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Nash-type inequalities and decay of semigroups of operators.

Patrick Maheux*,¹

*Fédération Denis Poisson, Département de Mathématiques MAPMO,
Université d'Orléans, F- 45067 Orléans, France*

Abstract

In that paper, we prove an equivalence between Nash-type inequalities and an exponential decay (in the sense of the definition 2.2) for symmetric submarkovian semigroups. This exponential decay generalizes the notion of spectral gap where this number is replaced by a function. We discuss different formulations of the decay associated to the usual Nash inequality in terms of Lyapunov-type functional. We apply this to different classes of ultracontractive semigroups as well as non-ultracontractive semigroups. In particular, we show that any ultracontractive semigroups always satisfy an exponential decay in the sense of 2.2. We treat different classes of examples, one of them containing the Ornstein-Uhlenbeck-type semigroup and Γ^* -semigroup. We apply our results to fractional powers of non-negative self-adjoint semigroup. We derive a simple criterium on the function characterizing the exponential decay to deduce ultracontractivity property and relations that must satisfy the ultracontractive bounds an heat kernel of the semigroup.

Key words: Submarkovian semigroup, Nash-type inequality, Functional decay of semigroup, Lyapunov functional, Ultracontractivity, Hypercontractivity, Log-Sobolev inequality, Ornstein-Uhlenbeck semi-group, Fractional powers, Spectral gap, Beckner's inequality, Heat kernel, Hardy-Littlewood-Sobolev inequalities, Γ^* -semigroup.

Mathematics Subject Classification (2010): 39B62

1. Introduction

Let (X, μ) be a σ -finite measure space and (T_t) a symmetric C_0 -semigroup on $L^2(X, \mu)$ with infinitesimal generator \mathcal{L} which can be extended as a contraction semigroup on $L^p(X)$, $1 \leq p \leq \infty$. We also write with the same notation

*Corresponding author at: Département de Mathématiques MAPMO, Université d'Orléans, F- 45067 Orléans, France.

Email address: patrick.maheux@univ-orleans.fr ()

(T_t) the semigroup acting on L^p . We shall consider more precisely symmetric sub-markovian semigroups that is: for any $0 \leq f \leq 1$ and for any $t > 0$, we have $0 \leq T_t f \leq 1$. So (T_t) extends as a contraction semigroup on L^p , $1 \leq p \leq +\infty$. In some cases, we shall assume that (T_t) is Markov that is $T_t 1 = 1$ for any $t > 0$ and μ can be a probability measure or not.

Recently, generalizations of the exponential spectral decay

$$(SG) \quad \|T_t f\|_2^2 \leq e^{-2\lambda t} \|f\|_2^2 \quad (1.1)$$

with $\lambda > 0$ has been extensively studied for semigroups (T_t) (see [W1] and also [W2] -[W7], [W-Z],[R-W] and references therein). See also the recent paper [C-G 2] proving equivalence between (SG) and L^p -analogues ($1 < p < +\infty$) when (T_t) is Markov in rather general situation with μ a probability measure.

One purpose, among the others of these papers, is to describe a function ξ such that

$$(GSG) \quad \|T_t f\|_p \leq \xi(t) \phi(f) \quad (1.2)$$

(for instance $\phi(f) = \|f\|_q$) where $\phi : \mathbb{R} \rightarrow [0, +\infty)$ is an homogeneous function of degree one and ξ a decreasing function on $(0, +\infty)$. This is a possible generalization of the L^2 - spectral gap (SG) corresponding to the case $\xi(t) = e^{-\lambda t}$, $p = 2$ and $\phi(f) = \|f\|_2$. In general situation, ξ can be weaker than exponential like polynomial for instance i.e. $\xi(t) = \frac{1}{t^\nu}$ and $\phi(f) = \|f\|_1$. For example, this is the case (by definition) for ultracontractive semigroups of polynomial decay i.e. for some $\nu > 0$ and for all $t > 0$,

$$\|T_t f\|_2 \leq \frac{c}{t^{\nu/2}} \|f\|_1. \quad (1.3)$$

Recall also that the spectral gap inequality (SG) is equivalent to Poincaré inequality

$$(P) \quad \lambda \|f\|_2^2 \leq (\mathcal{L}f, f). \quad (1.4)$$

The spectral gap is defined by

$$\lambda_0 = \inf\{(\mathcal{L}f, f) : \|f\|_2^2 = 1\}$$

that is the largest $\lambda > 0$ satisfying (1.4).

Other functional inequalities have been introduced to study (GSG). For instance, the so-called super-Poincaré (SP) inequality

$$(SP) \quad \|f\|_2^2 \leq s(\mathcal{L}f, f) + \beta(s)\phi(f), \quad s > 0 \quad (1.5)$$

with $\phi(f) = \|f\|_q^2$ and $q = 1$ or $q = +\infty$ (but other cases can be considered, see [W1]-[W5] and also [Z] and [A-B-D]). Another important case is super-Log-Sobolev inequality i.e.

$$\int_X f^2 \log \frac{f}{\|f\|_2} d\mu \leq t \mathcal{E}(f) + M(t) \|f\|_2^2$$

which is stronger than Log-Sobolev inequality of Gross (see [D],[D-S],[Bi-Ma],[BM1]).

It is well-known that super-Poincaré inequality is equivalent to a Nash-type inequality (NTI for short) and it has been proved by F-Y Wang that it is equivalent to the fact that the essential spectrum is empty, see [W0]. Indeed, (SP) clearly implies a (NTI) of the form

$$(NTI) \quad \Theta(\|f\|_2^2) \leq (\mathcal{L}f, f), \quad \phi(f) \leq 1$$

with $\Theta(x) = \sup_{t>0}(tx - t\beta(1/t))$. Usually $\phi(f) = \|f\|_1$ but also other cases can be considered. The original case of (NTI) is on \mathbb{R}^n with $\Theta(x) = cx^{1+2/\nu}$, $\nu > 0$, see [N],[C-L],[C-K-S]. Nash-type inequalities are used in different settings. See, for instance [C-K, H-K], for recent results on fractal sets and for jump processes.

Also in the case of super-Log-Sobolev inequality, it is proved in [Ma] that it implies Nash-type inequality with $\Theta(x) = x\mathcal{N}(\log x)$ with $\mathcal{N}(y) = \sup_{t>0}(ty/2 - tM(1/t))$, $y \in \mathbb{R}$. It is also shown in [Ma] that super-Log-Sobolev inequality and Nash-type inequality for Dirichlet forms are equivalent when Θ is given by such expression above.

Note that, (NTI) above formally also includes the case of Poincaré inequality (with $\Theta(x) = \lambda x$). So, we can guess that (NTI) can also be used to study divers notions of functional decay for semigroups enlarging the case of the spectral gap. We start by generalizing (NTI) in L^p -spaces as

$$(NTI)_{p,q} \quad \Theta(\|f\|_p^p) \leq (\mathcal{L}f, f_p), \quad \|f\|_q \leq 1.$$

for $1 \leq q < p < \infty$. So, $(NTI)_{2,1}$ corresponds to (NTI). Such inequality is used as a tool to study different types of functional decays: ultracontractivity, Sobolev inequalities, spectral gap ... It is also deduced from them, see for instance [Co],[BM1]. In this paper, we are mainly interested by the control of the form $\phi(f) = \|f\|_q$. In particular, we show that $(NTI)_{2,1}$ implies $(NTI)_{p,q}$ inequalities for some other couples (p, q) . Note that in general, it is necessary to introduce a control on f of the form $\phi(f) \leq 1$ because $\Theta(x)$ is not necessarily of the linear form cx .

In this paper, we shall study functional decay with another point of view. We introduce a new functional decay which also generalizes the spectral gap in L^p for sub-markovian symmetric semigroups. We shall show that this functional decay is equivalent to Nash-type inequality (see Th. 2.5 and Th. 2.8). Our approach, first, consists in noting that the spectral gap (SG) on L^p can be also written as

$$(EFD)_{p,q} \quad \mathcal{G}(\|T_t f\|_p^p) \leq e^{-pt} \mathcal{G}(\|f\|_p^p), \quad \|f\|_q \leq 1.$$

with $\mathcal{G}(x) = x^{1/\lambda}$ with λ the spectral gap. So, the spectral gap can be seen as a function, namely \mathcal{G} . The normalization above is artificial when \mathcal{G} is an

homogeneous function but it is necessary in other situations.

Note that the idea to see a dimension as a function has been previously introduced to study the generalization of isoperimetric dimension, see [C-G-L] for instance. In our case, the function \mathcal{G} can be seen as a generalization of the spectral dimension. Such function may exist even if there is no spectral gap in the usual sense, for instance with the Laplacian on \mathbb{R}^n (see Section 4.1).

Our aim is to study what we call exponential functional decay (EFD for short) described by inequality $(EFD)_{p,q}$ where \mathcal{G} is an increasing function (see definitions 2.2 and 2.3). In particular, we study $(EFD)_{p,q}$ in relation with (NTI). In fact, we show that these two functional inequalities $(EFD)_{p,q}$ and $(NTI)_{p,q}$ are equivalent when Θ has the following form $\Theta(x) = x\mathcal{N}(\log x)$. This assumption on Θ which includes many interesting cases will be made throughout all this paper. Note that it contains the case of the spectral gap with $\mathcal{N}(\log x) = \lambda$. We also give explicit formulas relating both functions Θ in $(NTI)_{p,q}$ and \mathcal{G} in $(EFD)_{p,q}$. The inequality $(EFD)_{p,q}$ can be interpreted as Lyapunov functional (with ϕ -control), see the comment after definition 2.2. We give several examples of function Θ where the function \mathcal{G} can be computed easily. In the course of the discussion, related to $(EFD)_{p,q}$, we shall see some relations with other functional inequalities like Sobolev, log-Sobolev, Sobolev-Orlicz inequalities, ultracontractivity, hypercontractivity ...

The contents of this paper is the following:

In Section 2, we introduce the definitions of $(NTI)_{p,q}$ and $(EFD)_{p,q}$ and the main theorems of this paper which show the equivalence between these two notions (Th. 2.5, 2.8).

In Section 3, we show or recall how Nash-type inequalities can be obtained from other functional inequalities. In particular, extending some results of [Bi-Ma] (see also [C-G-L]) we show that super L^p -log-Sobolev inequalities implies $(NTI)_{p,q}$ with $q = p - 1$ (Th.3.1). We prove that $(NTI)_{2,1}$ implies $(NTI)_{p,p/2}$ (Th. 3.3) with $p \geq 2$ and deduce $(EFD)_{p,q}$ with $q = p/2$ ($p \geq 2$) from $(EFD)_{2,1}$ or equivalently from $(NTI)_{2,1}$. We also apply our results to ultracontractive and hypercontractive semigroups (Th.3.2).

In Section 4, we give different family of examples of semigroups and compute explicitly the corresponding function \mathcal{G} . More explicitly,

Subsection 4.1: we deal with polynomial ultracontractive semigroups (in particular the heat semigroup on \mathbb{R}^n). We describe explicitly the function \mathcal{G} of $(EFD)_{p,q}$ in that case and give an application in terms of space-time mixed-norms and deduce a Hardy-Littlewood-Sobolev-type inequality.

Subsection 4.2: we generalize the preceding subsection by introducing more-

over a spectral gap. Of course, we recover the case with no spectral gap ($\lambda = 0$) letting $\lambda \rightarrow 0^+$.

Subsection 4.3: we study ultracontractive semigroups with one-exponential decay which corresponds to $\Theta(x) = x \log x^{1+1/\alpha}$, $\alpha > 0$ with x large (see definition 4.31).

Subsection 4.4: we consider the case of double-exponential case or more generally the corresponding Nash-type inequality $(NTI)_{2,1}$ with $\Theta(x) = x \log x (\log \log x)^{1/\gamma}$ with x large.

Subsection 4.5 is devoted to examples of Ornstein-Uhlenbeck semigroups-type. We also discuss hypercontractivity, Gross-type inequality and a modified super-Poincaré inequality adapted to this situation.

Subsection 4.6 deals with Γ^* -semigroup on \mathbb{R} . This case is closed to Ornstein-Uhlenbeck semigroups and share similar properties up to some points. In particular, we prove a Gross-type inequality.

In Section 5, we apply our theory to fractional power of generators of semigroups. We use in a crucial way a result of [BM1] which states that a Nash-type inequality always implies Nash-type inequality for the fractional powers. This enable us to deduce $(EFD)_{p,q}$ for these operators. In particular, we can apply this to fractional powers of the Laplacian on \mathbb{R}^n .

In the last Section 6, we revisit some inequalities as spectral gap, ultracontractivity from the point of our (EFD) inequalities. In particular, we study the case where the measure is a finite measure. We also reinterpret the implication $(NTI) \rightarrow$ ultracontractivity of [Co] in terms of boundedness of \mathcal{G} in (EFD). We also give a consequence of (EFD) on the best ultracontractive bound of the semigroup (if it exists) and also for the heat kernel related the function \mathcal{G} of $(EFD)_{2,1}$.

2. Nash-type inequality and functional decay

We introduce the following definition of a (p, q) -Nash-type inequality. Let f be a measurable function. We denote by $f_p = \text{sgn}(f)|f|^{p-1}$ for $1 < p < +\infty$ with $\text{sgn}(x) = \frac{x}{|x|}$, $x \neq 0$ and $\text{sgn}(0) = 0$.

Definition 2.1. *Let $1 \leq q \leq p < \infty$ and $2 \leq p$. We say that a generator \mathcal{L} of a L^2 -symmetric semigroup $T_t = e^{-t\mathcal{L}}$ of contraction on L^r , $1 \leq r \leq \infty$, satisfies a (p, q) -Nash-type inequality if there exists a non-decreasing function $\mathcal{N}_{p,q} : \mathbb{R} \rightarrow \mathbb{R}$ such that : for all $f \in \mathcal{D}_0$ with $\|f\|_q \leq 1$,*

$$\|f\|_p^p \mathcal{N}_{p,q}(\log \|f\|_p^p) \leq (\mathcal{L}f, f_p) \quad (2.1)$$

The set \mathcal{D}_0 is some subdomain of the domain \mathcal{D} of \mathcal{L} . We shall call the function $\mathcal{N}_{p,q}$ a (p, q) -Nash function for \mathcal{L} . By homogeneity argument, it is necessary to introduce some control in (2.1) on the function f , namely $\|f\|_q \leq 1$. If $\mathcal{N}_{p,q}$ has the form $\mathcal{N}_{p,q}(y) = c \exp(\alpha y)$, $\alpha > 0$, then (2.1) reads

$$c \|f\|_p^{(1+\alpha)p} \leq (\mathcal{L}f, f_p) \|f\|_q^{\alpha p}. \quad (2.2)$$

When $\mathcal{L} = \Delta$ on \mathbb{R}^n (or a Riemannian manifold), the inequality just above implies a Gagliardo-Nirenberg type inequalities. Thus (2.1) can be seen a generalization of such inequality. We shall not continue in that direction in this paper.

We now introduce a general definition of functional decay in normed spaces. We give a slightly different definition in L^p setting (see definition 2.3 below). In that paper, we shall not attempt to give results in its greatest generality. But some results also hold true in this abstract setting.

Definition 2.2. *Let X and Y two normed spaces with norms respectively $\|\cdot\|_X$ and $\|\cdot\|_Y$. We assume that $X \cap Y$ is non-empty. We suppose that we are given a continuous semigroup of operators (T_t) which is defined on X and Y . We say that (T_t) satisfies a decay inequality on X relatively to Y if there exists \mathcal{G} a non-decreasing function defined on $[0, \infty)$ with value in $[0, \infty)$ and $\lambda \geq 0$ such that, for all $t > 0$ and for all $\|f\|_Y \leq 1$,*

$$\mathcal{G}(\|T_t f\|_X) \leq e^{-\lambda t} \mathcal{G}(\|f\|_X). \quad (2.3)$$

We shall say that (T_t) satisfies a $(\mathcal{G}, \lambda, X, Y)$ -decay.

When $\lambda = 0$, we shall say that the decay is degenerate.

The decay of $t \longrightarrow \mathcal{G}(\|T_t f\|_X)$ is obvious if the semigroup is a contraction on X . So, we have got some gain when $\lambda > 0$. If the semigroup is a contraction on Y , the inequality (2.3) is equivalent to the fact that the map $\gamma_f(t) = \gamma(t) := e^{\lambda t} \mathcal{G}(\|T_t f\|_X)$ is non-increasing. Indeed, if γ is non-increasing then $\gamma(t) \leq \gamma(0)$ which is exactly (2.3). Conversely, if (2.3) is satisfied, then for all $s > 0$, $\|T_s f\|_Y \leq \|f\|_Y \leq 1$. We apply (2.3) to $T_s f$. By semigroup property,

$$e^{\lambda t} \mathcal{G}(\|T_{t+s} f\|_X) \leq \mathcal{G}(\|T_s f\|_X).$$

Therefore

$$e^{\lambda(t+s)} \mathcal{G}(\|T_{t+s} f\|_X) \leq e^{\lambda s} \mathcal{G}(\|T_s f\|_X)$$

i.e $\gamma(t+s) \leq \gamma(s)$, for any $t, s > 0$. Let $u(t, x) = T_t f(x)$ be the solution of the parabolic equation $\mathcal{L}u(t, x) = \frac{\partial u(t, x)}{\partial t}$, $u(0, x) = f(x)$, we shall call γ a (time-dependent) Lyapunov functional with Y -control for this equation.

For $t = 0$, the inequality (2.3) is an equality. Note also that if (\mathcal{G}, λ) satisfies (2.3) then $(\alpha \mathcal{G}^\beta, \lambda \beta)$ also satisfies (2.3) for all $\alpha, \beta > 0$.

We give an equivalent definition for semigroups acting on L^p -spaces more suitable for computations. Namely, we substitute $\mathcal{G}(x^p)$ to $\mathcal{G}(x)$. More explicitly,

Definition 2.3. *Let $1 \leq q \leq p \leq +\infty$. A continuous semigroup (T_t) acting on L^p is said to satisfy a (\mathcal{G}, λ) -decay inequality on L^p relatively to L^q if there exists \mathcal{G} a non-decreasing function defined on $[0, \infty)$ with value in $[0, \infty)$ and $\lambda \geq 0$ such that, for all $t > 0$ and for all $\|f\|_q \leq 1$,*

$$\mathcal{G}(\|T_t f\|_p^p) \leq e^{-\lambda t} \mathcal{G}(\|f\|_p^p). \quad (2.4)$$

Remark 2.4. 1. When $\lambda = 0$ and $\mathcal{G}(x) = x$, we recover the definition of a contraction on L^p .

2. When $\lambda > 0$, $p = 2$, $\mathcal{G}(x) = \sqrt{x}$, we recover the definition of the spectral gap λ on L^2 for the semigroup (T_t) .
3. $(EFD)_{p,q}$ that is (2.4) says that there exists a function of the L^p -norm of $T_t f$ which decays exponentially fast in t when f is in L^q . This motivates the expression *Exponential Functional Decay*, (EFD) in short.

In Section 6.2, we discuss the relation between the decay (2.4) and the property of ultracontractivity of (T_t) from L^q to L^p i.e.

$$\|T_t f\|_p^p \leq e^{pM(t)} \|f\|_q^p.$$

In the following theorem, we prove that $(NTI)_{p,q}$ satisfied by a generator \mathcal{L} of a symmetric semigroup of contractions is equivalent to $(EFD)_{p,q}$ for the semigroup. Moreover, we specify the relationship between the functions \mathcal{N} in (2.5) and \mathcal{G} in (2.6) below. Recall that we say that a semigroup is equicontinuous on L^q if $\sup_{t>0} \|T_t\|_{q,q} \leq M < \infty$. We recall also the definition of $f_p = \text{sgn}(f)|f|^{p-1}$ for $1 < p < +\infty$ with $\text{sgn}(x) = \frac{x}{|x|}$, $x \neq 0$ and $\text{sgn}(0) = 0$.

Theorem 2.5. *Denote by \mathcal{L} the generator of a symmetric submarkovian semigroup (T_t) . Let $1 \leq q < p < \infty$ and \mathcal{D} be the domain of \mathcal{L} . The two following statements (1) and (2) below are equivalent:*

1. *There exists $\mathcal{N} : \mathbb{R} \rightarrow (0, +\infty)$ a non-decreasing continuous function such that for all $f \in \mathcal{D}$ with $\|f\|_q \leq 1$,*

$$\|f\|_p^p \mathcal{N}(\log \|f\|_p^p) \leq (\mathcal{L}f, f_p). \quad (2.5)$$

2. *There exists $\mathcal{G} \in C^1((0, \infty), (0, \infty))$ an increasing function such that for all $t > 0$ and for all $f \in \mathcal{D}$ with $\|f\|_q \leq 1$,*

$$\mathcal{G}(\|T_t f\|_p^p) \leq e^{-pt} \mathcal{G}(\|f\|_p^p). \quad (2.6)$$

Moreover (2.5) implies (2.6) with $\mathcal{G} = \exp \circ \mathcal{F} \circ \log$ with the derivative of \mathcal{F} satisfying $\mathcal{F}' = 1/\mathcal{N}$.

Conversely (2.6) implies (2.5) with $\mathcal{N}(y) = \frac{\mathcal{G}(e^y)}{e^y \mathcal{G}'(e^y)}$, $y \in \mathbb{R}$.

3. Let (T_t) be a symmetric Markov semigroup (i.e $T_t 1 = 1$). The two statements (1) and (2) above are equivalent with the additional assumption $\int_X f d\mu = 0$ in (2.5) and (2.6).
4. Let (T_t) be a C_0 -semigroup of contraction on L^p which has an equicontinuous extension on L^q for some $1 \leq q < p < +\infty$. Let $M \geq 1$ such that, for any $t > 0$, $\|T_t f\|_q \leq M\|f\|_q$.
 - (a) Assume that (2.5) holds with $\|f\|_q \leq M$ then (2.6) holds true with $\|f\|_q \leq 1$.
 - (b) Conversely, if (2.6) holds true with $\|f\|_q \leq M_1$ then (2.5) holds with $\|f\|_q \leq M_1$.

We shall call a (\mathcal{G}, p, q) -decay the inequality (2.6) and $(NTI)_{p,q}$ the inequality (2.5). The exponential factor $\exp(-pt)$ of (2.6) doesn't depend on the Nash function \mathcal{N} .

Remark 2.6. 1. Assume \mathcal{G} be continuous. Let $\gamma(t) = e^{\lambda t} \mathcal{G}(\|T_t f\|_p^p)$. Then the two statements " γ is non-increasing" and

$$\mathcal{G}(\|T_t f\|_p^p) \leq e^{-\lambda t} \mathcal{G}(\|f\|_p^p), \quad \forall t > 0.$$

are equivalent (see the general remark after Definition 2.2).

2. For instance, if $p = 2$ and $q = 1$ and $\mathcal{N}(y) = \lambda > 0$, $y \in \mathbb{R}$, we recover a well-known result. The inequality (2.5) (with or without the condition $\int_X f d\mu = 0$), namely Poincaré inequality (1.4). We easily compute $\mathcal{G}(x) = x^{1/\lambda}$ ($x > 0$) and the corresponding decay is the exponential decay

$$\|T_t f\|_2^2 \leq e^{-2\lambda t} \|f\|_2^2, \quad t > 0. \quad (2.7)$$

The L^1 -control of f can be removed by homogeneity of the norm $\|\cdot\|_2$.

3. Note that, in practice, the function \mathcal{N} may be non-positive on some interval $]-\infty, a]$ with $a > 0$ or only bounded below by a negative constant. In that case, we change \mathcal{N} by $\delta + \mathcal{N}^+$ where \mathcal{N}^+ is the non-negative part of \mathcal{N} and $\delta > 0$. So the quadratic form $\mathcal{E}(f) = (\mathcal{L}f, f)$ is changed by $((\mathcal{L} + \delta)f, f)$ (see Section 4 for examples of applications). We give a general formulation of our theorem 2.5 taking into account of this fact in view of some applications (see Th. 2.8).

Proof of Theorem 2.5: We first prove that Nash-type inequality $(NTI)_{p,q}$ implies (\mathcal{G}, p, q) -decay inequality. We assume that \mathcal{N} satisfies the assumptions above. Fix $f \in \mathcal{D}$ with $\|f\|_q \leq 1$. Since (T_t) is a contraction on L^q , we have $\|T_t f\|_q \leq 1$ and $T_t f \in \mathcal{D}$ for any $t > 0$. We apply $(NTI)_{p,q}$ to $T_t f$,

$$\|T_t f\|_p^p \mathcal{N}(\log \|T_t f\|_p^p) \leq (\mathcal{L}T_t f, (T_t f)_p).$$

Let $\varphi(t) = \|T_t f\|_p^p$. We have $\varphi'(t) = -p(\mathcal{L}T_t f, (T_t f)_p)$ (see [V-S-C] p.15). It follows

$$\varphi(t) \mathcal{N}(\log \varphi(t)) \leq -\frac{1}{p} \varphi'(t)$$

which can be rewritten as

$$p \leq \frac{1}{\mathcal{N}(\log \varphi(t))} \left(\frac{-\varphi'(t)}{\varphi(t)} \right). \quad (2.8)$$

We integrate this inequality: let $s > 0$,

$$ps \leq \int_0^s \frac{d(-\log \varphi(t))}{\mathcal{N}(\log \varphi(t))}$$

By change of variable,

$$ps \leq \int_{\log \varphi(s)}^{\log \varphi(0)} \frac{dy}{\mathcal{N}(y)}.$$

Thus by definition of \mathcal{F} , we get

$$\mathcal{F}(\log \varphi(s)) \leq -ps + \mathcal{F}(\log \varphi(0)).$$

We conclude by taking the exponential on both sides of this inequality:

$$\mathcal{G}(\varphi(s)) \leq e^{-ps} \mathcal{G}(\varphi(0)).$$

We have proved the first implication.

Converse. We prove that (\mathcal{G}, p, q) -decay inequality implies Nash-type inequality $(NTI)_{p,q}$. We obtain this result simply by differentiation at $s = 0^+$ inequality (2.6) as follows. With the same notations as above, we write for all $s > 0$, $\varphi(s) = \|T_s f\|_p^p$ with $\|f\|_q \leq 1$. Thus (\mathcal{G}, p, q) -decay inequality is equivalent to

$$[\mathcal{G}(\varphi(s)) - \mathcal{G}(\varphi(0))] / s \leq [(e^{-ps} - 1) / s] \mathcal{G}(\varphi(0)).$$

We take the limit as s goes to zero and get

$$-p(\mathcal{L}f, f_p) \mathcal{G}'(\|f\|_p^p) \leq -p\mathcal{G}(\|f\|_p^p).$$

Therefore,

$$\mathcal{G}(\|f\|_p^p) / \mathcal{G}'(\|f\|_p^p) \leq (\mathcal{L}f, f_p).$$

We define the function \mathcal{N} as the solution of

$$x \mathcal{N}(\log x) = \mathcal{G}(x) / \mathcal{G}'(x), \quad x > 0.$$

That is $\mathcal{N}(y) = e^{-y} \mathcal{G}(e^y) / \mathcal{G}'(e^y)$, $y \in \mathbb{R}$. This proves the converse.

Now we prove statement (3) just by saying that, for a fixed $f \in \mathcal{D}$ satisfying $\int_X f d\mu = 0$, then $\int_X T_t f d\mu = 0, \forall t > 0$ since (T_t) is a (symmetric) Markov semigroup. We conclude the proof by the same arguments as above. The statement (4) follows with a similar proof as above. This completes the proof of Th.2.5.

Note that \mathcal{G} and \mathcal{F} are conjugate by the intertwining function \log .

Remark 2.7. The first implication holds true for \mathcal{H} (instead of \mathcal{F}) satisfying the following condition : for some $c > 0$ and all $a \leq b$,

$$c(\mathcal{H}(a) - \mathcal{H}(b)) \leq \mathcal{F}(a) - \mathcal{F}(b).$$

When it is not possible to find an explicite expression of \mathcal{F} , it is useful to find an explicit expression of \mathcal{H} comparable to \mathcal{F} in the following sense: there exists $c_1, c_2 > 0$ such that

$$c_1 (\mathcal{H}(a) - \mathcal{H}(b)) \leq \mathcal{F}(a) - \mathcal{F}(b) \leq c_2 (\mathcal{H}(a) - \mathcal{H}(b)), \quad a \leq b.$$

This allows us to keep track of the function F . Indeed, let $\mathcal{G}_{\mathcal{F}} := \exp \circ \mathcal{F} \circ \log$. So, if $\mathcal{G}_{\mathcal{F}}$ satisfies (2.6) then $\mathcal{G}_{c_1 \mathcal{H}} = (\mathcal{G}_{\mathcal{H}})^{c_1}$ also. Now, if $\mathcal{G}_{c_1 \mathcal{H}}$ satisfies (2.6) then $\mathcal{G}_{\frac{c_1}{c_2} \mathcal{F}} = (\mathcal{G}_{\mathcal{F}})^{\frac{c_1}{c_2}}$ also. Note that the Lyapunov exponent p in (2.6) with $\mathcal{G}_{\mathcal{F}}$ is replaced by $p' = \frac{c_2}{c_1} p$ in this process. (for an application, see subsection 4.6).

In order to deal with examples, we are now interested by the same kind of results but with a slightly weaker assumptions on Nash-function \mathcal{N} . Indeed, we need to treat examples where the function \mathcal{N} may not be positive on an interval of the form $] -\infty, a]$ for some $a > 0$ see subsection 4.3, 4.4. We have the following generalization of Th. 2.5.

Theorem 2.8. Assume that \mathcal{L} is the generator of a submarkovian semigroup (T_t) . Let \mathcal{N} be a non-decreasing continuous function and assume there exists $\alpha \in [0, +\infty)$ such that $\mathcal{N} \leq 0$ on $(-\infty, \log \alpha]$ and $\mathcal{N} > 0$ on $(\log \alpha, +\infty)$ and \mathcal{F} such that $\mathcal{F}' = 1/\mathcal{N}$ on $(\log \alpha, +\infty)$. We define $\mathcal{G} = \exp \circ \mathcal{F} \circ \log$ on the set $(\alpha, +\infty)$. The two following statements are equivalent:

1. For all $f \in \mathcal{D}$ with $\|f\|_q \leq 1$,

$$\|f\|_p^p \mathcal{N}(\log \|f\|_p^p) \leq (\mathcal{L}f, f_p). \quad (2.9)$$

2. For all $t > 0$ and for all $f \in \mathcal{D}$ with $\|f\|_q \leq 1$ such that $\alpha < \|T_t f\|_p^p$,

$$\mathcal{G}(\|T_t f\|_p^p) \leq e^{-pt} \mathcal{G}(\|f\|_p^p). \quad (2.10)$$

3. Analogue statements as in (3) and (4) of Th.(2.5) hold true.

Remark 2.9. Recall that $\|T_t f\|_p^p$ is a non-increasing and continuous function of t . If $\alpha < \|f\|_p^p$ then there exists $t_0 \in (0, \infty]$ depending on f such that for all $0 \leq t < t_0$, $\alpha < \|T_t f\|_p^p$ and for all $t \geq t_0$, $\|T_t f\|_p^p \leq \alpha$.

PROOF. The proof is similar to the proof of Theorem 2.5 so we only sketch the arguments. First, we show the implication "(1) \Rightarrow (2)". Let $f \in \mathcal{D}$ be such that $\|f\|_q \leq 1$. We can assume $\alpha < \|f\|_p^p$ if it is not the case there is nothing to prove (see Remark 2.9). Let t_0 be as in Remark 2.9. For all t and s such that $0 < t < s < t_0$, we have $\alpha < \|T_t f\|_p^p$ and the inequality (2.8) holds true. Now, we integrate over $(0, s]$ with respect to t and conclude in the same way as in

Theorem 2.5.

The converse is proved as follows. We can assume that $\alpha < \|f\|_p^p$. If this condition is not satisfied there is nothing to prove. Indeed, $\mathcal{N}(\log \|f\|_p^p) \leq 0$ and $(\mathcal{L}f, f_p) = \frac{-1}{p} \frac{d}{dt} \|T_t f\|_p^p \geq 0$ since the semigroup is a contraction on L^p . So, there exists $t_0 > 0$ such that for $0 < t < t_0$, the inequality (2.10) holds true. Thus we can take the derivative at $t = 0^+$ as in the proof of Theorem 2.5 and conclude. The proof is completed.

3. Related inequalities

We study the relationship between \mathcal{G} -decay and other families of inequalities. In particular, we recall (with proof or a sketch of the proof) that L^p -log-Sobolev inequality with parameter implies a L^p -Nash-type inequality (see [Bi-Ma] for the L^2 -version). We also recall that ultracontractivity property of the semigroup implies L^p -log-Sobolev inequality with parameter and therefore Nash-type inequality. We also prove that $(2, 1)$ -Nash-type inequality implies $(p, p-1)$ Nash-type inequality (with $p \geq 2$). L^p -log-Sobolev inequality with parameter will be also called super-log-Sobolev inequality.

Theorem 3.1. *Let $T_t = e^{-\mathcal{L}t}$ be a symmetric submarkovian semigroup. Assume that (T_t) satisfies L^p -log-Sobolev inequality with one parameter for a fixed $p \in [2, +\infty)$ that is:*

$$\int g^p \log g \, d\mu \leq t \frac{p}{2(p-1)} (\mathcal{L}g, g_p) + 2M(t)p^{-1} \|g\|_p^p + \|g\|_p^p \log \|g\|_p \quad (3.1)$$

for all $g \in \mathcal{D}_+ := \cup_{t>0} e^{-\mathcal{L}t}(L^1 \cap L^\infty)_+$ with $g_p = g^{p-1}$.

1. Then for all q such that $1 \leq q < p$ and $g \in \mathcal{D}_+$, satisfying $\|g\|_q \leq 1$, we have

$$\|g\|_p^p \mathcal{N}_{p,q}^\#(\log \|g\|_p^p) \leq (\mathcal{L}g, g_p) \quad (3.2)$$

where $\mathcal{N}_{p,q}^\#(y) = \frac{4(p-1)}{p^2} \mathcal{N}(\frac{qy}{p-q})$ with $\mathcal{N}(y) = \sup_{t>0} (ty/2 - tM(1/t))$, $y \in \mathbb{R}$.

2. Let \mathcal{F} such that $\mathcal{F}' = 1/\mathcal{N}$. We define $\mathcal{F}_{p,q}^\#(y) = \frac{p^2(p-q)}{4q(p-1)} \mathcal{F}(\frac{qy}{p-q})$, $y \in \mathbb{R}$ and $\mathcal{G}_{p,q}^\#(x) = \exp \circ \mathcal{F}_{p,q}^\# \circ \log(x)$. So,

$$\mathcal{G}_{p,q}^\#(x) = \left[\mathcal{G}(x^{\frac{q}{p-q}}) \right]^{\frac{p^2(p-q)}{4q(p-1)}},$$

with $\mathcal{G}(x) = \exp \circ \mathcal{F} \circ \log(x)$. Then for all $f \in \mathcal{D}_+$, $\|f\|_q \leq 1$ and for all $t > 0$, we have

$$\mathcal{G}_{p,q}^\#(\|T_t f\|_p^p) \leq e^{-pt} \mathcal{G}_{p,q}^\#(\|f\|_p^p). \quad (3.3)$$

or equivalently

$$\mathcal{G}(\|T_t f\|_p^{\frac{pq}{p-q}}) \leq e^{-\tilde{p}t} \mathcal{G}(\|f\|_p^{\frac{pq}{p-q}}) \quad (3.4)$$

with $\tilde{p} = \frac{4q(p-1)}{p(p-q)}$.

Note that $\mathcal{N}_{2,1}^\sharp(y) = \mathcal{N}(y)$ is the Legendre (or conjugate) transform of $t \rightarrow tM(1/t)$ evaluated at $y/2 \in \mathbb{R}$. Recall that the inequality (3.1) can be deduced from the same inequality with $p = 2$ (see [D] Lemma 2.2.5 p.67). Such inequality will be called log-Sobolev inequality with parameter or super-log-Sobolev inequality. Our theorem above says that a Nash-type inequality can be deduced from super-log-Sobolev inequality and consequently a \mathcal{G} -decay can be obtained from super-log-Sobolev inequality. Recall that (3.2) and (3.3) (or (3.4)) are equivalent by Th.2.5.

PROOF. Let $g \in \mathcal{D}_+$. So g is non-negative and $g \in L^s$ for any $s \in [1; +\infty]$. Let $2 \leq q < p < +\infty$. We assume that $\int g^q d\mu = 1$ then $d\nu = g^q d\mu$ is a probability measure. We apply Jensen inequality to the convex function $\Phi(x) = x \log x$ with the probability measure $d\nu$. We get

$$\begin{aligned} \int g^p \log g d\mu &= \frac{1}{p-q} \int g^{p-q} \log g^{p-q} d\nu \\ &= \frac{1}{p-q} \int \Phi(g^{p-q}) d\nu \geq \frac{1}{p-q} \Phi\left(\int g^{p-q} d\nu\right) = \frac{1}{p-q} \|g\|_p^p \log \|g\|_p^p. \end{aligned}$$

From this inequality and the assumption (3.1), we easily deduce for all $t > 0$:

$$\frac{q}{p-q} \|g\|_p^p \log \|g\|_p \leq \frac{pt}{2(p-1)} (\mathcal{L}g, g_p) + \frac{2M(t)}{p} \|g\|_p^p.$$

Hence,

$$\|g\|_p^p \left[\frac{2q(p-1)}{tp^2(p-q)} \log \|g\|_p^p - \frac{4(p-1)}{tp^2} M(t) \right] \leq (\mathcal{L}g, g_p).$$

For $\|g\|_q = 1$, our result (3.2) follows from optimization over $t > 0$ and definitions of \mathcal{N} and $\mathcal{N}_{p,q}^\sharp$.

We now prove that inequality (3.2) hold true when $\|g\|_q \leq 1$. We set $g = f/\|f\|_q$ with $f \in \mathcal{D}_+ := \cup_{t>0} e^{-\mathcal{L}t}(L^1 \cap L^\infty)_+$ and get from (3.2) applied to g :

$$\|f\|_p^p \mathcal{N}_{p,q}^\sharp \left(\log \frac{\|f\|_p^p}{\|f\|_q^p} \right) \leq (\mathcal{L}f, f_p). \quad (3.5)$$

Since \mathcal{N}_p^\sharp is non-decreasing and $\|f\|_q^p \leq 1$

$$\log \|f\|_p^p \leq \log \|f\|_p^p - \log \|f\|_q^p = \log \frac{\|f\|_p^p}{\|f\|_q^p}.$$

We deduce (3.2) for f . This proves the first statement.

To prove the second statement, we apply Th.2.5. Indeed, with the definition of $\mathcal{F}_{p,q}^\sharp(y) = \frac{p^2(p-q)}{4q(p-1)} \mathcal{F}\left(\frac{qy}{p-q}\right)$, $y \in \mathbb{R}$, we easily check that $(\mathcal{F}_{p,q}^\sharp)'(y) = \frac{1}{\mathcal{N}_{p,q}^\sharp(y)}$. Thus $\mathcal{G}_{p,q}^\sharp$ is given by $\mathcal{G}_{p,q}^\sharp = \exp \circ \mathcal{F}_{p,q}^\sharp \circ \log$ and the rest of the proof is a simple computation. This completes the proof.

Note that (3.5) and (3.2) are equivalent under the assumption $\|f\|_q \leq 1$ by homogeneity.

Corollary 3.2. *Let (T_t) be a symmetric submarkovian semigroup and assume (T_t) that satisfies the following ultracontractivity property: for all $t > 0$,*

$$\|T_t f\|_2 \leq e^{\tilde{M}(t)} \|f\|_1 \quad (3.6)$$

where \tilde{M} is monotonically decreasing continuous function of t . Then (3.1) is satisfied with $M = \tilde{M}$ and therefore (3.2)-(3.3)- (3.4) hold true.

We only sketch the proof of this corollary. Ultracontractivity property implies L^p -log-Sobolev inequalities with parameter by Th.2.2.3 and Lemma 2.2.6 of [D]. We apply Th.3.1 to conclude.

Note that (3.6) and (3.1) are not equivalent in general. There exists a semigroup satisfying (3.1) but not ultracontractive (see example 2.3.5 p. 73 of [D]).

In some concret situations, we are able to prove Nash-type inequality, but nor the property of ultracontractivity, nor the inequality of log-Sobolev with parameter are satisfied (see [BM1]). In case of existence of a $(2, 1)$ -Nash-type inequality, we state a similar result in L^p but with some restriction on the index q of the L^q -norm.

Theorem 3.3. *Let $T_t = e^{-\mathcal{L}t}$ be a symmetric submarkovian semigroup. Assume that there exists a non-decreasing function $\mathcal{N} : \mathbb{R} \rightarrow \mathbb{R}$ such that the following inequality holds true for all $f \in \mathcal{D} \cap L^1 \cap L^\infty$ with $\|f\|_1 \leq 1$,*

$$\|f\|_2^2 \mathcal{N}(\log \|f\|_2^2) \leq (\mathcal{L}f, f). \quad (3.7)$$

Then

1. For all $g \in \mathcal{D}_+ := \cup_{t>0} e^{-\mathcal{L}t}(L^1 \cap L^\infty)_+$, satisfying $\|g\|_{p/2} \leq 1$ with $p \geq 2$, we have

$$\|g\|_p^p \mathcal{N}_{p,p/2}(\log \|g\|_p^p) \leq (\mathcal{L}g, g_p) \quad (3.8)$$

where $\mathcal{N}_{p,p/2}(y) = \frac{4(p-1)}{p^2} \mathcal{N}(y)$, $y \in \mathbb{R}$.

2. Let $\mathcal{F}_{2,1}$ such that $\mathcal{F}'_{2,1} = 1/\mathcal{N}$. We define $\mathcal{F}_{p,p/2}(y) = \frac{p^2}{4(p-1)}\mathcal{F}_{2,1}(y)$, $y \in \mathbb{R}$ and $\mathcal{G}_{p,p/2} = \exp \circ \mathcal{F}_{p,p/2} \circ \log$. Then $\mathcal{G}_{p,p/2} = [\mathcal{G}_{2,1}]^{\frac{p^2}{4(p-1)}}$ with $\mathcal{G}_{2,1} = \exp \circ \mathcal{F}_{2,1} \circ \log$ and for all $f \in \mathcal{D}_+$, $\|f\|_{p/2} \leq 1$ and for all $t > 0$, we have:

$$\mathcal{G}_{p,p/2}(\|T_t f\|_p^p) \leq e^{-pt} \mathcal{G}_{p,p/2}(\|f\|_p^p). \quad (3.9)$$

or equivalently,

$$\mathcal{G}_2(\|T_t f\|_p^p) \leq e^{-p^* t} \mathcal{G}_2(\|f\|_p^p) \quad (3.10)$$

with $p^* = \frac{4(p-1)}{p}$. The converse also holds true i.e. (3.9) or (3.10) implies (3.8).

PROOF. Recall that $\mathcal{E}(f) = (\mathcal{L}f, f)$ is a Dirichlet form. So we have for $g \in \mathcal{D}$, $g \geq 0$:

$$\mathcal{E}(g^{p/2}) \leq \frac{p^2}{4(p-1)}(\mathcal{L}g, g_p). \quad (3.11)$$

(see [V-S-C] p.23, [D].p.67). Now, we set $f = g^{p/2}$ in (3.7), we easily deduce the inequality (3.8) from the inequality (3.7) and (3.11). We complete the proof by applying Th.2.5.

Note that \mathcal{N}_p of Th.3.3 and $\mathcal{N}_{p,q}^\#$ of Th.3.1 can be compared when $q = p/2$: $\mathcal{N}_p = \mathcal{N}_p^\# = \frac{4(p-1)}{p^2}\mathcal{N}$.

In this last theorem, we have only considered $(p, p/2)$ -Nash-type inequalities in Th.3.3 deduced from $(2, 1)$ -Nash-type inequality. The motivation comes from the fact that, on \mathbb{R}^n , for some operators \mathcal{L} , we use Fourier analysis to prove inequality (3.7) as in the original proof (see [N] and Section 4.6). Indeed, it is not clear if such analysis can be carry on the L^p -setting with $p \neq 2$. Of course, if a (p, q) -Nash-type inequality is available with $(p, q) \neq (2, 1)$, we directly apply Th.2.5.

4. Examples

In this section, we compute the function \mathcal{G} of the decay of Th. 2.5 or Th.2.8, $\mathcal{G}_p^\#$ of Th. 3.1 and \mathcal{G}_p of Th.3.3. Some examples are classical and some are new. We give a specific or simplified presentation of each of these inequalities and some consequences. We also mention the settings where such situations appear.

In subsection 4.1, we start with the classical Nash inequality with polynomial exponent in abstract setting (see [N], [C-L] for \mathbb{R}^n). In Section 4.2, we deal the same case with an additional spectral information. In subsection 4.3 and 4.4, we also study some families of examples, respectively what we call the one-exponential case and the double-exponential case (see [Bi-Ma] for other results on these examples).

We also consider the situation analogue to the Ornstein-Uhlenbeck semigroup (subsection 4.5). We study the Γ^* -semigroup in subsection 4.6 closed to Ornstein-Uhlenbeck semigroup. For explicit examples satisfying ultracontractive bounds of type (4.31) of section 4.3 or (4.37) of section 4.4, we refer the reader to [B2] where such semigroups appear in a natural way on the infinite dimensional torus for some Laplacians.

In Section 5, for all these examples above, we deduce inequalities of functional decay for fractional powers \mathcal{L}^β of the infinitesimal generator \mathcal{L} of the corresponding semigroup. But we shall not give complete details of the proofs to avoid a too lengthy paper.

4.1. $\mathcal{N}(y) = ce^{\gamma y}$, $\gamma, c > 0$.

This section deals with the classical Nash inequality. Namely, for all $f \in \mathcal{D} \cap L^1$ with $\|f\|_1 \leq 1$,

$$k_1 \|f\|_2^{2+\frac{4}{\nu}} \leq (\mathcal{L}f, f). \quad (4.1)$$

With our notation,

$$\|f\|_2^2 \mathcal{N}(\log \|f\|_2^2) \leq (\mathcal{L}f, f). \quad (4.2)$$

with $\mathcal{N}(y) = k_1 e^{\frac{2y}{\nu}}$, $y \in \mathbb{R}$.

Such inequality first appeared in [N] and it has been generalized for submarkovian semigroups, see [C-K-S], [V-S-C]. We start by recalling connections between polynomial ultracontractivity, Sobolev inequality and Nash inequality (4.1), see also Prop 6.5.

Throughout this section, we assume that (T_t) is submarkovian semigroup and its non-negative generator \mathcal{L} on L^2 with domain \mathcal{D} . It has been proved by Carlen-Kusuoka-Strook ([C-K-S]) that the following property, called polynomial ultracontractivity,

$$\|T_t f\|_2 \leq k_3 t^{-\nu/4} \|f\|_1, \quad \forall t > 0, \quad (4.3)$$

for some $\nu > 0$, is equivalent to the following Nash inequality: for all $f \in \mathcal{D}$, $\|f\|_1 \leq 1$,

$$k_1 \|f\|_2^{2+4/\nu} \leq (\mathcal{L}f, f), \quad (4.4)$$

for some constant $k_1 > 0$ (with the same ν).

For instance, $\mathcal{L} = \Delta$, the laplacian on the Euclidean space \mathbb{R}^n satisfies both inequalities (with independent proofs) with $\nu = n$, see [N] for (4.4). The best constants k_1 and This theory has been applied to sub-Laplacians on Lie groups, see [V-S-C] p.56 and p.108.

In the abstract theory, we distinguish two cases $0 < \nu \leq 2$ and $\nu \geq 2$ (ν need not be an integer). When $\nu > 2$, (4.4) (or (4.3)) is equivalent to the following L^2 -Sobolev inequality:

$$\|f\|_{2q_0}^2 \leq k_2(\mathcal{L}f, f) \quad (4.5)$$

with $q_0 = \frac{\nu}{\nu-2}$ with (see [V-S-C]). From this inequality we can also deduce (p, q) -Nash-type inequalities as shown in proposition 4.1 just below.

To summarize the situation, for sub-markovian semigroups, Sobolev inequality (4.5), ultracontractivity property (4.3) and Nash inequality (4.4) are equivalent when $\nu > 2$ (see [V-S-C]). For a direct proof of the equivalence between (4.5) and (4.4) using properties of Dirichlet forms see [B-C-L-S]. The best constants k_1 , k_2 and k_3 just above are known.

We first focus on the case $\nu > 2$, we prove that Nash-type inequalities are available for a large class of indices (p, q) (see Prop.4.1 just below). The (p, q) -Nash inequality of this section is the motivation of our general study in this paper. We deduce the corresponding \mathcal{G} -decay which can be reformulated as an improved contraction on L^p with constraint (see Cor.4.4 and Cor. 4.5). From this reformulation, we deduce some space-time mixed-norms inequalities for the semigroup in Cor.4.6 and, in particular, is related to Hardy-Littlewood-Sobolev type inequality.

In the next proposition, we explicit the \mathcal{G} -decay in terms of the Sobolev constant.

Proposition 4.1. *Assume that $T_t = e^{-t\mathcal{L}}$ is a submarkovian semigroup which satisfies Sobolev inequality (4.5) with $\nu > 2$. Let $1 \leq q < p < \infty$ with $p \geq 2$. We set $b = \frac{2q}{\nu(p-q)}$ (so $b > 0$), $c_p = \frac{4(p-1)}{k_2 p^2}$ with k_2 of (4.5) and $\tilde{\mathcal{N}}_{p,q}(y) = c_p \exp(by)$, $y \in \mathbb{R}$. Then we have*

$$\|f\|_p^p \tilde{\mathcal{N}}_{p,q}(\log \|f\|_p^p) \leq (\mathcal{L}f, f_p), \quad \|f\|_q \leq 1. \quad (4.6)$$

and the inequality (2.6) holds true with the corresponding function $\tilde{\mathcal{G}}_{p,q}(x) = \exp(-ax^{-b})$ ($x > 0$) with $a = \frac{1}{bc_p}$.

Recall that the inequalities (4.3)-(4.4)-(4.5) are equivalent when $\nu > 2$.

PROOF. These inequalities are deduced from (4.5) as follows: let $1 \leq q < p < \infty$ with $p \geq 2$. We set $f_p = f^{p-1}$. Changing f by $f^{p/2}$ with $f \in \mathcal{D}$ in Sobolev inequality (4.5) and using the fact that $(\mathcal{L}f, f)$ is a Dirichlet form, we get by (3.11),

$$\|f\|_{pq_0}^p \leq k_2 \frac{p^2}{4(p-1)} (\mathcal{L}f, f_p).$$

Now we apply Hölder inequality ($q_0 > 1$),

$$\|f\|_p \leq \|f\|_{pq_0}^\alpha \|f\|_q^{1-\alpha}$$

with $\frac{1}{\alpha} = \frac{1/pq_0 - 1/q}{1/p - 1/q} > 0$. It yields the following Nash-type inequality:

$$c_p (\|f\|_p^p)^{1/\alpha} \leq (\mathcal{L}f, f_p), \quad \|f\|_q \leq 1. \quad (4.7)$$

with $c_p = \frac{4(p-1)}{k_2 p^2}$. We apply the assertion "(1) \Rightarrow (2)" of Th.2.5 and deduce the expression of $\tilde{\mathcal{N}}_{p,q}$ in (4.6): $\tilde{\mathcal{N}}_{p,q}(y) = c_p \exp(by)$, $y \in \mathbb{R}$, with $b = \frac{1}{\alpha} - 1 = \frac{1}{p} \frac{1 - \frac{1}{q_0}}{\frac{1}{q} - \frac{1}{p}} = \frac{2q}{\nu(p-q)} > 0$. The function $\tilde{\mathcal{G}}_{p,q}$ of (2.6) is easily computed from $\tilde{\mathcal{N}}_{p,q}$ by the formulas given in 2) of Th.2.5. The proof is completed.

Because, the relationship between the best constant in Sobolev, Nash and ultracontractivity are not clear. We compute the \mathcal{G} function for each case. First, we present the \mathcal{G} -decay in terms of L^p -contraction of the semigroup with L^q -constant under Sobolev inequality.

Corollary 4.2. *Assume that the generator \mathcal{L} of the semigroup (T_t) satisfies Sobolev inequality (4.5) with $\nu > 2$. Let $2 \leq p < +\infty$ and $1 \leq q < p$. We set $\delta = \frac{p}{a} = \frac{8q(p-1)}{k_2 p^2 \nu(p-q)}$, and $1/\alpha_0 = \frac{1}{pb} = \frac{2pq}{\nu(p-q)}$ with constant k_2 in Sobolev inequality (4.5). Then for any $f \in L^p \cap L^q$ with $\|f\|_q \leq 1$ and any $t > 0$,*

$$\|T_t f\|_p \leq H_{p,q}(f, t) \|f\|_p \quad (4.8)$$

with

$$H_{p,q}(f, t) := \left[1 + \delta t \left(\frac{\|f\|_p}{\|f\|_q} \right)^{1/\alpha_0} \right]^{-\alpha_0}.$$

The inequality (4.8) is equivalent to the following \mathcal{G} -decay

$$\mathcal{G}(\|T_t f\|_p^p) \leq e^{-pt} \mathcal{G}(\|f\|_p^p).$$

with $\mathcal{G}(x) = \exp(-ax^{-b})$, $a = \frac{k_2 p^2 \nu(p-q)}{8q(p-1)}$ and $b = \frac{2q}{\nu(p-q)}$.

Remark 4.3. 1. We note that $H_{p,q}(f, t)$ depends on f only through the ratio of the norms $\frac{\|f\|_p}{\|f\|_q}$ so $H_{p,q}(f, t)$ is homogeneous of degree 0. 2. For all f and $t > 0$, we have $0 \leq H_{p,q}(f, t) \leq 1$ and $H_{p,q}(f, 0) = 1$. The map $t \rightarrow H_{p,q}(f, t)$ is non-increasing. For large t , we have the asymptotic estimate:

$$H_{p,q}(f, t) \sim \left(\frac{\|f\|_q}{\|f\|_p} \right) (\delta t)^{-\alpha_0}.$$

On one side, we recover that the semigroup is a contraction on L^p from the fact $H_{p,q}(f, t) \leq 1$. On the other side, from the following inequality, for any $t > 0$:

$$H_{p,q}(f, t) \leq \left(\frac{\|f\|_q}{\|f\|_p} \right) (\delta t)^{-\alpha_0}.$$

we recover the ultracontractivity property of the semigroup (T_t) from L^q to L^p :

$$\|T_t f\|_p \leq (\delta t)^{-\alpha_0} \|f\|_q.$$

If $p = 2$ we get the usual exponent $\alpha_0 = \frac{\nu}{4}$. So our \mathcal{G} -decay interpolates between L^p -contraction and ultracontractivity property of the semigroup.

PROOF. By Proposition 4.1, we have for any $f \in L^p \cap L^q$, $\|f\|_q \leq 1$ and any $t > 0$,

$$\mathcal{G}(\|T_t f\|_p^p) \leq e^{-pt} \mathcal{G}(\|f\|_p^p). \quad (4.9)$$

with $\mathcal{G}(x) = \exp(-ax^{-b})$, $a = \frac{k_2 p^2 \nu(p-q)}{8q(p-1)}$ and $b = \frac{2q}{\nu(p-q)}$. By taking the log and changing sign of (4.9), we deduce

$$pt + a\|f\|_p^{-pb} \leq a\|T_t f\|_p^{-pb}.$$

i.e.

$$\|T_t f\|_p \leq \|f\|_p \left[1 + \frac{p}{a} t \|f\|_p^{pb} \right]^{-1/pb}$$

which is clearly equivalent to (4.9). We change f by $f/\|f\|_q$ (renormalisation). We set $1/\alpha_0 = pb = \frac{2pq}{\nu(p-q)}$ and $\delta = \frac{p}{a}$ and conclude the proof.

Now, we deal with the general case $\nu > 0$ but with some restriction on (p, q) . When $\nu > 2$, we have recalled that (4.4) and (4.5) are equivalent. In the case $0 < \nu \leq 2$, the situation is different with respect to Sobolev inequality which cannot be defined by (4.5). But we can also apply Th. 3.1 and Th. 3.3 to compute explicitly the function \mathcal{G}_p^\sharp and also \mathcal{G}_p which corresponds to different controls of L^q -norm in these theorems. Of course, we can apply these results to the case $\nu > 2$. Below, we explicit these functions \mathcal{G}_p^\sharp and \mathcal{G}_p .

Now, we present the \mathcal{G} -decay in terms of L^p -contraction of the semigroup with L^q -constaint under ultracontractivity property of the semigroup (T_t) .

Corollary 4.4. *Under the assumptions of Th.3.1 with $M(t) = \log(k_3 t^{-\nu/4})$, $t > 0$ in (3.1) for some $\nu > 0$. Then the $\mathcal{G}_{p,q}^\sharp$ -function of decay of Th.3.1 is given by*

$$\mathcal{G}_{p,q}^\sharp(x) = \exp\left(-a'x^{-b'}\right), \quad x > 0, \quad (4.10)$$

with $a' = \frac{ek_3^{\frac{\nu}{4}} p^2(p-q)}{2q(p-1)}$, $b' = \frac{2q}{\nu(p-q)}$. We have, for all $t > 0$ and all $1 \leq q < p$, $2 \leq p < +\infty$, $f \in L^p \cap L^q$, $\|f\|_q \leq 1$,

$$\mathcal{G}_{p,q}^\sharp(\|T_t f\|_p^p) \leq e^{-pt} \mathcal{G}_{p,q}^\sharp(\|f\|_p^p). \quad (4.11)$$

or equivalently,

$$\|T_t f\|_p \leq H_{p,q}^\sharp(f, t) \|f\|_p. \quad (4.12)$$

where

$$H_{p,q}^\sharp(f, t) := \left[1 + \delta t \left(\frac{\|f\|_p}{\|f\|_q} \right)^{1/\alpha_0} \right]^{-\alpha_0}.$$

with $1/\alpha_0 = pb' = \frac{2pq}{\nu(p-q)}$ and $\delta = \frac{p}{a'} = \frac{2q(p-1)}{p(p-q)ek_3^{\frac{\nu}{4}}}$.

PROOF. We apply Th.3.1 with the assumption $M(t) = \log(k_3 t^{-\nu/4})$, $t > 0$ in (3.1). So, by computations, we obtain successively:

$$\mathcal{N}_2(y) := \sup_{t>0} (ty/2 - tM(1/t)) = \frac{\nu}{4ek_3^{\frac{4}{\nu}}} \exp\left(\frac{2}{\nu}y\right)$$

and the function \mathcal{F}_2 satisfying the condition $\mathcal{F}_2' = \frac{1}{\mathcal{N}_2}$ is given by

$$\mathcal{F}_2(y) = -2ek_3^{\frac{4}{\nu}} \exp\left(-\frac{2}{\nu}y\right).$$

From both formulas $\mathcal{F}_{p,q}^\sharp$ and $\mathcal{G}_{p,q}^\sharp$ in 2) of Th.3.1, we obtain (4.10). The inequality (4.12) is obtained as in Cor.4.2. This concludes the proof.

This result applies to the Laplacian on the Euclidean space \mathbb{R}^n . The decay function $\mathcal{G}_2(x)$ is given by

$$\mathcal{G}_2(x) = \mathcal{G}_2^\sharp(x) = \exp\left(-ax^{-2/n}\right), \quad x > 0. \quad (4.13)$$

with $a = \frac{e}{4\pi}$ by applying Cor.4.4 with $k_3 = (8\pi)^{-n/4}$ (see [D] p. 60). The best constant a is related to the best constant of Nash inequality: see the end of Section 6 Prop.??).

Note that, conversely, if $e^{-t\mathcal{L}}$ satisfies a $(\mathcal{G}, 2)$ -decay inequality with $\mathcal{G}(x) = \exp(-ax^{-b})$ with $a, b > 0$ then Nash inequality (4.4) is satisfied. So, combining Cor.4.4 and Th.2.5 we obtain a variant for the proof of polynomial ultracontractivity implies Nash inequality. This also shows that \mathcal{G} -decay is at the cross-road of Nash inequality, Sobolev inequality, ultracontractivity property.

We recall Varopoulos' result which asserts that polynomial ultracontractivity with $\nu > 2$, is equivalent to Sobolev inequality

$$\|f\|_{2\nu/\nu-2} \leq c(\mathcal{L}f, f)$$

in the setting of submarkovian semigroups.

In particular, Sobolev inequality is equivalent to the contraction inequality (4.12) or (4.16) below with $p = 2$ and $q = 1$ up to constants.

Now, we present the \mathcal{G} -decay in terms of L^p -contraction of the semigroup with L^q -constant under Nash inequality.

Corollary 4.5. *Assume that Nash-type inequality of polynomial decay (4.4) is satisfied with $\nu > 0$ for the generator \mathcal{L} of a submarkovian semigroup (T_t) . Then the \mathcal{G} -function of decay of Th.3.1 is given by*

$$\mathcal{G}_{p,p/2}(x) = \exp\left(\frac{-\nu p^2}{8k_1(p-1)} x^{-\frac{2}{\nu}}\right), \quad x > 0, \quad (4.14)$$

with k_1 of (4.4).

Then, for all $t > 0$ and all $f \in L^p \cap L^{p/2}$, $2 \leq p < \infty$,

$$\mathcal{G}_{p,p/2}(\|T_t f\|_p^p) \leq e^{-pt} \mathcal{G}_{p,p/2}(\|f\|_p^p), \quad \|f\|_{p/2} \leq 1, \quad (4.15)$$

or equivalently

$$\|T_t f\|_p \leq H_{p,p/2}(f, t) \|f\|_p, \quad \|f\|_{p/2} \leq 1, \quad (4.16)$$

with

$$H_{p,p/2}(f, t) := \left[1 + \delta t \left(\frac{\|f\|_p}{\|f\|_{p/2}} \right)^{1/\alpha_0} \right]^{-\alpha_0}$$

with $\delta = \frac{8k_1(p-1)}{p\nu}$, $1/\alpha_0 = \frac{2p}{\nu}$.

PROOF. We only sketch the proof. Nash inequality (4.4) can be written as

$$\|f\|_2^2 \mathcal{N}(\log \|f\|_2^2) \leq (\mathcal{L}f, f), \quad \|f\|_1 \leq 1$$

with $\mathcal{N}(y) = k_1 \exp(\frac{2}{\nu}y)$, $y \in \mathbb{R}$. We then apply Th.3.3 and compute $\mathcal{G}_{p,p/2}$ explicitly. The proof of (4.16) is similar to the one given in the proof of Cor.4.2.

From Cor.4.2 or Cor.4.4 or Cor.4.5, we can deduce mixed-norm estimates for the semi-group (T_t) . This is possible due to the fact that there is no singularity at $t = 0$ in the function $H_{p,q}(t)$ of (4.8). As particular cases, we deduce Hardy-Littlewood-Sobolev-type inequalities (see comments at the end of this subsection). Let $p, s, \alpha > 0$. We introduce the following mixed-norm

$$\|f\|_{p,s,\alpha} = \left[\int_0^\infty \|T_t f\|_p^s \frac{dt}{t^\alpha} \right]^{1/s}.$$

The space of functions f such that $\|f\|_{p,s,\alpha} < \infty$ will be denoted by

$$W_{p,s,\alpha} = L^s \left((0, \infty) \rightarrow L^p(X, d\mu), \frac{dt}{t^\alpha} \right).$$

This space is introduced to study the space-time regularity of the semigroup. But other time-weights could certainly be considered.

Corollary 4.6. 1. Assume that the semigroup (T_t) satisfies the following inequality for some (or any) $2 \leq p < +\infty$ and $1 \leq q < p$: there exists $\delta > 0$ and $\alpha_0 > 0$ such that, for any $f \in L^p \cap L^q$ with $\|f\|_q \leq 1$ and for any $t > 0$,

$$\|T_t f\|_p \leq H_{p,q}(f, t) \|f\|_p$$

with

$$H_{p,q}(f, t) := \left[1 + \delta t \left(\frac{\|f\|_p}{\|f\|_q} \right)^{1/\alpha_0} \right]^{-\alpha_0}.$$

Then for any $s > 0$ and any α such that $1 - s\alpha_0 < \alpha < 1$ there exists a constant $K > 0$ such that, for any $f \in L^p \cap L^q$,

$$\|f\|_{p,s,\alpha} \leq K \|f\|_p^{1-\theta} \|f\|_q^\theta \quad (4.17)$$

with $\theta = \frac{1-\alpha}{\alpha_0 s}$ and $K = \delta^{\frac{\alpha-1}{s}} \left(\int_0^\infty [1+u]^{-s\alpha_0} \frac{du}{u^\alpha} \right)^{1/s}$.

2. In particular, this result holds under the assumptions of Cor.4.2 or of Cor.4.4 or of Cor.4.5 (with the corresponding α_0 and δ).

The fact that $0 < \theta < 1$ shows that (4.17) is an interpolation result.

PROOF. It is straightforward and we only sketch the main steps. By assumption, we have for $f \in L^p \cap L^q$,

$$\|T_t f\|_p \leq \left[1 + \delta t \left(\frac{\|f\|_p}{\|f\|_q} \right)^{1/\alpha_0} \right]^{-\alpha_0} \|f\|_p.$$

We raise this inequality to the power $s > 0$ and integrate over $(0, \infty)$ w.r.t. the weight $\frac{dt}{t^\alpha}$.

$$\int_0^\infty \|T_t f\|_p^s \frac{dt}{t^\alpha} \leq \left(\int_0^\infty \left[1 + \delta t \left(\frac{\|f\|_p}{\|f\|_q} \right)^{1/\alpha_0} \right]^{-s\alpha_0} \frac{dt}{t^\alpha} \right) \|f\|_p^s.$$

Let $\eta = \delta \left(\frac{\|f\|_p}{\|f\|_q} \right)^{1/\alpha_0}$. We set $u = t\eta$, thus

$$\int_0^\infty \|T_t f\|_p^s \frac{dt}{t^\alpha} \leq \|f\|_p^s \eta^{\alpha-1} \left(\int_0^\infty [1+u]^{-s\alpha_0} \frac{du}{u^\alpha} \right).$$

The integral converges at $t = 0$ iff $\alpha < 1$ and at $t = \infty$ iff $s\alpha_0 + \alpha > 1$. We set $\theta = \frac{1-\alpha}{\alpha_0 s}$. The conditions just above corresponds exactly to $0 < \theta < 1$. Under that condition on θ , we get

$$\left[\int_0^\infty \|T_t f\|_p^s \frac{dt}{t^\alpha} \right]^{1/s} \leq K \|f\|_p \left(\frac{\|f\|_q}{\|f\|_p} \right)^\theta$$

with $K = \delta^{\frac{\alpha-1}{s}} \left(\int_0^\infty [1+u]^{-s\alpha_0} \frac{du}{u^\alpha} \right)^{1/s}$. This yields the result and completes the proof.

It appears that Cor.4.6 has some links with the Hardy-Littlewood-Sobolev (i.e. HLS) theory as stated in [V-S-C] Th. II.2.7 (ii) p.12. The HLS inequality reads as

$$\|G_{\gamma_0} f\|_p \leq c \|f\|_q$$

and holds under the assumption of ultracontractivity $M(t) = \log(k_3 t^{-\nu/4})$. The indices are related by the formula $\gamma_0 = \nu(\frac{1}{q} - \frac{1}{p}) > 0$ for $1 < q < +\infty$. The operator G_{γ_0} is defined by $G_{\gamma_0} f = \int_0^{+\infty} t^{\frac{\gamma_0}{2}-1} T_t f dt$. Cor.4.6 gives us with $s = 1$ and $\alpha = 1 - \frac{\gamma}{2}$, the inequality

$$\|G_{\gamma} f\|_p \leq K_{\theta} \|f\|_1^{1-\theta} \|f\|_q^{\theta} \quad (4.18)$$

with $0 < \gamma < \gamma_0$ and $\theta = \frac{\gamma p q}{\nu(p-q)} \in (0, 1)$. The HLS inequality corresponds to the limit case $\theta = 1$ (i.e. $\gamma = \gamma_0$). But unfortunately K_{θ} is not bounded as $\theta \rightarrow 1$ (with δ and α_0 as in Cor.4.4, for instance).

Conversely, HLS inequality implies (4.18). Indeed, let $0 < \gamma < \gamma_0$ with γ_0 as above and $2 \leq p < +\infty$, $1 \leq q < p$. There exists q_0 such that $q < q_0 < p$ and $\gamma = \nu(\frac{1}{q_0} - \frac{1}{p}) > 0$. By HLS inequality,

$$\|G_{\gamma} f\|_p \leq c \|f\|_{q_0}$$

and by Hölder inequality, with $\theta = \frac{\gamma p q}{\nu(p-q)}$,

$$\|f\|_{q_0} \leq \|f\|_p^{1-\theta} \|f\|_q^{\theta}$$

which proves (4.18) but with a constant K_{θ} independent of θ .

4.2. $\mathcal{N}(y) = c e^{\frac{2y}{\nu}} + \rho$.

This case of study will be motivated by the applications (see Theorem 4.8) for semigroups with polynomial ultracontractivity and spectral gap informations. Our results allow us to deal in an unified way the spectral decay

$$\rho \|f\|_2^2 \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}, \quad \|f\|_1 \leq 1. \quad (4.19)$$

and the Nash inequality

$$c_0 \|f\|_2^{2+4/\nu} \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}, \quad \|f\|_1 \leq 1. \quad (4.20)$$

combined in the form

$$c_0 \|f\|_2^{2+4/\nu} + \rho \|f\|_2^2 \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}, \quad \|f\|_1 \leq 1. \quad (4.21)$$

It corresponds to $\mathcal{N}(y) = c_0 e^{\frac{2y}{\nu}} + \rho$. Fortunately, in this case, the function \mathcal{G} can also be explicitly computed.

Theorem 4.7. *Let \mathcal{L} be the infinitesimal generator of a symmetric submarkovian semigroup. Assume that \mathcal{L} satisfies for some $\rho > 0$ and some $c > 0$ the following Nash-type inequality*

$$c_0 \|f\|_2^{2+4/\nu} + \rho \|f\|_2^2 \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}, \quad \|f\|_1 \leq 1. \quad (4.22)$$

That is $\mathcal{N}(y) = c_0 e^{\frac{2y}{\nu}} + \rho$ in (5.1). Then

$$\mathcal{G}(x) = x^{1/\rho} \left(x^{2/\nu} + \frac{\rho}{c_0} \right)^{\frac{-\nu}{2\rho}}, \quad x > 0. \quad (4.23)$$

and for all $t > 0$,

$$\|T_t f\|_2^2 \left(\|T_t f\|_2^{4/\nu} + \frac{\rho}{c_0} \right)^{-\nu/2} \leq e^{-2\rho t} \|f\|_2^2 \left(\|f\|_2^{4/\nu} + \frac{\rho}{c_0} \right)^{-\nu/2} \quad (4.24)$$

Conversely (4.24) implies (4.22) with the same constants c_0, ρ, ν (??).

Note that Theorem 4.7 generalize the case of section 4.1 (i.e. $\rho = 0$).

We have explicated only the case $p = 2$ and $q = 1$. Since the other (p, q) -decays are obtained from \mathcal{G} as in Th.3.1 and Th.3.3, it is immediate to formulate the corresponding general results. We do not give details to avoid a too long paper.

PROOF. We apply Theorem 2.5 and we compute explicitly \mathcal{G} of this theorem. Indeed $\mathcal{F}'(y) = \frac{1}{c_0 e^{\frac{2y}{\nu}} + \rho}, y \in \mathbb{R}$ then $\mathcal{F}(y) = \frac{y}{\rho} - \frac{\nu}{2\rho} \log(e^{\frac{2y}{\nu}} + \frac{\rho}{c_0})$. We get

$$\mathcal{G}(x) = \exp \circ \mathcal{F} \circ \log(x) = x^{1/\rho} \left(x^{2/\nu} + \frac{\rho}{c_0} \right)^{\frac{-\nu}{2\rho}}, \quad x > 0.$$

We now apply Theorem 4.7 to a family of semigroups which justifies the introduction of the case $\mathcal{N}(y) = c e^{\frac{2y}{\nu}} + \rho$ when $\nu > 0$. In fact, we introduce the family of semigroups which have both properties namely polynomial ultracontractivity (4.3) and spectral gap (4.19). See [V-S-C] chap.IX. for explicit examples. Examples can be construct from semigroups satisfying a Nash-type inequality of the form

$$c_0 \|f\|_2^{2+4/\nu} \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}, \quad \|f\|_1 \leq 1.$$

Indeed, by adding $\lambda \|f\|_2^2$ with $\lambda > 0$, we obtain (4.22) for the generator $\mathcal{L} + \lambda$ of the symmetric submarkovian semigroup $S_t^\lambda = e^{-\lambda t} e^{-t\mathcal{L}}$.

We now give an application of Theorem 4.7.

Theorem 4.8. *Let $e^{-t\mathcal{L}}$ be a symmetric submarkovian semigroup satisfying for some $\lambda > 0$ and $\nu > 0$,*

$$\|T_t f\|_2 \leq \frac{c}{t^{\nu/4}} e^{-\lambda t} \|f\|_1, \quad t > 0, \quad (4.25)$$

then

$$\mathcal{G}(x) = x^{1/\lambda} \left(x^{2/\nu} + \frac{\lambda}{c_0} \right)^{-\nu/2\lambda}, \quad x > 0. \quad (4.26)$$

with $c_0 = \frac{\nu}{4} c^{-\frac{4}{\nu}} e^{-1}$ and the following decay inequality holds true

$$\mathcal{G}(\|T_t f\|_2^2) \leq e^{-2t} \mathcal{G}(\|f\|_2^2), \quad t > 0. \quad (4.27)$$

Remark 4.9. The value ρ in $\mathcal{N}(y) = ce^{\frac{2y}{\nu}} + \rho$ associated to (4.22) plays the role of a spectral value λ of the exponential decay of the semigroup in L^2 i.e.

$$\|T_t f\|_2 \leq e^{-\lambda t} \|f\|_2 \quad (4.28)$$

We can apply this result when the semigroup satisfies (4.28) and

$$\|T_t f\|_2 \leq \frac{c}{t^{\nu/4}} \|f\|_1 \quad (4.29)$$

but with some loss on λ since for all $\varepsilon \in (0, 1)$,

$$\|T_t f\|_2^2 \leq c_\varepsilon^2 \frac{e^{-2(1-\varepsilon)\lambda t}}{t^{\nu/2}} \|f\|_1^2. \quad (4.30)$$

Indeed, by semigroup property, we get

$$\|T_t f\|_2 = \|T_{(1-\varepsilon)t} T_{\varepsilon t} f\|_2 \leq e^{-\lambda(1-\varepsilon)t} \|T_{\varepsilon t} f\|_2 \leq e^{-\lambda(1-\varepsilon)t} \frac{c_\varepsilon}{t^{\nu/4}} \|f\|_1$$

with $c_\varepsilon = \frac{c}{\varepsilon^{\nu/4}}$.

Proof of Theorem 4.8. We apply Prop.II.2 p.514 of [Co] with $m(t) = \frac{c^2}{t^{\nu/2}} e^{-2\lambda t}$ and compute $\tilde{\Theta}$ of this theorem given by the formula $\tilde{\Theta}(x) = x \mathcal{N}(\log x)$ where $\mathcal{N}(y) = \sup_{s>0} (sy - s \log m(1/2s))$. A simple computation gives us $\mathcal{N}(y) = \lambda + c_0 e^{\frac{2y}{\nu}}$ with $c_0 = \frac{\nu}{4} c^{-\frac{4}{\nu}} e^{-1}$. Thus we get (4.22) with $\rho = \lambda$ and c_0 above. By Theorem 4.7, we finish the proof.

Remark 4.10. Note that the constant c_0 , the exponent ν and the spectral value λ can be read on the function \mathcal{G} .

Above we have assumed some properties on the semigroup. We can also express this directly on the quadratic form \mathcal{E} . Indeed, we can mixed spectral gap and Nash-type inequality of the form (4.4) in the following way. Assume that there exists $c_1, \nu, \lambda_1 > 0$ such that

$$c_1 \|f\|_2^{2+4/\nu} \leq \mathcal{E}(f), \quad \|f\|_1 \leq 1$$

and

$$\lambda_1 \|f\|_2^2 \leq \mathcal{E}(f).$$

Then theses two inequalities are equivalent to the following one

$$\|f\|_2^2 \max \left(c_1 \|f\|_2^{4/\nu}, \lambda_1 \right) \leq \mathcal{E}(f), \quad \|f\|_1 \leq 1.$$

The function \mathcal{G} is easy to compute. We obtain

$$\mathcal{G}(x) = x^{1/\lambda_1} \quad \text{if } 0 < x \leq \left(\frac{\lambda_1}{c_1} \right)^{\nu/2}$$

and

$$\mathcal{G}(x) = \left(\frac{e\lambda_1}{c_1} \right)^{\frac{\nu}{2\lambda_1}} \exp \left(-\frac{\nu}{2\lambda_1} x^{-2/\nu} \right) \quad \text{if } x \geq \left(\frac{\lambda_1}{c_1} \right)^{\nu/2}.$$

4.3. $\mathcal{N}(y) = y_+^{1+1/\gamma}$.

Here again this case is motivated by semigroups satisfying the following ultracontractivity bound

$$\|T_t f\|_2^2 \leq c_0 e^{c_1/t^\gamma} \|f\|_1^2, \quad t > 0 \quad (4.31)$$

with c_0, c_1, γ positive constants.

Such semigroups appear naturally associated to some laplacians on the infinite dimensional torus (see [B2],[B1]).

We can also compute explicitly $\mathcal{N}(y) = \sup_{s>0} (sy - s \log m(1/2s))$, $y \in \mathbb{R}$ with $m(t) = c_0 e^{c_1/t^\gamma}$. A simple computation gives us

$$\mathcal{N}_a(y) = k (y - a)_+^{1+1/\gamma}, \quad a = \log c_0 \in \mathbb{R}$$

with k depending on γ and c_1 but not on c_0 . We denote by $f_+(y) = f(y)$ if $f(y) \geq 0$ and $f_+(y) = 0$ if $f(y) \leq 0$ and also $y_+ = y$ if $y \geq 0$ and $y_+ = 0$ if $y \leq 0$. We have the relation $\mathcal{N}_a(y) = \mathcal{N}_0(y - a)$ for all $a \in \mathbb{R}$. So it is enough to deal with the case \mathcal{N}_0 . We compute explicitly \mathcal{G}_0 associated to \mathcal{N}_0 . We get

$$\mathcal{G}_0(x) = \exp\left(-\frac{\gamma}{k} [\log x]^{-1/\gamma}\right), \quad x > 1. \quad (4.32)$$

We deduce

$$\mathcal{G}_a(x) = \mathcal{G}_0(x/c_0) = \exp\left(-\frac{\gamma}{k} [\log(x/c_0)]^{-1/\gamma}\right), \quad x > c_0. \quad (4.33)$$

By applying Theorem 2.8 with $\alpha = c_0$, we have the following result.

Proposition 4.11. *Let $a \in \mathbb{R}$ and $c_0 > 0$ such that $a = \log c_0$. Assume that the following inequality holds true:*

$$\|f\|_2^2 \mathcal{N}_a(\log \|f\|_2^2) \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}(\mathcal{L}), \quad \|f\|_1 = 1, \quad (4.34)$$

with $\mathcal{N}_a(y) = k(y - a)_+^{1+1/\gamma}$. Then, for all $x > c_0$,

$$\mathcal{G}_a(x) = \exp\left(-\frac{\gamma}{k} [\log(x/c_0)]^{-1/\gamma}\right). \quad (4.35)$$

4.4. $\mathcal{N}(y) = y_+(\log_+ y_+)^{1/\gamma}$

We now consider the following family of Nash-type inequalities. Assume that the generator \mathcal{L} of a symmetric submarkovian semigroup satisfies for some $\gamma > 0$ the following inequality

$$c\|f\|_2^2 (\log \|f\|_2^2)_+ [\log_+ ((\log \|f\|_2^2)_+)]^{1/\gamma} \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}, \quad \|f\|_1 \leq 1. \quad (4.36)$$

that is $\mathcal{N}(y) = y_+(\log_+ y_+)^{1/\gamma}$, $\gamma > 0$.

Such inequalities appear in natural way when one considers semigroups satisfying a double-exponential decay of the following form

$$\|T_t f\|_2^2 \leq m(t) \|f\|_1^2, \quad t > 0, \quad (4.37)$$

with $m(t) = c_0 \exp(c_1 \exp(c_2/t^\gamma))$, for some $c_0, c_1, c_2, \gamma > 0$ (see the end of this section for a more detailed discussion on this example). Here again, such semigroups appear naturally associated to some laplacians on the infinite dimensional torus (see [B2],[B1]).

Under the assumption (4.36), we have explicitly the expression of the decay function \mathcal{G} . We get two expressions depending on $\gamma = 1$ or not. The theorem 2.8 applies in this case with $\alpha = e$. We just describe the function \mathcal{G} .

Proposition 4.12. *Assume that the inequality*

$$\|f\|_2^2 \mathcal{N}(\log \|f\|_2^2) \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}(\mathcal{L}), \quad \|f\|_1 = 1, \quad (4.38)$$

holds true with $\mathcal{N}(y) = y_+(\log y_+)^{1/\gamma}$. Then, for all $x > e$,

1. If $\gamma \neq 1$, the function \mathcal{G} of (2.10) is given by

$$\mathcal{G}(x) = \exp\left(\frac{\gamma}{\gamma-1} (\log \log x)^{1-1/\gamma}\right). \quad (4.39)$$

2. If $\gamma = 1$, the function \mathcal{G} of (2.10) is given by

$$\mathcal{G}(x) = \log(\log x). \quad (4.40)$$

Note that the function \mathcal{G} is bounded if and only if $0 < \gamma < 1$.

PROOF. We compute the function \mathcal{G} by Theorem 2.8. Let $\mathcal{F}'(y) = \frac{1}{y(\log y)^{1/\gamma}}$ for $y > 1$ (i.e. $\alpha = e$ in Theorem 2.8). Then if $\gamma \neq 1$, we have

$$\mathcal{F}(y) = \frac{\gamma}{\gamma-1} (\log y)^{1-1/\gamma}$$

If $\gamma = 1$, we get $\mathcal{F}(y) = \log(\log y)$, $y > 1$.

We deduce the expression of \mathcal{G} by the formula $\mathcal{G}(x) = \exp o\mathcal{F} o \log(x)$, $x > e$. The proof is completed.

We now discuss when $\mathcal{N}(y) = y_+(\log_+ y_+)^{1/\gamma}$ (or some variants) appears for some ultracontractive semigroups. Let's consider semigroups satisfying (4.37) with $m(t) = c_0 \exp(c_1 \exp(c_2/t^\gamma))$, for some $c_0, c_1, c_2, \gamma > 0$. Because it seems not possible to compute explicitly \mathcal{N} , we prove an estimate for \mathcal{N} . We note that we are in the situation where Theorem 2.8 applies since \mathcal{N} is only known for large value of y (essentially we study the behavior of the function \mathcal{N} when $y \rightarrow +\infty$). For small value of y , we set $\mathcal{N}(y) = 0$.

With the constants $c_0, c_1, c_2, \gamma > 0$ defined above, let $z = y - \log c_0$. We now prove that there exists $z_1 > c_1$ and constants $k, k' > 0$ such that

$$kz \left(\log \frac{z}{c_1}\right)^{1/\gamma} \leq \mathcal{N}(z + \log c_0) \leq k'z \left(\log \frac{z}{c_1}\right)^{1/\gamma}. \quad (4.41)$$

Indeed, $\mathcal{N}(y) = \sup_{s>0} h_y(s)$ with

$$h_y(s) = sy - sm(1/2s) = s(y - \log c_0) - c_1 s \exp(c_2 2^\gamma s^\gamma).$$

Let $k_2 = c_2 2^\gamma$ then $h'(s) = z - c_1 e^{k_2 s^\gamma} (1 + k_2 \gamma s^\gamma)$. If $z \leq c_1$ the supremum is obtained at $s = 0^+$ since h is decreasing. Thus $\mathcal{N}(y) = 0$ for $y \leq \log c_0 + c_1$. If $z > c_1$ the supremum is obtained at $s = s_0$ satisfying $h'(s_0) = 0$ since h' is positive for $s < s_0$ and negative for $s > s_0$. We get the following relation between z and s_0 :

$$z = c_1 e^{k_2 s_0^\gamma} + c_1 k_2 \gamma s_0^\gamma e^{k_2 s_0^\gamma}$$

We deduce from this relation that $z \rightarrow +\infty$ iff $s_0 \rightarrow +\infty$. Then we deduce the two following asymptotics for z and s_0 large:

$$z \sim (c_1 k_2 \gamma) s_0^\gamma \exp(k_2 s_0^\gamma)$$

from which we deduce

$$h(s_0) \sim z s_0 \tag{4.42}$$

and

$$\log(z/c_1) \sim k_2 s_0^\gamma$$

from which, with (4.42), we deduce (4.41).

4.5. $\mathcal{N}(y) = cy$

Here, we consider the following Nash-type inequality

$$c \|f\|_2^2 \log \|f\|_2^2 \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}, \|f\|_1 \leq 1. \tag{4.43}$$

with Nash function $\mathcal{N}(y) = cy$, $y \in \mathbb{R}$ ($c > 0$) and the associated \mathcal{G} -decay of the corresponding semigroup.

We motivate this study by two examples: the Ornstein-Uhlenbeck semigroup and the Γ^* -semigroup (see definition below). They satisfy Gross' inequality just below:

$$\int_X f^2 \log \frac{f^2}{\|f\|_2^2} d\mu \leq c_{LS} (\mathcal{L}f, f). \tag{4.44}$$

See for instance, [Gr1, Gr2, W5?] and Section 4.6. It is well-known that the generator $\mathcal{L} = \Delta - x \cdot \nabla$ of the Ornstein-Uhlenbeck semigroup on $L^2(\mathbb{R}^n, d\mu)$ with the gaussian measure $d\mu(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx$ satisfies Gross' inequality (4.44) with $c_{LS} = 2$ (see [Gr1], [A-S]). The extremal functions of Gross' inequality (let's say in one-dimensional case) are exactly $f_\lambda(x) = e^{\lambda x}$, $\lambda \in \mathbb{R}$, but normalized functions $f_\lambda / \|f_\lambda\|_1$ are no longer optimal for (4.43) (except the trivial case f_0).

For the Γ^* -semigroup the inequality (4.44) is satisfied for an infinite measure (Lebesgue measure): see (4.66) below.

Gross' inequality (4.44) always implies inequality (4.43). It is deduced by the same convexity argument as in Theorem 3.1. The inequality (4.43) is a priori weaker than Log-Sobolev inequality (4.44). In fact we prove in Proposition 4.13 below that (4.43) and (4.44) are equivalent for Dirichlet forms on probability space (but in this process the best constants are lost). See also [Ma], [BM1] for a discussion on such an equivalence in more general cases. Note that, for Ornstein-Uhlenbeck semigroup, the space is a probability space but inequalities of the form (4.43) also appear for infinite measures, see for instance the Γ^* -semigroup treated in Section 4.6.

We first start by a simple result. Recall that if μ is a probability measure, Gross' inequality (4.43) implies Poincaré inequality ([A-S] p.6, [Gr2] Th. 2.5). In next proposition, we go further in the spirit of Section 1 and 2 of [A-S].

We set $\mu(f) = \int f d\mu$ and $\text{Var}_\mu(f) = \|f - \mu(f)\|_2^2$. For a Dirichlet form \mathcal{E} on $L^2(\mu)$, we denote by T_t the associated symmetric (sub-)markov semigroup and \mathcal{L} the generator of T_t : $\mathcal{E}(f) = (\mathcal{L}f, f)_\mu$, $f \in \mathcal{D}(\mathcal{L})$ (See [Fu]). For entropy of f , we set: $\text{Ent}_\mu(f) := \int f \log f d\mu - (\int f d\mu) \log(\int f d\mu)$.

Proposition 4.13. *Assume that μ is a probability measure. Let \mathcal{E} be a Dirichlet form with generator \mathcal{L} such that $\mathbf{1} \in \mathcal{D}(\mathcal{E})$ and for any $h \in \mathcal{D}(\mathcal{E})$ we have $\mathcal{E}(\mathbf{1}, h) = 0$ (i.e. μ is invariant). Suppose that there exists $c_1 > 0$ such that for all $f \in \mathcal{D}(\mathcal{E})$, $\|f\|_1 \leq 1$:*

$$\|f\|_2^2 \log \|f\|_2^2 \leq c_1 \mathcal{E}(f). \quad (4.45)$$

Then

1. For any $f \in \mathcal{D}(\mathcal{E})$,
$$\text{Var}_\mu(f) \leq c_1 \mathcal{E}(f). \quad (4.46)$$

2. For any $f \in L^2$,
$$\text{Var}_\mu(T_t f) \leq e^{-2t/c_1} \text{Var}_\mu(f). \quad (4.47)$$

3. For any $f \in \mathcal{D}(\mathcal{E})$:

$$\int_X f^2 \log \frac{f^2}{\|f\|_2^2} d\mu \leq c'_1 (\mathcal{L}f, f). \quad (4.48)$$

Conversely, the inequality (4.48) implies (4.43) with $c_1 = c'_1$.

4. If moreover, P_t is of diffusion type,

$$\text{Ent}_\mu(P_t f) \leq e^{-4t/c'_1} \text{Ent}_\mu(f). \quad (4.49)$$

As corollary, under the assumptions just above, Nash-type inequality (4.45) and Gross' inequality (4.48) are equivalent up to constants.

PROOF. **1.** Let $g \in \mathcal{D}(\mathcal{E})$ such that $\|g\|_\infty \leq 1/2$ and $\mu(g) = 0$. For any $0 < \varepsilon < 1$, set $f = 1 + \varepsilon g$. So $f \geq 0$, $\|f\|_1 = 1$ and $\|f\|_2^2 = 1 + \varepsilon^2 \|g\|_2^2 \geq 1$. By (4.45),

$$\log(1 + \varepsilon^2 \|g\|_2^2) \leq c_1 \varepsilon^2 \mathcal{E}(g).$$

Dividing by ε^2 and taking the limit as $\varepsilon \rightarrow 0^+$, we get the result under the assumption $\|g\|_\infty \leq 1/2$. By homogeneity, we deduce (4.46) for any $g \in \mathcal{D}(\mathcal{E}) \cap L^\infty$ with $\mu(g) = 0$. Now, let $h \in \mathcal{D}(\mathcal{E})$ and set $g_n = (h \wedge n) \vee (-n)$ for $n \in \mathbb{N}$. Since \mathcal{E} is a Dirichlet form then $g_n \in \mathcal{D}(\mathcal{E})$, $g_n \in L^\infty$ and $\mathcal{E}(g_n) \leq \mathcal{E}(h)$. Thus, we have $\text{Var}_\mu(g_n) \leq c_1 \mathcal{E}(h)$. We conclude by taking the limit as $n \rightarrow \infty$.

2. The proof is well-known (see [A–S] p.30 for instance). Note that during the course of the proof if μ is invariant then $\mu(f) = 0$ implies $\mu(T_t f) = 0$.

3. Under the hypothesis (4.45), it is proved in [Ma] (see also [B–C–L–S]) using the cut-off method that there exist $c'_1, c'_2 > 0$ such that the defective log-Sobolev inequality holds true:

$$\text{Ent}_\mu(f^2) := \int_X f^2 \log \frac{f^2}{\|f\|_2^2} d\mu \leq c'_1 \mathcal{E}(f) + c'_2 \|f\|_2^2.$$

Now, by Rothaus' lemma (See [Ro] p.310 or [A–S] Lemma 4.3.8):

$$\text{Ent}_\mu(f^2) \leq 2 \text{Var}_\mu(f) + \text{Ent}_\mu(\tilde{f}^2)$$

with $\tilde{f} = f - \mu(f)$. So, we get

$$\begin{aligned} \text{Ent}_\mu(f^2) &\leq 2 \text{Var}_\mu(f) + c'_1 \mathcal{E}(\tilde{f}) + c'_2 \|\tilde{f}\|_2^2 \\ &\leq (2 + c'_2) \text{Var}_\mu(f) + c'_1 \mathcal{E}(f) \leq ((2 + c'_2)c_1 + c'_1) \mathcal{E}(f) \end{aligned}$$

by using Poincaré inequality (4.46) and $\mathcal{E}(\tilde{f}) = \mathcal{E}(f)$. We conclude Gross' inequality (4.48) with $c'_1 = ((2 + c'_2)c_1 + c'_1)$.

4. This is a classical result (see [A–S] p.35).

Remark 4.14. Note that Gross' inequality implies (4.45) with $c_1 = 2$ and implies also Poincaré inequality with $c_1/2 = 1$ and they are the best constants. But under (4.45), we only get c_1 for Poincaré inequality instead of $c_1/2$ and consequently the best constants are lost.

Now, we prove some relations between Nash-type inequality (4.43) and other inequalities: hypercontractivity-like and super-Poincaré-like inequalities. In fact, hypercontractivity-like inequality introduced here is nothing else that \mathcal{G} -decay for Nash function $\mathcal{N}(y) = cy$. To simplify, we just present the case $p = 1, q = 2$ of the \mathcal{G} -decay. A generalization to (p, q) \mathcal{G} -decay can certainly be adapted, but we shall not give details.

Theorem 4.15. *Let \mathcal{E} be a Dirichlet form with generator \mathcal{L} and μ a probability measure. The following statements are equivalent:*

1. *There exists $c_1 > 0$ such that for all $f \in \mathcal{D}(\mathcal{E})$, $\|f\|_1 \leq 1$:*

$$c_1^{-1} \|f\|_2^2 \log \|f\|_2^2 \leq \mathcal{E}(f). \quad (4.50)$$

2. *There exists $t_0 \in (0, \infty]$, $c_2 > 0$ and $\alpha_2 : (0, t_0) \rightarrow (0, 1)$ non-increasing of class C^1 , $\alpha_2(0^+) = 1$, $\alpha_2'(0^+)$ exists and $\alpha_2'(0^+) < 0$ such that for any f , $\|f\|_1 \leq 1$:*

$$\|e^{-tc_2 \mathcal{L}} f\|_2^2 \leq \|f\|_2^{2\alpha_2(t)} \quad (4.51)$$

3. *There exists $t_0 \in (0, \infty]$, $c_3 > 0$ and $\alpha_3 : (0, t_0) \rightarrow (0, 1)$ non-increasing of class C^1 , $\alpha_3(0^+) = 1$, $\alpha_3'(0^+)$ exists and $\alpha_3'(0^+) < 0$ such that for any $f \in \mathcal{D}(\mathcal{E})$, $\|f\|_1 \leq 1$:*

$$\|f\|_2^2 \leq 2tc_3 \mathcal{E}(f) + \|f\|_2^{2\alpha_3(t)}. \quad (4.52)$$

4. *For any $f \in \mathcal{D}(\mathcal{E})$:*

$$\int_X f^2 \log \frac{f^2}{\|f\|_2^2} d\mu \leq c(\mathcal{L}f, f). \quad (4.53)$$

Moreover, (1) implies (2) with $c_2 = c_1$, $t_0 = \infty$, $\alpha_2(t) = \gamma(t) = e^{-2t}$. (2) implies (3) with $c_3 = c_2$ and $\alpha_3 = \alpha_2$. (3) implies (1) with $c_1 = \frac{2c_3}{-\alpha_3'(0^+)}$.

Recall that (4.53) is equivalent to hypercontractivity (see [Gr1] Section 3). Here, we particularly focus on new formulations (4.51) and (4.52).

Note that (2) implies (3) and (3) implies (1) for any function α_2 (respectively α_3) satisfying the conditions of (2) (respectively (3)) without the assumptions that \mathcal{E} is a Dirichlet form and μ is a probability measure (See the proof below). In particular if $\alpha_3'(0^+) = -2$ the constant c_1 and c_3 are the same. The inequality (4.52) is a new type of super-Poincaré inequality. Compare for instance with the following super-Poincaré which has been extensively studied:

$$\|f\|_2^2 \leq 2tc_3 \mathcal{E}(f) + a(t)\|f\|_2^2, \quad \|f\|_1 \leq 1. \quad (4.54)$$

There are many results about super Poincaré and examples in Wang's work: see for instance (but non exhaustive list) [W0]-[W6],[W-Z] and the book [W7]. See also recent works of [Z] and [C-G 1] (via Lyapunov functions).

Note also that if μ is an ergodic probability measure that is $\lim_{t \rightarrow +\infty} T_t f(x) = \int_X f d\mu$. Then for $f \geq 0$, we have $\|f\|_1 = \int_X f d\mu \leq \|T_t f\|_2$ since $t \rightarrow \|T_t f\|_2$ is non-increasing. Hence, the condition $1 \leq \|T_t f\|_2$ is automatically satisfied for any $t > 0$ when $\|f\|_1 = 1$ for which our discussion can be reduced.

The inequality (4.51) can be seen as a variant of the hypercontractivity property. Indeed, under the assumption of Proposition 4.15, inequality (4.51) or (4.52) is equivalent to Gross' inequality and also to hypercontractivity (see p.64-65 of ([Gr2])).

PROOF. (1) *implies* (2). We apply Theorem 2.8 with $(c_1 \mathcal{L}f, f)$ as Dirichlet form and $\mathcal{N}(y) = y$ with $y > 0$ so $\mathcal{G}(x) = \log x, x > 1$. Therefore, for all $t > 0$, for all f such that $\|f\|_1 \leq 1$ and $1 < \|T_t f\|_2^2$ where $T_t = e^{-tc_1 \mathcal{L}}$:

$$\log \|T_t f\|_2^2 \leq e^{-2t} \log \|f\|_2^2. \quad (4.55)$$

Assume that $\|f\|_1 = 1$. Since μ is a probability measure: $1 = \|f\|_1 \leq \|f\|_2$. If $\|T_t f\|_2^2 \leq 1$ then (4.55) is also satisfied. Hence, for any $f \in L^1 \cap L^2$:

$$\|T_t f\|_2^2 \leq \|f\|_2^{2\gamma(t)} \|f\|_1^{2(1-\gamma(t))}$$

with $\gamma(t) = e^{-2t}$. Since $1 - \gamma(t) > 0$, we conclude (2) with $t_0 = \infty$, $c_2 = c_1$ and $\alpha_2 = \gamma$.

Note that (2) *implies* (1) by the converse part of Thm 2.8.

(2) *implies* (3). Let $f \in \mathcal{D}(\mathcal{E})$ with $\|f\|_1 \leq 1$. Set $T_t = e^{-tc_2 \mathcal{L}}$. We have

$$\|f\|_2^2 = (f - T_{2t}f, f) + \|T_t f\|_2^2$$

since (T_t) is symmetric with (\cdot, \cdot) the inner product of $L^2(\mu)$. We bound the second term by

$$(f - T_{2t}f, f) = \int_0^{2t} -\frac{d}{ds}(T_s f, f) ds = c_2 \int_0^{2t} (\mathcal{L}T_s f, f) ds \leq 2tc_2(\mathcal{L}f, f)$$

because the function $s \rightarrow (\mathcal{L}T_s f, f)$ is non-increasing. Assuming (2), we deduce (3) with $\alpha_3 = \alpha_2$ and $c_3 = c_2$ by the inequality

$$\|f\|_2^2 \leq 2tc_2(\mathcal{L}f, f) + \|T_t f\|_2^2.$$

(3) *implies* (1). Under the assumptions of (3), we have

$$\left(\|f\|_2^{2\alpha_3(0)} - \|f\|_2^{2\alpha_3(t)} \right) / 2t \leq c_3 \mathcal{E}(f).$$

Taking the limit, we get

$$\frac{-\alpha_3'(0)}{2c_3} \|f\|_2^2 \log \|f\|_2^2 \leq \mathcal{E}(f).$$

In L^p -norm and without assuming that μ is a probability measure, we have the following result.

Proposition 4.16. *Let $2 \leq p < \infty$ and $1 \leq q < p$. Assume that inequality (4.43) holds true then: For all f such that $\|f\|_{p/2} \leq 1$ and for all $t > 0$ satisfying $1 \leq \|T_t f\|_p^p$, we get*

$$\log \|T_t f\|_p^p \leq e^{-pt} \log \|f\|_p^p. \quad (4.56)$$

or equivalently,

$$\|T_t f\|_p^p \leq \|f\|_p^{p\gamma(t)} \quad (4.57)$$

with $\gamma(t) = e^{-pt}$. Conversely when $p = 2$, (4.56) implies (4.43).

PROOF. The proof follows the same arguments of Thm 4.15 using, moreover Thm 3.3.

Suggested by the above considerations, we revisit the equivalence between Gross' inequality, hypercontractivity, Beckner's interpolated inequality (between Gauss and Poincaré) and another modified Nash-type inequality which is introduced here. This modified Nash-type inequality implies Nash-type inequality (4.52) above. All these inequalities are equivalent under usual assumptions on the semigroup.

Theorem 4.17. *Let μ be a probability measure and \mathcal{E} be a Dirichlet form and $c_0 > 0$.*

The following statements are equivalent:

1. (Hypercontractivity). *For any $1 < p < \infty$ and any $t > 0$, we set $q(t) = e^{\frac{4t}{c_0}}(p-1) + 1$,*

$$\|T_t f\|_{q(t)} \leq \|f\|_p. \quad (4.58)$$

2. (Modified Nash-type inequality). *For any $t > 0$,*

$$\|f\|_2^2 \leq 2t \mathcal{E}(f) + \|f\|_{p(t)}^2 \quad (4.59)$$

with $p(t) = 1 + e^{-\frac{4t}{c_0}}$.

3. (Interpolated inequalities). *For any $p \in (1, 2)$,*

$$\|f\|_2^2 - \|f\|_p^2 \leq \frac{c_0}{2} \log \left(\frac{1}{p-1} \right) \mathcal{E}(f). \quad (4.60)$$

4. (Gross' inequality).

$$\int_X f^2 \log \frac{f^2}{\|f\|_2^2} d\mu \leq c_0 \mathcal{E}(f). \quad (4.61)$$

From modified Nash-type inequality (4.59), we can deduce the new type of Poincaré inequality above (4.52). Indeed, by interpolation we have under the assumption $\|f\|_1 \leq 1$:

$$\|f\|_p \leq \|f\|_1^{1-\theta(t)} \|f\|_2^{\theta(t)} \leq \|f\|_2^{\theta(t)}$$

with $\theta(t) = 2(1 - 1/p) = \frac{4e^{-\frac{4t}{c_0}}}{1+e^{-\frac{4t}{c_0}}} = \frac{4}{1+e^{\frac{4t}{c_0}}}$. We deduce for $\|f\|_1 \leq 1$:

$$\|f\|_2^2 \leq 2t \mathcal{E}(f) + \|f\|_2^{2\theta(t)}$$

with $\theta(t)$ different from $\alpha_3(t/c_3) := \gamma(t/c_0) = e^{-2t/c_0}$ obtained in (4.52) and deduce from (4.50) with $c_1 = c_0$. In particular, $\theta(t) \sim 2\alpha_3^2(t/c_3)$ when t is

closed to zero.

To prove (4.60), we do not use a specific spectral theory as in Bekner's results [Bec1] for the generator of Ornstein-Uhlenbeck process or the Poisson semigroup on the unit sphere. To work in our abstract setting, we use the semigroup approach (see (4.62 below) and deduce interpolated Poincaré inequalities also by hypercontractivity property as in [Bec1]. So, a priori we cannot expect to get an optimal bound. Indeed, our bound in (4.60) is $\frac{c_0}{2} \log \left(\frac{1}{p-1} \right)$ and for the Ornstein-Uhlenbeck semigroup $c_0 = 2$ (see [A-S] p.25), we have $2 - p$ as a bound in (4.60). It is easily shown that for any $1 < p < 2$,

$$2 - p < \log \left(\frac{1}{p-1} \right) = \log \left(1 + \frac{2-p}{p-1} \right).$$

Consequently, our bound (4.60) is not optimal for Ornstein-Uhlenbeck semigroup and only asymptotically optimal when p goes to 2 since $\log \left(\frac{1}{p-1} \right) \sim 2 - p$. But it is enough to deduce Gross's inequality in the same way as in [Bec1].

PROOF. (1) implies (2): For any $t > 0$, any $f \in \mathcal{D}(\mathcal{E})$,

$$\|f\|_2^2 \leq 2t\mathcal{E}(f) + \|T_t f\|_2^2. \quad (4.62)$$

Let $p = p(t) = 1 + e^{-\frac{4t}{c_0}} \in (1, 2)$ then $q(t) = 2$. We apply hypercontractivity and conclude.

(2) implies (3): Let $1 < p < 2$. There exists a unique $t > 0$ such that $p = p(t) = 1 + e^{-\frac{4t}{c_0}}$. Then $2t = \frac{c_0}{2} \log \left(\frac{1}{p-1} \right)$. So, from (2) we get (3).

(3) implies (4). The proof is similar to Beckner's proof [Bec1]. We have

$$(\|f\|_2^2 - \|f\|_p^2) / (2 - p) \leq \frac{c_0}{2(2 - p)} \log \left(\frac{1}{p-1} \right) \mathcal{E}(f).$$

We get the result by taking the limit as p approaches 2 and by the fact that $\log \left(\frac{1}{p-1} \right) \sim 2 - p$.

(4) implies (1). The proof is well-known, see [Gr1, Gr2] see also [A-S] p. 38.

As a consequence of Wang's theorem ([W5] Th 1.1) and Theorem 4.17 above, we get:

Corollary 4.18. *Let (M, g) be a Riemannian manifold with dv_g the riemannian volume. There exists V a smooth function on $M \setminus D$ with D a closed set of measure zero such that $d\mu = \frac{e^{-V}}{Z} dv_g$ with $Z = \int_M e^{-V} dv_g$ satisfies all the equivalent inequalities (4.58)-(4.61) with Dirichlet form $\mathcal{E}(f) = \int_M |\nabla f|^2 d\mu$.*

PROOF. For the Dirichlet form $\mathcal{E}(f) = \int_M |\nabla f|^2 d\mu$, F-Y Wang proved that there exists a measure μ as mentioned in the Corollary above such that (4.61) holds true.

Other results concerning, in particular, super Poincaré inequality can be drawn from Th 1.2 and Th. 1.3 of [W5]. But to avoid a too long paper we shall not give details.

4.6. Γ^* -semigroup

In that section, we deal with the following Nash function

$$\mathcal{N}(y) = k_1 \log(1 + k_2 e^{2y}), \quad y \in \mathbb{R}, \quad k_1, k_2 > 0.$$

Note that $\mathcal{N}(y)$ is close to cy for large y . So, this situation has some similarity with the preceding section. We give an example (with an infinite measure μ) which motivates this study (see below). We begin by a general result.

Proposition 4.19. *Assume that inequality*

$$k_1 \|f\|_2^2 \log(1 + k_2 \|f\|_2^4) \leq (\mathcal{L}f, f). \quad (4.63)$$

holds true for the generator of (T_t) a sub-markovian semigroup. Let $2 \leq p < \infty$ and \mathcal{G} defined by

$$\mathcal{G}(x) = \left[\log(1 + k_2 x^2) \exp\left(\frac{-1}{\log(1 + k_2 x^2)}\right) \right]^{\frac{1}{2k_1}}, \quad x > 0.$$

Then, for all f such that $\|f\|_{p/2} \leq 1$ and for all $t > 0$ with $1 \leq \|T_t f\|_p^p$, we have

$$\mathcal{G}(\|T_t f\|_p^{2p}) \leq e^{-pt} \mathcal{G}(\|f\|_p^{2p}).$$

Conversely, this inequality is equivalent to Nash inequality (4.63).

PROOF. We apply Theorem 2.8 and Remark 2.7. For $a \leq b$, we define \mathcal{F} such that

$$\mathcal{F}(b) - \mathcal{F}(a) = \int_a^b \frac{1}{\mathcal{N}(x)} dx = k_1^{-1} \int_{\log(1+k_2 e^{2a})}^{\log(1+k_2 e^{2b})} \frac{e^y}{2y(e^y - 1)} dy.$$

The right-hand side of this equality is obtained by the change of variable $y = \log(1 + k_2 e^{2x})$. From the inequality

$$\frac{e^y}{(e^y - 1)} \leq 1 + 1/y, \quad y > 0,$$

we deduce

$$\mathcal{F}(b) - \mathcal{F}(a) \leq \frac{1}{2k_1} [\log y - 1/y]_{\log(1+k_2 e^{2a})}^{\log(1+k_2 e^{2b})} = \mathcal{H}(b) - \mathcal{H}(a).$$

where we set

$$\mathcal{H}(y) = \frac{1}{2k_1} \left[\log \log (1 + k_2 e^{2y}) - \frac{1}{\log (1 + k_2 e^{2y})} \right], \quad y \in \mathbb{R}.$$

Now, we can compute explicitly the \mathcal{G} -function associated to \mathcal{H} :

$$\mathcal{G}(x) = \exp o\mathcal{H}o \log(x) = \left[\log(1 + k_2 x^2) \exp \left(-\frac{1}{\log(1 + k_2 x^2)} \right) \right]^{\frac{1}{2k_1}}, \quad x > 0.$$

We easily conclude the proof of this proposition applying Theorem 2.8 and Remark 2.7. This completes the proof.

Before expliciting this result to $(\mu_t^*)_{t>0}$ the Γ^* -semigroup on \mathbb{R} , we recall the definition and some facts about this semigroup and its infinitesimal generator. From now on, we use the following definition of the Fourier transform

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{-2i\pi x \cdot y} dy.$$

This definition is convenient due to the fact that it is an isometry on $L^2(\mathbb{R}^n, dx)$ (dx : Lebesgue measure). Let μ_t be the Γ -convolution semigroup on \mathbb{R} given in terms of its Fourier transform:

$$\hat{\mu}_t(x) = \int_{\mathbb{R}} e^{-2i\pi x \cdot y} d\mu_t(y) = e^{-t\psi_0(x)}$$

with

$$\psi_0(x) = \frac{1}{2} \ln(1 + 4\pi^2 x^2) + i \arctan(2\pi x).$$

The measure $d\mu_t$ has density μ_t with respect to the Lebesgue measure:

$$\mu_t(y) = \chi_{\mathbb{R}^+}(y) \frac{1}{\Gamma(t)} y^{t-1} e^{-y}.$$

We construct the Γ^* -semigroup $(\mu_t^*)_{t>0}$ by symmetrisation of $(\mu_t)_{t>0}$: $\mu_t^* = \mu_t * \check{\mu}_t$ where $\check{\mu}_t(u) = \mu_t(\check{u})$ and $\check{u}(x) = u(-x)$. Then we have

$$\hat{\mu}_t^*(x) = e^{-t\psi(x)} = e^{-t \ln(1 + 4\pi^2 x^2)}.$$

Thus, $\psi(x) = \ln(1 + 4\pi^2 x^2)$ is the symbol of the Γ^* -semigroup and the corresponding quadratic form is

$$(\mathcal{L}f, f) = \lim_{t \rightarrow 0^+} (f - \mu_t^* \star f, f)_{L^2} = \int_{\mathbb{R}} \log(1 + 4\pi^2 x^2) |\hat{f}(x)|^2 dx.$$

The generator is formally denoted by $\mathcal{L} = \log(1 + \Delta)$ because the Fourier symbol of \mathcal{L} is $\ln(1 + 4\pi^2 x^2)$ and the symbol of Δ is $4\pi^2 x^2$. (See [B-F]).

Now, we list some important properties of the density μ_t^* on \mathbb{R} . We have explicetly the density

$$\mu_t^*(x) = \mu_t * \check{\mu}_t(x) = \frac{1}{2^{2t-1} \Gamma(t)^2} \int_x^\infty (s^2 - x^2)^{t-1} e^{-s} ds.$$

This implies:

1. $\mu_t^*(x) \sim \frac{1}{2^t \Gamma(t)} |x|^{t-1} e^{-|x|}$, as $x \nearrow \infty$.
2.
$$\mu_t^*(x) \sim \begin{cases} c(t) |x|^{2t-1}, & 0 < t < 1/2 \\ \tilde{c} \ln \frac{1}{|x|}, & t = 1/2. \end{cases}, \quad \text{as } x \searrow 0.$$
3. $\mu_t^*(x) \in C_0(\mathbb{R})$, $t > 1/2$.

Hence by the formula $\|\mu_t^*\|_{1 \rightarrow \infty} = \sup_{x \in \mathbb{R}} \mu_t^*(x) = \mu_t^*(0)$

$$\|\mu_t^*\|_{1 \rightarrow \infty} = \infty, \quad 0 < t \leq 1/2,$$

and

$$\|\mu_t^*\|_{1 \rightarrow \infty} = \frac{\Gamma(2t-1)}{2^{2t-1} \Gamma(t)^2}, \quad t > 1/2,$$

where $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$. We deduce the large time behavior of the operator norm

$$\|\mu_t^*\|_{1 \rightarrow \infty} \sim \frac{t^{-3/2}}{\pi}, \quad \text{as } t \nearrow \infty$$

Note that this semi-group is not ultracontractive in the sense that (T_t) is bounded from L^1 to L^∞ for any $t > 0$. It is only bounded in the interval $(\frac{1}{2}, +\infty)$. So, the usual theory of ultracontractivity doesn't apply. To quantify the regularisation property of this operator, we use Nash-type inequality.

In [BM2], it is proved that the generator \mathcal{L} satisfies the following Nash-type inequality on \mathbb{R} : for all $\varepsilon > 0$ and for all $f \in \mathcal{D}$, $\|f\|_1 \leq 1$,

$$k_1 \|f\|_2^2 \log(1 + k_2 \|f\|_2^4) \leq (\mathcal{L}f, f).$$

with $k_1 = \frac{\varepsilon}{\varepsilon+2}$ and $k_2 = \frac{4\pi^2}{(2+\varepsilon)^2}$ (see below).

Corollary 4.20. *Let (T_t) be the Γ^* -semigroup on the real line. Let $2 \leq p < \infty$. For all f such that $\|f\|_{p/2} \leq 1$ and for all $t > 0$ with $1 \leq \|T_t f\|_p^p$, we have*

$$\begin{aligned} & \log(1 + k_2 \|T_t f\|_p^{2p}) \exp\left(\frac{-1}{\log(1 + k_2 \|T_t f\|_p^{2p})}\right) \\ & \leq e^{-2pk_1 t} \log(1 + k_2 \|f\|_p^{2p}) \exp\left(\frac{-1}{\log(1 + k_2 \|f\|_p^{2p})}\right). \end{aligned} \quad (4.64)$$

with $k_1 = \frac{\varepsilon}{\varepsilon+2}$ and $k_2 = \frac{4\pi^2}{(2+\varepsilon)^2}$ for any $\varepsilon > 0$.

PROOF. The inequality (4.63) is satisfied (see [BM2]). Then we apply Proposition 4.19.

We now discuss some aspects relating (4.63) and Gross' inequality in the higher dimensional case for the generator \mathcal{L} of the Γ^* -semigroup.

Let n be an integer such that $n \geq 1$. Let $a > 0$. We consider the operator $\mathcal{L}_a = \log(I + a^2 \Delta)$ with Δ the usual non-negative Laplacian on \mathbb{R}^n (I the identity operator on L^2) defined by Fourier analysis

$$(\mathcal{L}_a f, f) = \int_{\mathbb{R}^n} \log(1 + 4\pi^2 a^2 |y|^2) |\hat{f}(y)|^2 dy$$

with dy the n -dimensional Lebesgue measure. The Γ^* -semigroup generated by \mathcal{L}_a is denoted by $T_t = e^{-t\mathcal{L}_a} = (I + a^2 \Delta)^{-t}$, $t > 0$. The generator \mathcal{L}_a satisfies: for any $\varepsilon > 0$,

$$(1 - \Omega_n \varepsilon^n) \|f\|_2^2 \log \left(1 + 4\pi^2 \varepsilon^2 \|f\|_2^{4/n} \right) \leq (\mathcal{L} f, f). \quad (4.65)$$

with Ω_n the Lebesgue measure of the unit ball of \mathbb{R}^n (see [BM2]).

By the method of truncation ([B-C-L-S]), we obtain in [BM2] an inequality closed to Gross' inequality: for any $\varepsilon > 0$, $\rho > 1$ and $f \in \mathcal{D}(\mathcal{L})$:

$$c_1 \int_{\mathbb{R}^n} f^2(x) \log \left(1 + c_2 \left(\frac{f^2(x)}{\|f\|_2^2} \right)^{2/n} \right) dx \leq (\mathcal{L} f, f).$$

with $c_1 = (1 - \Omega_n \varepsilon^n) \frac{(\rho-1)^2}{\rho^4}$ and $c_2 = 4\pi^2 \varepsilon^2 (\rho-1)^{4/n} \rho^{-12/n}$. By changing f by $a^{n/2} f(ax)$, we obtain:

$$c_1 \int_{\mathbb{R}^n} f^2(x) \log \left(1 + c_2 a^2 \left(\frac{f^2(x)}{\|f\|_2^2} \right)^{2/n} \right) dx \leq \int_{\mathbb{R}^n} \log(1 + 4\pi^2 a^2 |y|^2) |\hat{f}(y)|^2 dy.$$

Note that such inequality is not completely surprising due to the Fourier symbol $\log(1 + 4\pi^2 a^2 |y|^2)$ of the generator \mathcal{L}_a . Note also that the dimension n also appears in our inequalities through the constants c_1, c_2 and the exponent in the integral.

Let a_n define by

$$a_n = \inf \left\{ \frac{1}{2\pi\varepsilon} \left(\frac{\rho^3}{\rho-1} \right)^{2/n} ; 1 < \rho, 0 < \varepsilon < (1/\Omega_n)^{1/n} \right\} = \frac{\Omega_n^{1/n}}{2\pi} \left(\frac{27}{4} \right)^{2/n}$$

Assume $a > a_n > 0$ then there exist $0 < \varepsilon < (1/\Omega_n)^{1/n}$ and $\rho > 0$ such that $c_2 a^2 > 1$ and $c_1 > 0$. In that case, we obtain a stronger form of Gross's inequality for \mathcal{L}_a :

$$c_1 \log(c_2 a^2) \|f\|_2^2 + \frac{2c_1}{n} \int_{\mathbb{R}^n} f^2(x) \log \left(\frac{f^2(x)}{\|f\|_2^2} \right) dx \leq (\mathcal{L}_a f, f) \quad (4.66)$$

because the term $c_1 \log(c_2 a^2) \|f\|_2^2$ is non-negative.

It is known that there is no spectral gap for $\mathcal{L}_a = \log(I + a^2\Delta)$ and, of course, we cannot deduce Poincaré inequality for \mathcal{L}_a . It is due to the fact that μ is an infinite measure: first the entropy $\int_{\mathbb{R}^n} f^2(x) \log\left(\frac{f^2(x)}{\|f\|_2^2}\right) dx$ is not necessarily *non-negative* since dx is not a probability measure ! Take for instance: $f(x) = \sqrt{p_t(x)} = \frac{1}{(4\pi t)^{n/4}} e^{-\frac{|x|^2}{4t}}$ with $t > \frac{1}{4\pi}$. Also, the usual proof use to deduce Poincaré inequality from Log-Sobolev inequality is the fact that μ is a probability measure. This is an important difference with Gross' inequality for probability measure.

Despite this aspect, when $c_2 a^2 > 1$, we deduce the following Log-Sobolev inequality for \mathcal{L}_a :

$$\frac{2c_1}{n} \int_{\mathbb{R}^n} f^2(x) \log\left(\frac{f^2(x)}{\|f\|_2^2}\right) dx \leq (\mathcal{L}_a f, f). \quad (4.67)$$

Indeed, the utmost left term in (4.66) is non-negative. Note that our inequality (4.66) can be read as an improved Log-Sobolev inequality when $c_2 a^2 > 1$:

$$\frac{2c_1}{n} \int_{\mathbb{R}^n} f^2(x) \log\left(\frac{f^2(x)}{\|f\|_2^2}\right) dx \leq ((\mathcal{L}_a f, f) - c_1 \log(c_2 a^2) \|f\|_2^2), \quad (4.68)$$

or analogously the operator $\mathcal{L}_a - c_1 \log(c_2 a^2)I$ satisfies a Log-Sobolev inequality (I is the identity operator).

Other operators like $\mathcal{L} = [\log(a + b\Delta^\alpha)]^\beta$, $0 < \alpha, \beta < 1$ can certainly be treated in a similar way. See [BM2] where a larger class of operators is treated on \mathbb{R}^n and on some other groups.

5. Fractional powers and functional decay

In [BM1], it is proved that a (2,1)-Nash type inequality for the operator \mathcal{L} with Nash function \mathcal{N} is equivalent to a (2,1)-Nash-type inequality for the operator \mathcal{L}^β , β with Nash function \mathcal{N}^β (up to multiplicative constants). More explicitly with the notation introduced above,

Theorem 5.1. ([BM1] Th. 1.4) *Let (X, μ) be a measure space with σ -finite measure μ . Let \mathcal{L} be a non-negative self-adjoint operator with domain $\mathcal{D}(\mathcal{L}) \subset L^2(X, \mu)$. Suppose that the semigroup $T_t = e^{-t\mathcal{L}}$ acts as a contraction on $L^1(X, \mu)$ and satisfies the following Nash type inequality*

$$\|f\|_2^2 \mathcal{N}(\log \|f\|_2^2) \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}(\mathcal{L}), \quad \|f\|_1 = 1, \quad (5.1)$$

where $\mathcal{N} : \mathbb{R} \rightarrow [0, +\infty)$ is a non-decreasing function which tends to infinity at infinity. Then, for any $\beta > 0$, the following Nash-type inequality holds for some constants $c_1, c_2 > 0$,

$$c_1 \|f\|_2^2 \mathcal{N}^\beta(\log(c_2 \|f\|_2^2)) \leq (\mathcal{L}^\beta f, f), \quad \forall f \in \mathcal{D}(\mathcal{L}^\beta), \quad \|f\|_1 = 1. \quad (5.2)$$

If $(\mathcal{L}f, f)$ is a Dirichlet form then $(\mathcal{L}^\beta f, f)$ is also a Dirichlet form when $0 \leq \beta \leq 1$ or equivalently, the subordinated semigroup $T_t = e^{-t\mathcal{L}^\beta}$ is a submarkovian semigroup (see [M-R] [Fu],[V-S-C]). Thus, we shall restrict our attention to $0 < \beta \leq 1$. We have the following corollary of Th.3.3,

Corollary 5.2. *Suppose that the semigroup $T_t = e^{-t\mathcal{L}}$ is a submarkovian semigroup which satisfies the following Nash type inequality*

$$\|f\|_2^2 \mathcal{N}(\log \|f\|_2^2) \leq (\mathcal{L}f, f), \quad \forall f \in \mathcal{D}(\mathcal{L}), \quad \|f\|_1 = 1, \quad (5.3)$$

where $\mathcal{N} : \mathbb{R} \rightarrow (0, +\infty)$ is a non-decreasing function which tends to infinity at infinity. For any $\beta > 0$, let $\mathcal{G}_{2,1,\beta} = \exp \circ \mathcal{F} \circ \log$ with $\mathcal{F}' = 1/\mathcal{N}_\beta$ and $\mathcal{N}_\beta(y) = c_1 \mathcal{N}^\beta(\log c_2 + y)$, $y \in \mathbb{R}$. Let $2 \leq p < \infty$, we set $\mathcal{G}_{p,p/2,\beta} = [\mathcal{G}_{2,1,\beta}]^{\frac{p^2}{4(p-1)}}$. Then, for all $f \in \mathcal{D}$, $\|f\|_{p/2} \leq 1$ and for all $t > 0$, we have

$$\mathcal{G}_{p,p/2,\beta}(\|T_t^{(\beta)} f\|_p^p) \leq e^{-pt} \mathcal{G}_{p,p/2,\beta}(\|f\|_p^p). \quad (5.4)$$

Note that $(T_t^{(\beta)})$ is related to the semigroups (T_t) by a subordination formula, but it is not clear how to deduce (5.4) from the same inequality valid with $\beta = 1$. Thus, the equivalence between \mathcal{G} -decay and Nash-type inequality allow us to find \mathcal{G} -decay for $(T_t^{(\beta)})$.

As an example, we first mention the fractional powers of the Laplacean $\mathcal{L} = \Delta$ in the Euclidean space \mathbb{R}^n . The following $(2, 1)$ -Nash-type inequality is satisfied ($\nu = n$),

$$c_0 \|f\|_2^{2+4\beta/n} \leq (\Delta^\beta f, f), \quad \|f\|_1 \leq 1. \quad (5.5)$$

is equivalent to the \mathcal{G} -decay with

$$\mathcal{G}_{p,p/2,\beta}(x) = \exp\left(\frac{-np^2}{8c_0\beta(p-1)} x^{-2\beta/n}\right)$$

obtained by the formula in (5.4) of Corollary 5.2.

Indeed, we can also use the sharp function \mathcal{G}_p^\sharp of Th. 3.1 since Nash-type inequality is equivalent to the ultracontractive bound

$$\|T_t^{(\beta)} f\|_2 \leq c_1 t^{-n/2\beta} \|f\|_1, \quad t > 0.$$

Thus $M(t) = \log(c_1 t^{-n/2\beta})$ and the computation of $\mathcal{G}_{p,q}^\sharp$ in Th. 3.1 is left as an exercise. Results of the same form can be written in the setting of some Lie groups (see [V-S-C]).

6. Revisiting functional decays

In this section, we revisit different functional decays. The first one is the (SGP) problem mentioned in the introduction. The second one is the ultracontractivity property of semigroups. In this last case, our result is a reformulation

of a result of [Co]. We also relate the \mathcal{G} -function with the ultracontractive bound and deduce a similar relation that must satisfies the heat kernel of the semigroup and the function \mathcal{G} .

6.1. Spectral decay in L^p

In the first part of this section, we show how a \mathcal{G} -decay implies a spectral bound on L^p of the following form:

$$\|T_t f\|_p^p \leq \psi(t) \|f\|_p^p$$

under some additional assumption on the measure μ (see [W0]-[W5] and equivalence between L^p -decays when $\psi(t) = e^{-\lambda t}$ see [C-G 2]).

Proposition 6.1. *Assume that the semigroup (T_t) satisfies (2.4) i.e. for some $1 \leq q < p < \infty$, there exists \mathcal{G} a bijection from $[0, \infty)$ into $[0, \infty)$ and $\lambda \geq 0$ such that, for all $t > 0$ and for all $\|f\|_q \leq 1$,*

$$\mathcal{G}(\|T_t f\|_p^p) \leq e^{-\lambda t} \mathcal{G}(\|f\|_p^p). \quad (6.1)$$

If the measure μ is finite, we have the following decay on L^p :

$$\|T_t f\|_p^p \leq \psi(t) \|f\|_p^p \quad (6.2)$$

with $\psi(t) = \mu(X)^\gamma \mathcal{G}^{-1}(e^{-\lambda t} \mathcal{G}(\mu(X)^{-\gamma}))$ and $\gamma = \frac{p}{q} - 1$.

PROOF. Assume that $\mu(X) < +\infty$. Let f such that $\|f\|_p^p = \mu(X)^{-\gamma}$ with $\gamma = \frac{p}{q} - 1$. Let $\alpha, \beta \geq 1$ be such that $1/\alpha + 1/\beta = 1$ with $\beta = p/q$. By Hölder inequality, $\|f\|_q^q \leq \mu(X)^{1/\alpha} \|f\|_p^{p/\beta} = 1$. Hence,

$$\|T_t f\|_p^p \leq \mathcal{G}^{-1}(e^{-\lambda t} \mathcal{G}(\mu(X)^{-\gamma})).$$

because \mathcal{G}^{-1} is increasing. We conclude by homogeneity on f .

6.2. \mathcal{G} -decay and (p, q) -Ultracontractivity

Under the condition that \mathcal{G} is bounded, we prove that the semigroup is ultracontractive.

Let $1 \leq q < p \leq \infty$. We say that a semigroup (T_t) is (p, q) -ultracontractive if it is bounded from L^q into L^p for each t , i.e

$$\|T_t f\|_p^p \leq e^{M(t)} \|f\|_q^p$$

where $M(t)$ is non-increasing and finite for all $t > 0$. Our cases of interest is M a non-increasing function with $M(0^+) = +\infty$. From Theorem 2.5, we have the following corollary (see also [Co] Prop.II.1).

Theorem 6.2. Let $\mathcal{G} : (\alpha, +\infty), \alpha \geq 0$ be a continuous increasing function satisfying $\mathcal{G}(\alpha^+) = 0^+$. Assume that $T_t = e^{-t\mathcal{L}}$ is a submarkovian semigroup satisfying the functional decay (2.4): for all $f \in \mathcal{D}, \|f\|_q \leq 1, \|T_t f\|_p^p > \alpha$,

$$\mathcal{G}(\|T_t f\|_p^p) \leq e^{-\lambda t} \mathcal{G}(\|f\|_p^p). \quad (6.3)$$

for some $\lambda > 0$ and assume that \mathcal{G} is a bounded function. Let $\mathcal{F} = \log \circ \mathcal{G} \circ \exp$. Let $c = \sup \mathcal{F}$ then $c - \mathcal{F}$ is invertible and (T_t) satisfies the following ultracontractivity property: for all $t > 0$ and for all $f \in \mathcal{D}$ such that $\|f\|_q \leq 1$ and $\|T_t f\|_p^p > \alpha \|f\|_q^p$,

$$\|T_t f\|_p^p \leq e^{(c-\mathcal{F})^{-1}(\lambda t)} \|f\|_q^p \quad (6.4)$$

In particular, this result applies for $\alpha = 0$ without lower bound condition on $\|T_t f\|_p^p$.

For $p = 2, q = 1$, this corollary is essentially a reformulation of Prop. II.1 of [Co]p.512 for the class of symmetric submarkovian semigroups. But Corollary 6.2 certainly holds true under the assumption that (T_t) is a semigroup acting on L^p and equicontinuous on L^1 and L^∞ as in [Co]. The assumption \mathcal{G} bounded is exactly the integrability of $1/\mathcal{N}$ at infinity i.e $\int^\infty \frac{1}{\mathcal{N}(x)} dx < \infty$ or, in the formulation of [Co], $\int^\infty \frac{1}{\Theta(x)} dx < \infty$ where Θ satisfies

$$\Theta(\|f\|_2^2) \leq (\mathcal{L}f, f)$$

with the relation $\Theta(x) = x \mathcal{N}(\log x)$.

PROOF. Let \mathcal{F} defined as in the corollary then $c - \mathcal{F} : (\log \alpha, +\infty)$ is a decreasing bijection with $c = \sup \mathcal{F}$ then $c - \mathcal{F} \geq 0$. From functional decay inequality, we deduce the following inequality: for $f \in \mathcal{D}$ with $\|f\|_q \leq 1, \|T_t f\|_p^p > \alpha$,

$$\mathcal{F}(\log \|T_t f\|_p^p) \leq -\lambda t + c$$

or equivalently

$$\lambda t \leq (c - \mathcal{F})(\log \|T_t f\|_p^p)$$

But $(c - \mathcal{F})$ is invertible and decreasing. Hence,

$$\|T_t f\|_p^p \leq e^{(c-\mathcal{F})^{-1}(\lambda t)}$$

The proof is completed by homogeneity.

Corollary 6.3. Let $1 \leq q < p < \infty$ and \mathcal{L} the generator of a submarkovian symmetric semigroup satisfying the (p, q) -Nash-type inequality (2.1) with Nash-function \mathcal{N} . Let \mathcal{F} such that $\mathcal{F}' = \frac{1}{\mathcal{N}}$. If \mathcal{F} is bounded i.e $\int^\infty \frac{1}{\mathcal{N}(x)} dx$ is finite. Then, the (p, q) -ultracontractivity property (6.4) holds true.

We shall show \mathcal{G} -decay in L^2 relatively to L^1 also gives us informations on the heat kernel evaluated on the diagonal $h_t(x, x)$. In particular, when the semigroup considered is a Markov symmetric semigroup, see (6.6) below. In the stronger case that the semigroup is ultracontractive, we shall see that the function \mathcal{G} and the ultracontractive bound are related by a similar inequality, see (6.5) below. From now on, we assume that T_t is given by a kernel h_t i.e.

$$T_t f(x) = \int_X h_t(x, y) f(y) d\mu(y)$$

with $0 \leq h_t(x, y) = h_t(y, x)$ finite for all $x, y \in X, t > 0$. We assume that $\int_X h_t(x, y) d\mu(y) \leq 1, t > 0$.

This representation holds true at least for ultracontractive semigroups on open sets of \mathbb{R}^n (See for instance [A] p. 45, 58, 155), for Ornstein-Uhlenbeck semigroup.... Here, we derive on-diagonal informations on the heat kernel h_t and on the best bound of the $L^2 - L^1$ ultracontractivity property from \mathcal{G} -decay.

Theorem 6.4. 1. *Let (T_t) be a symmetric sub-markovian semigroup satisfying a \mathcal{G} -decay in L^2 relatively to L^1 with \mathcal{G} continuous and non-decreasing. Assume that the semigroup is ultracontractive i.e. $e^{M(t)} = \sup\{\|T_t f\|_2 : \|f\|_1 \leq 1\} < \infty$. Then*

$$\mathcal{G}\left(e^{2M(t+s)}\right) \leq e^{-2s} \mathcal{G}\left(e^{2M(t)}\right). \quad (6.5)$$

2. *Let (T_t) be a symmetric Markov semigroup. Assume that (T_t) has a density h_t and satisfies a \mathcal{G} -decay in L^2 relatively to L^1 . Then, for any $x \in X, t > 0, s \geq 0$:*

$$\mathcal{G}(h_{s+t}(x, x)) \leq e^{-s} \mathcal{G}(h_t(x, x)). \quad (6.6)$$

Note that, in the second statement, the semigroup need not be ultracontractive (that is $\sup_x h_t(x, x) < \infty$) and the function \mathcal{G} is any function satisfying the \mathcal{G} -decay. Of course, the inequalities (6.5) and (6.6) can be rewritten with the function \mathcal{F} defining $\mathcal{G} = \exp \circ \mathcal{F} \circ \log$. This is left to the reader.

PROOF. 1. From the \mathcal{G} -decay assumption, we have for any $t > 0, s \geq 0, \|f\|_1 \leq 1$, then $\|T_t f\|_1 \leq 1$ and

$$\mathcal{G}(\|T_{t+s} f\|_2^2) \leq e^{-2s} \mathcal{G}(\|T_t f\|_2^2).$$

Since \mathcal{G} is non-decreasing,

$$\mathcal{G}(\|T_t f\|_2^2) \leq \mathcal{G}(e^{2M(t)}).$$

In the L-H-S of the inequality, we take a maximizing sequence (f_n) to approximate the $L^1 - L^2$ -norm of T_t and use the continuity of \mathcal{G} to conclude the first statement.

2. Let $x \in X, t > 0$ be fixed and set $f(y) = h_{t/2}(x, y)$. Then $\|f\|_1 \leq 1$, $\|f\|_2^2 = h_t(x, x)$ and $\|T_{s/2}f\|_2^2 = h_{s+t}(x, x)$ for any $s \geq 0$. The inequality, for any $s \geq 0$:

$$\mathcal{G}(\|T_{s/2}f\|_2^2) \leq e^{-s} \mathcal{G}(\|f\|_2^2).$$

reads

$$\mathcal{G}(h_{s+t}(x, x)) \leq e^{-s} \mathcal{G}(h_t(x, x)).$$

The proof is completed.

It is not difficult to show that $\mathcal{G}(x) := \exp(\frac{-1}{4\pi}x^{-2/n}), x > 0$ is a function which satisfies inequality (6.6) for the heat kernel $h_t(x, y) = \frac{1}{(4\pi t)^{n/2}}$ on \mathbb{R}^n . In fact, inequality (6.6) is an equality for all $x \in \mathbb{R}^n$ because $\mathcal{G}(h_u(y, y)) = e^{-u}, u > 0, y \in \mathbb{R}^n$ (Compare with (4.13)).

Now, we explicate the best constant for the \mathcal{G} -decay on \mathbb{R}^n for the heat semigroup in the following sense. Let $a > 0$ such that, for all $f \in \mathcal{G}$ with $\|f\|_1 \leq 1$ and for all $t > 0$,

$$(D_a) : \quad \mathcal{G}_a(\|T_t f\|_2^2) \leq e^{-2t} \mathcal{G}_a(\|f\|_2^2),$$

with $\mathcal{G}_a(x) = \exp(-ax^{-2/n}), x > 0$. By subsection 4.1, there exists $\alpha > 0$ such that (D_α) is valid. Note that if $b > a > 0$ and (D_a) holds true then (D_b) also holds true. This obvious if we write (D_a) in the equivalent form

$$\mathcal{G}_1(\|T_t f\|_2^2) \leq e^{-2t/a} \mathcal{G}_1(\|f\|_2^2).$$

In the next proposition, we explicate the best constant of (D_a) .

Proposition 6.5. *Let $T_t = e^{-t\Delta}$ be the heat semigroup on \mathbb{R}^n with generator $\mathcal{L} = \Delta$. The lower bound $\alpha = \inf\{a > 0, (D_a) \text{ holds true}\}$ is attained and $\alpha = \frac{nN_n}{2} > 0$ with N_n (see [C-L]) the best constant of Nash inequality:*

$$\|f\|_2^{2+4/n} \leq N_n(\Delta f, f), \quad \|f\|_1 \leq 1. \quad (6.7)$$

Then, we have for any $f \in L^1 \cap L^2, \|f\|_1 \leq 1$ and any $t > 0$,

$$\mathcal{G}_1(\|T_t f\|_2^2) \leq e^{-4t/nN_n} \mathcal{G}_1(\|f\|_2^2).$$

As a consequence, Nash inequality (6.8) is equivalent to the following optimal inequality, for any $f \in L^1 \cap L^2, \|f\|_1 \leq 1$ and any $t > 0$,

$$\frac{4t}{nN_n} + \|f\|_2^{-4/n} \leq \|T_t f\|_2^{-4/n}, \quad (6.8)$$

The inequality (6.8) seems to be new in the literature as far as the author knows. Remark that this inequality is expressed with the semigroup and not the generator. Note that it improves the contraction property of the semigroup (T_t) in L^2 and is possible because of function f is in L^p for any $1 \leq p \leq 2$ and the semigroup regularizes from L^p to L^2 .

PROOF. Let $\alpha = \inf\{a > 0, (D_a) \text{ holds true}\}$. We have seen in the introduction of this paper that (D_a) is equivalent to the fact that $I_a(t) = e^{2t} \mathcal{G}_a(\|T_t f\|_2^2)$ is non-increasing i.e. $I'_a(t) \leq 0$ for any $t > 0$. An explicit computation gives us

$$I'_a(t) = 2e^{2t} \mathcal{G}_a(\|T_t f\|_2^2) - 2e^{2t} \left(\frac{2a}{n}\right) \|T_t f\|_2^{-2-4/n} \mathcal{G}_a(\|T_t f\|_2^2) (\Delta T_t f, T_t f).$$

So, $I'_a(t) \leq 0$ if and only if

$$\|T_t f\|_2^{2+4/n} \leq \frac{2a}{n} (\Delta T_t f, T_t f). \quad (6.9)$$

Then the infimum α in (D_a) corresponds to the infimum of $\frac{2\alpha}{n}$ in Nash inequality i.e. $\frac{2\alpha}{n} = N_n$ and conversely. The proof is completed.

Note that (6.8) implies for any $n \geq 1$, $N_n \geq \frac{1}{2\pi n}$. Indeed, we have the optimal inequality $\|T_t f\|_2^2 \leq \left(\frac{1}{8\pi t}\right)^{n/2} \|f\|_1^2$ and by (6.8), we get $\|T_t f\|_2^2 \leq \left(\frac{nN_n}{4\pi t}\right)^{n/2} \|f\|_1^2$. This gives the lower bound. But the asymptotic behavior of N_n is well-known $N_n \sim \frac{2}{\pi en}$ [Bec2], [Bec3] p.253.

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