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► **To cite this version:**

Ouerdia Arkoun. Sequential adaptive estimators in nonparametric autoregressive models. 2010. hal-00465587v1

HAL Id: hal-00465587

<https://hal.science/hal-00465587v1>

Preprint submitted on 27 Apr 2010 (v1), last revised 9 Nov 2010 (v2)

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Sequential adaptive estimators in nonparametric autoregressive models

Ouerdia Arkoun *

April 27, 2010

Abstract

We construct a sequential adaptive procedure for estimating the autoregressive function at a given point in nonparametric autoregression models with Gaussian noise. We make use of the sequential kernel estimators. The optimal adaptive convergence rate is given as well as the upper bound for the minimax risk.

Key words: Adaptive estimation, kernel estimator, minimax, nonparametric autoregression.

AMS (2000) Subject Classification : primary 62G07,62G08; secondary 62G20.

1 Introduction

Our problem is the following. Suppose we observe data from the model :

$$y_k = S(x_k)y_{k-1} + \xi_k, \quad 1 \leq k \leq n, \quad (1.1)$$

where $x_k = k/n$, $(\xi_k)_{k \in \{1, \dots, n\}}$ are random variables independent and identically distributed by standard Gaussian.

The model (1.1) is a generalization of autoregressive processes of the first order. In Dahlhaus (1996a) the process (1.1) is considered with the function S having a parametric form. Moreover, the paper of Dahlhaus (1996b) studies spectral properties of the stationary process (1.1) with the nonparametric function S .

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This paper deals with a nonparametric estimation of the autoregressive function S at a fixed point $z_0 \in]0; 1[$, when the smoothness of S is unknown. Assume that the function S belongs to a strong Hölder class but its regularity β is unknown. The goal is to find an adaptive convergence rate for adaptive which we construct a sequential adaptive estimator. Since β is unknown, this rate will differ here from the convergence rate obtained otherwise.

Many studies have been devoted to the search of optimal rate of convergence or an asymptotically efficient estimator when one or more parameters of the model are assumed to be unknown, in particular the regularity of the function to estimate. This case, called adaptive, was first led to results on the adaptive minimax convergence rate as in Efroïmovich and Pinsker (1984) for a model of white Gaussian noise, Härdle and Marron (1985) for a regression model and Efroïmovich (1985) for estimating a density.

Belitser (2000a) considers the model (1.1) with Lipschitz conditions, proposes a recursive estimator and study the problem non-adaptive of estimation. Using the quadratic risk, the author establishes the convergence rate.

In Galtchouk and Pergamenshchikov (2005b) the authors describe a method for the sequential problem of nonparametric estimation process with drift diffusion coefficient. In Lepskiï (1990) the author considers the adaptive problem in a model with white Gaussian noise, he estimate the signal belonging to a given Hölderian class $\Sigma(m+\alpha, L)$, where $m+\alpha$ and L are known constants. Fourdrinier, Konev and Pergamenshchikov (2009) propose a truncated sequential procedure which allows to consider the problem of estimating the parameter of the autoregressive process of first order with dependent noise.

Galtchouk and Pergamenshchikov (2001) obtain an adaptive minimax convergence rate, and adaptive estimator for the convergence rate of the drift diffusion, belonging to Hölderian class.

In this paper we consider the adaptive case with β unknown. Our construction is based on the method proposed in Lepskiï (1990) and Galtchouk and Pergamenshchikov (2001). The sequential procedure is the one found in Borisov and Konev (1977) but in the parametric case. The procedure of Lepskiï applies to estimators having the property that the tail of the distribution has the same asymptotic behavior of a Gaussian random variable, this procedure usually used in the i.i.d Gaussian case. For our problem the non sequential kernel estimator does not have the above property, however the sequential approach succeeds in the model (1.1) by performing the adaptive procedure of Lepskiï.

For the constant $H > 0$ such that for $0 \leq \alpha_H \leq 1$

$$\sum_{j=1}^{\tau_H-1} Q(u_j) y_{j-1}^2 + \alpha_H Q(u_{\tau_H}) y_{\tau_H-1}^2 = H,$$

where τ_H is the stopping time defined as follows

$$\tau_H = \inf\{1 \leq k \leq n : \sum_{j=1}^k Q(u_j) y_{j-1}^2 \geq H\}. \quad (1.2)$$

Note

$$A_k = \sum_{j=1}^k Q(u_j) y_{j-1}^2 \quad \text{with} \quad u_j = \frac{x_j - z_0}{h_n}.$$

Thus the kernel estimator is written as follows

$$S_{H,h_n}^*(z_0) = \frac{1}{H} \left(\sum_{j=1}^{\tau_H-1} Q(u_j) y_{j-1} y_j + \alpha_H Q(u_{\tau_H}) y_{\tau_H-1} y_{\tau_H} \right) \mathbf{1}_{(A_n \geq H)}, \quad (1.3)$$

where the kernel $Q(\cdot)$ is the indicator function on the interval $[-1; 1]$. Such an estimator is very convenient to calculate the quantity $\mathbf{E} |S_{H,h_n}^*(z_0) - S(z_0)|$.

We describe in detail the statement of the problem is given in section 2. In section 3 we prove the result of an asymptotic lower bound of adaptive minimax risk. Section 4 is devoted to prove the asymptotic upper bound of risk to the kernel estimator (1.3). Section 5 the appendix contains some technical results. Finally, we illustrate the obtained results by numerical examples.

2 Statement of the problem

The problem is to estimate the S at a fixed point $z_0 \in]0, 1[$, i.e. the value $S(z_0)$. For any estimate $\tilde{S}_n = \tilde{S}_n(z_0)$ (i.e. any measurable with respect to the observations $(y_k)_{1 \leq k \leq n}$ function), the risk is defined on the neighborhood $\mathcal{H}^{(\beta)}(z_0, K, \varepsilon)$ by

$$\mathcal{R}_n(\tilde{S}_n) = \sup_{\beta \in [\beta_*, \beta^*]} \sup_{S \in \mathcal{H}^{(\beta)}(z_0, K, \varepsilon)} N(\beta) \mathbf{E}_S |\tilde{S}_n(z_0) - S(z_0)|, \quad (2.1)$$

where $N(\beta) = \left(\frac{n}{\ln n} \right)^{\beta/(2\beta+1)}$ corresponds to the convergence rate of adaptive estimators on class $\mathcal{H}^{(\beta)}(z_0, K, \varepsilon)$ and \mathbf{E}_S is the expectation taken with respect to the distribution \mathbf{P}_S of the vector (y_1, \dots, y_n) in (1.1) corresponding to the function S .

We consider the model (1.1) where $S \in \mathbf{C}_1([0, 1], \mathbb{R})$ is the unknown function. Our goal to estimate the autoregressive function S at a fixed point z_0 . To obtain a stable (uniformly with respect to the function S) model (1.1) we assume that for some fixed $0 < \varepsilon < 1$ the unknown function S belongs to the *stability set*

$$\Gamma_\varepsilon = \{S \in \mathbf{C}_1(]0, 1], \mathbb{R}) : \|S\| \leq 1 - \varepsilon\}, \quad (2.2)$$

where $\|S\| = \sup_{0 < x \leq 1} |S(x)|$. Here $\mathbf{C}_1[0, 1]$ is the Banach space of continuously differentiable $]0, 1[\rightarrow \mathbb{R}$ functions. For fixed constants $K > 0$ and $0 < \beta \leq 1$ we define the corresponding *stable local Hölder class* at the point z_0 as

$$\mathcal{H}^{(\beta)}(z_0, K, \varepsilon) = \{S \in \Gamma_\varepsilon : \Omega^*(z_0, S) \leq K\} \quad (2.3)$$

with

$$\Omega^*(z_0, S) = \sup_{x \in [0, 1]} \frac{|S(x) - S(z_0)|}{|x - z_0|^\beta}.$$

The regularity β is supposed to be unknown but the interval $[\beta_*; \beta^*]$ is considered known.

First we give the lower bound for the minimax risk. We shows that with the convegence rate $N(\beta)$ the lower bound for the minimax risk is strictly positive.

Theorem 2.1. *The risk adaptive admits the following lower bound*

$$\liminf_{n \rightarrow \infty} \inf_{\tilde{S}_n} \mathcal{R}_n(\tilde{S}_n) \geq \frac{1}{4},$$

where the infimum is taken over all estimators \tilde{S}_n .

Now we give the upper bound for the maximal risk of the sequential adaptive estimator defined in (1.3). Taking into account that β is unknown, can not use this estimator because the bendwith h_n depends of β . That is why we partition the interval $[\beta_*; \beta^*]$ to follow a procedure of Lepskiï. Let us set

$$d_n = n / \ln n \quad \text{et} \quad h(\beta) = \left(\frac{1}{d_n} \right)^{\frac{1}{2\beta+1}}. \quad (2.4)$$

We define the grid on the interval $[\beta_*; \beta^*]$ with the points :

$$\beta_k = \beta_* + \frac{k}{m}(\beta^* - \beta_*), \quad k = 0, \dots, m \quad \text{with} \quad m = \lfloor \ln d_n \rfloor + 1. \quad (2.5)$$

We denote

$$N_k = N(\beta_k) \quad \text{et} \quad h_k = h(\beta_k),$$

and

$$\omega(h_j) = \max_{0 \leq k \leq j} \left(|S_{h_j}^* - S_{h_k}^*| - \frac{\lambda}{N_{k+1}} \right).$$

We define the optimal index of the bendwith as

$$\hat{k} = \inf \left\{ 0 \leq j \leq m : \omega(h_j) \geq \frac{\lambda}{N_j} \right\} - 1. \quad (2.6)$$

We note that $\omega(h_0) = -\lambda/N_1$, thus $\widehat{k} \geq 0$. The positive parameter λ is chosen as $\lambda > K + e\sqrt{4 + \frac{4}{2\beta_* + 1}}$.

The adaptive estimator is now defined as

$$\widehat{S}_n = S_{H, \widehat{h}}^* \quad \text{with} \quad \widehat{h} = h_{\widehat{k}}. \quad (2.7)$$

The following result gives the upper bound for the maximal risk of the sequential adaptive estimator defined above.

Theorem 2.2. *For all $0 < \varepsilon < 1$, we have*

$$\limsup_{n \rightarrow \infty} \mathcal{R}_n(\widehat{S}_n) < \infty, \quad (2.8)$$

thus \widehat{S}_n is an adaptive estimator to convergence rate.

3 Lower bound

We shows that with this appropriate rate $N(\beta)$, the lower bound of minimax risk is strictly positive.

Proof of Theorem 2.1

To simplify notations we denote $N(\beta_*) = N_*$, $N(\beta^*) = N^*$ and $h(\beta_*) = h_*$.

We denote

$$S(y) = \frac{1}{N_*} V\left(\frac{y - z_0}{h_*}\right),$$

where V is a function of C^∞ class with compact support $[-1, 1]$ as

$$\int_{-1}^1 V^2(u) du = \frac{\bar{\beta}}{2} \quad \text{with} \quad \bar{\beta} = \frac{\beta^* - \beta_*}{(2\beta^* + 1)(2\beta_* + 1)},$$

$V(0) = 1$ and $V(u) = 0$ for $|u| \geq 1$.

It is easy to show that for all real K , large enough, $S \in \mathcal{H}^{(\beta_*)}(z_0, K, \varepsilon)$. Note that for all S the measure \mathbf{P}_S is equivalent to the measure \mathbf{P}_0 , where \mathbf{P}_0 is the distribution of vector (y_1, \dots, y_n) in (1.1) corresponding to function $S_0 = 0$. It is easy to see that in this case the density of Radon-Nikodym can be written

$$\begin{aligned} \rho_n &:= \frac{d\mathbf{P}_0}{d\mathbf{P}_S}(y_1, \dots, y_n) \\ &= \exp \left\{ -\frac{1}{2} \sum_{k=1}^n (y_k^2 - (y_k - S(x_k)y_{k-1})^2) \right\} \\ &= \exp \left(-\varsigma_n \eta_n - \frac{1}{2} \varsigma_n^2 \right), \end{aligned}$$

with

$$\varsigma_n^2 = \frac{1}{d_n h_*} \sum_{k=1}^n V^2 \left(\frac{x_k - z_0}{h_*} \right) y_{k-1}^2 \quad \text{and} \quad \eta_n = \frac{1}{\sqrt{d_n h_*} \varsigma_n} \sum_{k=1}^n V \left(\frac{x_k - z_0}{h_*} \right) y_{k-1} \xi_k.$$

We define

$$\tau(S) = 1 - S^2(z_0). \quad (3.1)$$

According to Lemma 5.2 it comes

$$\begin{aligned} \mathbf{P}_S - \lim_{n \rightarrow \infty} \frac{d_n}{n} \varsigma_n^2 &= \mathbf{P}_S - \lim_{n \rightarrow \infty} \left(\frac{1}{n h_*} \sum_{k=1}^n V^2 \left(\frac{x_k - z_0}{h_*} \right) y_{k-1}^2 \right) \\ &= \mathbf{P}_S - \lim_{n \rightarrow \infty} \frac{1}{\tau(S)} \int_0^1 V^2 \left(\frac{x - z_0}{h_*} \right) dx \\ &= \int_{-1}^1 V^2(u) du = \frac{\bar{\beta}}{2} = \varsigma_*^2, \end{aligned}$$

because $\tau(S) = 1 - \frac{1}{N_*^2}$.

Furthermore, using a central limit theorem for martingales (cf. Lemma 5.6), it is easy to see that under the measure \mathbf{P}_S

$$\eta_n \implies \mathcal{N}(0, 1) \quad \text{when} \quad n \rightarrow \infty.$$

In fact, we can rewrite η_n as follows :

$$\eta_n = \sqrt{\frac{n}{d_n}} \frac{\varsigma_*}{\varsigma_n} \sum_{k=1}^n u_{k,n},$$

with

$$u_{k,n} = \frac{1}{\varsigma_* \sqrt{n h_*}} V \left(\frac{x_k - z_0}{h_*} \right) y_{k-1} \xi_k.$$

Let us consider the first condition of lemma 5.6. To verify it, suffices to show

$$\mathbf{E}_S \sum_{k=1}^n \mathbf{E}_S(u_{k,n}^2 \mathbf{1}_{(|u_{k,n}| > \varepsilon)} | \mathcal{F}_{k-1,n}) \xrightarrow[n \rightarrow \infty]{} 0.$$

We have

$$\begin{aligned} \mathbf{E}_S \sum_{k=1}^n \mathbf{E}_S(u_{k,n}^2 \mathbf{1}_{(|u_{k,n}| > \varepsilon)} | \mathcal{F}_{k-1,n}) &= \sum_{k=1}^n \mathbf{E}_S(u_{k,n}^2 \mathbf{1}_{(|u_{k,n}| > \varepsilon)}) \\ &= \frac{1}{\varsigma_*^2 n h_*} \sum_{k=k_*}^{k=k_*^*} V^2 \left(\frac{x_k - z_0}{h_*} \right) \mathbf{E}_S(y_{k-1}^2 \xi_k^2 \mathbf{1}_{(|u_{k,n}| > \varepsilon)}), \end{aligned} \quad (3.2)$$

where

$$k_* = [nz_0 - nh_n] + 1 \quad \text{and} \quad k^* = [nz_0 + nh_n], \quad (3.3)$$

with

$$\begin{aligned} \mathbf{E}_S(y_{k-1}^2 \xi_k^2 \mathbf{1}_{(|u_{k,n}| > \varepsilon)}) &\leq \sqrt{\mathbf{E}_S y_{k-1}^4 \mathbf{E}_S \xi_k^4} \sqrt{\mathbf{P}_S(|u_{k,n}| > \varepsilon)} \\ &\leq \sqrt{\mathbf{E}_S y_{k-1}^4 \mathbf{E}_S \xi_k^4} \sqrt{\frac{1}{\varepsilon^2} \mathbf{E}_S u_{k,n}^2} \\ &\leq C_1 \sqrt{\frac{\mathbf{E}_S y_{k-1}^2 \xi_k^2}{nh_*}} \leq \frac{C_2}{\sqrt{nh_*}}, \end{aligned}$$

where C_1 and C_2 are constants independent of n . So the term (3.2) is bounded by

$$\mathbf{E} \sum_{k=1}^n \mathbf{E}(u_{k,n}^2 \mathbf{1}_{(|u_{k,n}| > \varepsilon)} | \mathcal{F}_{k-1,n}) \leq \frac{C_3}{nh_*} \sum_{k=k_*}^{k^*} \frac{1}{\sqrt{nh_*}},$$

where C_3 is a new constant, the latter term tends to zero as $n \rightarrow \infty$.

The second condition, is easily checked by

$$\begin{aligned} \sum_{k=1}^n \mathbf{E}_S(u_{k,n}^2 | \mathcal{F}_{k-1,n}) &= \frac{1}{\zeta_*^2 n h_*} \sum_{k=1}^n V^2 \left(\frac{x_k - z_0}{h_*} \right) \mathbf{E}(y_{k-1}^2 \xi_k^2 | \mathcal{F}_{k-1,n}) \\ &= \frac{1}{\zeta_*^2 n h_*} \sum_{k=1}^n V^2 \left(\frac{x_k - z_0}{h_*} \right) y_{k-1}^2 \\ &= \frac{d_n \zeta_n^2}{n \zeta_*^2} \xrightarrow{n \rightarrow \infty} 1. \end{aligned}$$

Let us denote $\theta_n = N_* |\tilde{S}_n|$, we have

$$\begin{aligned} \mathcal{R}_n(\tilde{S}_n) &\geq \max \left(\mathbf{E}_{S_0} N_* |\tilde{S}_n|, \mathbf{E}_S N_* |\tilde{S}_n - S(z_0)| \right) \\ &= \max \left(\mathbf{E}_{S_0} \frac{N_*}{N_*} |\theta_n|, \mathbf{E}_S |1 - \theta_n| \right) \\ &\geq \frac{1}{2} \mathbf{E}_S \left(\frac{N_*}{N_*} |\theta_n| \frac{d\mathbf{P}_0}{d\mathbf{P}_S}(y) + |1 - \theta_n| \right) \end{aligned} \quad (3.4)$$

We set $\gamma_n = \frac{N^*}{N_*}$, we can bounded (3.4) from the following :

$$\mathcal{R}_n(\tilde{S}_n) \geq \frac{1}{2} \mathbf{E}_S (\gamma_n \rho_n |\theta_n| + |1 - \theta_n|).$$

Let $B_n = \{\eta_n \leq 0\}$ and $C_n = \{\frac{d_n}{n} \varsigma_n^2 < \bar{\beta}\}$. Clearly, when $B_n \cap C_n$ is realized, we have

$$\gamma_n \rho_n \geq \exp\{\bar{\beta} \ln d_n - \frac{\bar{\beta}}{2} \frac{n}{d_n}\},$$

this last expression tends to ∞ as n tends to ∞ . This means that for n sufficiently large

$$\begin{aligned} \mathcal{R}_n(\tilde{S}_n) &\geq \frac{1}{2} \mathbf{E}_S \mathbf{1}_{B_n \cap C_n} (\gamma_n \rho_n |\theta_n| + |1 - \theta_n|) \\ &\geq \frac{1}{2} \mathbf{E}_S \mathbf{1}_{B_n \cap C_n} (|\theta_n| + 1 - |\theta_n|) \\ &= \frac{1}{2} \mathbf{P}_S(B_n \cap C_n). \end{aligned} \tag{3.5}$$

Since,

$$\begin{aligned} \mathbf{P}_S(B_n \cap C_n) &= \mathbf{P}_S(B_n) - \mathbf{P}_S(B_n \cap C_n^c), \\ \mathbf{P}_S(B_n \cap C_n^c) &\leq \mathbf{P}_S(C_n^c) = \mathbf{P}_S\left(\frac{d_n}{n} \varsigma_n \geq \bar{\beta}\right) \end{aligned}$$

and

$$\frac{d_n}{n} \varsigma_n \xrightarrow[n \rightarrow \infty]{\mathbf{P}_S} \frac{\bar{\beta}}{2},$$

hence

$$\mathbf{P}_S(C_n^c) \xrightarrow[n \rightarrow \infty]{} 0.$$

As $\mathbf{P}_S(B_n) = 1/2$, we deduce that $\mathbf{P}_S(B_n \cap C_n) \xrightarrow[n \rightarrow \infty]{} 1/2$.

Passing to the limit as $n \rightarrow \infty$ in (3.5), we obtain the Theorem. \square

4 Sequential adaptive estimation (upper bound)

Proof of Theorem 2.2

We proceed following a method based on sequential analysis. First, we rewrite the estimation error as follows:

$$S_{H,h}^*(z_0) - S(z_0) = -S(z_0) \mathbf{1}_{(A_n < H)} + B_H(h) \mathbf{1}_{(A_n \geq H)} + \frac{1}{\sqrt{H}} \zeta_H(h) \mathbf{1}_{(A_n \geq H)}, \tag{4.1}$$

where

$$B_H(h) = \frac{1}{H} \left(\sum_{j=1}^{\tau_H-1} Q(u_j) (S(x_j) - S(z_0)) y_{j-1}^2 + \alpha_H Q(u_{\tau_H}) (S(x_{\tau_H}) - S(z_0)) y_{\tau_H-1}^2 \right)$$

and

$$\zeta_H(h) = \frac{1}{\sqrt{H}} \left(\sum_{j=1}^{\tau_H-1} Q(u_j) y_{j-1} \xi_j + \alpha_H Q(u_{\tau_H}) y_{\tau_H-1} \xi_{\tau_H} \right).$$

Note that the first term in the right quantity of (4.1) is studied in Lemma 5.3. We can show directly that for every $S \in \mathcal{H}^{(\beta)}(z_0, K, \varepsilon)$

$$|B_H(h)| \leq Kh^\beta \quad (4.2)$$

and also, using Lemma 5.5 we have

$$\sup_{n \geq 1} \sup_{h_* \leq h \leq h^*} \mathbf{E}_S |\zeta_H(h)| < \infty, \quad (4.3)$$

where $h_* = h(\beta_*)$ and $h^* = h(\beta^*)$. Now we choose $H = nh$ and

$$\iota = \inf\{k \geq 0 : \beta_k \geq \beta\} - 1.$$

This means

$$\beta_\iota < \beta \leq \beta_{\iota+1} \quad \text{and} \quad h_\iota < h(\beta) \leq h_{\iota+1}.$$

In the sequel, we note $S_h^*(z_0) = S_{H,h}^*(z_0)$. We have now

$$|S_{h_\iota}^*(z_0) - S(z_0)| \leq \mathbf{1}_{(A_n(h_\iota) < nh_\iota)} + K(h(\beta_\iota))^\beta + \frac{1}{\sqrt{nh_\iota}} |\zeta_H(h_\iota)|$$

and

$$|S_{h_{\iota-1}}^*(z_0) - S(z_0)| \leq \mathbf{1}_{(A_n(h_{\iota-1}) < nh_{\iota-1})} + K(h(\beta_{\iota-1}))^\beta + \frac{1}{\sqrt{nh_{\iota-1}}} |\zeta_H(h_{\iota-1})|.$$

Inequality (4.3) implies

$$\limsup_{n \rightarrow \infty} \sup_{\beta_* \leq \beta \leq \beta^*} N(\beta) \sup_{S \in \mathcal{H}^{(\beta)}(z_0, K, \varepsilon)} \mathbf{E}_S \varpi(\iota, z_0) < \infty, \quad (4.4)$$

where

$$\varpi(\iota, z_0) = |S_{h_{\iota-1}}^*(z_0) - S(z_0)| + |S_{h_\iota}^*(z_0) - S(z_0)|.$$

Now considering the estimator \widehat{S}_n , we can write

$$|\widehat{S}_n(z_0) - S(z_0)| \leq I_1 + I_2 + \varpi(\iota, z_0), \quad (4.5)$$

where

$$I_1 = |\widehat{S}_n(z_0) - S(z_0)| \mathbf{1}_{\{\widehat{k} \geq \iota+1\}} \quad \text{and} \quad I_2 = |\widehat{S}_n(z_0) - S(z_0)| \mathbf{1}_{\{\widehat{k} \leq \iota-2\}}.$$

We focus now on the first term in this inequality. We have

$$|\widehat{S}_n(z_0) - S(z_0)| \mathbf{1}_{\{\widehat{k} \geq \iota+1\}} \leq |S_{\widehat{h}}^*(z_0) - S_{h_\iota}^*(z_0)| \mathbf{1}_{\{\widehat{k} \geq \iota+1\}} + |S_{h_\iota}^*(z_0) - S(z_0)| \mathbf{1}_{\{\widehat{k} \geq \iota+1\}}.$$

Moreover,

$$\begin{aligned} |S_{\widehat{h}}^*(z_0) - S_{h_\iota}^*(z_0)| \mathbf{1}_{\{\widehat{k} \geq \iota+1\}} &\leq \omega(h_{\widehat{k}}) \mathbf{1}_{\{\widehat{k} \geq \iota+1\}} + \frac{\lambda}{N_{\iota+1}} \\ &\leq \frac{\lambda}{N_{\widehat{k}}} \mathbf{1}_{\{\widehat{k} \geq \iota+1\}} + \frac{\lambda}{N_{\iota+1}} \leq \frac{2\lambda}{N_{\iota+1}} \leq \frac{2\lambda}{N(\beta)}. \end{aligned}$$

This implies directly

$$\limsup_{n \rightarrow \infty} \sup_{\beta_* \leq \beta \leq \beta^*} N(\beta) \sup_{S \in \mathcal{H}^{(\beta)}(z_0, K, \varepsilon)} \mathbf{E}_S I_1 < \infty. \quad (4.6)$$

We establish now a bound for the second term of (4.5):

$$I_2 \leq \left(\mathbf{1}_{(A_n(h_{\widehat{k}}) < nh_{\widehat{k}})} + K(h(\beta_{\widehat{k}}))^\beta + \frac{1}{\sqrt{nh_{\widehat{k}}}} \zeta^* \right) \mathbf{1}_{\{\widehat{k} \leq \iota-2\}},$$

where

$$\zeta^* = \max_{1 \leq j \leq m} |\zeta_{H_j}(h_j)|. \quad (4.7)$$

Note that

$$\{\widehat{k} \leq \iota - 2\} = \bigcup_{j=1}^{\iota-1} \{\omega(h_j) \geq \lambda/N_j\}.$$

Moreover,

$$\begin{aligned} \{\omega(h_j) \geq \lambda/N_j\} &= \bigcup_{l=0}^{j-1} \left\{ |S_{h_j}^*(z_0) - S_{h_l}^*(z_0)| \geq \lambda/N_j + \lambda/N_{l+1} \right\} \\ &\subseteq \bigcup_{l=0}^{j-1} \left(\{|S_{h_j}^*(z_0) - S(z_0)| \geq \lambda/N_j\} \cup \{|S_{h_l}^*(z_0) - S(z_0)| \geq \lambda/N_{l+1}\} \right). \end{aligned} \quad (4.8)$$

We denote for $j \leq \iota - 1$

$$N_j(h_j)^\beta \leq \exp\left\{-\frac{\ln d_n}{(2\beta^* + 1)m}\right\} \leq 1.$$

For $l \leq \iota - 1$

$$N_{l+1}(h_l)^\beta \leq \exp\left\{-\frac{\ln d_n}{(2\beta^* + 1)m}\right\} \leq 1$$

and

$$\frac{N_l}{N_{l+1}} \geq \exp\left\{-\frac{\ln d_n}{m}\right\} = e^{-1}.$$

In the first term of (4.8), by Lemma 5.2 we prove that for n sufficiently large and for $\lambda > K + e\sqrt{4 + \frac{4}{2\beta_* + 1}}$ we have

$$\begin{aligned} \{|S_{h_j}^*(z_0) - S(z_0)| \geq \lambda/N_j\} &\subseteq \left\{ K(h_j)^\beta + \frac{1}{\sqrt{nh_j}} |\zeta_n(h_j)| \geq \lambda/N_j \right\} \\ &\subseteq \left\{ |\zeta_n(h_j)| \geq \sqrt{nh_j} \left(\frac{\lambda}{N_j} - K(h_j)^\beta \right) \right\}. \end{aligned}$$

We just have to note that $(1/d_n)^{\beta/(2\beta+1)} \sqrt{nh} = \sqrt{n/d_n}$ and the last inclusion becomes

$$\{|S_{h_j}^*(z_0) - S(z_0)| \geq \lambda/N_j\} \subseteq \left\{ |\zeta_n(h_j)| \geq (\lambda - K) \sqrt{\frac{n}{d_n}} \right\}.$$

Similarly for the second term in (4.8) we obtain

$$\{|S_{h_l}^*(z_0) - S(z_0)| \geq \lambda/N_{l+1}\} \subseteq \left\{ |\zeta_n(h_l)| \geq (\lambda - K)/e \sqrt{\frac{n}{d_n}} \right\}.$$

Finally,

$$\{\widehat{k} \leq \iota - 2\} \subseteq \{\zeta^* \geq \lambda_1 \sqrt{n/d_n}\},$$

with $\lambda_1 = (\lambda - K)/e$. So

$$I_2 \leq \mathbf{1}_{(A_n(h_{\widehat{k}}) < nh_{\widehat{k}})} + \frac{K}{N(\beta)} + \frac{1}{\sqrt{nh_*}} \zeta^* \mathbf{1}_{\{\zeta^* \geq \lambda_1 \sqrt{n/d_n}\}}. \quad (4.9)$$

Using Lemma 5.2 for $t \geq 2$, one can easily estimate the first term of this inequality by

$$\begin{aligned} \mathbf{P}_S(A_n(h_{\widehat{k}}) < nh_{\widehat{k}}) &= \sum_{l=1}^m \mathbf{P}_S(A_n(h_l) < nh_l, \widehat{k} = l) \\ &\leq \sum_{l=1}^m \mathbf{P}_S(A_n(h_l) < nh_l) \\ &= \sum_{l=1}^m \mathbf{P}_S \left(\frac{1}{\tau(S)} \int_{-1}^1 Q(u) du + \Delta_n(Q, h_l) < 1 \right) \\ &= \sum_{l=1}^m \mathbf{P}_S \left(\Delta_n(Q, h_l) < 1 - \frac{2}{\tau(S)} \right) \\ &\leq \sum_{l=1}^m \mathbf{P}_S (|\Delta_n(Q, h_l)| > 1) \\ &\leq \sum_{l=1}^m \mathbf{E}_S \Delta_n^{2t}(Q, h_l) \leq ([\ln d_n] + 1) C_1 R^{2t} (h^*)^{2t\beta}. \end{aligned}$$

Consider now the last term on the right side of inequality (4.9):

$$\begin{aligned}
\mathbf{E}_S \zeta^* \mathbf{1}_{\{\zeta^* \geq \lambda_1 \sqrt{\ln n}\}} &= \int_0^{+\infty} \mathbf{P}_S(\zeta^* \mathbf{1}_{\{\zeta^* \geq \lambda_1 \sqrt{\ln n}\}} \geq z) dz \\
&= \int_0^{+\infty} \mathbf{P}_S(\zeta^* \geq z, \zeta^* \geq \lambda_1 \sqrt{\ln n}) dz \\
&= \lambda_1 \sqrt{\ln n} \mathbf{P}_S(\zeta^* \geq \lambda_1 \sqrt{\ln n}) + \int_{\lambda_1 \sqrt{\ln n}}^{+\infty} \mathbf{P}_S(\zeta^* \geq z) dz.
\end{aligned}$$

Using (4.7) and Lemma 5.5, we have

$$\begin{aligned}
\mathbf{P}_S(\zeta^* \geq z) &= \mathbf{P}_S(\max_{1 \leq j \leq m} |\zeta_n(h_j)| \geq z) \\
&= \sum_{j=1}^m \mathbf{P}_S(|\zeta_n(h_j)| \geq z) \\
&\leq 2m e^{-z^2/8}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{E}_S \zeta^* \mathbf{1}_{\{\zeta^* \geq \lambda_1 \sqrt{\ln n}\}} &\leq 2m \lambda_1 \sqrt{\ln n} e^{-\frac{1}{8} \lambda_1^2 \ln n} + 2m \int_{\lambda_1 \sqrt{\ln n}}^{+\infty} e^{-z^2/8} dz \\
&\leq 2m \lambda_1 \sqrt{\ln n} e^{-\frac{1}{8} \lambda_1^2 \ln n} + 2m \int_{\lambda_1 \sqrt{\ln n}}^{+\infty} z e^{-z^2/8} dz \\
&\leq (\lambda_1 \sqrt{\ln n} + 4) 2m n^{-\lambda_1^2/8}.
\end{aligned}$$

This implies the inequality (2.8) and Theorem (2.2). \square

5 Appendix

In this section we study the properties of stationary processes in the model (1.1).

Lemma 5.1. *For all $t \in \mathbb{N}^*$ and $0 < \varepsilon < 1$, the random variables in (1.1) satisfy the following inequality*

$$r^* = \sup_{n \geq 1} \sup_{0 \leq k \leq n} \sup_{S \in \Gamma_\varepsilon} \mathbf{E}_S y_k^{2t} < \infty. \quad (5.1)$$

Proof.

Assume that $y_0 = 0$. The model (1.1) become

$$y_k = \sum_{i=1}^k \prod_{l=i+1}^k S(x_l) \xi_i,$$

deduced with $S \in \Gamma_\varepsilon$ and for all $1 \leq k \leq n$

$$y_k^{2t} \leq \left(\sum_{j=1}^k (1-\varepsilon)^{k-j} |\xi_j| \right)^{2t}.$$

Moreover, the Hölder inequality with $p = 2t$,

$$\begin{aligned} y_k^{2t} &\leq \left(\sum_{j=1}^k (1-\varepsilon)^{k-j} \right)^{2t-1} \left(\sum_{j=1}^k (1-\varepsilon)^{k-j} \xi_j^{2t} \right) \\ &\leq \left(\frac{1}{\varepsilon} \right)^{2t-1} \left(\sum_{j=1}^k (1-\varepsilon)^{k-j} \xi_j^{2t} \right). \end{aligned}$$

It follows that

$$\mathbf{E}_S y_k^{2t} \leq \frac{(2t)!}{2^t t!} \left(\frac{1}{\varepsilon} \right)^{2t}.$$

Hence Lemma 5.1. \square

We introduce the following notation

$$\Delta_n(f, h) = \frac{1}{nh} \sum_{k=1}^n f(u_k) y_{k-1}^2 - \frac{1}{\tau(S)} \int_{-1}^1 f(u) du.$$

Lemma 5.2. *Let f be a function twice continuously differentiable in $[-1, 1]$, as $f(u) = 0$ for $|u| > 1$. Then for all $t \in \mathbb{N}^*$*

$$\limsup_{n \rightarrow \infty} \sup_{h_* \leq h \leq h^*} \sup_{R > 0} \frac{1}{R^{2t} h^{2t\beta}} \sup_{\|f\|_1 \leq R} \sup_{S \in \mathcal{H}^\beta(z_0, K, \varepsilon)} \mathbf{E}_S \Delta_n^{2t}(f, h) \leq C_1, \quad (5.2)$$

where $\|f\|_1 = \|f\| + \|\dot{f}\|$ and $C_1 = 2^{4t} K^{2t} (r^*)^2$.

Proof. First rewrite

$$\sum_{k=1}^n f(u_k) y_{k-1}^2 = T_n + a_n, \quad (5.3)$$

where

$$T_n = \sum_{k=k_*}^{k^*} f(u_k) y_k^2 \quad \text{and} \quad a_n = \sum_{k=k_*}^{k^*} (f(u_k) - f(u_{k-1})) y_{k-1}^2 - f(u_{k^*}) y_{k^*}^2,$$

integers k^* and k_* are defined in (3.3). Substituting into model (1.1) gives us

$$T_n = I_n(f) + \sum_{k=k_*}^{k^*} f(u_k) S^2(x_k) y_{k-1}^2 + M_n,$$

where

$$I_n(f) = \sum_{k=k_*}^{k^*} f(u_k) \quad \text{and} \quad M_n = \sum_{k=k_*}^{k^*} f(u_k) (2S(x_k) y_{k-1} \xi_k + \eta_k)$$

with $\eta_k = \xi_k^2 - 1$. Noting

$$C_n = \sum_{k=k_*}^{k^*} (S^2(x_k) - S^2(z_0)) f(u_k) y_{k-1}^2 \quad \text{and} \quad D_n = \sum_{k=k_*}^{k^*} f(u_k) (y_{k-1}^2 - y_k^2),$$

we obtain

$$\frac{1}{nh} T_n = \frac{1}{\tau(S)} \frac{I_n(f)}{nh} + \frac{1}{\tau(S)} \frac{H_n}{nh} \quad (5.4)$$

with $H_n = M_n + C_n + S^2(z_0) D_n$. Moreover, it is easy to see that

$$\begin{aligned} \frac{I_n(f)}{nh} &= \int_{-1}^1 f(t) dt + \sum_{k=k_*}^{k^*} \int_{u_{k-1}}^{u_k} f(u_k) dt - \int_{-1}^1 f(t) dt \\ &= \sum_{k=k_*}^{k^*} \int_{u_{k-1}}^{u_k} (f(u_k) - f(t)) dt + \int_{u_{k_*-1}}^{u_{k^*}} f(t) dt - \int_{-1}^1 f(t) dt. \end{aligned}$$

Recall that $\|f\| + \|\dot{f}\| \leq R$. Then

$$\left| \frac{1}{nh} \sum_{k=k_*}^{k^*} f(u_k) - \int_{-1}^1 f(t) dt \right| \leq \frac{R}{nh}.$$

The definition (3.1) implies that for any $S \in \Gamma_\varepsilon$

$$\varepsilon^2 \leq \tau(S) \leq 1. \quad (5.5)$$

Taking into account (5.3) and the lower bound for $\tau(S)$ given in (5.5), we prove that

$$\left| \frac{T_n}{nh} - \frac{1}{\tau(S)} \int_{-1}^1 f(t) dt \right| \leq \frac{1}{\varepsilon^2} \left(\frac{R}{nh} + \frac{M_n}{nh} + \frac{C_n}{nh} + \frac{D_n}{nh} \right). \quad (5.6)$$

We note that M_n is the last term of the martingale $(G_j)_{k_* \leq j \leq k^*}$ of square integrable, where

$$G_j = \sum_{k=k_*}^j f(u_k) (2S(x_k) y_{k-1} \xi_k + \eta_k).$$

So by applying the Burkholder inequality, it comes

$$\begin{aligned} \mathbf{E}_S \left(\frac{1}{nh} M_n \right)^{2t} &\leq \frac{A_{2t}^{2t}}{(nh)^{2t}} \mathbf{E}_S \left(\sum_{k=k_*}^{k^*} f^2(u_k) (2S(x_k) y_{k-1} \xi_k + \eta_k)^2 \right)^t \\ &\leq A_{2t}^{2t} \frac{R^t}{(nh)^{t+1}} \mathbf{E}_S \sum_{k=k_*}^{k^*} (2S(x_k) y_{k-1} \xi_k + \eta_k)^{2t} \\ &\leq \frac{R^t}{(nh)^t} 2^{4t-2} A_{2t}^{2t} \left(\frac{(2t)!}{2^{2t} t!} \left(2r^* + \frac{(2t)!}{2^{2t} t!} \right) + 1 \right) \end{aligned}$$

where $A_{2t} = 18(2t)^{3/2}/(2t-1)^{1/2}$ and r^* is given in (5.1). Since, $|S(x_k) - S(z_0)| \leq K|x_k - z_0|^\beta$ for all $S \in \mathcal{H}^\beta(z_0, K, \varepsilon)$ and applying then the Hölder inequality for $p = 2t$ and $q = 2t/(2t-1)$, we obtain

$$\begin{aligned} \frac{1}{(nh)^{2t}} \mathbf{E}_S C_n^{2t} &\leq \frac{1}{(nh)^{2t}} \left(\sum_{k=k_*}^{k^*} |(S^2(x_k) - S^2(z_0))|^{q \mathbf{1}_{|u_k| \leq 1}} \right)^{2t/q} \sum_{k=k_*}^{k^*} f^{2t}(u_k) \mathbf{E}_S y_{k-1}^{4t} \\ &\leq 2^{4t} R^{2t} K^{2t} (r^*)^2 h^{2t\beta}. \end{aligned}$$

Now consider the last term on the right side of inequality (5.4). Can be written D_n as

$$D_n = \sum_{k=k_*}^{k^*} ((f(u_k) - f(u_{k-1})) y_{k-1}^2 + f(u_{k_*-1}) y_{k_*-1}^2 - f(u_{k^*}) y_{k^*}^2).$$

Since $\|f\| + \|\dot{f}\| \leq R$ we have

$$\mathbf{E}_S D_n^{2t} \leq 2^{4t-2} R^{2t} \mathbf{E}_S \left(\frac{1}{nh} \sum_{k=k_*}^{k^*} y_{k-1}^{4t} + y_{k^*}^{4t} + y_{k_*-1}^{4t} \right) \leq 2^{4t} R^{2t} (r^*)^2.$$

Similarly we find a bound for the second term on the right of the expression (5.2). Hence Lemma 5.2. \square

Lemma 5.3. *For any $t \geq 1$ the stopping time τ_H defined in (1.2) satisfies the following properties, for $H = nh$*

$$\mathbf{P}_S(\tau_H > n) \leq C_1 (Rh)^{2t\beta},$$

where C_1 is defined in (5.2).

Proof.

Taking into account that $\tau(S) \leq 1$ we obtain

$$\begin{aligned} \mathbf{P}_S(\tau_H > n) &= \mathbf{P}_S\left(\frac{1}{nh} \sum_{k=1}^n Q(u_k) y_{k-1}^2 < \frac{H}{nh}\right) \\ &= \mathbf{P}_S\left(\frac{1}{\tau(S)} \int_{-1}^1 Q(u) du + \Delta_n(Q, h) < 1\right) \\ &= \mathbf{P}_S\left(\Delta_n(Q, h) < 1 - \frac{2}{\tau(S)}\right) \\ &\leq \mathbf{P}_S(|\Delta_n(Q, h)| > 1) \leq \mathbf{E}_S \Delta_n^{2t}(Q, h) \leq C_1 R^{2t} h^{2t\beta}, \end{aligned}$$

this last inequality comes from Lemma 5.2 \square

To prove Lemma 5.5, we need the following lemma proved in Liptser and Shiryaev (1978) p.234-235.

Lemma 5.4. Consider the Wiener process $W = (W_t, \mathcal{F}_t), t \geq 0$, given a probability space and is a random process $f = (f_t, \mathcal{F}_t), t \geq 0$, such that :

$$(1) \quad P \left(\int_0^T f_t^2 dt < \infty \right) = 1, \quad 0 < T < \infty,$$

$$(2) \quad P \left(\int_0^\infty f_t^2 dt = \infty \right) = 1.$$

Then the random process $z = (z_s, \Gamma_s), s \geq 0$, with $z_s = \int_0^{\tau_s} f_t dW_t, \Gamma_s = \mathcal{F}_{\tau_s}$, where $\tau_s = \inf(t : \int_0^t f_u^2 du > s)$, is the Wiener process.

Lemma 5.5. For all $z \geq 2$ and $H > 0$,

$$\mathbf{P}_S(|\zeta_H(h)| > z) \leq 2e^{-z^2/8}. \quad (5.7)$$

Proof. The Brownian motion $(W_t)_{t \geq 0}$ is a stochastic process whose disjoint increments are independent as $W_{t+s} - W_t$ follows a Gaussian distribution with zero mean and variance s . So in our case we can write

$$\xi_k = W_k - W_{k-1} \sim \mathcal{N}(0, 1).$$

We recall that

$$\zeta_H = \frac{1}{\sqrt{H}} \left(\sum_{j=1}^{\tau_H-1} Q(u_j) y_{j-1} \xi_j + \alpha_H Q(u_{\tau_H}) y_{\tau_H-1} \xi_{\tau_H} \right) \mathbf{1}_{(A_n \geq H)}.$$

So

$$\mathbf{P}_S(|\zeta_H| > z \mathbf{1}_{(A_n \geq H)}) = \mathbf{P}_S(|\zeta_H| > z, \mathbf{1}_{(A_n \geq H)}) = \mathbf{P}_S(|\tilde{\zeta}_H| > z, \mathbf{1}_{(A_n \geq H)}),$$

where

$$\tilde{\zeta}_H(h) = \frac{1}{\sqrt{H}} \left(\sum_{k=1}^{\tilde{\tau}_H-1} \delta_k \xi_k + \tilde{\alpha}_{\tilde{\tau}_H} \delta_{\tilde{\tau}_H} \xi_{\tilde{\tau}_H} \right)$$

and

$$\sum_{k=1}^{\tilde{\tau}_H-1} \delta_k^2 + \tilde{\alpha}_{\tilde{\tau}_H}^2 \delta_{\tilde{\tau}_H}^2 = H,$$

with $\delta_k = Q(u_k) y_{k-1} \mathbf{1}_{(k \leq k^*)} + \mathbf{1}_{(k > k^*)}$ and

$$\tilde{\tau}_H = \inf \left\{ k \geq 1 : \sum_{j=1}^k \delta_j^2 \geq H \right\}.$$

One can see that

$$\mathbf{P}_S(|\tilde{\zeta}_H(h)| > z) = \mathbf{P}_S \left(\frac{1}{\sqrt{H}} \left| \int_0^{\tilde{\tau}_H} f_t dW_t \right| > z \right),$$

where

$$f_t = \sum_{j=1}^{\infty} \delta'_j \mathbf{1}_{[j-1, j]}(t)$$

with

$$\delta'_j = \begin{cases} \delta_j & j < \tilde{\tau}_H \\ \tilde{\alpha}_{\tilde{\tau}_H} \delta_{\tilde{\tau}_H} & j = \tilde{\tau}_H \\ 0 & j > \tilde{\tau}_H. \end{cases} \quad (5.8)$$

Indeed,

$$\begin{aligned} \int_0^{\tilde{\tau}_H} f_t \, dw_t &= \sum_{j=1}^{\tilde{\tau}_H} \int_{j-1}^j f_t \, dw_t = \sum_{j=1}^{\tilde{\tau}_H} \delta'_j [w_j - w_{j-1}] \\ &= \sum_{j=1}^{\tilde{\tau}_H} \delta'_j \xi_j = \sum_{j=1}^{\tilde{\tau}_H-1} \delta_j \xi_j + \tilde{\alpha}_{\tilde{\tau}_H} \delta_{\tilde{\tau}_H} \xi_{\tilde{\tau}_H}. \end{aligned}$$

We set

$$g_t = \sum_{j=1}^{\infty} \delta''_j \mathbf{1}_{[j-1, j]}(t)$$

with

$$\delta''_j = \begin{cases} \delta_j & j < \tilde{\tau}_H \\ \sqrt{\tilde{\alpha}_{\tilde{\tau}_H}} \delta_{\tilde{\tau}_H} & j = \tilde{\tau}_H \\ 0 & j > \tilde{\tau}_H, \end{cases} \quad (5.9)$$

it comes,

$$\begin{aligned} \int_0^{\tilde{\tau}_H} g_t^2 \, dt &= \sum_{j=1}^{\tilde{\tau}_H} \int_{j-1}^j g_t^2 \, dt \\ &= \sum_{j=1}^{\tilde{\tau}_H-1} \delta_j^2 + \tilde{\alpha}_{\tilde{\tau}_H} \delta_{\tilde{\tau}_H}^2 = H. \end{aligned}$$

By lemma 5.4, we obtain

$$\eta = \frac{1}{\sqrt{H}} \int_0^{\tilde{\tau}_H} g_t \, dW_t \sim \mathcal{N}(0, 1).$$

Or,

$$\begin{aligned}
& \mathbf{P}_S \left(\frac{1}{\sqrt{H}} \left| \int_0^{\tilde{\tau}_H} f_t dW_t \right| > z \right) \\
& \leq \mathbf{P}_S \left(\frac{1}{\sqrt{H}} \left| \int_0^{\tilde{\tau}_H} g_t dW_t \right| > \frac{z}{2} \right) + \mathbf{P}_S \left(\frac{1}{\sqrt{H}} \left| \int_0^{\tilde{\tau}_H} (f_t - g_t) dW_t \right| > \frac{z}{2} \right) \\
& \leq \mathbf{P}_S \left(|\eta| > \frac{z}{2} \right) + \mathbf{P}_S \left(\frac{1}{\sqrt{H}} |\sqrt{\alpha_{\tilde{\tau}_H}} - \alpha_{\tilde{\tau}_H}| |\delta_{\tilde{\tau}_H} \xi_{\tilde{\tau}_H}| > \frac{z}{2} \right) \\
& \leq \mathbf{P}_S \left(|\eta| > \frac{z}{2} \right) + \mathbf{P}_S \left(\frac{1}{\sqrt{H}} \sqrt{\alpha_{\tilde{\tau}_H}} |\delta_{\tilde{\tau}_H}| |\xi_{\tilde{\tau}_H}| > \frac{z}{2} \right) \\
& = \mathbf{P}_S \left(|\eta| > \frac{z}{2} \right) + \mathbf{P}_S \left(\frac{1}{H} \alpha_{\tilde{\tau}_H} \delta_{\tilde{\tau}_H}^2 \xi_{\tilde{\tau}_H}^2 > \frac{z^2}{4} \right) \\
& \leq \mathbf{P}_S \left(|\eta| > \frac{z}{2} \right) + \mathbf{P}_S \left(\xi_{\tilde{\tau}_H}^2 > \frac{z^2}{4} \right). \tag{5.10}
\end{aligned}$$

As η is a standard Gaussian random variable we can write for all $z \geq 2$

$$\begin{aligned}
\mathbf{P}_S \left(|\eta| > \frac{z}{2} \right) &= \sqrt{\frac{2}{\pi}} \int_{z/2}^{+\infty} e^{-t^2/2} dt \\
&\leq \sqrt{\frac{2}{\pi}} \int_{z/2}^{+\infty} t e^{-t^2/2} dt = \sqrt{\frac{2}{\pi}} e^{-z^2/8}.
\end{aligned}$$

We can write the second term of (5.10) as

$$\begin{aligned}
\mathbf{P}_S \left(\xi_{\tilde{\tau}_H}^2 > \frac{z^2}{4} \right) &= \sum_{l=1}^{+\infty} \mathbf{P}_S \left(\xi_l^2 > \frac{z^2}{4}, \tilde{\tau}_H = l \right) \\
&= \sum_{l=1}^{+\infty} \mathbf{P}_S \left(\xi_l^2 > \frac{z^2}{4}, \sum_{j=1}^{l-1} \delta_j^2 < H, \sum_{j=1}^l \delta_j^2 \geq H \right) \\
&= \sum_{l=1}^{+\infty} \mathbf{P}_S \left(|\xi_l| > \frac{z}{2} \right) \mathbf{P}_S(\tilde{\tau}_H = l) \\
&\leq \sqrt{\frac{2}{\pi}} e^{-z^2/8} \sum_{l=1}^{+\infty} \mathbf{P}_S(\tilde{\tau}_H = l) = \sqrt{\frac{2}{\pi}} e^{-z^2/8}.
\end{aligned}$$

So for any $z \geq 2$, (5.10) implies

$$\mathbf{P}_S \left(\frac{1}{\sqrt{nh}} \left| \int_0^{\tilde{\tau}_H} f_t dw_t \right| > z \right) \leq 2 \sqrt{\frac{2}{\pi}} e^{-z^2/8}.$$

□

Lemma 5.6. (Helland , 1981, pp. 80-82)

Let $(u_{k,n})_{1 \leq k \leq n}$ be a "martingale difference" defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and filtrations $\{\mathcal{F}_{k,n}, k \in \mathbb{N}\}$ of \mathcal{F} , $n \in \mathbb{N}^*$ such that $u_{k,n}$ is $\mathcal{F}_{k,n}$ -measurable. Assume that the following two conditions are satisfied:

$$\sum_{k=1}^n \mathbf{E}(u_{k,n}^2 \mathbf{1}_{(|u_{k,n}| > \varepsilon)} | \mathcal{F}_{k-1,n}) \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0, \quad \text{for all } \varepsilon > 0,$$

$$\sum_{k=1}^n \mathbf{E}(u_{k,n}^2 | \mathcal{F}_{k-1,n}) \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 1.$$

Therefore

$$\sum_{k=1}^n u_{k,n} \Longrightarrow \mathcal{N}(0, 1).$$

6 Numerical simulations

We illustrate the obtained results by the following simulation which is established using Scilab.

The purpose is to estimate at a given point z_0 the function S defined over $[0; 1]$ by $S(x) = |x - z_0|^\beta$. We check that such a function belongs to $\mathcal{H}^{(\beta)}(z_0, K, \varepsilon)$ when $K \geq 1$. The values of z_0 and β are arbitrary, which permits to the user the freedom of those choice. We take $z_0 = 0.8$, then $\beta_* = 0.6$ as a value lower regularity and $\beta^* = 0.8$ as the higher value.

We simulated n data from the function $S(x) = |x - z_0|^\beta$ for $\beta = 0.7$. We obtained a estimation in constructing the estimator \hat{S}_n defined in (2.7) with the procedure of Lepskiï who gives us the optimal window for the index \hat{k} defined in (2.6).

By varying the number of observations n , we obtain different estimations listed in the following table:

n	1000	5000	10000	30000	50000	100000
\hat{S}_n	0.180	0.086	0.103	0.090	0.039	0.05

In performing these simulations we noticed that the index \hat{k} given by the procedure of Lepskiï was always $m - 1$ is the number of grid points on $[\beta_*, \beta^*]$ except the ends.

This feature was expected since it is a priori a large number of observations for this index is strictly less than $m - 1$ by the definition (2.6).

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