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## NOTE ON THE CHEN-LIN RESULT WITH LI-ZHANG METHOD.

SAMY SKANDER BAHOURA

**Abstract:** We give a new proof of Chen-Lin result with Li-Zhang method.

### 1. INTRODUCTION AND RESULTS.

We set  $\Delta = -\partial_{11} - \partial_{22}$  the geometric Laplacian on  $\mathbb{R}^2$ .

On an open set  $\Omega$  of  $\mathbb{R}^2$ , with a smooth boundary, we consider the following problem:

$$(P) \quad \begin{cases} \Delta u = Ve^u & \text{in } \Omega. \\ 0 < a \leq V \leq b < +\infty \end{cases}$$

The previous equation is called, the Prescribed Scalar Curvature, in relation with conformal change of metrics. The function  $V$  is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of the previous type were studied by many authors. We can see in [B-M], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on  $V$ , for example we suppose  $V \geq 0$  and  $V \in L^p(\Omega)$  or  $Ve^u \in L^p(\Omega)$  with  $p \in [1, +\infty]$ .

We can see in [B-M] the following important Theorem,

**Theorem A (Brezis-Merle).** *If  $(u_i)_i$  and  $(V_i)_i$  are two sequences of functions relatively to the problem (P) with,  $0 < a \leq V_i \leq b < +\infty$ , then, for all compact set  $K$  of  $\Omega$ ,*

$$\sup_K u_i \leq c = c(a, b, m, K, \Omega) \quad \text{if} \quad \inf_{\Omega} u_i \geq m.$$

A simple consequence of this theorem is that, if we assume  $u_i = 0$  on  $\partial\Omega$  then, the sequence  $(u_i)_i$  is locally uniformly bounded.

If, we assume  $V$  with more regularity, we can have another type of estimates,  $\sup + \inf$ . It was proved, by Shafirir, see [S], that, if  $(u_i)_i, (V_i)_i$  are two sequences of functions solutions of the previous equation without assumption on the boundary and,  $0 < a \leq V_i \leq b < +\infty$ , then we have the following interior estimate:

$$C(a/b) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

We can see in [C-L], an explicit value of  $C(a/b) = \sqrt{\frac{a}{b}}$ .

Now, if we suppose  $(V_i)_i$  uniformly Lipschitzian with  $A$  the Lipschitz constant, then,  $C(a/b) = 1$  and  $c = c(a, b, A, K, \Omega)$ , see Brézis-Li-Shafirir [B-L-S]. This result was extended for Hölderian sequences  $(V_i)_i$  by Chen-Lin, see [C-L]. Also, we can see in [L], an extension of the Brezis-Li-Shafirir to compact Riemann surface without boundary. We can see in [L-S] explicit form,  $(8\pi m, m \in \mathbb{N}^*$  exactly), for the numbers in front of the Dirac masses, when the solutions blow-up.

On open set  $\Omega$  of  $\mathbb{R}^2$  we consider the following equation:

$$\Delta u_i = V_i e^{u_i} \quad \text{on } \Omega.$$

$$0 < a \leq V_i \leq b < +\infty, \quad |V_i(x) - V_i(y)| \leq A|x - y|^s, \quad 0 < s < 1, \quad x, y \in \Omega.$$

**Theorem B**(Chen-Lin). For all compact  $K \subset \Omega$  and all  $s \in ]0, 1[$  there is a constant  $c = c(a, b, A, s, K, \Omega)$  such that,

$$\sup_K u_i + \inf_{\Omega} u_i \leq c, \quad \forall i.$$

Here we try to prove the previous theorem by the moving-plane method and Li-Zhang method.

We argue by contradiction, and we want to proof that:

$$\exists R > 0, \text{ such that } 4 \log R + \sup_{B_R(0)} u + \inf_{B_{2R}(0)} u \leq c = c(a, b, A),$$

Thus, by contardition we can assume:

$$\exists (R_i)_i, (u_i)_i \text{ } R_i \rightarrow 0, 4 \log R_i + \sup_{B_{R_i}(0)} u_i + \inf_{B_{2R_i}(0)} u_i \rightarrow +\infty,$$

The blow-up analysis

Let  $x_0 \in \Omega$ , we want to prove the theorem locally around  $x_0$ , we use the previous assertion with  $x_0 = 0$ . The classical blow-up analysis gives the existence of the sequence  $(x_i)_i$  and a sequence of functions  $(v_i)_i$  such that:

We set,

$$\sup_{B_{R_i}(0)} u_i = u_i(\bar{x}_i),$$

$$s_i(x) = 2 \log(R_i - |x - \bar{x}_i|) + u_i(x), \text{ and } s_i(x_i) = \sup_{B_{R_i}(\bar{x}_i)} s_i, \quad l_i = \frac{1}{2}(R_i - |x_i - \bar{x}_i|).$$

Also, we set:

$$v_i(x) = u_i[x_i + xe^{-u_i(x_i)/2}] - u_i(x_i), \quad \bar{V}_i(x) = V_i[x_i + xe^{-u_i(x_i)/2}],$$

Then,

$$\Delta v_i = \bar{V}_i e^{v_i},$$

$$v_i \leq 2 \log 2, \quad v_i(0) = 0,$$

$$v_i \rightarrow v = \log \frac{1}{(1 + [V(0)/8]|x|^2)^2} \text{ converge uniformly on each compact set of } \mathbb{R}^2$$

with  $V(0) = \lim_{i \rightarrow +\infty} V_i(x_i)$ .

The classical elliptic estimates and the classical Harnack inequality, we can prove the previous uniform convergence on each compact of  $\mathbb{R}^2$ .

The Kelvin transform and the moving-plane method: Li-Zhang method.

For  $0 < \lambda < \lambda_1$ , we define:

$$\Sigma_\lambda = B(0, l_i M_i) - B(0, \lambda).$$

First, we set :

$$\bar{v}_i^\lambda = v_i^\lambda - 4 \log |x| + 4 \log \lambda = v_i \left( \frac{\lambda^2 x}{|x|^2} \right) + 4 \log \frac{\lambda}{|x|},$$

$$\bar{V}_i^\lambda = \bar{V}_i \left( \frac{\lambda^2 x}{|x|^2} \right),$$

$$M_i = e^{u_i(x_i)/2},$$

and,

$$w_\lambda = \bar{v}_i - \bar{v}_i^\lambda.$$

Then,

$$\Delta \bar{v}_i^\lambda = \bar{V}_i^\lambda e^{\bar{v}_i^\lambda},$$

$$\min_{|y|=R_i M_i} \bar{v}_i^\lambda = u_i(x_i + r\theta) - u_i(x_i) + 2u_i(x_i) \geq \inf_{\Omega} u_i + u_i(x_i) \rightarrow +\infty,$$

and,

$$\Delta(v_i - v_i^\lambda) = \bar{V}_i(e^{v_i} - e^{v_i^\lambda}) + (\bar{V}_i - \bar{V}_i^\lambda)e^{v_i^\lambda},$$

We have the following estimate:

$$|\bar{V}_i - \bar{V}_i^\lambda| \leq A M_i^{-s} |x|^s \left| 1 - \frac{\lambda^2}{|x|^2} \right|^s,$$

We take an auxiliary function  $h_\lambda$  :

Because,  $v_i(x^\lambda) \leq C(\lambda_1) < +\infty$ , we have,

$$h_\lambda = C_1 M_i^{-s} \lambda^2 (\log(\lambda/|x|)) + C_2 M_i^{-s} \lambda^{2+s} \left[ 1 - \left( \frac{\lambda}{|x|} \right)^{2-s} \right], \quad |x| > \lambda,$$

with  $C_1, C_2 = C_1, C_2(s, \lambda_1) > 0$ ,

$$h_\lambda = M_i^{-s} \lambda^2 (1 - \lambda/|x|) \left( C_1 \frac{\log(\lambda/|x|)}{1 - \lambda/|x|} + C_2' \right),$$

with,  $C_2' = C_2'(s, \lambda_1) > 0$ . We can choose  $C_1$  big enough to have  $h_\lambda < 0$ .

**Lemma 1:** There is an  $\lambda_{k,0} > 0$  small enough, such that, for  $0 < \lambda < \lambda_{k,0}$ , we have:

$$w_\lambda + h_\lambda > 0.$$

We have,

$$f(r, \theta) = v_i(r\theta) + 2 \log r,$$

then,

$$\frac{\partial f}{\partial r}(r, \theta) = \langle \nabla v_i(r\theta) | \theta \rangle + \frac{2}{r},$$

According to the blow-up analysis,

$$\exists r_0 > 0, C > 0, |\langle \nabla v_i(r\theta) | \theta \rangle| \leq C, \text{ for } 0 \leq r < r_0,$$

Then,

$$\exists r_0 > 0, C' > 0, \frac{\partial f}{\partial r}(r, \theta) > \frac{C'}{r}, \quad 0 < r < r_0,$$

if  $0 < \lambda < |y| < r_0$ ,

$$w_\lambda(y) + h_\lambda(y) = v_i(y) - v_i^\lambda(y) + h_\lambda(y) > C(\log |y| - \log |y^\lambda|) + h_\lambda(y),$$

by the definition of  $h_\lambda$ , we have, for  $C, C_0 > 0$  and  $0 < \lambda \leq |y| < r_0$ ,

$$w_\lambda(y) + h_\lambda(y) > (|y| - \lambda) \left[ C \frac{\log |y| - \log |y^\lambda|}{|y| - \lambda} - \lambda^{1+s} C_0 M_i^s \right],$$

but,

$$|y| - |y^\lambda| > |y| - \lambda > 0, \text{ and } |y^\lambda| = \frac{\lambda^2}{|y|},$$

thus,

In the first step of the lemma 2, we have,

$$v_i \geq \min v_i = C_i^1, \quad v_i^\lambda(y) \leq C_1(\lambda_1, r_0), \quad \text{if } r_0 \leq |y| \leq R_i M_i,$$

Thus, in  $r_0 \leq |y| \leq R_i M_i$  and  $\lambda \leq \lambda_1$ , we have,

$$w_\lambda + h_\lambda \geq C_i - 4 \log \lambda + 4 \log r_0 - C' \lambda_1^{2+s}$$

then, if  $\lambda \rightarrow 0$ ,  $-\log \lambda \rightarrow +\infty$ , and,

$$w_\lambda + h_\lambda > 0, \quad \text{if } \lambda < \lambda_0^i, \lambda_0^i \text{ (small), and } r_0 < |y| \leq R_i M_i,$$

By the maximum principle and the Hopf boundary lemma, we have:

Lemma 2: Let  $\tilde{\lambda}_k$  be a positive number such that:

$$\tilde{\lambda}_k = \sup\{\lambda < \lambda_1, \quad w_\lambda + h_\lambda > 0 \text{ in } \Sigma_\lambda.\}$$

Then,

$$\tilde{\lambda}_k = \lambda_1.$$

The blow-up analysis gives the following inequality for the boundary condition,

For  $y = |y|\theta = R_i M_i \theta$ , we have,

$$\begin{aligned} & w_{\lambda^i}(|y| = R_i M_i) + h_{\lambda^i}(|y| = R_i M_i) = \\ = & u_i(x_i + R_i \theta) - u_i(x_i) - v_i(y^{\lambda^i}, |y| = R_i M_i) - 4 \log \lambda + 4 \log(R_i M_i) + C(s, \lambda_1) M_i^{-s} \lambda^{2+s} \left[1 - \left(\frac{\lambda}{R_i M_i}\right)^{2-s}\right], \\ & 4 \log \bar{R}_i + u_i(x_i) + \inf_{\Omega} u_i \rightarrow +\infty, \end{aligned}$$

which we can write,

$$w_{\lambda^i}(|y| = R_i M_i) + h_{\lambda^i}(|y| = R_i M_i) \geq \min_{\Omega} u_i + u_i(x_i) + 4 \log R_i - C(s, \lambda_1) \rightarrow +\infty,$$

because,  $0 < \lambda \leq \lambda_1$ ,

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