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NOTE ON THE CHEN-LIN RESULT WITH LI-ZHANG METHOD.

SAMY SKANDER BAHOURA

Abstract: We give a new proof of Chen-Lin result with Li-Zhang method.

1. INTRODUCTION AND RESULTS.

We set $\Delta = -\partial_{11} - \partial_{22}$ the geometric Laplacian on \mathbb{R}^2 .

On an open set Ω of \mathbb{R}^2 , with a smooth boundary, we consider the following problem:

$$(P) \quad \begin{cases} \Delta u = Ve^u & \text{in } \Omega. \\ 0 < a \leq V \leq b < +\infty \end{cases}$$

The previous equation is called, the Prescribed Scalar Curvature, in relation with conformal change of metrics. The function V is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of the previous type were studied by many authors. We can see in [B-M], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on V , for example we suppose $V \geq 0$ and $V \in L^p(\Omega)$ or $Ve^u \in L^p(\Omega)$ with $p \in [1, +\infty]$.

We can see in [B-M] the following important Theorem,

Theorem A (Brezis-Merle). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω ,*

$$\sup_K u_i \leq c = c(a, b, m, K, \Omega) \quad \text{if} \quad \inf_{\Omega} u_i \geq m.$$

A simple consequence of this theorem is that, if we assume $u_i = 0$ on $\partial\Omega$ then, the sequence $(u_i)_i$ is locally uniformly bounded.

If, we assume V with more regularity, we can have another type of estimates, $\sup + \inf$. It was proved, by Shafrir, see [S], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C(a/b) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

We can see in [C-L], an explicit value of $C(a/b) = \sqrt{\frac{a}{b}}$.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see Brézis-Li-Shafrir [B-L-S]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [C-L]. Also, we can see in [L], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [L-S] explicit form, $(8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses, when the solutions blow-up.

On open set Ω of \mathbb{R}^2 we consider the following equation:

$$\Delta u_i = V_i e^{u_i} \quad \text{on } \Omega.$$

$$0 < a \leq V_i \leq b < +\infty, \quad |V_i(x) - V_i(y)| \leq A|x - y|^s, \quad 0 < s < 1, \quad x, y \in \Omega.$$

Theorem B((Chen-Lin). For all compact $K \subset \Omega$ and all $s \in]0, 1[$ there is a constant $c = c(a, b, A, s, K, \Omega)$ such that,

$$\sup_K u_i + \inf_{\Omega} u_i \leq c, \quad \forall i.$$

Here we try to prove the previous theorem by the moving-plane method and Li-Zhang method.

We argue by contradiction, and we want to proof that:

$$\exists R > 0, \text{ such that } 4 \log R + \sup_{B_R(0)} u + \inf_{B_{2R}(0)} u \leq c = c(a, b, A),$$

Thus, by contardition we can assume:

$$\exists (R_i)_i, (u_i)_i \text{ } R_i \rightarrow 0, \quad 4 \log R_i + \sup_{B_{R_i}(0)} u_i + \inf_{B_{2R_i}(0)} u_i \rightarrow +\infty,$$

The blow-up analysis

Let $x_0 \in \Omega$, we want to prove the theorem locally around x_0 , we use the previous assertion with $x_0 = 0$. The classical blow-up analysis gives the existence of the sequence $(x_i)_i$ and a sequence of functions $(v_i)_i$ such that:

We set,

$$\sup_{B_{R_i}(0)} u_i = u_i(\bar{x}_i),$$

$$s_i(x) = 2 \log(R_i - |x - \bar{x}_i|) + u_i(x), \text{ and } s_i(x_i) = \sup_{B_{R_i}(\bar{x}_i)} s_i, \quad l_i = \frac{1}{2}(R_i - |x_i - \bar{x}_i|).$$

Also, we set:

$$v_i(x) = u_i[x_i + xe^{-u_i(x_i)/2}] - u_i(x_i), \quad \bar{V}_i(x) = V_i[x_i + xe^{-u_i(x_i)/2}],$$

Then,

$$\Delta v_i = \bar{V}_i e^{v_i},$$

$$v_i \leq 2 \log 2, \quad v_i(0) = 0,$$

$$v_i \rightarrow v = \log \frac{1}{(1 + [V(0)/8]|x|^2)^2} \text{ converge uniformly on each compact set of } \mathbb{R}^2$$

with $V(0) = \lim_{i \rightarrow +\infty} V_i(x_i)$.

The classical elliptic estimates and the classical Harnack inequality, we can prove the previous uniform convergence on each compact of \mathbb{R}^2 .

The Kelvin transform and the moving-plane method: Li-Zhang method.

For $0 < \lambda < \lambda_1$, we define:

$$\Sigma_\lambda = B(0, l_i M_i) - B(0, \lambda).$$

First, we set :

$$\bar{v}_i^\lambda = v_i^\lambda - 4 \log |x| + 4 \log \lambda = v_i \left(\frac{\lambda^2 x}{|x|^2} \right) + 4 \log \frac{\lambda}{|x|},$$

$$\bar{V}_i^\lambda = \bar{V}_i \left(\frac{\lambda^2 x}{|x|^2} \right),$$

$$M_i = e^{u_i(x_i)/2},$$

and,

$$w_\lambda = \bar{v}_i - \bar{v}_i^\lambda.$$

Then,

$$\Delta \bar{v}_i^\lambda = \bar{V}_i^\lambda e^{\bar{v}_i^\lambda},$$

$$\min_{|y|=R_i M_i} \bar{v}_i^\lambda = u_i(x_i + r\theta) - u_i(x_i) + 2u_i(x_i) \geq \inf_{\Omega} u_i + u_i(x_i) \rightarrow +\infty,$$

and,

$$\Delta(v_i - v_i^\lambda) = \bar{V}_i(e^{v_i} - e^{v_i^\lambda}) + (\bar{V}_i - \bar{V}_i^\lambda)e^{v_i^\lambda},$$

We have the following estimate:

$$|\bar{V}_i - \bar{V}_i^\lambda| \leq A M_i^{-s} |x|^s |1 - \frac{\lambda^2}{|x|^2}|^s,$$

We take an auxiliary function h_λ :

Because, $v_i(x^\lambda) \leq C(\lambda_1) < +\infty$, we have,

$$h_\lambda = C_1 M_i^{-s} \lambda^2 (\log(\lambda/|x|)) + C_2 M_i^{-s} \lambda^{2+s} [1 - (\frac{\lambda}{|x|})^{2-s}], \quad |x| > \lambda,$$

with $C_1, C_2 = C_1, C_2(s, \lambda_1) > 0$,

$$h_\lambda = M_i^{-s} \lambda^2 (1 - \lambda/|x|) (C_1 \frac{\log(\lambda/|x|)}{1 - \lambda/|x|} + C_2'),$$

with, $C_2' = C_2'(s, \lambda_1) > 0$. We can choose C_1 big enough to have $h_\lambda < 0$.

Lemma 1: There is an $\lambda_{k,0} > 0$ small enough, such that, for $0 < \lambda < \lambda_{k,0}$, we have:

$$w_\lambda + h_\lambda > 0.$$

We have,

$$f(r, \theta) = v_i(r\theta) + 2 \log r,$$

then,

$$\frac{\partial f}{\partial r}(r, \theta) = \langle \nabla v_i(r\theta) | \theta \rangle + \frac{2}{r},$$

According to the blow-up analysis,

$$\exists r_0 > 0, C > 0, |\nabla v_i(r\theta) | \theta \rangle| \leq C, \text{ for } 0 \leq r < r_0,$$

Then,

$$\exists r_0 > 0, C' > 0, \frac{\partial f}{\partial r}(r, \theta) > \frac{C'}{r}, \quad 0 < r < r_0,$$

if $0 < \lambda < |y| < r_0$,

$$w_\lambda(y) + h_\lambda(y) = v_i(y) - v_i^\lambda(y) + h_\lambda(y) > C(\log |y| - \log |y^\lambda|) + h_\lambda(y),$$

by the definition of h_λ , we have, for $C, C_0 > 0$ and $0 < \lambda \leq |y| < r_0$,

$$w_\lambda(y) + h_\lambda(y) > (|y| - \lambda) [C \frac{\log |y| - \log |y^\lambda|}{|y| - \lambda} - \lambda^{1+s} C_0 M_i^s],$$

but,

$$|y| - |y^\lambda| > |y| - \lambda > 0, \text{ and } |y^\lambda| = \frac{\lambda^2}{|y|},$$

thus,

In the first step of the lemma 2, we have,

$$v_i \geq \min v_i = C_i^1, \quad v_i^\lambda(y) \leq C_1(\lambda_1, r_0), \quad \text{if } r_0 \leq |y| \leq R_i M_i,$$

Thus, in $r_0 \leq |y| \leq R_i M_i$ and $\lambda \leq \lambda_1$, we have,

$$w_\lambda + h_\lambda \geq C_i - 4 \log \lambda + 4 \log r_0 - C' \lambda_1^{2+s}$$

then, if $\lambda \rightarrow 0$, $-\log \lambda \rightarrow +\infty$, and,

$$w_\lambda + h_\lambda > 0, \quad \text{if } \lambda < \lambda_0^i, \lambda_0^i \text{ (small), and } r_0 < |y| \leq R_i M_i,$$

By the maximum principle and the Hopf boundary lemma, we have:

Lemma 2: Let $\tilde{\lambda}_k$ be a positive number such that:

$$\tilde{\lambda}_k = \sup\{\lambda < \lambda_1, \quad w_\lambda + h_\lambda > 0 \text{ in } \Sigma_\lambda.\}.$$

Then,

$$\tilde{\lambda}_k = \lambda_1.$$

The blow-up analysis gives the following inequality for the boundary condition,

For $y = |y|\theta = R_i M_i \theta$, we have,

$$\begin{aligned} & w_{\lambda^i}(|y| = R_i M_i) + h_{\lambda^i}(|y| = R_i M_i) = \\ & = u_i(x_i + R_i \theta) - u_i(x_i) - v_i(y^{\lambda^i}, |y| = R_i M_i) - 4 \log \lambda + 4 \log(R_i M_i) + C(s, \lambda_1) M_i^{-s} \lambda^{2+s} \left[1 - \left(\frac{\lambda}{R_i M_i}\right)^{2-s}\right], \end{aligned}$$

$$4 \log \bar{R}_i + u_i(x_i) + \inf_{\Omega} u_i \rightarrow +\infty,$$

which we can write,

$$w_{\lambda^i}(|y| = R_i M_i) + h_{\lambda^i}(|y| = R_i M_i) \geq \min_{\Omega} u_i + u_i(x_i) + 4 \log R_i - C(s, \lambda_1) \rightarrow +\infty,$$

because, $0 < \lambda \leq \lambda_1$,

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