Super Logarithmic Sobolev inequalities and Nash-type inequalities for sub-markovian symmetric semigroups

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Abstract

[1] We study relationships between Logarithmic Sobolev inequalities with one parameter of Davies-Simon type, energy-entropy inequality, Nash-type inequality and Sobolev-type inequalities. The inequalities of Sobolev-type apply in the general setting of symmetric sub-Markovian semigroups (and some generalizations). We provide several examples of application of theses results for ultracontractive semigroups but also for some non-ultracontractive like Ornstein-Ulhenbeck semigroup.

1 Introduction

The main motivation of this work is to prove equivalence theorems for symmetric sub-Markovian semigroups, generalizing the case of polynomial ultracontractivity, between decay of the semigroup and functional inequalities satisfied by the Dirichlet form associated to the generator of the semigroup.

For general (let say symmetric sub-Markov) semigroups, E-B.Davies and B.Simon proved a Logarithmic Sobolev inequality with parameter for general ultracontractive semigroups and some partial converses (see[DS],[Ma]). By a direct approach, T.Coulhon ([C]) obtained a Nash-type inequality from the ultracontractivity property of the semigroup and gave also partial converse (See [C],[BCS],[Ma]).

In this paper, we show that, from Logarithmic Sobolev inequality, we can also deduce Nash-type inequality and obtain the same Nash function as in [C]. Since ultracontractivity implies Logarithmic Sobolev inequality, we obtain an indirect proof of Coulhon's result. But one advantage of our approach is that for non-ultracontractive semigroups but satisfying a Logarithmic Sobolev inequality, we can deduce a Nash-type inequality for these semigroups. This can be applied as well as to the Ornstein-Ulhlenbveck hypercontractive semigroup or to Davies-Simon counter-example (See [D] Example 2.3.5. and [DS]. Remark 1 p.359) or to Γ-type semigroups.

Our results have closed relationships with F-Sobolev inequalities obtained by F-Y-Wang (See [W1] and also the book [W2]). In these papers, the assumption used is Super-Poincaré inequality (Poincaré inequality with one parameter) closed to Logarithmic Sobolev inequality with parameter.

In the case of ultracontractivity, we apply this to several classes of decay of semigroups of particular interest: one-exponential and double-exponential classes (See below for definitions). The F-Sobolev obtained for the polynomial class has a weaker formulation than the L^p-Sobolev inequality but, in fact, is equivalent.

To summarize, we study relationships between Logarithmic Sobolev inequalities with one parameter of Davies-Simon-type, energy-entropy inequality, Nash-type inequality and Sobolev-type inequalities (also called F-Sobolev). The inequalities of Sobolev-type applies in the general setting of symmetric sub-Markovian semigroups (and some generalizations). We give several examples of application of theses results for ultracontractive semigroups as well as for some non-ultracontractive semigroups.

Let's describe the setting and recall some well-known facts.

Let $(T_t)_{t>0}$ be a symmetric sub-Markovian semigroup defined on $L^2(X,\mu)$ with (X,μ) a σ-finite measure space and $\mathcal{L}$ as infinitesimal generator. We denote by $\mathcal{E}(f) = (\mathcal{L}f, f)$ the associated Dirichlet form (See for instance [Fu] for the general theory). Following [D], a semigroup is said to be ultracontractive if

$$\|T_t f\|_\infty \leq a(t)\|f\|_2, \quad t > 0, \tag{1.1}$$

[1]Several years ago, a French version was available but with a limited diffusion.

with $a : (0, +\infty) \to (0, +\infty)$ a positive decreasing function. In other words, T_t sends L^2 into L^∞ for any $t > 0$. We shall say that the semigroup is (at most) of polynomial type if we can take $a(t) = a_n(t) = c\, t^{-\frac{n}{4}}, t > 0$ with $n > 0$ (not necessarily an integer). Note also that, with this terminology if (T_t) is in the class a then it is also in the class b whenever $a \leq b$.

In a well-known paper [V] (See also [VSC]), N.Varopoulos has shown that polynomial ultracontractivity (1.1), with $n > 2$, is equivalent to an L^2-Sobolev inequality (in the setting of symmetric sub-Markovian semigroups) that is

$$\|f\|_q^2 \leq c\, \mathcal{E}(f) \tag{1.2}$$

with $q = \frac{2n}{n-2}$, for all $f \in \mathcal{D}$ where $\mathcal{D}$ is the domain of the Dirichlet form $\mathcal{E}$.

In their paper [CKS], E. Carlen, S.Kusuoka and D. Strook have shown that ultracontractivity property (1.1) with $a = a_n$ is also equivalent to the following Nash inequality:

$$\|f\|_2^{2+\frac{4}{n}} \leq c\, \mathcal{E}(f) \|f\|_1^{\frac{4}{n}}, \quad f \in \mathcal{D} \tag{1.3}$$

using old ideas of Nash ([CKS],[N]). See also [N] for the proof of this inequality in $\mathbb{R}^n$.

Also independently , E.B. Davies et B. Simon ([DS]) following ideas of L.Gross ([G]), have introduced Logarithmic Sobolev inequalities with one parameter of the following form:

$$\int_X f^2 \ln f \, d\mu \leq t\mathcal{E}(f) + M(t)\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{1.4}$$

for any $t > 0$ and for any $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

They proved that if the semigroup is ultracontractive (1.1) then (1.4) is satisfied avec $M(t) = \log a(t)$. Clearly, Sobolev inequality with parameter (LSP for short) allows to deal with with more general classes of ultracontractivity. A complete converse doesn't hold in general but for some classes of a, the converse holds true (up to constants) (See [DS] p.359) see also [Ma]). For instance in the polynomial class $\{a_n, n > 0\}$ (See [D]). It is important to note that we pass from the polynomial ultracontractivity with $a_n(t) = c\, t^{-\frac{n}{4}}$ to LSP and vice-versa with exactly the same exponent n (But the constant c may change). Another important class we are interested in is the one-exponential class $a(t) = e^{c(1/t^\alpha + 1)}, \alpha > 0$, we also have a converse with no loss on the exponent α (See [Ma] for instance). Examples of such behavior of the heat kernel really ocure (at least for small time t). Contrary to the one-exponential case, the double exponential class $a(t) = e^{e^{1/t^\alpha}}, \alpha > 0$ has no converse. But we can show that if (1.4) is satisfied with $0 < \alpha < 1$ then the known techniques allow us only to prove that the semigroup is also in the double-exponential class but with another index index $\alpha' = \frac{\alpha}{1-\alpha}$. Moreover in this double-exponential class, a converse cannot hold. Indeed, there exists a counter-example due to Davies and Simon with the "boundary" case $\alpha = 1$ (See [DS] remark 1 p.359).

We now summarize the results mentioned above. Some part of these equivalences have been generalized (See Coulhon ???) but we shall only describe them in the common setting of symmetric sub-Markovian semigroups.

Theorem 1.1 *Let $(T_t)_{t>0}$ be a symmetric sub-Markovian semigroup on $L^2(X, \mu)$ with (X, μ) σ-finite measure space. We denote by $\mathcal{E}$ the Dirichlet form associated to the semigroup (T_t). Let $n > 2$. The following statements are equivalent*

1. *For any $f \in L^2$ and for any $t > 0$,*

$$\|T_t f\|_\infty \leq c_1\, t^{-\frac{n}{4}} \|f\|_2. \tag{1.5}$$

2. For any $f \in \mathcal{D}$,

$$\|f\|^2_{\frac{2n}{n-2}} \leq c_2\, \mathcal{E}(f). \tag{1.6}$$

3. For any $f \in \mathcal{D} \cap L^1$,

$$\|f\|_2^{2+\frac{4}{n}} \leq c_3\, \mathcal{E}(f) \|f\|_1^{\frac{4}{n}}. \tag{1.7}$$

4. For any $t > 0$ and for any $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$,

$$\int_X f^2 \ln f \, d\mu \leq t\mathcal{E}(f) + M(t)\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{1.8}$$

avec $M(t) = \log(c_4\, t^{-\frac{n}{4}})$.

5. For any $n > 0$, (1.5),(1.7), (1.8) are equivalent.

The aim of this paper is to find a general formulation of this theorem. The fundamental fact is to determine a general version of the Sobolev inequality (1.6). Indeed by the theorem above, only the polynomial decay can be dealt by this inequality. In that paper, we obtain such inequality for the general setting. We shall call this inequality a Sobolev-type inequality closed to energy-entropy inequality as we shall show in Section 2 ??? (See [B]). Unfortunately, in the case of polynomial decay, this Sobolev-type inequality is not the Sobolev inequality (1.6) above. It is apparently weaker but in fact it is equivalent to Sobolev inequality as already noted in [B],[BCL] (See also Section 3)..

We also introduce the so-called energy-entropy inequalities, in a natural way, from Log-Sobolev inequalities with parameter. They appear as optimization of these Log-Sobolev inequalities with parameter.

About tools used in the proof, we shall apply cut-off method widely developed in [BCLS] for generalized Nash inequalities (See Theorem 2.9).

The content of the paper is the following:

1. Introduction.
2. Relationships between functional inequalities.
3. Equivalence theorems between some functional inequalities.
4. Applications to some family ultracontractive semigroups.
5. Applications to some non- ultracontractive semigroups.
6. Application to some sub-laplacians on Lie groups.

Now we give more details about the results of each sections above.

In Section 2, we study diverses general relationships between Logarithmic Sobolev inequalites, energy-entropy inequalities, Nash-type inequalities and Sobolev-type inequalities.

In Section 3, we state the main Theorem of equivalence between these functional inequalities.

In Section 4, we provide some applications of the main results for several important classes of ultracontractivity. These classes are the motivation of this work. We, particularly, study the one-exponential class and the double-exponential class (See section 3 for definitions). For the particular class of double-exponential class, we shall make some comments about the fact that there is no equivalence between this class of behavior and the corresponding Log-Sobolev inequality. It will be interesting to clarify the limit case to this equivalence.

In Section 5, we show how we can apply our results to non-ultracontractive semigroups. In particular, we study Davies-Simon counter-example satisfying a Log-Sobolev inequality but not ultracontractive (See [DS] p.359).

In Section 6, we apply our results in the explicit setting of sub-laplacians on Lie groups. (citer Lohou et Mustapha).

2 Relationships between some functional inequalities

In that section, we study relationships between Logarithmic Sobolev inequalities (with one paramater), the optimized form of theses inequalities (Energy-Entropy inequalities), Nash-type inequalities and weakened Sobolev-type inequalities (See definition (2.22) below).

We begin by studying relationships between Logarithmic Sobolev inequalities (with one paramater), the optimized form of theses inequalities i.e. Energy-Entropy inequalities. Here, we do not need to assume that we are dealing with a Dirichlet form.

We recall the defintion of Nash-type inequality (See [C]). By convexity lemma (2.3), we show that an Energy-Entropy inequality (2.3) always implies a Nash-type inequality. We introduce weak form of Sobolev inequality which are the main subject of this paper. In [**?**], a Super-Poincaré inequality satisfied by an operator A (or Poincaré inequality with parameter) has been introduced to study the spectrum of A.

We also show we easily obtain a LSP from a F-Sobolev inequality. For some implications, our arguments are elementary arguments of convexity (Jensen inequality) and are independant of the Dirichlet form involved in these inequalities. In some implication, the Dirichlet form can be replaced by a homogeneous form of degree two which well behaves with respect to cutt-off method (See [BCLS])
In the last part of this section, we state the main result of that paper a Nash-type inequality always implies F-Sobolev inequality with a similar from.
This will show that all these inequalities are equivalent in the abstract setting of semigroups. In particular, we shall follow the constants appearing in the main theorem 3.

In the main Theorem 2.9, the assumptions we need on the form W which allows us to get a F-Sobolev inequality are the following Let $\mathcal{D}$ the domain of $\mathcal{W}$ in L^2. We assume that there exists $\rho > 1$ tel que pour tout $k \in \mathbb{Z}$, $f_{\rho,k} = (f - \rho^k)^+ \wedge \rho^k(\rho - 1) \in \mathcal{D}$ and a constant $A(\rho) > 0$ tels que :

$$\sum_{k\in\mathbb{Z}} \mathcal{W}(f_{\rho,k}) \leq A(\rho)\mathcal{W}(f). \tag{2.1}$$

We now present an easy relationship between a LSP and Entropy-Energy inequality.

Proposition 2.1 *Let $\mathcal{W}$ be a non-negative map homogeneous of degree 2 on its domain $\mathcal{D} \subset L^2$.*

a) We assume that the following LSP is satisfied by $\mathcal{W}$: for any $t > 0$ and any $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$,

$$\int_X f^2 \ln f \, d\mu \leq t\mathcal{W}(f) + M(t)\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{2.2}$$

with M a real-valued function defined $]0, +\infty)$.

For any $t \in \mathbb{R}$, we set $A_0(t) = \sup_{s>0}(st - sM(1/s))$ and $A_0^+ = \max(A_0, 0)$. We assume that $A_0(t)$ is finite for any $t \in \mathbb{R}$. Then the following inequality is satisfied : for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$ with $\|f\|_2 = 1$,

$$A_0\left(\int_X f^2 \ln f \, d\mu\right) \leq \mathcal{W}(f) \tag{2.3}$$

Or equivalently,

$$A_0^+\left(\int_X f^2 \ln f \, d\mu\right) \leq \mathcal{E}(f). \tag{2.4}$$

b)Conversely, if (2.3) or (2.4) is satisfied then (2.2) holds true

Proof : Let $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$, $\|f\|_2 = 1$. Then by setting $s = 1/t$f or any $t > 0$, (2.2) implies

$$s(\int_X f^2 \ln f \, d\mu) - sM(1/s) \leq \mathcal{W}(f). \tag{2.5}$$

We deduce (2.3) by optimizing over $s > 0$. The converse is then obvious. The equivalence between (2.3) et (2.4) follows for the fact that $\mathcal{W}$ is non-negative. The proof is completed.

Following [BCL], we shall call (2.3) Entropy-Energy inequality. We denote $Ent_{2,\mu}(f) = \int_X f^2 \ln f \, d\mu$.

The proposition just above make use of the so-called Legendre transform of a function b defined by the formula

$$A(t) = \sup_{s>0}(st - b(s)), \qquad t \in \mathbb{R}. \tag{2.6}$$

The function A is also called the conjugate function when b is a N-function (voir [A]). Thus A_0 is nothing else than the Legendre transform of $b(s) = s\,M(1/s)$. In particular, this function is non-decreasing. We also note that $M(s) = s\,b(\frac{1}{s})$ is defined by the same relation as b with respect to M. Note that we do not assume that M is continuous and strictly increasing as in [D]. But we assume that $A_0(t)$ is finite for all $t \in \mathbb{R}$. This implies the conditions $\lim_{t\mapsto+\infty} A_0(t)/t = +\infty$ and $\lim_{t\mapsto 0} M(t) = +\infty$.

In that paper, this technique will be called relation-optimization. This is our first method. The second method used is the cut-off method widely used in [BCLS].

Remark 2.2 *1- The hypthesis (2.2)implies that $A_0(t)$ is finite for any $t \in \mathcal{M} = \{Ent_{2,\mu}(f), f \in \mathcal{D} \cap L^1 \cap L^\infty, f \geq 0, \|f\|_2 = 1\}$. Moreover is $\mathcal{M}$ is not bounded above in $\mathbb{R}$ then the assumption $A_0(t)$* finite for any $t \in \mathbb{R}$ *is always satisfied since A_0 is non-decreasing*

2- In some cases, $M(t)$ is bounded below i.e. for all $t > 0$, $M(t) \geq C_0$ thus $A_0(t) \leq 0$ et $A_0^+(t) = 0$ pour $t \leq C_0$.

3- The fact that $\mathcal{E}$ is a Dirichlet from plays no rôle in this proposition. But these inequalities are usually obtained in the setting of Dirichlet forms.

The preceding proposition and the next lemma allows us to deduce Nash-type inequality under LSP assumption. The Nash-type inequality is analogue to the one obtained in [C]. This lemma relies also on a convexity argument.

Lemma 2.3 *Pour tout $f \in L^1 \cap L^\infty$ avec $f \geq 0$, on a :*

$$\|f\|_2^2 \ln\left(\frac{\|f\|_2^2}{\|f\|_1}\right) \leq \int_X f^2 \ln f \, d\mu. \tag{2.7}$$

Proof : A simple proof consists in applying Jensen's inequality to the convex function $\Phi(t) = t \ln t, t > 0$ assuming that $d\nu = f d\nu$ is a probability measure.

An alternative proof, in the same spirit of Proposition 2.1 is as follows. We set $B(t) = t \ln t$, $t > 0$ and $B^*(s) = e^{(s-1)}$, $s \in \mathbb{R}$. We have $B(t) = \sup_{s\in\mathbb{R}}(st - B^*(s))$. We deduce for all $s \in \mathbb{R}$ the following inequality μ-a.e.,

$$sf^2 - B^*(s)f \leq fB(f). \tag{2.8}$$

Integrating with respect to the measure μ, we get:

$$s\|f\|_2^2 - B^*(s)\|f\|_1 \leq \int_X fB(f) \, d\mu. \tag{2.9}$$

For any non-zero f, we can, write this inequality as

$$\|f\|_1 \left(s\frac{\|f\|_2^2}{\|f\|_1} - B^*(s)\right) \leq \int_X fB(f) \, d\mu. \tag{2.10}$$

Now we optimize over s and use the relation between B^* and B. So,

$$\|f\|_1 B\left(\frac{\|f\|_2^2}{\|f\|_1}\right) \leq \int_X fB(f) \, d\mu. \tag{2.11}$$

We conclude the lemma by expliciting B.

We note that we don't need to know explicitly B^* in the proof but only its existence. So this lemma can be generalized to any function B given by a Legendre transform of some function B^*. We shall not need such generaliy in that paper..

From the Lemma (2.3), we obtain information on the set $\mathcal{M}$ above. The lower bound

$$Ent_{2,\mu}(f) \geq \ln(1/\|f\|_1), \quad 0 \leq f \in L^1 \cap L^\infty, \|f\|_2 = 1 \tag{2.12}$$

shows that in general $\mathcal{M}$ is unbounded above. When μ is a finite measure, Hölder's inequality

$$\|f\|_1 \leq \|f\|_2 \, \mu(X)$$

shows that $\mathcal{M}$ is bounded below by $-\frac{1}{2}\log\mu(X)$. In particularr, $\mathcal{M}$ is bounded below by zéro when μ is a probability measure.

By definition, we say that the functional $\mathcal{W}$ satisfies a Nash-type inequality (or generalized Nash inequality) if there exists a $\Theta : [0, +\infty[\longrightarrow [0, +\infty[$ such that for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$ with $f \geq 0$ and $\|f\|_1 = 1$,

$$\Theta(\|f\|_2^2) \leq \mathcal{W}(f). \tag{2.13}$$

In [C], for a Dirichlet from $\mathcal{E}$, the hypthesis on f is solely $f \in \mathcal{D} \cap L^1$ instead of $f \in \mathcal{D} \cap L^1 \cap L^\infty$. If Θ is left-continuous then we can replace the assumption on f of (2.13) by $f \in \mathcal{D} \cap L^1$. Indeed, let $f \in \mathcal{D} \cap L^1$ with $f \geq 0$ tehn, for $n \in \mathbb{N}$, $f_n = f \wedge n \in \mathcal{D} \cap L^1 \cap L^\infty$ and$\mathcal{E}(f_n) \leq \mathcal{E}(f)$ since $\mathcal{E}$ is a Dirichlet form. we deduce

$$\Theta(\|f\|_2^2) = \Theta(\lim_n \|f_n\|_2^2) = \lim_n \Theta(\|f_n\|_2^2) \leq \lim_n \mathcal{E}(f_n) \leq \mathcal{E}(f).$$

We cal also remove the assumption $f \geq 0$ because $\mathcal{E}(|\ g\ |) \leq \mathcal{E}(g)$ for any $g \in \mathcal{D}$ (See [Fu] Thm1.4.1(e)).

In the next proposition, we get Nash-type inequality where we can precise the function Θ of (2.13) under the assumption of Energy-Entropy inequality (2.3.

Proposition 2.4 *Supposons que, pour tout $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$ et $\|f\|_2 = 1$, l'inégalité suivante est satisfaite:*

$$A_1\left(\int_X f^2 \ln f \, d\mu\right) \leq \mathcal{W}(f) \tag{2.14}$$

pour une fonction A_1 croissante. Alors, on a

$$\|f\|_2^2 \, A_1(\ln \frac{\|f\|_2}{\|f\|_1}) \leq \mathcal{W}(f) \tag{2.15}$$

pour tout $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$.

Inequality (2.15) is a Nash-type inequality since we can write it in the form (2.13) where

$$\Theta(x) = x A_1(\frac{1}{2} \ln x), \quad x > 0. \tag{2.16}$$

Proof : Let $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$ and $\|f\|_2 = 1$. We apply Lemma (2.3):

$$\ln\left(\frac{1}{\|f\|_1}\right) \leq \int_X f^2 \ln f \, d\mu. \tag{2.17}$$

and since the function A_1 is non-decreasing,

$$A_1\left(\ln \frac{1}{\|f\|_1}\right) \leq A_1\left(\int_X f^2 \ln f \, d\mu\right). \tag{2.18}$$

So for any $g \in \mathcal{D} \cap L^1 \cap L^\infty$, $g \geq 0$ with $g \neq 0$, we set $f = g/\|g\|_2$. We deduce from the preceding inequality

$$\|g\|_2^2 A_1\left(\ln \frac{\|g\|_2}{\|g\|_1}\right) \leq \mathcal{W}(g) \tag{2.19}$$

due to the hypothesis (2.14). The proof is completed

We have the following corollary :

Corollary 2.5 *Let $\mathcal{W}$ as above such that the LSP (2.2) is satisfied so we have the following Nash-type inequality, for any $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$:*

$$\|f\|_2^2 \, A_0(\ln \frac{\|f\|_2}{\|f\|_1}) \leq \mathcal{W}(f) \tag{2.20}$$

with function A_0 defined in Proposition 2.1.

A So the corollary assets that, if LSP holds true then NTI (2.13) also holds with:

$$\Theta(x) = x \, A_0(\frac{1}{2} \ln x), \quad x > 0. \tag{2.21}$$

Proof : From Proposition (2.1), LSP (2.2) implies Entropy-Energy inequality (2.3). So (2.14) is satisfied with $A_1 = A_0$. We deduce the result by applying Proposition 2.4.

By definition, we say that the Dirichlet form $\mathcal{E}$satsifies a weak Sobolev inequality if there exists anon-decreasing function B such that

$$\int_X f^2 B\left(\ln \frac{f}{\|f\|_2}\right) d\mu \leq \mathcal{W}(f) \tag{2.22}$$

for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$.

Under the assumption that B is the Legendre transform of some function, we prove that a weak Sobolev inequality implies a LSP.

We begin by showing that, under a more general convexity assumption on B that a weak Sobolev inequality always implies an Energy-Entropy inequality. .

Proposition 2.6 *Assume taht there exists a function $A_2 : \mathbb{R} \longrightarrow \mathbb{R}$ such that for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$, the following inequality is satisfied*

$$\int_X f^2 A_2\left(\ln f/\|f\|_2\right) d\mu \leq \mathcal{W}(f). \tag{2.23}$$

Then

1. *If A_2 is convex, we have*

$$A_2\left(\int_X f^2 \ln f \, d\mu\right) \leq \mathcal{W}(f). \tag{2.24}$$

   with $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$ et $\|f\|_2 = 1$.

2. *In particular, if A_2 is the Legendre of some function b i.e*

$$A_2(t) = \sup_{s>0}(st - b(s)), \quad t \in \mathbb{R}$$

   Then the following LSP is satisfied

$$\int_X f^2 \ln f \, d\mu \leq t\mathcal{W}(f) + M_0(t)\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{2.25}$$

   pour tout $t > 0$ et $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$ avec $M_0(t) = tb(1/t), t > 0$.

Remark 2.7 *1. None bounded function is the Legendre transform of some function. Indeed, if there exists $b(s)$ and a constant M tsuch that $st - b(s) \leq M$ for any $t, s > 0$. Letting t goes to infinity, we get a contradiction.*

2. In case 1)of Proposition 2.6 and if additionaly $A_2^*(t) = \sup_{s\in\mathbb{R}}(st - A_2(s))$is finite for $t \in \mathbb{R}$, then we can write a LSP with $A_2^{**}(t) = \sup_{t\in\mathbb{R}}(st - A_2^*(t))$, $s \in \mathbb{R}$ instead of A_2 in 2) of Proposition 2.6. Indeed, $A_2^{**}(s) \leq A_2(s)$, $s \in \mathbb{R}$. Then we get (2.25) with $M_0(t) = tA_2^*(1/t)$ pour tout $t > 0$.

Proof :

1. Assume that A_2 is convex. We set $d\nu = f^2 d\mu$ with $\|f\|_2 = 1$ and $f \geq 0$ then $d\nu$ is a probability measure. By Jensen's inequality, (2.23) implies

$$A_2\left(\int_X \ln f \, d\nu\right) \leq \int_X A_2(\ln f) \, d\nu. \tag{2.26}$$

   Therefore

$$A_2\left(\int_X f^2 \ln f \, d\mu\right) \leq \mathcal{W}(f) \tag{2.27}$$

   which is the expected Energy-Entropy inequality.

2. Since Legendre are convex function, we deduce (2.24). We then apply part b) of Proposition 2.1.

Now we are interested by the converse of the preceding proposition. This is the main work of this paper. More precisely, we prove that LSP implies an Orlicz-Sobolev-type inequality (2.22). In fact, we show that NTI imlies (2.22). Until now, properties of the functional $\mathcal{W}$ we used were possibly positivity and homogeneity of order two. Now we shall assume for $\mathcal{W}$ to satisfy a good behavior with respect to cut-method Of course, this good behavior is satisfied by Dirichlet form as shown in the next lemma.

Lemma 2.8 *(See [Al]). Let $\mathcal{E}$ be a Dirichlet form on L^2 with domain $\mathcal{D}$. Let $0 \leq f \in \mathcal{D} \subset L^2$. We set for any $\rho > 1$ and any $k \in \mathbb{Z}$, $f_{\rho,k} = (f - \rho^k)^+ \wedge \rho^k(\rho - 1)$ then $f_{\rho,k} \in \mathcal{D}$ and*

$$\sum_{k\in\mathbb{Z}} \mathcal{E}(f_{\rho,k}) \leq \mathcal{E}(f). \tag{2.28}$$

This a particular case of Corollaire 2.3 of [BCLS] where $\sqrt{6}\mathcal{E}(f)$ is improved by $\mathcal{E}(f)$. We give here a proof [2].

Proof : To simplify notation, we shall denote $\mathcal{E}(f) = \mathcal{E}(f, f)$. The starting point is the following important remark. Let g, h non-negatives functions in the domain $\mathcal{D}$. Then, if h is equal to a constant λ on the support of g, supp$g = \{x \in X : g(x) \neq 0\}$ and if $0 \leq h \leq \lambda$ then $\mathcal{E}(h, g) \geq 0$. Indeed, for any $\epsilon > 0$, we have $(h + \epsilon g) \wedge \lambda = h$. So,

$$\mathcal{E}(h) = \mathcal{E}((h + \epsilon g) \wedge \lambda) \leq \mathcal{E}(h + \epsilon g) = \mathcal{E}(h) + 2\epsilon\mathcal{E}(h, g) + \epsilon^2\mathcal{E}(g).$$

Therefore $\mathcal{E}(h, g) \geq 0$.

The proof of the lemma is the following for any $n \in \mathbb{N}$, we set $f_\rho^n = f \wedge \rho^{n+1}$, then $f_\rho^n = \sum_{k=-\infty}^{n} f_{\rho,k}$ and $f_\rho^n = \sum_{k=-n}^{n} f_{\rho,k} + f_\rho^{-(n+1)}$. We apply the remark just above with $h = f_\rho^{-(n+1)}$, $\lambda = \rho^{-n}$ and $g = f_{\rho,k}$ for $k = -n$ to n. Thus $\mathcal{E}(f_{\rho,k}, f_\rho^{-(n+1)}) \geq 0$.

In the same way, for any $(p, k) \in \mathbb{Z}^2$ with $p < k$, we again apply the remark above now with $h = f_{\rho,p}$, $\lambda = \rho^p(\rho - 1)$ and $g = f_{\rho,k}$. It yields $\mathcal{E}(f_{\rho,p}, f_{\rho,k}) \geq 0$. Finally, we get

$$\mathcal{E}(f) \geq \mathcal{E}(f_\rho^n) = \mathcal{E}(\sum_{k=-n}^{n} f_{\rho,k}) + 2\sum_{k=-n}^{n} \mathcal{E}(f_{\rho,k}, f_\rho^{-(n+1)}) + \mathcal{E}(f_\rho^{-(n+1)}). \tag{2.29}$$

Thus

$$\mathcal{E}(f) \geq \sum_{k,p=-n}^{n} \mathcal{E}(f_{\rho,p}, f_{\rho,k}) = \sum_{k=1}^{n} \mathcal{E}(f_{\rho,k}, f_{\rho,k}) + \sum_{k\neq p, k,p\geq -n} \mathcal{E}(f_{\rho,p}, f_{\rho,k}) \tag{2.30}$$

en particulier

$$\mathcal{E}(f) \geq \sum_{k=-n}^{n} \mathcal{E}(f_{\rho,k}, f_{\rho,k}). \tag{2.31}$$

The lemma is proved by passing to the limit $n \rightarrow$.

[2] I thank G.Allain for mentioning to me this result and the reference ([Al])

Theorem 2.9 *Assume that the following NTI is satisfied with function a non-decreasing* $A_3 : \mathbb{R} \longrightarrow \mathbb{R}$,

$$\|f\|_2^2 A_3\left(\frac{\|f\|_2}{\|f\|_1}\right) \leq \mathcal{E}(f) \tag{2.32}$$

for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$. *Assume that there exists* $\rho > 1$ *and* $D = D_\rho : [0, +\infty) \longrightarrow [0, +\infty)$ *a non-decreasing function satisfying*

$$H(\rho, k) := \rho^2 D(\rho^{k+1}) - D(\rho^k) \leq \left(\frac{\rho - 1}{\rho}\right)^2 A_3^+\left(\rho^k \frac{(\rho - 1)}{\rho^2}\right). \tag{2.33}$$

Then for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$, $\|f\|_2 = 1$, *we have*

$$\int_X f^2 D_\rho(f)\, d\mu \leq \mathcal{E}(f). \tag{2.34}$$

Remark 2.10 *Condition (2.33) is rather a technical condition coming from the cut-off method used in the proof.*

Corollary 2.11 *Assume that NTI (2.32) is satisfied. For any* $\rho > 1$, *we set :*

$$V_\rho(x) = \frac{(\rho - 1)^2}{\rho^4} A_3\left(\frac{(\rho - 1)}{\rho^3} x\right). \tag{2.35}$$

Then

$$\int_X f^2 V_\rho^+(f)\, d\mu \leq \mathcal{E}(f) \tag{2.36}$$

for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$,$\|f\|_2 = 1$.

Proof : We apply Theorem 2.9 with $D_\rho = V_\rho^+$.

In the course of the proof of Theorem 2.9, we shall use in a crucial way of the property (2.28) of the Lemma 2.8 satisfied by Dirichlet forms.

Proof of Theorem 2.9 : With some variations, we essentially follows the cut-off method developed and used intensively in [BCLS].

Let $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$,$\|f\|_2 = 1$. For any $\rho > 1$ and any tout $k \in \mathbb{Z}$, we set $f_{\rho,k} = (f - \rho^k)^+ \wedge \rho^k(\rho - 1)$. Then $f_{\rho,k} \in \mathcal{D}$ and $f = \sum_{k \in \mathbb{Z}} f_{\rho,k}$ a.e.. Moreover, $\|f_{\rho,k}\|_2 \leq \|f\|_2 = 1$ et $\|f_{\rho,k}\|_1 \leq \|f\|_1$.

For any $k \in \mathbb{Z}$, we have the following estimates

$$\frac{\|f_{\rho,k}\|_2}{\|f_{\rho,k}\|_1} \geq \rho^{(k-1)}(\rho - 1). \tag{2.37}$$

Indeed, by Hölder's inequality

$$\|f_{\rho,k}\|_1 \leq \|f_{\rho,k}\|_2\, \mu(\Omega_{\rho,k})^{\frac{1}{2}}. \tag{2.38}$$

with $\Omega_{\rho,k} = \{f \geq \rho^k\}$ and since

$$[\rho^k(\rho - 1)]^2 \mu(\Omega_{\rho,k+1}) = \int_{\Omega_{\rho,k+1}} f_{\rho,k}^2 \leq \|f_{\rho,k}\|_2^2 \leq 1. \tag{2.39}$$

So the measure of the level set $\Omega_{\rho,k}$ is bounded as follows

$$\mu(\Omega_{\rho,k})^{\frac{1}{2}} \leq \left[\rho^{k-1}(\rho-1)\right]^{-1} \tag{2.40}$$

Using (2.38) and (2.40), we egt (2.37).

We now apply (2.32) to $f_{\rho,k}$, we get

$$\|f_{\rho,k}\|_2^2 A_3\left(\frac{\|f_{\rho,k}\|_2}{\|f_{\rho,k}\|_1}\right) \leq \mathcal{E}(f_{\rho,k}) \tag{2.41}$$

Because A_3 is non-decreasing, $\mathcal{E}(f)$ is non-negative and by inequalities (2.37) and (2.39), we have for any $k \in \mathbb{Z}$ et tout $\rho > 1$,

$$(\rho^k(\rho-1))^2 \mu(\Omega_{\rho,k+1}) A_3^+(\rho^{k-1}(\rho-1)) \leq \mathcal{E}(f_{\rho,k}). \tag{2.42}$$

This fondamental estimate on the level sets $\Omega_{\rho,k}$ allows us to estimate the integral $\int_X \Phi(f)\, d\mu$ for some functionΦ. by the Dirichlet from $\mathcal{E}$ under the constaint $\|f\|_2 = 1$.

The second steps consists in discretising the following integral

$$\mathcal{I} = \int_X f^2 D(f)\, d\mu.$$

For any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$, $\|f\|_2 = 1$, the following series converges for any $\rho > 1$,

$$\sum_{k\in\mathbb{Z}} (\rho^{k+1})^2 D(\rho^{k+1}) \mu(\Omega_{\rho,k}) < \infty. \tag{2.43}$$

We now prove this fact. The following series summing over the positive integers is convergent because this is a finite sum.

$$\sum_{k\in\mathbb{N}} (\rho^{k+1})^2 D(\rho^{k+1}) \mu(\Omega_{\rho,k}) < \infty. \tag{2.44}$$

Indeed $f \in L^\infty$ implies that there exists an integer $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, $\mu(\Omega_{\rho,k}) = 0$. For the series summing over the negative integers, we write this series as

$$\sum_{k\in\mathbb{Z}^-} \rho^{k+1} \left[D(\rho^{k+1})\rho^{k+1}\right] \mu(\Omega_{\rho,k}). \tag{2.45}$$

Since $f \in L^1$, we have $\rho^{k+1}\mu(\Omega_{\rho,k}) \leq C_\rho \|f\|_1$ pour tout $k \in \mathbb{Z}$ with $C_\rho = \frac{\rho^2}{\rho-1}$. This bounds by above the preceding series by

$$C_\rho \|f\|_1 D(\rho) \left(\sum_{k\in\mathbb{N}} \rho^{-k+1}\right) = \frac{\rho^4}{(\rho-1)^2} D(\rho)\|f\|_1. \tag{2.46}$$

We get the bound

$$\mathcal{I} \leq \sum_{k\in\mathbb{Z}} (\rho^k)^2 H(\rho,k) \mu(\Omega_{\rho,k}) \tag{2.47}$$

with

$$H(\rho,k) = \rho^2 D(\rho^{k+1}) - D(\rho^k).$$

Finally, using assumption (2.33) and the inequality, we get (2.42) :

$$\int_X f^2 D(f)\, d\mu \leq \sum_{k\in\mathbb{Z}} \mathcal{E}(f_{\rho,k}). \tag{2.48}$$

But the right-hand side is bounded by $\mathcal{E}(f)$ by Lemme 2.8.

This completes the proof of the theorem.

The following corollary allows us to quantify the lost in the implication between weak Sobolev inequality and LSP. Indeed, corollaries 2.5 and 2.11 says that LSP always implies a weak Sobolev inequality. More precesely, we have

Corollary 2.12 *If the folllowing LSP is satisied*

$$\int f^2 \ln f\, d\mu \leq t\mathcal{E}(f) + M(t)\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{2.49}$$

for any $t > 0$ and any $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.
Then we have the following weak Sobolev inequality, for any $\rho > 1$,

$$\int f^2 W_\rho(\log f)\, d\mu \leq \mathcal{E}(f) \tag{2.50}$$

for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$, $\|f\|_2 = 1$, with

$$W_\rho(t) = \sup_{s>0}(st - s\tilde{M}_\rho(1/s)), \qquad t \in \mathbb{R} \tag{2.51}$$

where

$$\tilde{M}_\rho(t) = M(c_{1,\rho}t) + c_{2,\rho} \tag{2.52}$$

and $c_{1,\rho} = \frac{(\rho-1)^2}{\rho^4}$, $c_{2,\rho} = \log \frac{\rho^3}{(\rho-1)}$.

We shall note that $0 < c_{1,\rho} \leq \frac{1}{16}$ and $c_{2,\rho} \geq \log 6,75$.

Proof : Under the assumption LSP and by Corollary 2.5, we have

$$\|f\|_2^2\, A_0(\ln \frac{\|f\|_2}{\|f\|_1}) \leq \mathcal{E}(f) \tag{2.53}$$

for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$,$\|f\|_2 = 1$, with

$$A_0(t) = \sup_{s>0}(st - sM(1/s)), \qquad t \in \mathbb{R}.$$

So the hypothesis (2.32) of Corollaire 2.11 is satisfied and $A_3(t) = A_0(\log t)$, $t > 0$. Corollary 2.11 gives us

$$\int f^2 W_\rho(\log f)\, d\mu = \int f^2 V_\rho(f)\, d\mu \leq \mathcal{E}(f) \tag{2.54}$$

for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$, $\|f\|_2 = 1$. We have set

$$W_\rho(\log t) = V_\rho(t) = \frac{(\rho-1)^2}{\rho^4} A_0(\log t + \log \frac{(\rho-1)}{\rho^3}).$$

(See (2.35)). We write W_ρ in the form

$$W_\rho(t) = \frac{(\rho-1)^2}{\rho^4} A_0(t + \log \frac{(\rho-1)}{\rho^3})$$

for any $t \in \mathbb{R}$.

If we denote Legendre transform of b by $L(b)(t) = \sup_{s>0}(st - b(s))$, $t \in \mathbb{R}$, we have the following elementary properties:

$$\lambda L(b)(t) = L(\lambda b(\frac{\cdot}{\lambda}))(t)$$

for any $\lambda > 0$. And also

$$L(b)(t+\gamma) = L(b(s) - s\gamma)(t)$$

for any $\gamma \in \mathbb{R}$.

Then it is easy to deduce that for $\lambda = \lambda_\rho = \frac{(\rho-1)^2}{\rho^4}$, $\gamma = \gamma_\rho = \log \frac{(\rho-1)}{\rho^3}$,

$$W_\rho(t) = L(\tilde{b}_\rho)(t)$$

with $\tilde{b}_\rho(s) = \lambda b(s/\lambda) - s\gamma$. We recall $b(s) = sM(1/s)$. If we set $\tilde{M}_\rho$ such that $\tilde{b}_\rho(s) = s\tilde{M}_\rho(1/s)$ therefore

$$\tilde{M}_\rho(t) = M(\lambda t) - \gamma$$

which gives exactly the expression of $\tilde{M}_\rho$ and completes the proof.

Conversly, from the weak Sobolev inequality (2.50) and Proposition 2.6, we can write the Entropy-Energy inequality (2.24) with $A_2 = W_\rho$, $\rho > 1$ and deduce (2.25) with$M_0 = \tilde{M}_\rho$. Now we can state the following corollary

Corollary 2.13 *If (2.50) is satisfied with W_ρ of the form (2.51) for some $\rho > 1$the we have LSP inequality with $\tilde{M}_\rho$,*

$$\int f^2 \ln f \, d\mu \leq t\mathcal{E}(f) + \tilde{M}_\rho(t)\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{2.55}$$

for any $t > 0$ and $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$. The function $\tilde{M}_\rho$ is given by the formula (2.52).

We denote (2.55) by $(LSP)_{\tilde{M}_\rho}$.

The sequence of implications $(LSP)_M \Longrightarrow$ weak Sobolev inequality with $W_\rho) \Longrightarrow (LSP)_{\tilde{M}_\rho}$ change the function M by the function $\tilde{M}_\rho$. This function is obtained by change of the parameter t with $\lambda_\rho t$ and by translation γ_ρ.

In some cases, $\tilde{M}_\rho$ stays in the same class as $M(t)$. For instance, let $\delta > 0$ and $\mathcal{P}_\delta = \{M(t) = \log(C_0 t^{-\delta}), C_0 > 0\}$. If $M \in \mathcal{P}_\delta$ then $\tilde{M}_\rho(t) = \log(C_0 C_\rho t^{-\delta})$ with $C_\rho = \frac{\rho^{4\delta+3}}{(\rho-1)^{2\delta+1}}$. So $\tilde{M}_\rho \in \mathcal{P}_\delta$. By definition of the class $\mathcal{P}_\delta$, the accepted lost comes from the constant C_0 but not on the exponent $\delta > 0$. We shall call this class $\mathcal{P}_\delta$ the polynomial class of exponent δ. This terminology comes from the fact that ultracontractivity property

$$\|T_t f\|_\infty \leq C_0 t^{-\delta} \|f\|_2$$

implies $(LSP)_M$ according to Theorem 2.2.3 of [D].

We can state the following corollary which allows us to built functions D in terms of A_3 and of the parameter $\rho > 1$:

Corollary 2.14 *Assume that NTI (2.32) is satisfied. For $\rho > 1$, we set*

$$E_\rho(t) = (\rho-1)^2 \sum_{k=0}^{\infty} \frac{1}{\rho^{2(k+2)}} A_3^+ \left(\frac{(\rho-1)t}{\rho^{k+3}} \right) \tag{2.56}$$

Then we have for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$, $\|f\|_2 = 1$,

$$\int f^2 E_\rho(f)\, d\mu \leq \mathcal{E}(f). \tag{2.57}$$

Moreover inequality (2.33) is an equality for any $k \in \mathbb{N}$ and $\rho > 1$.

Proof : First note that the series defining E_ρ is absolutely convergent since A_3^+ is non-negative, non-decreasing and $\rho > 1$. We apply Theorem 2.9 with $D = E_\rho$ which satisfies

$$\rho^2 D(\rho t) - D(t) = \frac{(\rho-1)^2}{\rho^2} A_3^+ \left(t \frac{(\rho-1)}{\rho^2} \right). \tag{2.58}$$

In the particular case, $t = \rho^k$, (2.33) is satisfied. This completes the proof

Remark 2.15 *1. Note that for any $\rho > 1$, we can built a function D_ρ which satisfies (2.33) (In fact an equality).*

2. The function V_ρ^+ of Corollary 2.11 is in fact the first term of the series defining E_ρ.

Anagously to Corollary 2.12, we can state the following corollary which improves inequality (2.50) with a better constant C_ρ^1 in (2.52).

Corollary 2.16 *Under the assumption (2.49), the inequality (2.50) is satisfied with $M_\rho(t) = M(G(\rho)t) - \frac{I(\rho)}{G(\rho)}$ instead of $\tilde{M}_\rho$ in (2.51) with constants $G(\rho)$ and $I(\rho)$ given by*

$$G(\rho) = \frac{(\rho-1)}{\rho^2(\rho+1)}$$

and

$$I(\rho) = G(\rho)\log(\rho-1) - (\log\rho)\frac{3\rho^2-2}{\rho^2(\rho+1)^2}.$$

Moreover, $G(\rho) > C_\rho^1$ et $-I(\rho)/G(\rho) > C_\rho^2$.

Remark 2.17 *In applications, we can choose $\rho = \rho_0$ such that $G(\rho_0) = \sup_{\rho>1} G(\rho)$. Indeed, $G(\rho) < 1$ for $\rho > 1$ and in the examples, the function M is non-increasing with a singularity when $t \longrightarrow 0^+$ of the type $\lim_{t\longrightarrow 0^+} M(t) = +\infty$. The lost on the constant C_2^ρ is of less importance for applications.*

Proof : Set $F_\rho(t) = \sup_{s>0}(st - sM_\rho(1/s)), t \in \mathbb{R}$. It is enough to show that for all $t > 0$:

$$F_\rho(\log t) \leq E_\rho(t)$$

and apply inequality (2.57) of Corollary 2.14. Details are left to the reader.

3 Equivalence theorems between some functional inequalities.

In that section, we state the main theorem of that paper. More precisely, we discuss the relationship between LSP, Energy-entropy inequality, Nash-type inequality and weak Sobolev inequality. Legendre transform plays an important role in the description in these inequalities.

The main interest of this result is that we do not assume that the semigroup is ultracontractive. This is important for application where ultracontractivity property is not available.

In the next section, we first provide applications for some classes of ultracontractivity (one-exponential and double-exponential , see defginitions below).
For particular relations between two such inequalities, we refer to Section 2.

Theorem 3.1 *Let $\mathcal{E}$ be a Dirichlet form of domain $\mathcal{D} \in L^2(X,\mu)$. The following statements are equivalent:*

(LSP) There exists $M :]0,+\infty[\longrightarrow \mathbb{R}$ such that $\sup_{s>0}(sr - sM(1/s))$ is finite for all $r \in \mathbb{R}$ and which satisfies, for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$ for all $t > 0$:

$$\int f^2 \log f \, d\mu \leq t\mathcal{E}(f) + M(t)\|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \tag{3.1}$$

(EE) There exists $b_0 :]0,+\infty[\longrightarrow \mathbb{R}$ such that $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$ et $\|f\|_2 = 1$:

$$B_0\Big(\int f^2 \log f \, d\mu\Big) \leq \mathcal{E}(f) \tag{3.2}$$

with $B_0(t) = \sup_{s>0}(st - b_0(s))$.

(NTI) There exists $b_1 :]0,+\infty[\longrightarrow \mathbb{R}$ such that $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$:

$$\|f\|_2^2 B_1\left(\log\Big(\frac{\|f\|_2}{\|f\|_1}\Big)\right) \leq \mathcal{E}(f) \tag{3.3}$$

with $B_1(t) = \sup_{s>0}(st - b_1(s))$.

(WS) There exits $b_2 :]0,+\infty[\longrightarrow \mathbb{R}$ such that $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$ et $\|f\|_2 = 1$:

$$\int f^2 B_2(\log f) \, d\mu \leq \mathcal{E}(f) \tag{3.4}$$

with $B_2(t) = \sup_{s>0}(st - b_2(s))$.

Proof :

- "(LSP) implies (EE) " with $b_0(s) = sM(\frac{1}{s})$ from a) of Proposition 2.1 .
- "(EE) implies (LSP)" with $M(s) = sb_0(\frac{1}{s})$ from b) of Proposition 2.1.

We note that the assumption of finitness of sup defining B_0 for any $t \in \mathbb{R}$ is useless because B_0 is only evaluated at the value $\int f^2 \log f \, d\mu$. The assumption insures that the value $B_0(\int f^2 \log f \, d\mu)$ is finite.

We set $T(M)(s) = sM(\frac{1}{s})$ then $T^{-1} = T$ and T is the tranformation which passes from e M to b_0 in the first application and from b_0 to M in the seconde application.

• "(EE) implies (NTI) " with $b_1 = b_0$. So, $B_1 = B_0$. This results of Proposition 2.4.

• "(NTI) implies (WS) " . We apply Corollary 2.11. The hypothesis (2.32) is satisfied with $A_3(t) = B_1(\log t), t > 0$. By this Corollary

$$\int f^2 V_\rho(f) \, d\mu \leq \mathcal{E}(f) \tag{3.5}$$

with $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $f \geq 0$ and $\|f\|_2 = 1$.

We can choose B_2in the following way,

$$B_2(\log t) = V_\rho(t) = \frac{(\rho-1)^2}{\rho^4} A_3(\frac{(\rho-1)}{\rho^3} t)$$

for some fixed $\rho > 1$. We get the following relation between B_1 and B_2 : $B_2(r) = \frac{(\rho-1)^2}{\rho^4} B_1(r + \log \frac{(\rho-1)}{\rho^3})$, $r \in \mathbb{R}$.

With the same relations Legendre transforms given in the course of the proof of Corollary 2.12, B_2 can be written as $B_2(r) = \lambda B_1(r + \gamma)$ with $\lambda = \frac{(\rho-1)^2}{\rho^4}$ and $\gamma = \log \frac{(\rho-1)}{\rho^3}$. Therefore

$$B_2(r) = \lambda L(b_1)(r + \gamma) = L(\lambda b_1(\frac{\cdot}{\lambda}) - s\gamma).$$

So, B_2 is the Legendre transform of b_2 when we set

$$b_2(s) = \lambda b_1(\frac{\cdot}{\lambda}) - s\gamma.$$

Inequality (WS) is satsified with B_2 above. This conclude this implication.

We note that a natural way to choose ρ is to minimize $\frac{\rho^4}{(\rho-1)^2}$ which is the factor in b_1. This expression is minimal for $\rho = 2$. So, $b_2(s) = \frac{1}{16} b_1(16s) + 3s \log 2$.

• "(WS) implies (EE) " with $b_0 = b_2$ and $B_0 = B_2$ by Proposition 2.6 1).

This completes the proof of this theorem.

We note that in the sequence of implications $(LSP) \Longrightarrow (EE) \Longrightarrow (NTI) \Longrightarrow (WS) \Longrightarrow (LSP)$, the function M from (LSP) is transform in $\tilde{M}$ by the relation :

$$\tilde{M}(t) = \log \lambda_2 + M(\frac{t}{\lambda_1}) \tag{3.6}$$

with $\lambda_1 = 16$, $\lambda_2 = 8$. By exponentiation of these functions appearing in the ultracontractivity property, we get

$$e^{\tilde{M}(t)} = \lambda_2 e^{M(\frac{t}{\lambda_1})} \tag{3.7}$$

This theorem also allows us to give another proof of LSP under ultracontractivity assumption. This proof avoids the complex interpolation argument used in [D] (see p. 65). Indeed, for an ultracontractive semigroup we apply Theorem ?? of T.Coulhon (See [C] p.514) which gives a NTI with function θ related to A which is a Legendre transform. The proof of [C] used the log-convexity of the semigroup. In a second step, we apply Theorem 3.1 from which we deduce (SW). It is easy to deduce (LSP) when A is a Legendre tansform which is the case.

In some sense, the implication LSP $\Rightarrow$ NTI generalize theorem ??? of [C]. Indeed, a LSP can be satisfied without ultracontractivity property (See Section ??).

Before going to applications, we compare Theorem 3.1 to Theorem 1.1 in the particular case of polynomial ultracontractivity i.e. $M(t) = \log c - \frac{n}{4}\log t$.
First, we defined several kind of decay of ultracontractivity for semigroups.

Definition 1 : We say that a semigroup (T_t) has, respectively, a polynomial decay ,a one-exponential or double-exponential decay if :

$$(p) \qquad a(t) = c_1 t^{\frac{-n}{4}}, \quad n > 0$$

ou

$$(oe) \qquad a(t) = c_1 \exp(\frac{c_2}{t^\gamma}), \quad \gamma > 0$$

ou

$$(de) \qquad a(t) = c_1 \exp(c_2 \exp(\frac{c_3}{t^\alpha})), \quad \alpha > 0$$

in the inequality

$$\|T_t f\|_\infty \leq a(t)\|f\|_2, \forall t > 0. \tag{3.8}$$

At the level of LSP, we translate these definitions as follows

Definition 2 : We say that a Dirichlet form $\mathcal{E}$ satisfies a LSP, respectively, of polynomial type ,of one- exponential type or double-exponential type if it satisfies

$$\int f^2 \ln f \, d\mu \leq t\mathcal{E}(f) + M(t)\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{3.9}$$

for any $t > 0$ and any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, with $M(t) = \log a(t)$ where $a(t)$ satisfies (p) ou (oe) ou (de) just above.

This definition is coherent with the implication "ultracontractivity implies LSP" given by Theorem 2.2.3 in [**?**] since (3.8) implies (3.9) with $M(t) = \log a(t)$. Recall that the converse implication doesn't hold in general.

Proposition 3.2 *(Polynomial decay) Let $n > 0$ be fixed (not necessarily an integer). The following inequalities are equivalent.*

(LSPp) There exists $k_1 > 0$ such that

$$\int_X f^2 \ln f \, d\mu \leq t\mathcal{E}(f) + \log(k_1 t^{\frac{-n}{4}})\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{3.10}$$

for any $t > 0$ and any $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

(EEp) There exists $k_2 > 0$ such that

$$\exp\left(\frac{4}{n}\int_X f^2 \ln f \, d\mu\right) \leq k_2 \mathcal{E}(f) \tag{3.11}$$

with $\|f\|_2 = 1$, $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

(NTp) There exists $k_3 > 0$ *such that*

$$\|f\|_2^{2+\frac{4}{n}} \leq k_3 \mathcal{E}(f) \|f\|_1^{\frac{4}{n}} \tag{3.12}$$

with $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

(WSp) There exists $k_4 > 0$ *such that*

$$\int_X f^{2+\frac{4}{n}} \, d\mu \leq k_4 \mathcal{E}(f) \|f\|_2^{\frac{4}{n}} \tag{3.13}$$

with $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

The constants k_i, $i = 1 \cdots 4$ *doesn't depend on* f.

All these inequalities are equivalent to the polynomial deacy of the corresponding semigroup. The inequality (EEp) can be re-written as

$$\int_X f^2 \ln f \, d\mu \leq \log \left(k_2' \mathcal{E}(f)^{\frac{n}{4}} \right) \tag{3.14}$$

with $\|f\|_2 = 1$ and $k_2' = k_2^{\frac{n}{4}}$.

This inequality is called Energy-Entropy inequality more appropriate than the usual expression "weak Sobolev inequality" (See [B]), [BCL]). The inequality (WSp) is not the usual L^p-Sobolev ($p = \frac{2n}{n-2}$) (See (1.2)). This inequality seems to be weaker but in fact is equivalent to Sobolev inequality(See ???cite or section). Indeed the (WSp) can be deduce Hölder inequality. More precisely,

$$\|f\|_q^q \leq \|f\|_p^2 \|f\|_2^{\frac{4}{n}} \tag{3.15}$$

with $q = 2 + \frac{4}{n}$. So we deduce (3.13) from (1.2).

The weakness of this inequality is only appearent since it is equivalent to NTI of "polynomial type" which is equivalent to the L^p-Sobolev inequality by Varopoulos 's result (See Theorem ?? in [VSC]). We can see that the L^q-norm induced by (WSp) is given by . $q = 2 + \frac{4}{n}$. Compare with L^p-Sobolev inequality where $p = 2 + \frac{4}{n-2} > q$ (When $n > 2$).

Note that we do not assume $n > 2$ in (Wsp). This inequality has been used by J. Moser [Mo] in $\mathbb{R}^n$ as mentioned in [BCLS].

4 Applications to some classes of ultracontractive semigroups

The main interest of the main Theorem 3.1 is to formulate Sobolev-type inequality generalizing the L^p-inequality:

$$||F||_p^2 \leq c\mathcal{E}(f)$$

with $p = \frac{2n}{n-2}$.

By Varopoulos' result such Sobolev inequality is equivalent to polynomial ultracontractivity. So generalization of this Sobolev inequality need to introduce a formulation less restrictive than L^p

norm with p related to some " dimension" n and, consequently, we have to weaken this inequality to take into account of different behavior or semigroups.

In this section, we give several examples of applications of the main theorem 3.1 These examples corresponds to natural examples which appear in the infinite dimensional torus $\mathbb{T}^\infty$ for instance but which can also be built on the real line (See Bendikov-Maheux 2). Many more examples could be found but we restrict to some classes of examples. Our theorem of equivalence will be applied following two points of view. The first point of view is: if the semigroup satisfies an ultracontractivity property with an explicit bound and we shall consider several classes of ultracontractivity. The second point of view is when no ultracontractivity is satisfied by the semigroup but a LSP is satisfied.

4.1 One-exponential class

Recall that a symmetric sub-marovian semigroup (T_t) satisfies a one-exponential decay if there exist $c_1, c_2 > 0$, $\alpha > 0$ such that, for any $t > 0$,

$$\|T_t f\|_\infty \leq c_1 \exp(\frac{c_2}{t^\alpha})\|f\|_2. \tag{4.1}$$

This definition is equivalent to the definition of one-exponential LSP. We shall apply Theorem 3.1 to this class of decay.

Theorem 4.1 *(One exponential)*
We fix $\gamma > 0$. The following statements are equivalent:

(LSPoe) There exist $c_1, c_2 > 0$ such that

$$\int_X f^2 \ln f \, d\mu \leq t\mathcal{E}(f) + [c_1 + c_2 t^{-\gamma}]\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{4.2}$$

for any $t > 0$ and any $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

(EEoe) There exist $c_3, c_4 > 0$ such that

$$\left[\int_X f^2 \ln \frac{f}{c_3} \, d\mu\right]_+^{1+\frac{1}{\gamma}} \leq c_4 \mathcal{E}(f) \tag{4.3}$$

with $\|f\|_2 = 1$ and $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

(NTIoe) There exist $c_5, c_6 > 0$ such that

$$\|f\|_2^2 \left[\log\left(\frac{\|f\|_2}{c_5}\right)\right]_+^{1+\frac{1}{\gamma}} \leq c_6 \mathcal{E}(f) \tag{4.4}$$

with $\|f\|_1 = 1$ and $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

(WSoe) There exist $c_7, c_8 > 0$ such that

$$\int_X f^2 \left[\log\left(\frac{f}{c_7}\right)\right]_+^{1+\frac{1}{\gamma}} d\mu \leq c_8 \mathcal{E}(f) \tag{4.5}$$

with $\|f\|_2 = 1$ and$0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

Constants c_i, $i = 1 \cdots 8$ doesn't depend on f.

Proof: From Theorem 3.1, it is enough to compute (or estimate) the function $B_0(t) = \sup_{s>0}(st - b_0(s))$ with $b_0(s) = sM(\frac{1}{s})$ and $M(t) = c_1 + c_2 t^{-\gamma}$. The computation shows that there exist $\beta_1, \beta_2 > 0$ such that

$$B_0(t) = \beta_1 [t - \log \beta_2]_+^{1+\frac{1}{\gamma}} \tag{4.6}$$

with β_2 which doesn't depend on c_1. It is easy to complete the proof.

The inequality (WS) essentially says that if $0 \leq f \in \mathcal{D}$ then f is in an Orlicz space of type $L^2(\log L_+)_+^{1+\frac{1}{\gamma}}$. We note that Gross' inequality or LSP is expressed with the Zygmund space $L \log L$. This space is universal in the sense that it doesn't depend on the semigroup.

In the case of polynomial ultracontractivity, we have seen that the space involved is $L^{2+\frac{4}{n}}$. So it was expected that in the case of one-exponential the space involved is intermediate between $L \log L$ and $L^{2+\frac{4}{n}}$, namely $L^2(\log L_+)_+^{1+\frac{1}{\gamma}}$.

4.2 Double-exponential class

We recall the definition of the class of double-exponential type for a symmetric submarkovian semigroup (T_t):

There exist $c_1, c_2, c_3 > 0$, $\alpha > 0$ such that, for any $t > 0$:

$$\|T_t f\|_\infty \leq c_1 \exp(c_2 \exp(\frac{c_3}{t^\alpha})) \tag{4.7}$$

But it can be more interesting to use the definition of LSP of double-exponential for a wider range of applications. Recall that LSP of double-exponential is not equivalent to the double-exponential ultracontractivity. Indeed, in the next section, we shall apply our result to a non-ultracontractive semigroup but satisfying a double-exponential LSP. We set $D_\alpha(t) = t_+ \, [\log t_+]_+^{\frac{1}{\alpha}}$.

Theorem 4.2 *(Double exponential)*
We fix $\alpha > 0$. The following statements are equivalent,

(LSPd) There exist $d_1, d_2 > 0$ such that

$$\int_X f^2 \ln f \, d\mu \leq t\mathcal{E}(f) + d_1 \exp(d_2 t^{-\alpha}) \|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{4.8}$$

for any $t > 0$ and any $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

(EEd) There exist $d_3, d_4 > 0$ such that

$$d_3 D_\alpha(d_4^{-1} \int_X f^2 \ln f \, d\mu) \leq \mathcal{E}(f) \tag{4.9}$$

with $\|f\|_2 = 1$ and $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

(NTId)There exist $d_5, d_6 > 0$ such that

$$d_5 \|f\|_2^2 D_\alpha(d_6^{-1} \log \|f\|_2) \leq \mathcal{E}(f) \tag{4.10}$$

with $\|f\|_1 = 1$ and $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

(Wsd) There exist $d_7, d_8 > 0$ such that

$$d_7 \int_X f^2 D_\alpha \left(d_8^{-1} \log \left(\frac{f}{8\|f\|_2} \right) \right) d\mu \leq \mathcal{E}(f) \tag{4.11}$$

with $\|f\|_2 = 1$ and $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$.

Constants d_i, $i = 1 \cdots 8$ doesn't depend on f.

Remark 4.3 *As in the preceding examples, we didn't try to get the best constants.*

Proof: We apply Theorem 3.1. It is enough to estimate B_0 of this theorem. But the computation of B_0 seems to be more diffucult. We have,
for any $\beta > 1$ and any $t \in \mathbb{R}$,

$$(\beta - 1) d_1 d_2^{-\frac{1}{\alpha}} D_\alpha(\frac{t}{c_1 \beta}) \leq B_0^+(t) \leq c_1 c_2^{-\frac{1}{\alpha}} D_\alpha(\frac{t}{c_1}) \tag{4.12}$$

where $B_0^+(t) = \sup(B_0(t), 0)$.

We note that the upper bound doesn't depend of β. Because β can as close as we want of 1 by above the estimate (4.12) is optimal up to constants.

The WS can be written more explicitely,

$$\int_X f^2 \log \left(\frac{f}{8\|f\|_2} \right) \left[\log \left(\frac{1}{d_8} \log_+ \left(\frac{f}{8\|f\|_2} \right) \right) \right]_+^{\frac{1}{\alpha}} d\mu \leq k \mathcal{E}(f) \tag{4.13}$$

with $k = \frac{d_7}{d_8}$. The underlying space is symbolically

$$L^2 \log L \left[\log \log_+ L \right]_+^{\frac{1}{\alpha}}. \tag{4.14}$$

We recall that at the autors' knowledge, we don't know if LSPd implies double-ultracontractivity when $0 < \alpha < 1$ with the *same* exponent (See section ??). Indeed, the method of proof associated to NTI give some lost in the exponent. The new exponent is $\alpha' = \frac{\alpha}{1-\alpha}$ with $0 < \alpha < 1$.

4.3 Another example

We consider the case where $M(t) = c_1 + c_2 \left[\log \frac{c_3}{t} \right]_+^{\frac{1}{\alpha}}$ in LSP (1.4). Recall that the ultracontractive bound

$$||T_t f||_\infty \leq e^{c_1 + c_2 \left[\log \frac{c_3}{t} \right]_+^{\frac{1}{\alpha}}}, \quad t > 0,$$

implies LSP with $M(t)$ just above. In such a situation, we have

Theorem 4.4 *Let $\alpha > 0$. Assume that, for any $t > 0$ and for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, we have the following LSP,*

$$\int_X f^2 \log \left(\frac{f}{||f||_2} \right) d\mu \leq t\, \mathcal{E}(f) + \left(c_1 + c_2 \left[\log \frac{c_3}{t} \right]_+^{\frac{1}{\alpha}} \right) ||f||_2^2.$$

We have the following weak Sobolev inequality, for any $f \in \mathcal{D} \cap L^1 \cap L^\infty$, $||f||_2^2 = 1$,

$$\int_X f^2\, W_\rho \left(\log f \right) d\mu \leq \mathcal{E}(f)$$

for any $\rho > 1$, with

$$W_\rho(x) \sim \left(\frac{k_2}{k_3}\right)^\alpha e^{-1}\alpha^{-1}\left(\frac{x-k_1}{k_3}\right)^{1-\alpha}\exp\left(\frac{x-k_1}{k_2}\right)^\alpha$$

as $x \to +\infty$, where $k_1 = c_1 + c_{2,\rho}, k_2 = c_2, k_3 = \frac{c_3}{c_{1,\rho}}$ with $c_{1,\rho}, c_{2,\rho}$ as in Corollary 2.12.

Note that $M(t) = c_1$ for any $t \geq c_3$. To the behavior of $M(t)$ as t goes to zero corresponds to the behavior of $W_\rho(x)$ as x goes to infinity. It is due to the inversion $t = \frac{1}{s}$ in the formula

$$W_\rho(x) = \sup_{s>0}\left(sx - s\tilde{M}_\rho\left(\frac{1}{s}\right)\right)$$

of Corollary 2.12.

Proof: By Corollary 2.12, we just have to estimate

$$W_\rho(x) = \sup_{s>0} H_x(s), \quad x \in \mathbb{R}$$

with $H_x(s) = s(x-k_1) - sk_2\left[\log k_3 s\right]_+^{\frac{1}{\alpha}}$ where k_1, k_2, k_3 are in Theorem 4.4. Letting $v = k_3 s$, we get

$$W_\rho(x) = \sup_{v>0}\left(v\left(\frac{x-k_1}{k_3}\right) - v\frac{k_2}{k_3}\left[\log v\right]_+^{\frac{1}{\alpha}}\right).$$

Let $a = \frac{x-k_1}{k_3}$ and $c = \frac{k_2}{k_3}$. So

$$W_\rho(x) = \text{Max}\left(\sup_{0<1\leq 1} I_x(v), \sup_{v>1} I_x(v)\right) = \text{Max}\left(a, \sup_{v>1} I_x(v)\right)$$

with $I_x(v) = av$ if $0 < s \leq 1$ and $I_x(v) = av - cv(\log v)^{\frac{1}{\alpha}}$ if $v > 1$. For $v > 1$, the derivative $I_x'(v) = a - cK(v)$ with $K(v) = (\log v)^{\frac{1}{\alpha}} + \frac{1}{\alpha}(\log v)^{\frac{1}{\alpha}-1}$.

If $0 < \alpha \leq 1$, $K : [1, +\infty[\to [0, +\infty[$ is strictly increasing so bijective. Therefore, there exists $v_0 \geq 1$ such that $\frac{a}{c} = K(v_0)$ and I_x is increasing on $[1, v_0]$ and decreasing on $[v_0, +\infty[$. So the supremum is realized at v_0.

If $\alpha > 1$, K is decreasing from 1 to its minimum and increasing from its minimum to infinity. So there are two extremum for I_x. The first one denoted by v_- which give a local minimum for I_x and is bounded uniformly with respect to x and $v_+ > v_-$ which gives a maximum for I_x and tends to infinity as x goes to infinity.

Let $v_1 = v_0$ if $\alpha \leq 1$ and $v_1 = v_+$ if $\alpha > 1$. So $I_x'(v_1) = 0$ i.e. $\frac{a}{c} = (\log v_1)^{\frac{1}{\alpha}} + \frac{1}{\alpha}(\log v_1)^{\frac{1}{\alpha}-1}$ gives the relation $a - c(\log v_1)^{\frac{1}{\alpha}} = \frac{c}{\alpha}(\log v_1)^{\frac{1}{\alpha}-1}$. Thus

$$I_x(v_1) = v_1\frac{c}{\alpha}(\log v_1)^{\frac{1}{\alpha}-1}$$

and also the estimates

$$\left(\frac{a}{c}\right)^\alpha \sim \log v_1$$

and

$$e^{-1}e^{\left(\frac{a}{c}\right)^\alpha} \sim v_1.$$

It yields

$$I_x(v_1) = \frac{c^\alpha}{\alpha}e^{-1}e^{\left(\frac{a}{c}\right)^\alpha}a^{1-\alpha}.$$

For x large i.e. a large, we deduce

$$W_\rho(x) \sim \frac{c^\alpha}{\alpha}\left(\frac{x-k_1}{k_3}\right)^{1-\alpha} e^{-1} e^{\left(\frac{x-k_1}{k_2}\right)^\alpha}.$$

The proof is completed.

5 Examples

In this section, we provide explicit examples of semigroup in the classes one-exponential and double-exponential ultracontractivity (See the preceding subsection for definitions or Section 4). These examples comes from the existing literature. First, we start by recalling some results of A. Bendikov [3] [B2]. These examples are convolution symmetric semigroups of the infinite dimensional torus $\mathbb{T}^\infty$. Similar examples can built on the real line [BM2], [BCS].

5.1 Brownian semi-groups on $\mathbb{T}^\infty$.

Let $X = \mathbb{T}^\infty$ be the infinite dimensional torus and $\mathbb{Z}^{(\infty)} = \{\theta = (\theta_k)_{k=1}^\infty, \theta_k \in \mathbb{Z}, ...\} \subset \mathbb{Z}^\infty$ its dual group (See [BF], [B3]). The compact group $\mathbb{T}^\infty$ is endowed of its normalized Haar measure denoted by $d\mu$. For a given sequence of positive numbers $\mathcal{A} = \{a_k\}_{k=1}^\infty$. we denote by g_t the Brownian semigroup on the one-dimensional torus $\mathbb{T}$. That is g_t are a semigroup of probability measures on $\mathbb{T}$, we shall also denote by g_t their density with respect to the normalized Haar measure on $\mathbb{T}$. Then we can consider the infinite product $\mu_t^{\mathcal{A}} = \otimes_{k=1}^\infty g_{ta_k}$. This defines a convolution probability semigroup on $\mathbb{T}^\infty$ (See Section 3 of [B2] and also [B3]). The Fourier transform of $\mu_t = \mu_t^{\mathcal{A}}$ is given by

$$\hat{\mu}_t(\theta) = e^{-t\Psi(\theta)}, \quad \Psi(\theta) = \sum_{k=1}^\infty a_k \theta_k^2, \quad \theta \in \mathbb{Z}^{(\infty)}.$$

The counting function associated to the coefficients

$$\mathcal{N}^{\mathcal{A}}(x) = \mathrm{Card}\{a_k \le x\}, \quad x > 0.$$

plays a fundamental rôle in the analysis of the behavior of the semigroup $\mu_t^{\mathcal{A}}$.
Indeed, there exists a continuous density for $\mu_t^{\mathcal{A}}$ for any $t > 0$ with respect to Haar measure on $\mathbb{T}^\infty$ iff $\log \mathcal{N}^{\mathcal{A}}(x) = o(x)$ pour $x \mapsto +\infty$ (See [BF] P. ??? [] Berg?). We also denote by $\mu_t = \mu_t^{\mathcal{A}}(x)$ this density and 0 the neutral element of $\mathbb{T}^\infty$. We have the following formula

$$\log \mu_t(0) = \int_0^\infty \mathcal{N}^{\mathcal{A}}(x) G(xt) \frac{dx}{x}$$

where G is a function independent of the sequence $\mathcal{A}$ (See [B2] Section 2.3). The value at 0 of $\mu_t^{\mathcal{A}}$ is related to the property of ultracontractivity as follows

$$\|T_t\|_{1,\infty} = \mu_t(0)$$

where (T_t) is the associated semigroup of operators (Let say on L^2) to μ_t. We denote by $M(t) = \log \mu_t(0)$. Then the following LSP is satisfied

$$\int_{\mathbb{T}^\infty} f^2 \log f \, d\mu \le t\mathcal{E}(f) + M(t), \quad t > 0, \quad \|f\|_2^2 = 1$$

[3] The authors thank A.Bendikov for mentioning the existence of such examples during theconference of Anoya 1995 ? Creete)

with $\|f\|_2^2 = \int_{\mathbb{T}^\infty} |f|^2 d\mu$. We denote by $\mathcal{E}(f) = \mathcal{E}_{\mathcal{A}}(f) = \int_{\mathbb{T}^\infty} \mathcal{L}_{\mathcal{A}} f(x) f(x)\, d\mu(x)$ the associated Dirichlet form where $\mathcal{L}_{\mathcal{A}} = \sum_{k=1}^{\infty} a_k \partial_k^2$ is (formally) the generator of $\mathcal{E}_{\mathcal{A}}$. With the help of the Fourier transform, we also have $\mathcal{E}(f) = \sum_{k \in \mathbb{Z}^{(\infty)}}^{\infty} a_k \theta_k^2 |\hat{f}(\theta_k)|^2$. So a particular choice of the sequence $\mathcal{A}$ determines the behavior of the counting function $\mathcal{N}^{\mathcal{A}}$ from which we deduce the behavior of $\mu_t(0)$ for small time t.

The following examples deserve to illustrate both type of ultracontractivity mentioned above and descrived in 4.1 and 4.2. These examples come from [B2].

5.2 One-exponential decay on $\mathbb{T}^\infty$

With notations and formulas above. Let $0 < \alpha$ and $a_k = k^{1/\alpha}, k \geq 1$ then $\mathcal{N}_{\mathcal{A}}(x) = [x^\alpha] \sim x^\alpha, x \mapsto \infty$. We apply Theorem (ii): there exists $c_1 > 0$ such that

$$\log \mu_t^{\mathcal{A}}(0) \sim c_1 t^{-\alpha}, \quad t \mapsto 0$$

Ultracontractivity property is obtained by the relation

$$\|T_t\|_{1 \to \infty} = \mu_t(0)$$

where $\|T_t\|_{1\to\infty}$ is the norm $L^1 - L^\infty$ of the operator T_t. So we have

Theorem 5.1 *[B2]. Let $a_k = k^{1/\alpha}, k \geq 1$. then for any $0 < \epsilon < 1$, there exists $t(\epsilon) > 0$ such for any $t \in]0, t(\epsilon)[$,*

$$e^{c_-(\epsilon) t^{-\alpha}} \leq \|T_t\|_{1\to\infty} \leq e^{c_+(\epsilon) t^{-\alpha}}$$

with $c_\pm(\epsilon) = (1 \pm \epsilon) c_1$.

Then we deduce the following functional inequalities,

Theorem 5.2 *Let $0 < \alpha$ and $a_k = k^{1/\alpha}, k \geq 1$. Then there exist $c_7, c_8 > 0$ such that, for any $0 \leq f \in \mathcal{D}(\mathcal{L}_{\mathcal{A}}) \cap L^1 \cap L^\infty$ with $\|f\|_2 = 1$,*

$$\int_{\mathbb{T}^\infty} f^2 \left[\log \left(\frac{f}{c_7} \right) \right]_+^{1+\frac{1}{\gamma}} d\mu \leq c_8 (\mathcal{L}_{\mathcal{A}} f, f) \tag{5.15}$$

5.3 Double-exponential decay on $\mathbb{T}^\infty$

Similarly to the preceding section, we build explicitly a convolution semigroup on $\mathbb{T}^\infty$ satisfying the double-exponential ultracontractivity for a convenient choice of $\mathcal{A} = (a_k)_k$.

For any fixed $\alpha > 0$, we set $a_k = (\log k)^{1/\beta}, k \geq 2$ et $a_1 = 1/2$ with $\beta = \alpha/(\alpha+1)$. Then $\log \mathcal{N}^{\mathcal{A}}(x) \sim x^\beta, x \mapsto \infty$. We have the following asymptotic estimate [B2](Thm 3.27),

$$\log \log \mu_t(0) \sim c_\alpha t^{-\alpha}, \quad t \mapsto 0, \quad c_\alpha = \alpha^\alpha/(\alpha+1)^{\alpha+1}.$$

In terms of ultracontractivity, it is expressed by the following inequalities,

Theorem 5.3 *(Thm 3.27 p.59 [B2]). For any $0 < \epsilon < 1$, there exists $t(\epsilon) > 0$ such that for any $t \in]0, t(\epsilon)[$,*

$$\exp(\exp(c_-(\alpha,\epsilon) t^{-\alpha})) \leq \|T_t\|_{1,\infty} \leq \exp(\exp(c_+(\alpha,\epsilon) t^{-\alpha}))$$

with $c_\pm(\alpha,\epsilon) = (1 \pm \epsilon) c_\alpha$.

5.4 Convolution semigroups on the real line

The first family of examples give the existence of convolution semigroups on the real line satisfying (**??**). In fact, $||T_t||_{1,\infty}$ is computed explicitly.

Proposition 5.4 *For any $\nu > 0$, there exists a convolution semigroup of probability $(\mu_t)_{t>0}$ on $\mathbb{R}$ such that the corresponding semigroup of operators $(T_t)_{t>0}$ satisfies*

$$||T_t||_{1,\infty} = \frac{1}{t}\exp(t^{-\nu})$$

for any $t > 0$. Consequently, as t tends to 0 :

$$\log ||T_t||_{1,\infty} \sim (t^{-\nu})$$

and for any $\epsilon > 0$ and any $\eta > 0$, there exist $C_{\epsilon,\eta} > 0$ such that

$$||T_t||_{1,\infty} \leq C_{\epsilon,\eta}\exp(\eta t^{-\epsilon} + t^{-\nu}).$$

In particular, for any $\eta > 0$:

$$||T_t||_{1,\infty} \leq C_{\eta}\exp((1+\eta)t^{-\nu})$$

This family of examples comes from [HJ]. The approach for the treatment of these examples comes from the potential theory on abelian locally compact groups (See [BF] and also the sketch of the proof below). On $\mathbb{R}^n$, by tensorization of measures of convolution, we get for any $\nu_i > 0, i = 1...n$, we can built convolution symmetric semigroup satisfying,

$$||T_t||_{1,\infty} = \frac{1}{t^n}\exp(\sum_{i=1}^{n} t^{-\nu_i})$$

By the same argument of the construction of W.Hoh et N.Jacob, we easily show

Proposition 5.5 *For any $\nu \geq 1$, there exists a symmetric convolution semigroup such that*

$$||T_t||_{1,\infty} = \exp(t^{-\nu}) - 1$$

for any $t > 0$.

Proof : We recall the construction of such semigroups (See [HJ]).

Let $a(t) = \exp(t^{-\nu}) - 1$. We look for a non-decreasing function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $f(x) > 0$ for any $x > 0$and such that $a(t) = \mathcal{L}f(t) = \int_0^{+\infty} e^{-xt} f(x)\,dx$ is the Laplace transform of f. For any $\nu > 1$, we find

$$f(x) = f_\nu(x) = \sum_{k=1}^{\infty} \frac{x^{\nu k-1}}{k!\Gamma(\nu k)}$$

Then the semigroup is built in the following way, we set $F(x) = \int_0^x f(u)\,du$, F is continuous, strictly increasing and convex with $F(0) = 0$ and $F(+\infty) = +\infty$. Let F^{-1} the inverse, we set $\psi(\xi) = F^{-1}(|\xi|/\pi)$ then ψ is an even non-decreasing concave continuous function. From Proposition 10.6 of [BF] : ψ is *negative definite* (See p.39 of [BF] for the definition and the next pages for properties).

We then define the convolution semigroup $(\mu_t)_{t>0}$ by setting for its Fourier transform

$$\hat{\mu}_t(\xi) = \exp(-t\psi(\xi))$$

for $\xi \in R$ (See Theorem 8.3 of [BF]). We then check that the norm satisfies

$$||T_t||_{1,\infty} = \mathcal{L}f(t), \quad t > 0.$$

We shall note that this construction of convolution semigroup applies to other type of ultracontractivity. On $\mathbb{R}^n$ and for any $\nu_i > 0, i = 1...n$, by tensorization of semigroups there exists a convolution semigroup

$$||T_t||_{1,\infty} = \prod_{i=1}^{n} (\exp(t^{-\nu_i}) - 1)$$

5.5 Semigroups generated by very degenerate differential operators

(Examples of semigroups with one-exponential ultracontractivity for small time).

In [FL], P.Florchinger et R. Léandre studied the density of very degenerate diffusions. We first start by recalling some of their results and apply Theorems of Section **??** to these diffusions.

On $\mathbb{R}^2$, we consider the generator A define by

$$A = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}g^2(x)\frac{\partial^2}{\partial y^2} + h(x)\frac{\partial}{\partial y}$$

with h and g bounded continuous functions. We suppose for any $x \in \mathbb{R}^2$:

$$g(x) > C\exp(-\frac{c}{|x|^\alpha})$$

with $\alpha \in [0, 2[$.
We have the following uniform bound $p_t((x, y), (x', y'))$ for p_t the heat kernel associated to A (See [FL]) :

Theorem 5.6 *For any (x, y) and $(x', y') \in \mathbb{R}^2$ and every $t > 0$:*

$$p_t((x, y), (x', y')) \leq \frac{1}{t}\exp(\frac{C_0}{t^\gamma})$$

Moreover if $\alpha \in [0, 1[$ then $\gamma < 1$ and γ tends to infinity as α tends to 2.

This upper bound is essentially optimal when $h = 0$ and g is exponentially flat that is

$$c_1\exp(-\frac{c_2}{|x|^\alpha}) < g(x) < c_3\exp(-\frac{c_4}{|x|^\alpha})$$

with $\alpha \in]0, 2[$. Then

$$p_t((0, 0), (0, 0)) \geq \exp(\frac{c_1}{t^{\frac{\alpha+\gamma\alpha}{2}}})\exp(\frac{-c_2}{t^\gamma}) \tag{5.16}$$

If $\gamma < \frac{\alpha}{2-\alpha}$, we have the following lower bound

$$p_t((0, 0), (0, 0)) \geq \exp(\frac{c_1}{t^{\frac{\alpha+\gamma\alpha}{2}}}) \tag{5.17}$$

In the sequel, we assume that $h = 0$. We set $Q(f) = (Af, f)_{L^2(dxdy)}$ then Q is a Dirichlet form (See [Fu]). We denote by T_t the associated sub-marovian semigroup.

We have LSP for Q,

Theorem 5.7 *For any $\epsilon > 0$ and all $f > 0$ with $f \in \mathcal{D}(Q) \cap L^1 \cap L^\infty$ then*

$$\int_{\mathbb{R}^2} f^2 \ln f\, dxdy \leq \epsilon Q(f) + F(\epsilon)\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{5.18}$$

with $\|f\|^2 = \int_{\mathbb{R}^2} h^2(x, y)\, dxdy$ and $F(\epsilon) = \frac{C}{\epsilon^\gamma} - \frac{1}{2}\ln\epsilon + C$.

Proof : It seems that we can avoid to reproduce probabilistic computations using the Brownian bridge Ilof [FL] p.139, by using an interpolation theorem ((Riesz-Thorin) since T_t is a contraction on L^∞. We then obtain two inequalities

$$||T_t f||_\infty \leq \sup_{X,Y\in\mathbb{R}^2} p_t(X,Y)||f||_1 \leq \frac{1}{t}\exp(\frac{C_0}{t^\gamma})||f||_1$$

and

$$||T_t f||_{infty} \leq ||f||_\infty$$

Then by interpolation ([**?**], p. 3), we get: for any $1 \leq p \leq \infty$,

$$||T_t f||_\infty \leq \frac{1}{t^{\frac{1}{p}}}\exp(\frac{C_0}{pt^\gamma})\,||f||_p \tag{5.19}$$

Thus we can explicite the constants appearing in (A.5)p.140 of [FL]. In the particular case $p = 2$, we obtain the $L^{2,\infty}$-norm of the semigroup: $T_t$is bounded above by $\frac{1}{\sqrt{t}}\exp(\frac{C_0}{2t^\gamma})$. We deduce the expected theorem by applying 2.2.3 de [**?**].
From that LSP, we can get that the semigroup H_t^L generated by

$$L = \frac{\partial}{\partial x}(a_{1,1}(x,y)\frac{\partial}{\partial x}) + \frac{\partial}{\partial x}(a_{1,2}(x,y)\frac{\partial}{\partial y}) + \frac{\partial}{\partial y}(a_{2,1}(x,y)\frac{\partial}{\partial x}) + \frac{\partial}{\partial y}(a_{2,2}(x,y)\frac{\partial}{\partial y})$$

where the matrix a is such that

$$a(x,y) \geq \begin{pmatrix} 1 & 2 \\ 0 & g(x) \end{pmatrix}$$

possesses a density h_t^L uniformly bounded above by $\frac{1}{t}\exp(\frac{C}{t^\gamma})$. Indeed, the operator L is formally self-adjoint on $L^2(dxdy)$ and the Dirichlet forms satisfy :

$$Q_L \geq Q_A$$

So LSP (5.18) is also satisfied by Q_L. By applying the converse (See [FL],[**?**]), we deduce that H_t^L satsifies

$$||H_t^L f||_\infty \leq \frac{C}{\sqrt{t}}\exp(\frac{C}{t^\gamma})||f||_2 \tag{5.20}$$

Or more generally,

$$||H_t^L f||_\infty \leq \frac{C_1}{t^{\frac{1}{p}}}\exp(\frac{C_2}{pt^\gamma})||f||_p \tag{5.21}$$

for any $1 \leq p \leq +\infty$. By symmetry of the semigroup H_t^L, we obtain from (5.21) with $p = 1$ that the density h_t^L is bounded above $\frac{C}{t}\exp(\frac{C}{t^\gamma})$.
We easily deduce from(5.21),

$$||H_t^L f||_\infty \leq \kappa_1\exp(\frac{\kappa_2}{pt^\gamma})||f||_p, \quad t > 0. \tag{5.22}$$

So the hypothesis of Theorem l **??** are satisfied. We have

Theorem 5.8 *For any function $f \geq 0$ and $f \in \mathcal{D}(Q) \cap L^p$.*
1)

$$\int f^2\left(\ln(\frac{f}{2\kappa||f||_p})\right)_+^{\frac{1}{\gamma}} dxdy \leq C_2\left((Lf,f) + ||f||_p^2\right) \tag{5.23}$$

2) For any $nm \geq 1$ and any $\nu > 0$,

$$\int f^2 \left(\ln(\frac{f}{\nu ||f||_p} + m) \right)^{\frac{1}{\gamma}} \, dxdy \leq C_3 \left((Lf, f) + ||f||_p^2 + ||f||_2^2 \right) \tag{5.24}$$

In particular if $p = 2$,

$$\int f^2 \left(\ln(\frac{f}{\nu ||f||_2} + m) \right)^{\frac{1}{\gamma}} \, dxdy \leq C_4 \left(Q(f) + ||f||_2^2 \right) \tag{5.25}$$

We note that the interesting situation is when $\alpha = \frac{1}{\gamma} \geq 1$. This is the case when $0 < \alpha < 1$ by [FL]. Indeed, we easily deduce (5.25),

$$\int f^2 \ln(\frac{f}{||f||_2}) \, dxdy \leq C \left(Q(f) + ||f||_2^2 \right) \tag{5.26}$$

We get what is called the local version of LSP (i.e. the L^2-norm of f ???) of LSP (5.18) for some fixed ϵ fixé. This is explained by the fact we use only use the one-exponential behavior for small time in this example.

6 Applications to some non-ultracontractive semigroups

In this section, we recall the example of Davies and Simon (See [DS] (Section 6.Remarque 1 p.359)) of the semigroup which satisfies no ultracontractivity property but satisfies a double exponential LSP with $\alpha = 1$

We also recall of an example of Kavian-Kerkyacharian-Roynette [KKR] for which explicit computations of lower and upper bound for the heat kernel $h_t(x, y)$ proves that the semigroup is not ultracontractive.

7 Application to some sub-laplacians on Lie groups

Let G be a real connected Lie group. We assume that G is unimodular i.e. the Haar measure dg is left and right invariant:

$$\int_G f(gx) \, dx = \int_G f(xg) \, dx = \int_G f(x) \, dx \tag{7.1}$$

for any $g \in G$ and $f \in C_0^\infty(G)$.

We consider $X = (X_1, \cdots, X_k)$ a familly of k left-invariant vector fields i.e.

$$X_i f_g = (X_i f)_g, \qquad \text{avec} \qquad f_g(x) = f(gx) \tag{7.2}$$

for any $x, g \in G$ and $f \in C_0^\infty(G)$. We assume that X satisfied the Hörmander condition that is X generates the Lie algebra of G. The operator $\Delta = -\sum_{i=1}^k X_i^2$ called sub-laplacian is sub-elliptic. We note $H_t = e^{-t\Delta}, t \geq 0$ the symmetric sub-markovian semigroup associated to l Δ (See [VSC]). The semigroup H_t admit a kernel h_t (called heat kernel) for the right convolution symmetric for the Haar measure such that

$$H_t f(x) = \int_G h_t(y^{-1}x) f(y) \, dy = f * h_t(x) \tag{7.3}$$

Moreover $0 < h_t \in C_0^\infty(\mathbb{R}^{+*} \times G)$.

The semigroup H_t satisfies the ultracontractive bound

$$\|H_t f\|_{1,\infty} = h_t(e) \tag{7.4}$$

where $\|f\|_p = \int_G |f|^p \, dg$ and e is the neutral element of G.
To apply our theory we just need to bound above the heat kernel at the origin. For that purpose, we recall a theorem due to N.Varopoulos under the assumption of exponential growth of the volume function. We denote by $\rho(x, y)$ the Carnot-Carathéodory distance between two points x and $y \in G$ and $B(x, t)$ the metric ball associated to d (See [VSC]). By invariance of the metric we have $B(x, t) = xB(e, t)$ where xA denotes the product of $x \in G$ and the set $A \subset G$. By invariance of the Haar measure, we have for all $x \in G$,

$$V(x, t) = \int_G \mathcal{X}_{B(x,t)}(g) \, dg = V(e, t) = V(t) \tag{7.5}$$

where $\mathcal{X}_A$ denote the characteristic function of the set A. The behavior of the volume $V(t)$ is given by a dichotomy result of Y.Guivarc'h,

- G has a polynomial volume growth i.e. there exists $D \in \mathbb{N}$ such that

$$V(t) \sim t^D, \quad t \longrightarrow +\infty. \tag{7.6}$$

- Or G has exponential volume growth i.e.

$$V(t) \sim e^t, \quad t \longrightarrow +\infty. \tag{7.7}$$

the following results come from [VSC] (See also references there in).

Theorem 7.1 *Let G be a connected real Lie group of exponential volume growth. Then*

(i) There exits $c_1 > 0$ such that for any $0 < t \leq 1$,

$$c_1^{-2} t^{-d/2} \leq h_t(e) \leq c_1^2 t^{-d/2}. \tag{7.8}$$

(ii) There exist $\kappa_0, \kappa_1, c_2, c_3 > 0$ such that for any $t \geq 1$,

$$\kappa_0^2 \exp(-\kappa_1 t^{1/3}) \leq h_t(e) \leq c_2^2 \exp(-2c_3 t^{1/3}) \tag{7.9}$$

(iii) For any $0 < t \leq 1$:

$$\|H_t f\|_\infty \leq c_1^2 t^{-d/2} \|f\|_1 \tag{7.10}$$

and

(iv) For any $t \geq 1$:

$$\|H_t f\|_\infty \leq c_2^2 \exp(-2c_3 t^{1/3}) \|f\|_1. \tag{7.11}$$

We deduce (iii) from (ii). The integer d is called the local dimension. This exponent is related to the volume behavior for small time, let say $V(t)$ with $0 < t \leq 1$ (See [VSC]). The estimates (7.10) and (7.11) are optimal up to constants. Indeed, we have the lower bounds (7.8) and 7.9)and the relation,

$$\|H_t\|_{1 \to +\infty} = h_t(e) \tag{7.12}$$

From Theorem 3.1, we get the following WS inequality

Theorem 7.2 *Let G, Δ, d be as in Theorem 7.1. Then there exist $t_0 = t_0(c_1, d, c_2, c_3) > 0$ and $\lambda_3 > 0, \lambda_4 > 0$ such that for any $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$,$\|f\|_2 = 1$,*

$$\int_G f^2[-\log(\frac{f}{\lambda_4})]^{-2}\mathcal{X}\, dg + \int_G f^{2+4/d}(1-\mathcal{X})\, dg \leq \lambda_3 \int_G |\nabla f|^2\ dg \tag{7.13}$$

with $|\nabla f|^2 = \sum_{i=1}^k |X_i f|^2$ and $\mathcal{X} = \mathcal{X}_{\{f \leq t_0\}}$

and also

$$\int_G f^2 W(f)\, dg \leq \lambda_3 \int_G |\nabla f|^2\ dg \tag{7.14}$$

with $W(f) = [-\log(f/\lambda_4)]^{-2}\,\mathcal{X} + f^{4/d}(1-\mathcal{X})$

Note that to the different behavior of the heat kernal (small or large time) corresponds to different control by $W(f)$ a logarithmic one when $f(x)$ is small ($0 \leq f \leq t_0$) and a polynomial behavior when $f(x)$ is large ($f \geq t_0$).

The small value of $f(x)$ correspond to large value of t in (7.11) and large value of f correspond small value of t in (7.10). This phenomenon is in fact general (See details of proof of Theorem 2.9 above).

Constants λ_3 and λ_4 depend also on c_1, d, c_2, c_3. The value of λ_4 is such that $t_0/\lambda_4 < 1$, there no singularity concerning the logarithm.
In fact, (7.13) is equivalent to ultracontractivity bound (7.10) and (7.11) for different constants.

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