])^N$ in the sense of the first order hierarchy is the solution $\rho$ (modulo a permutation of pairs $(b_k, d_k)$, $k \in \{1, \ldots, N\}$) with*

$$(b_1, d_1) = \ldots = (b_{m(\eta)}, d_{m(\eta)}) = (b_*^{(1)}(\eta), \eta),$$
$$(b_{m(\eta)+1}, d_{m(\eta)+1}) = \ldots = (b_{N-1}, d_{N-1}) = (b_*^{(2)}(\eta), 1-\eta),$$
$$(b_N, d_N) = (b_*^{(3)}(\eta), \rho(\eta)).$$

*Sketch of proof:* Let $(d_1, \ldots, d_N)$ be fixed. The Lagrange multipliers method applied to the function $h : (b_1, \ldots, b_N) \to \sum_{k=1}^N \frac{d_k}{b_k}$ leads to the solution $(\widehat{b}_1, \ldots, \widehat{b}_N)$, where $\widehat{b}_k = \frac{B\sqrt{d_k}}{\sum_{j=1}^N \sqrt{d_j}}$ $k \in \{1, \ldots, N\}$.

Since $h$ is convex, $(\widehat{b}_1, \ldots, \widehat{b}_N)$ gives the minimal value of $h$, that is $\frac{1}{B}(\sum_{k=1}^N \sqrt{d_k})^2$. Now consider the function $H : (d_1, , \ldots, d_N) \to \frac{1}{B}(\sum_{k=1}^N \sqrt{d_k})^2$ to minimize under the constraint $\sum_{k=1}^N d_k = D$. The function $H$ being concave, minimal solutions are to be found among the solutions lying at the border of the bounding, viz. among the solutions such that $d_1 = \ldots = d_{m(\eta)} = \eta, d_{m(\eta)+1} = \ldots = d_{N-1} = 1-\eta, d_N = D - m(\eta)\eta - (N - m(\eta) - 1)(1-\eta)$. Hence the above statement.

Other solutions are less efficient. Some of them may be equivalent in the sense of the first order hierarchy, viz. they may give the same value to the function $F$. ■

In order to decide between two solutions $\rho = (b_1, d_1, \ldots, b_N, d_N)$ and $\rho' = (b'_1, d'_1, \ldots, b'_N, d'_N)$, which are equivalent in the sense of the first order

hierarchy, we then use the second order hierarchy. In such a case, we have to compare the "global" dispersion of values around the ${}^kT_2$'s and the ${}^kT_3$'s for solution $\rho$ to the "global" dispersion of values around the ${}^kT_2$'s and the ${}^kT_3$'s for solution $\rho'$. More precisely, $\rho = (b_1, d_1, \ldots, b_N, d_N)$ is said to be better than $\rho' = (b'_1, d'_1, \ldots, b'_N, d'_N)$ if and only if

$$\sum_{k=1}^{N} ({}^kv_{22} + {}^kv_{33}) < \sum_{k=1}^{N} ({}^kv'_{22} + {}^kv'_{33}), \tag{7}$$

where ${}^kv_{ii}$ denotes the term $(i,i)$ in the diagonal of the matrix ${}^kV$ given by (4). In other words, $\rho$ is said to be better than $\rho'$ iff (globally) the sum of the conditional variances corresponding to $\rho$ is strictly smaller than the one which corresponds to $\rho'$.

Here are the analytic expression of ${}^kv_{22}$ and ${}^kv_{33}$:

PROPOSITION 2

$$\begin{aligned}
{}^kv_{22} &= \frac{1}{d_{k^2}}[2b_k + b_k d_k + d_{k^2}(9 - 8\Gamma) - (b_k + d_k(3 - 2\Gamma))^2], \\
{}^kv_{33} &= \frac{1}{b_{k^2}}[d_k(18 - 24\Gamma + 8\Gamma^2) + b_{k^2} + b_k d_k(9 - 8\Gamma) \\
&\quad - (b_k + d_k(3 - 2\Gamma))^2].
\end{aligned}$$

Note that the analytic study of ${}^kv_{22}$ and ${}^kv_{33}$ is not easy, and hence, it is not possible to give directly the analytic expression of an optimal solution in the second order hierarchy. By contrast, a simple programming software easily computes a numerical comparison through inequality (7) for deciding between two equivalent solutions $\rho$ and $\rho'$ (equivalent in the sense of the first order hierarchy).

### 4.2. Example 2

In the deadlock problem, (studied in [4]), good functioning properties are given. We use our stochastic model as follows: the model is a network of $N (N \geq 4)$ Markov processes with four states (active, idle, terminated, blocked) where the transition matrix of the $k^{th}$ processor is

$${}^kP = \begin{pmatrix} \frac{1}{N} & \frac{N-1}{N} & 0 & 0 \\ \beta_k & \frac{1}{N} & \theta_k & \frac{N-1}{N} - \beta_k - \theta_k \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

under the constraints

$$\begin{cases} (\beta_k, \theta_k) \in ]0,1[^2 \\ \beta_k + \theta_k < \dfrac{N-1}{N} \\ \sum_{k=1}^{N} \beta_k - 1 = 0 \\ \sum_{k=1}^{N} \theta_k - 1 = 0. \end{cases}$$

The ergodic class is the singleton {4} and the transient class is the set {1,2,3}. In the present problem, $\rho_k = (\beta_k, \theta_k)$, $k \in \{1, \ldots, N\}$. The matrix ${}^kW$; which is the restriction of ${}^kP$ to the transient states, is here

$${}^kW = \begin{pmatrix} \frac{1}{N} & \frac{N-1}{N} & 0 \\ \beta_k & \frac{1}{N} & \theta_k \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned} {}^k\Lambda &= (I - {}^kW)^{-1} \\ &= \frac{1}{\frac{N-1}{N} - \beta_k - \theta_k} \begin{pmatrix} 1 & 1 & \theta_k \\ \frac{\beta_k + \theta_k}{\frac{N-1}{N}} & \frac{1}{N} & \theta_k \\ 1 & 1 & \frac{N-1}{N} - \beta_k \end{pmatrix}. \end{aligned}$$

Here, the criterion of choice amounts to act on the sum $\sum_{k=1}^{N} \sum_{j=1}^{3} {}^kS_{2j}$ of the mean sojourn times in the states 1,2,3, respectively, when starting from the state 2, so as to delay the entering in state 4 as long as possible. This defines the guide function

$$F(\beta_1, \theta_1, \ldots, \beta_N, \theta_N) = \sum_{k=1}^{N} \frac{\beta_k + \frac{2N-1}{N}\theta_k + \frac{N-1}{N}}{(N-1)(\frac{N-1}{N} - \beta_k - \theta_k)}$$

to be maximized under the above constraints. As we shown in [4], there are no best solutions, but there exits a set $\mathcal{A}$ of advisable solutions, which complementary set is the following set $\mathcal{B}$ of bad solutions:

PROPOSITION 3: *$\mathcal{B}$ is the equivalent set (in the sense of the first order hierarchy) of solutions $\rho = (\beta_1, \theta_1, \ldots, \beta_k, \theta_k, \ldots, \beta_N, \theta_N)$, where*

$$\theta_k = -\frac{2N+1}{3N-2}\beta_k + \frac{5N-1}{N(3N-2)}.$$

*Fairness (viz. the solution $\rho$ where $(\beta_k, \theta_k) = (\frac{1}{N}, \frac{1}{N})$) belongs to $\mathcal{B}$.*

*Sketch of proof:* Indeed, let the following change of variables,

$$\begin{cases} u_k = \beta_k + \theta_k \\ t_k^2 = (N-1)(\theta_k + 2). \end{cases}$$

Substituting, the guide function of choice can be put in the form

$$\mathcal{F}(u_1, t_1, \ldots, u_N, t_N) = -\frac{N}{N-1} + \frac{1}{N-1} \sum\nolimits_{k=1}^{N} \frac{t_k^2}{(N-1) - N u_k}.$$

$\mathcal{F}$ has thus to be optimized under the constraints

$$\forall k \in \{1, \ldots, N\}, \quad \begin{cases} u_k < \dfrac{N-1}{N} \\ 2(N-1) < t_k^2 < 3(N-1), \end{cases}$$

and

$$\begin{cases} \sum_{k=1}^{N} u_k = 2 & (\mathcal{L}_1) \\ \sum_{k=1}^{N} t_k^2 = (2N+1)(N-1) & (\mathcal{L}_2) \end{cases}$$

The use of Lagrange multipliers, $\lambda$ for $\mathcal{L}_1$ and $\mu$ for $\mathcal{L}_2$, yields

$$\frac{\partial \mathcal{F}}{\partial u_k} + \lambda \frac{\partial \mathcal{L}_1}{\partial u_k} + \mu \frac{\partial \mathcal{L}_2}{\partial u_k} = \frac{N t_k^2}{[(N-1) - N u_k]^2} + \lambda = 0, \quad k \in \{1, \ldots, N\},$$
$$\frac{\partial \mathcal{F}}{\partial t_k} + \lambda \frac{\partial \mathcal{L}_1}{\partial t_k} + \mu \frac{\partial \mathcal{L}_2}{\partial t_k} = \frac{2 t_k^2}{[(N-1) - N u_k]} + 2\mu t_k = 0, \quad k \in \{1, \ldots, N\}.$$

This implies

$$\frac{\lambda}{\mu} = \frac{N t_k^2}{[(N-1) - N u_k]} = \frac{\sum_{k=1}^{N} t_k^2}{(N-1) - \sum_{k=1}^{N} u_k} = \frac{(2N+1)(N-1)}{N-3},$$

and hence,

$$u_k = -\frac{N-3}{(2N+1)(N-1)} t_k^2 + \frac{N-1}{N}, \quad k \in \{1, \ldots, N\}.$$

Thus,

$$\theta_k = -\frac{2N+1}{3N-2} \beta_k + \frac{5N-1}{N(3N-2)}.$$

Now, let $g = \sum_{k=1}^{N} g_k$, where $g_k = \frac{t_k^2}{(N-1)-Nu_k}$. The function $g_k$ is convex, since the Hessian

$$\nabla^2 g_k = \begin{pmatrix} \dfrac{2}{(N-1)-Nu_k} & \dfrac{2Nt_k}{[(N-1)-Nu_k]^2} \\ \dfrac{2Nt_k}{[(N-1)-Nu_k]^2} & \dfrac{2N^2t_k}{[(N-1)-Nu_k]^3} \end{pmatrix}$$

corresonds to a positive semi-definition quadratic form. Therefore, $g$ is also convex. Consequently, every $\rho \in \mathcal{B}$ minimizes function $F$, and $\rho$ is a bad solution in the sense of the first order hierarchy. ■

Let us examine now the second order hierarchy. When two solutions $\rho = (\beta_1, \theta_1, \ldots, \beta_N, \theta_N)$ and $\rho' = (\beta'_1, \theta'_1, \ldots, \beta'_N, \theta'_N)$ are found to be equivalent in the sense of the first order hierarchy (for example, solutions of $\mathcal{B}$), one has to turn to the second order hierarchy to decide between them. As the criterion of choice in the first order hierarchy introduces the mean sojourn times in all transient states, (starting from the state 2), formula (6) is used in the second order hierarchy. More precisely, for the $k$-th processor, the conditional variance of the sum of the sojourn random times in the states 1, 2 and 3 (from state 2) is given by the second term of the vector ${}^kV^*$ (given) in (6). Let ${}^kv_2^*$ denote this term. We consider the expression $\sum_{k=1}^{N} {}^kv_2^*$ as an element of comparison; in other words, the solution $\rho = (\beta_1, \theta_1, \ldots, \beta_N, \theta_N)$ is preferred to the solution $\rho' = (\beta'_1, \theta'_1, \ldots, \beta'_N, \theta'_N)$ if and only if

$$\sum_{k=1}^{N} {}^kv_2^* < \sum_{k=1}^{N} {}^kv_2'^*.$$

The analytic expression of ${}^kv_2^*$ is then useful. Here is the expression obtained by use of a programming software:

PROPOSITION 4

$${}^kv_2^* = \sum_{k=1}^{N} \frac{N^4(3\beta_k + 8\theta_k - \beta_k\theta_k) - N^3(\beta_k + 12\theta_k + \beta_k^2 + \theta_k^2 + 1)}{(1 + N(\beta_k + \theta_k - 1))^2(N-1)^2} + \frac{-N^2(3\beta_k - 4\theta_k + \beta_k\theta_k) + N(\beta_k - 5) + 2}{(1 + N(\beta_k + \theta_k - 1))^2(N-1)^2}.$$

As mentioned above, a simple programming software easily computes a numerical comparison for deciding between two equivalent solutions $\rho$ and $\rho'$.

## 5. CONCLUSION

Generating the first order hierarchy, our models allows a reasoned choice of solutions for the functioning of some distributed-algorithms. This choice by first order hierarchy, which possibly leads to equivalent solutions, is extended by second order hierarchy. Thus, we developped the necessary theoretical tools to the second order hierarchy, illustrated by the examples in Section 4.

Note that, the analytical comparison between two solutions in the second order hierarchy is not always possible. However, it could be easily done numerically by a programming software (*i.e.* Maple).

Other problems already studied by our model: namely the dining philosphers problem, the mutual exclusion problem, the multi-way-Rendezvous problem, could be easily extended by using the same method (with adaptations to the various context) formalized in the present article.

## REFERENCES


1. D. Brand and P. Zafiropulo, On communicating finite-state machines, *Journal of the ACM*, 1983, *30* (2), pp. 323-342.
2. A. Bui, Analyse du fonctionnement d'un type d'algorithmes distribués, *Rev. R. de Math. Pures et Appl.*, 1995, *40* (7-8), pp. 583-593.
3. A. Bui, *Etude analytique d'algorithmes distribués de routage*, Thèse de doctorat, Université Paris-7, 1994.
4. M. Bui, Tuning Distributed Control Algorithms for Optimal Functioning, *J. of Global Optimization*, 1992, *2*, pp. 177-199.
5. M. Bui, Résolution du Problème de Multi-Rendez-Vous à l'aide d'un modèle d'algorithme distribué, *RAIRO*, 1993, *27* (2), pp. 249-264.
6. T. L. Casavant and J. G. Kuhl, A communicating finite automata approach to modeling distributed computation and its application to distributed decision making, *IEEE Trans. on Computers*, 1990, *39* (5), pp. 628-639.
7. J. G. Kemeny and J. L. Snell, *Finite Markov chains*, Springer, 1983.
8. B. Rozoy, On distributed languages and models for distributed computation, *Res. Rept. No. 563*, University Paris-11, 1990.
9. K. M. Chandry and J. Misra, The Drinking Philosophers Problem, *ACM TOPLAS*, 1984, *6* (4), pp. 632-646.
10. G. Gao and G. V. Bochmann, An Virtual Ring Algorithm for the Distributed Implementation of Multi-Rendez-Vous, *Tech. Rept. No. 675, Dpt IRO*, Université de Montréal, 1990.
11. S. P. Rana and D. K. Banerji, An optimal distributed solution to the dining philosophers, *Res. Rept.*, Waystate University, Detroit, Michigan, 1987.
12. A. Segall, Distributed Network Protocols, *IEEE transactions on Information Theory*, 1983, *29* (1), pp. 23-25.
13. W. D. Tajibnapis A Correctness Proof of a Topology Information Maintenance Protocol for a Distributed Computer Network, *Communications of the ACM*, 1977, *20* (7), pp. 477-485.