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SYMBOLIC EXTENSIONS FOR NONUNIFORMLY ENTROPY EXPANDING MAPS

DAVID BURGNET

ABSTRACT. We call nonuniformly entropy expanding map any C^1 map defined on a compact manifold whose ergodic measures with positive entropy have only nonnegative Lyapunov exponents. We prove that a C^r nonuniformly entropy expanding map T with $r > 1$ has a symbolic extension and we give an explicit upper bound of the symbolic extension entropy in terms of the positive Lyapunov exponents by following the approach of T.Downarowicz and A.Maass[14].

1. INTRODUCTION

Given a continuous map $T : X \rightarrow X$ on a compact metrizable space X one can wonder if this topological dynamical system admits a symbolic extension, i.e. a topological extension, which is a subshift over a finite alphabet. The topological symbolic extension entropy $h_{sex}(T) = \inf\{h_{top}(Y, S) : (Y, S) \text{ is a symbolic extension of } (X, T)\}$ estimates how the dynamical system (X, T) differs from a symbolic extension from the point of view of entropy. The question of the existence of symbolic extensions leads to a deep theory which was developed mainly by M.Boyle and T.Downarowicz, who related the entropy of symbolic extensions with the convergence of the entropy of (X, T) computed at finer and finer scales [4].

Dynamical systems with symbolic extensions have necessarily finite topological entropy, because the topological entropy of a factor is less than or equal to the topological entropy of the extension and the topological entropy of a subshift over a finite alphabet is finite. Joe Auslander asked if the opposite was true : does every finite entropy system have a symbolic extension? M.Boyle answered this question negatively by constructing a zero dimensional dynamical system with finite topological entropy but without symbolic extension. Nonetheless it was proved by M.Boyle, D.Fiebig, U.Fiebig [6] that asymptotically h -expansive dynamical systems with finite topological entropy admit principal symbolic extensions, i.e. which preserve the entropy of invariant measures. Following Y.Yomdin [23], J.Buzzi showed that C^∞ maps on a compact manifold are asymptotically h -expansive [10]. In particular such maps admit principal symbolic extensions. Recall that uniformly

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hyperbolic dynamical systems are expansive. It is also known that partially hyperbolic dynamical systems with a central bundle splitting into one dimension subbundles [19] [12] are h -expansive. Therefore all these dynamical systems are asymptotically h -expansive and then admit principal symbolic extensions. On the other hand \mathcal{C}^1 maps without symbolic extensions have been built in several works by using generic arguments [17] [1] or with an explicit construction [8].

We say that a map $T : M \rightarrow M$ defined on a compact manifold is \mathcal{C}^r with $r > 1$ when T is $[r]$ times differentiable ¹ and the $[r]$ derivative of T is $r - [r]$ Hölder. T.Downarowicz and A.Maass [14] have recently proved that \mathcal{C}^r maps of the interval $f : [0, 1] \rightarrow [0, 1]$ with $1 < r < +\infty$ have symbolic extensions. More precisely they showed that $h_{sex}(f) \leq \frac{r \log \|f'\|_\infty}{r-1}$. The author of this paper built explicit examples [8] proving this upper bound to be sharp. Similar \mathcal{C}^r examples with large symbolic extension entropy have been previously built by T.Downarowicz and S.Newhouse for diffeomorphisms in higher dimensions [17] by using generic arguments on homoclinic tangencies. The author of this paper proved newly that \mathcal{C}^2 surface local diffeomorphisms have symbolic extensions [7]. The existence of symbolic extensions for general \mathcal{C}^r maps with $1 < r < +\infty$ is still an open question. It was conjectured in [17] that :

Conjecture 1. *Let $T : M \rightarrow M$ be a \mathcal{C}^r map, with $r > 1$, on a compact manifold M of dimension d . Then*

$$h_{sex}(T) \leq h_{top}(T) + \frac{dR(T)}{r-1}$$

where $R(T)$ is the dynamical Lipschitz² constant of T , that is $R(T) := \lim_{n \rightarrow +\infty} \frac{\log^+ \|DT^n\|}{n}$.

A \mathcal{C}^1 map $T : M \rightarrow M$ on a compact manifold M will be called *nonuniformly entropy expanding* if any ergodic measure with nonzero entropy has only nonnegative Lyapunov exponents³. In this paper we prove the conjecture for \mathcal{C}^r nonuniformly entropy expanding maps with $r > 1$ up to a factor d , i.e. $h_{sex}(T) \leq h_{top}(T) + \frac{d^2 R(T)}{r-1}$ (See Corollary 1).

It follows from Ruelle's inequality that \mathcal{C}^1 maps of the interval are nonuniformly entropy expanding maps. Therefore Theorem 4 generalizes the result of T.Downarowicz and A.Maass [14]. We do not know how large the class of nonuniformly entropy expanding map in higher dimensions is. Does it contain a \mathcal{C}^r open set for some $r \in \mathbb{Z}^+$? Do "Alves-Viana like maps" belong to this class [21]? Anyway, we think the results presented in this paper can be considered as a first step

¹Throughout this paper $[x]$ denotes the integer part of x for all real numbers x .

² $R(T)$ does not depend on the riemannian metric $\|\cdot\|$ on M .

³For usual nonuniformly expanding maps [2] which are well adapted to the study of SRB measures it is required that Lebesgue almost all points have only nonnegative Lyapunov exponents.

in the proof of Conjecture 1 (especially the Main Theorem which applies to general \mathcal{C}^r maps).

We give now a class of nontrivial examples of a nonuniformly entropy expanding map. Let $T : N \rightarrow N$ be a \mathcal{C}^r isometry on a compact Riemannian manifold N of dimension d . Then any \mathcal{C}^r skew product on $N \times [0, 1]$ of the form $T_g(x, y) = (Tx, g(x, y))$ with positive entropy is nonuniformly entropy expanding : any ergodic measure ν has d zero Lyapunov exponents and by Ruelle's inequality the first exponent must be positive when the entropy of ν is positive. Finally observe that the set of $g \in \mathcal{C}^r(N \times [0, 1])$ such that T_g has positive entropy contains a \mathcal{C}^0 open subset of $\mathcal{C}^r(N \times [0, 1])$. Indeed if $f : [0, 1] \rightarrow [0, 1]$ is a \mathcal{C}^r map of the interval with positive entropy then it admits a horseshoe which is persistent under small \mathcal{C}^0 perturbations [16]. Therefore there exists a \mathcal{C}^0 neighborhood \mathcal{V} of f in $\mathcal{C}([0, 1])$ such that if $g(x, \cdot) \in \mathcal{V}$ for all $x \in N$ then $h_{top}(T_g) > 0$.

2. PRELIMINARIES

In the following we denote by $\mathcal{M}(X, T)$ the set of invariant Borel probability measures of the dynamical system (X, T) and by $\mathcal{M}_e(X, T)$ the subset of ergodic measures. We endow $\mathcal{M}(X, T)$ with the weak star topology. Since X is a compact metric space, this topology is metrizable. We denote by *dist* a metric on $\mathcal{M}(X, T)$. It is well known that $\mathcal{M}(X, T)$ is a compact convex metric space whose extreme points are exactly the ergodic measures. Moreover if $\mu \in \mathcal{M}(X, T)$ there exists a unique Borel probability measure M_μ on $\mathcal{M}(X, T)$ supported on $\mathcal{M}_e(X, T)$ such that for all Borel sets B we have $\mu(B) = \int \nu(B) dM_\mu(\nu)$. This is the so called ergodic decomposition of μ . A Borel function $f : \mathcal{M}(X, T) \rightarrow \mathbb{R}$ is said to be harmonic if $f(\mu) = \int_{\mathcal{M}_e(X, T)} f(\nu) dM_\mu(\nu)$ for all $\mu \in \mathcal{M}(X, T)$. It is a well known fact that affine upper semicontinuous functions are harmonic.

If f is a Borel function defined on $\mathcal{M}_e(X, T)$, the harmonic extension of f is the function defined on $\mathcal{M}(X, T)$ by :

$$\bar{f}(\mu) := \int_{\mathcal{M}_e(X, T)} f(\nu) dM_\mu(\nu)$$

It is easily seen that \bar{f} is harmonic.

2.1. Symbolic extension entropy function and entropy structures. A symbolic extension of (X, T) is a subshift (Y, S) of a full shift on a finite number of symbols, along with a continuous surjection $\pi : Y \rightarrow X$ such that $T\pi = \pi S$. Given a symbolic extension $\pi : (Y, S) \rightarrow (X, T)$ we consider the extension entropy $h_{ext}^\pi : \mathcal{M}(X, T) \rightarrow \mathbb{R}^+$ defined for all $\mu \in \mathcal{M}(X, T)$ by :

$$h_{ext}^\pi(\mu) = \sup_{\pi_*\nu = \mu} h(\nu)$$

where π_* is the map induced on measures by π .

Then the symbolic extension entropy function, $h_{sex} : \mathcal{M}(X, T) \rightarrow \mathbb{R}^+$, is :

$$h_{sex} = \inf h_{ext}^\pi$$

where the infimum is taken over all the symbolic extensions of (X, T) . By convention, if (X, T) does not admit any symbolic extension we simply put $h_{sex} \equiv +\infty$. Recall that in the Introduction we have defined the topological symbolic extension entropy $h_{sex}(T)$ as the infimum of the topological entropies of the symbolic extensions of (X, T) . We also put $h_{sex}(T) = +\infty$ when there are no such extensions.

2.2. Newhouse local entropy. Let us first recall some usual notions related to the entropy of dynamical systems (we refer to [22] for a general introduction to entropy). Consider a continuous map $T : X \rightarrow X$ with (X, d) a compact metric space. Let $n \in \mathbb{Z}^+$ and $\delta > 0$. A subset E of X is called a (n, δ) separated set when for all $x, y \in E$ there exists $0 \leq k < n$ such that $d(f^k x, f^k y) > \delta$.

We recall now the "Newhouse local entropy". We fix some finite open cover \mathcal{V} of X , a point $x \in X$, a number $\delta > 0$, an integer n , and a Borel set $F \subset X$. By \mathcal{V}^n we will denote the open cover consisting of all the open sets of the form $V_0 \cap T^{-1}V_1 \cap \dots \cap T^{-n+1}V_{n-1}$, where $V_i \in \mathcal{V}$ for each $i = 0, 1, \dots, n-1$. We define :

$$H(n, \delta | F, \mathcal{V}) := \log \max \{ \#E : E \text{ is a } (n, \delta) \text{ separated set in } F \cap V^n \text{ with } V^n \in \mathcal{V}^n \}$$

$$h(\delta | F, \mathcal{V}) := \limsup_{n \rightarrow +\infty} \frac{1}{n} H(n, \delta | F, \mathcal{V})$$

$$h(X | F, \mathcal{V}) := \lim_{\delta \rightarrow 0} h(\delta | F, \mathcal{V})$$

Then for any ergodic measure ν we put :

$$h^{New}(X | \nu, \mathcal{V}) := \lim_{\sigma \rightarrow 1} \inf_{\nu(F) > \sigma} h(X | F, \mathcal{V})$$

Finally we extend the function $h^{New}(X | \cdot, \mathcal{V})$ to $\mathcal{M}(X, T)$ by the harmonic extension. Given a sequence of finite open covers $(\mathcal{V}_k)_{k \in \mathbb{Z}^+}$ whose diameter is converging to 0 and with \mathcal{V}_{k+1} finer than \mathcal{V}_k for all $k \in \mathbb{Z}^+$, we consider the sequence $\mathcal{H}^{New} = (h_k^{New})_{k \in \mathbb{Z}^+} := (h - h^{New}(X | \cdot, \mathcal{V}_k))_{k \in \mathbb{Z}^+}$. T.Downarowicz proved that this sequence defines an entropy structure⁴ [13] for homeomorphisms and the author of this paper extends the result to the noninvertible case [9].

⁴In particular $h^{New}(X | \cdot, \mathcal{V}_k)$ converges pointwise to zero when k goes to infinity.

2.3. Estimate theorem. One of the main tools introduced in [14] is the so-called Estimate Theorem. We can roughly resume the statement as follows : in order to estimate the symbolic extension entropy function one only needs to bound the local entropy of an ergodic measure near an invariant one by the difference of the values of some convex upper semicontinuous function on $\mathcal{M}(X, T)$ at these two measures.

Theorem 1. (*Downarowicz-Maass*)[14] *Let (X, T) be a dynamical system with finite topological entropy and fix some $r > 1$. Let g be an upper semicontinuous convex positive function on $\mathcal{M}(X, T)$ such that for every $\gamma > 0$ and $\mu \in \mathcal{M}(X, T)$ there exist $\tau_\mu > 0$ and a finite open cover $\mathcal{V}_\mu > 0$ such that for every ergodic measure ν with $\text{dist}(\nu, \mu) < \tau_\mu$ it holds that :*

$$(1) \quad h^{New}(M|\nu, \mathcal{V}_\mu) \leq g(\mu) - g(\nu) + \gamma$$

Then there exists a symbolic extension $\pi : (Y, S) \rightarrow (X, T)$ satisfying $h_{ext}^\pi = \bar{g}$. In particular $h_{sex} \leq h + \bar{g}$.

We will apply this theorem to smooth dynamical systems where the map g is related with the Lyapunov exponents of invariant measures.

2.4. Ruelle's inequality. Given a compact riemannian manifold $(M, \|\cdot\|)$ of dimension d and an integer $k \leq d$, we consider the vector bundle $\Lambda^k TM$ over M whose fiber at $x \in M$ is the space of k -forms w_x on the cotangent space T_x^*M . It inherits a norm from the riemannian structure of M as follows : $\|w_x\| = \sup |w_x(e_1, \dots, e_k)|$ where the supremum is taken over all the orthonormal families (e_1, \dots, e_k) of $(T_x M)^k$. A \mathcal{C}^1 map T on M induces naturally a map $DT^{\wedge k}$ on $\Lambda^k TM$ defined by $D_x T^{\wedge k}(w_x)(v_1, \dots, v_k) = w_x(D_x T v_1, \dots, D_x T v_k)$ for any $w_x \in \Lambda^k T_x M$ and any k -tuple (v_1, \dots, v_k) of $(T_x M)^k$. The operator norm $\|D_x T^{\wedge k}\| = \sup_{\|w_x\| \leq 1} \|D_x T^{\wedge k}(w_x)\|$ is simply the supremum of the k -volumes of the ellipsoids $D_x T(D_k)$ over all the k -disks D_k of the tangent space with unit k -volume. For $k = n$ it coincides with the jacobian $\text{Jac}_x(T)$ of T at x . Let $\|D_x T^\wedge\| = \max_{k=1, \dots, d} \|D_x T^{\wedge k}\|$. For all $k = 1, \dots, d$ the cocycle $(x, n) \mapsto \log \|(D_x T^n)^{\wedge k}\|$ is subadditive so that given an ergodic measure ν one can define "the k -volume growth" of the action of DT on TM for ν as the limit $\lim_n \frac{1}{n} \log \|(D_x T^n)^{\wedge k}\|$ for ν -generic points x . For $k = d$ the cocycle is in fact additive and the d -volume growth of DT coincides with $\int \log \text{Jac}_x(T) d\nu(x)$. The k -volume growth of DT is related to the Lyapunov exponents as follows :

Theorem 2. (*Oseledets*)[18][20] *Let $T : M \rightarrow M$ be a \mathcal{C}^1 map defined on a compact riemannian manifold $(M, \|\cdot\|)$ of dimension d . Let ν be an ergodic measure and $+\infty > \chi_1(\nu) \geq \dots \geq \chi_d(\nu) \geq -\infty$ its Lyapunov exponents. Then for ν almost all x :*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|(D_x T^n)^{\wedge k}\| = \sum_{i=1}^k \chi_i(\nu) \text{ for every } 0 \leq k \leq d$$

and thus

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log^+ \|(D_x T^n)^{\wedge}\| = \sum_{i=1}^d \chi_i^+(\nu).$$

Observe that we have in particular $\int \log \text{Jac}_x(T) d\nu(x) = \sum_{i=1}^d \chi_i(\nu)$ for all ergodic measures ν . The affine function $g : \mathcal{M}(M, T) \rightarrow [-\infty, +\infty[$ defined by $g(\mu) = \int \log \text{Jac}_x(T) d\mu(x)$ for all invariant measures μ is upper semicontinuous. Therefore⁵ $g^+ := \max(g, 0)$ is an upper semicontinuous convex function.

In the following we are interested in the entropy of ergodic measures. We recall the Ruelle's inequality which states that the entropy is bounded from above by "the maximal volume growth of DT " :

Theorem 3. (*Ruelle's inequality*) *Let $T : M \rightarrow M$ be a \mathcal{C}^1 map on a compact manifold M of dimension d then for all ergodic measures ν :*

$$h(\nu) \leq \sum_{i=1}^d \chi_i^+(\nu).$$

3. STATEMENTS

We first state our Main theorem which holds for general \mathcal{C}^r maps with $r > 1$:

Main Theorem . *Let $T : M \rightarrow M$ be a \mathcal{C}^r map, with $r > 1$, on a compact manifold of dimension d . Let μ be an invariant measure and fix some $\gamma > 0$. Then there exist $\tau_\mu > 0$ and a finite open cover $\mathcal{V}_\mu > 0$ such that for every ergodic measure ν with $\text{dist}(\nu, \mu) < \tau_\mu$ it holds that :*

$$(2) \quad h^{New}(M|\nu, \mathcal{V}_\mu) \leq \frac{d(g^+(\mu) - g^+(\nu))}{r - 1} - \sum_{i=1}^d \chi_i^-(\nu) + \gamma$$

where $g^+(\xi) = \max(\int \log \text{Jac}_x(T) d\xi(x), 0)$ for all invariant measures ξ .

If we assume moreover that T is a nonuniformly entropy expanding map then the conclusion of the Main theorem can be rewritten as

$$h^{New}(M|\nu, \mathcal{V}_\mu) \leq \frac{d\left(\sum_{i=1}^d \chi_i^+(\mu) - \sum_{i=1}^d \chi_i^+(\nu)\right)}{r - 1} + \gamma$$

⁵In the following we use the notations $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

Indeed if ν is an ergodic measure with nonzero entropy then all its Lyapunov exponents are by assumption nonnegative. Therefore $g(\nu) = \int \log \text{Jac}_x(T) d\nu(x) = \sum_{i=1}^d \chi_i(\nu) = \sum_{i=1}^d \chi_i^+(\nu) = g^+(\nu)$. Moreover $g^+(\mu) \leq \sum_{i=1}^d \chi_i^+(\mu)$ for all invariant measures μ .

By the Estimate theorem (Theorem 1) we conclude that :

Theorem 4. *Let $T : M \rightarrow M$ be a C^r nonuniformly entropy expanding map, with $r > 1$, defined on a compact manifold M of dimension d . Then there exists a symbolic extension $\pi : (Y, S) \rightarrow (X, T)$ such that $h_{ext}^\pi = h + \frac{d}{r-1} \overline{\sum_{i=1}^d \chi_i^+}$. In particular,*

$$h_{sex} \leq h + \frac{d}{r-1} \overline{\sum_{i=1}^d \chi_i^+}.$$

Then it follows from the usual variational principle for the entropy and the obvious inequality $\chi_1(\nu) \leq R(T)$ for all ergodic measures ν that⁶ :

Corollary 1. *Let $T : M \rightarrow M$ be a C^r nonuniformly entropy expanding map, with $r > 1$, defined on a compact manifold M of dimension d . Then there exists a symbolic extension (Y, S) of (X, T) such that $h_{top}(S) \leq h_{top}(T) + \frac{d^2 R(T)}{r-1}$. In particular,*

$$h_{sex}(T) \leq h_{top}(T) + \frac{d^2 R(T)}{r-1}.$$

Let $T : M \rightarrow M$ be a C^1 map on a compact manifold M . We call n -invertible branch any set $A_n \subset M$ such that for each $0 \leq k < n$ the set $T^k A_n$ is open and the map $T|_{T^k A_n}$ is a diffeomorphism onto $T^{k+1} A_n$. Any connected component of the set $\{x : \text{Jac}_x(T^n) \neq 0\}$ of noncritical points of T^n is an n -invertible branch. Such n -invertible branches will be called maximal n -invertible branches. In dimension one n -invertible branches coincide with the branches of monotonicity of T^n .

The proof of the Main Theorem goes as follows :

- we first prove a Ruelle's inequality which bounds from above the entropy in the invertible branches by the sum of the negative Lyapunov exponents;
- then, given a C^r map, we count the number of invertible branches with a large jacobian;
- finally we bound the Newhouse local entropy of ergodic measures as in [14].

⁶In fact it is easily seen that the following variational principle holds : $\sup_{\nu \in \mathcal{M}_e(M, T)} \chi_1^+(\nu) = R(T)$.

4. INVERSE RUELLE'S INEQUALITY

To estimate the entropy of a \mathcal{C}^1 map in the invertible branches we introduce the following quantity. We fix a number $\delta > 0$, an integer n , a n -invertible branch A_n and a Borel set $F \subset M$. We define :

$$H^{inv}(n, \delta|F, A_n) := \log \max\{\#E : E \text{ is a } (n, \delta) \text{ separated in } F \cap A_n\}$$

$$H^{inv}(n, \delta|F) := \sup_{A_n} H^{inv}(n, \delta|F, A_n)$$

$$h^{inv}(\delta|F) := \limsup_{n \rightarrow +\infty} \frac{1}{n} H^{inv}(n, \delta|F)$$

$$h^{inv}(M|F) := \lim_{\delta \rightarrow 0} h^{inv}(\delta|F)$$

Then for any ergodic measure ν we put :

$$h^{inv}(\nu) := \lim_{\sigma \rightarrow 1} \inf_{\nu(F) > \sigma} h^{inv}(M|F)$$

We prove in this section the following "inverse Ruelle's inequality" of independent interest :

Theorem 5. *Let $T : M \rightarrow M$ be a $\mathcal{C}^{1+\eta}$ map with $\eta > 0$. Then for all ergodic measures ν ,*

$$h^{inv}(\nu) \leq - \sum_{i=1}^d \chi_i^-(\nu).$$

Clearly $h^{inv}(\nu)$ is less than or equal to the usual Kolmogorov-Sinai entropy $h_T(\nu)$. When T is a diffeomorphism $h^{inv}(\nu)$ is equal to $h_T(\nu)$. It is well known that $h_T(\nu) = h_{T^{-1}}(\nu)$ and thus Theorem 5 follows in this case from the usual Ruelle's inequality. When T is a local diffeomorphism $h^{inv}(\nu)$ is bigger than or equal to $h_k^{New}(\nu)$ for large k . In the one dimensional case it follows easily from the total order on \mathbb{R} that the cardinality of a (n, δ) separated set lying in a given monotone branch of T^n is bounded from above by $\frac{n}{\delta}$ and then $h^{inv}(\nu)$ is zero.

To prove the usual Ruelle's inequality one relates the maximal volume growth of DT which is equal to the sum of the positive Lyapunov exponents, to the maximal volume growth of T , that is the maximal volume growth of disks of the riemannian manifold $(M, \|\cdot\|)$. It is then convenient to work with the exponential map of M to make the connection between these two quantities. We recall the basic properties of the exponential map which we use in the present paper. We refer to [15] for a definition and further developments. We denote by ∂ the distance induced on M by the riemannian structure and by $\exp_x : T_x M \rightarrow M$ the exponential map at $x \in M$. The derivative of \exp_x at the origin of $T_x M$ is the identity map so

that by the Inverse function theorem the restriction of \exp_x on the ball of the tangent space at x centered at the origin with radius r is a diffeomorphism onto its image for small $r > 0$. The radius of injectivity R_{inj} of the compact riemannian manifold M is the largest $r > 0$ such that the previous property holds for all $x \in M$. Furthermore the exponential map \exp_x maps bijectively the ball of the tangent space at x centered at the origin with radius $r < R_{inj}$, denoted by $B_x(0, r) := \{v \in T_x M, \|v\|_x < r\}$ onto the ball of M centered at x with radius $r > 0$ denoted by $B(x, r) := \{y \in M, \partial(x, y) < r\}$. The global exponential map $\exp : TM \rightarrow M$ defined by $\exp(x, v) = \exp_x(v)$ is \mathcal{C}^1 . In particular there exists $R < R_{inj}$ such that $\|D_y \exp_x\| < 2$ and $\|D_z(\exp_x^{-1})\| < 2$ for all $y \in B_x(0, R)$, for all $z \in B(x, R)$ and for all $x \in M$.

We introduce now some geometrical tools which will be useful in the proof of our Ruelle's inequality. For each $\alpha > 0$ let M_α be a subset of M which meets any ball of radius $\frac{\alpha}{16}$. One can assume that $\alpha^d \# M_\alpha$ is bounded above by a constant $C(M)$ depending only on M . For example consider a finite atlas $\mathcal{A} = \{\Phi_1, \dots, \Phi_K\}$ such that the local charts $\Phi_i :]0, 1[^d \rightarrow M$ satisfy $\|D\Phi_i\| \leq 1$ for all $i = 1, \dots, K$. For all $\beta > 0$ let L_β be the set defined by $L_\beta = \{k\beta \in]0, 1[^d, k \in \mathbb{Z}^+\}$. Then the set $M_\alpha = \bigcup_{i=1, \dots, K} \Phi_i(L_{\frac{\alpha}{16\sqrt{d}}})$ meets any ball of radius $\frac{\alpha}{16}$. For each subset S of M let $\text{Cov}(\alpha, S)$ be a subset of M_α with minimal cardinality such that the balls of radius $\frac{\alpha}{2}$ centered at the points of $\text{Cov}(\alpha, S)$ cover S . Let $x \in M$ and $E \subset T_x M$ be an ellipsoid centered at the origin of $T_x M$. We denote by $\|E\|^\wedge$ the supremum of the k -volumes of $E \cap V$ over all the vector subspaces V of $T_x M$ of dimension k . Let $\|E\|^\wedge = \max_{k=1, \dots, d} \|E\|^\wedge$. With these notations we have then $\|D_y T(B_y(0, 1))\|^\wedge = \|D_y T\|^\wedge$ for all $y \in M$.

Lemma 1. *Let $R > \alpha_1, \alpha_2 > 0$. Let $x \in M$ and let $E \subset T_x M$ be an ellipsoid centered at the origin of $T_x M$ such that $\alpha_1 E \subset B_x(0, \frac{R}{2})$, then*

$$\# \text{Cov}(\alpha_2, \exp_x(\alpha_1 E)) \leq P(\|E\|^\wedge + 1) \left(\frac{\max(\alpha_1, \alpha_2)}{\alpha_2} \right)^d$$

with a constant P depending only on d .

PROOF : Since M_{α_2} meets any ball of radius less than $\frac{\alpha_2}{16}$ and $\|D_y(\exp_x^{-1})\| < 2$ for all $y \in B(x, R)$ the set $\exp_x^{-1}(M_{\alpha_2})$ meets any ball of radius less than $\frac{\alpha_2}{8}$ which is included in $B_x(0, R)$. The ellipsoid $\alpha_1 E$ can be covered by at most $(\|E\|^\wedge + 1)$ cubes of size α_1 and therefore by at most $(\|E\|^\wedge + 1) \left(\left[\frac{8 \max(\alpha_1, \alpha_2) \sqrt{d}}{\alpha_2} \right] + 1 \right)^d$ cubes of size $\frac{\alpha_2}{8\sqrt{d}}$. Such a cube intersecting $\alpha_1 E$ is included in a subball of $B_x(0, R)$ of radius $\frac{\alpha_2}{8}$ and therefore in a subball of $B_x(0, R)$ of radius $\frac{\alpha_2}{4}$ centered at a point of $\exp_x^{-1}(M_{\alpha_2})$. This last subball is mapped by \exp_x in a ball of M of radius

$\frac{\alpha_2}{2}$ centered at a point of M_{α_2} because $\|D_y \exp_x\| < 2$ for all $y \in B_x(0, R)$. We conclude that $\text{Cov}(\alpha_2, \exp_x(\alpha_1 E)) \leq (\|E\|^\wedge + 1) \left(\left[\frac{8 \max(\alpha_1, \alpha_2) \sqrt{d}}{\alpha_2} \right] + 1 \right)^d$. \square

Lemma 2. *Let ν be an ergodic measure with $\int \log \text{Jac}_x T d\nu(x) > -\infty$. Then $(\frac{1}{n} \log^+ \|(D_x T^n)^{-1}\|^\wedge)_{n \in \mathbb{Z}^+}$ converges to $-\sum_{i=1}^d \chi_i^-(\nu)$ for ν almost all x and $\frac{1}{n} \int \log^+ \|(D_x T^n)^{-1}\|^\wedge d\nu(x)$ converges to $-\sum_{i=1}^d \chi_i^-(\nu)$ when n goes to infinity.*

PROOF : It is well known that invertible $d \times d$ matrices, endowed with the operator norm $\|\cdot\|$ induced by the euclidean norm, satisfy the relations $\|A\| \|A^{-1}\| \geq 1$ and $|\det(A)| \leq \frac{\|A\|^{d-1}}{\|A^{-1}\|}$. Therefore $+\infty > (d-1) \log \|DT\|_\infty - \int \log \|(D_x T)^{-1}\| d\nu(x) \geq \int \log \text{Jac}_x(T) d\nu(x) > -\infty$ and then the map $x \mapsto \log \|(D_x T)^{-1}\|$ is ν integrable. Let $(\bar{M}, \bar{\nu})$ be the natural extension of (M, ν) . The invertible cocycle $\bar{x} = (\dots x_{-1}, x_0, x_1, \dots) \mapsto D_{x_0} T$ over the natural extension is integrable because $\int \log^+ \|D_{x_0} T\| d\bar{\nu}(\bar{x}) = \int \log^+ \|D_x T\| d\nu(x) < +\infty$ and $\int \log^+ \|(D_{x_0} T)^{-1}\| d\bar{\nu}(\bar{x}) = \int \log^+ \|(D_x T)^{-1}\| d\nu(x) < +\infty$. This cocycle has the same Lyapunov exponents as T and the sequence $\frac{1}{n} \int \log^+ \|(D_{x_{-n}} T^n)^{-1}\|^\wedge d\bar{\nu}(\bar{x}) = \frac{1}{n} \int \log^+ \|(D_x T^n)^{-1}\|^\wedge d\nu(x)$ converges to $-\sum_{i=1}^d \chi_i^-(\nu)$ when n goes to infinity according to the cocycle invertible version of Oseledec's Theorem [20]. Now by Kingman's subadditive ergodic theorem applied to the subadditive sequence of integrable functions $x \mapsto \log^+ \|(D_x T^n)^{-1}\|^\wedge$ the limit $\lim_{n \rightarrow +\infty} \frac{1}{n} \log^+ \|(D_x T^n)^{-1}\|^\wedge$ exists for ν -almost all x and coincides with the limit of the integrals $\lim_{n \rightarrow +\infty} \frac{1}{n} \int \log^+ \|(D_x T^n)^{-1}\|^\wedge d\nu(x)$. \square

It is convenient in the next proofs to use the following vocabulary :

Definition 1. *Let $S \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^+$. We say that a sequence of n positive integers $\mathcal{K}_n := (k_1, \dots, k_n)$ misses the value S if $\frac{1}{n} \sum_{i=1}^n k_i \leq S$.*

The number of sequences of n positive integers missing the value S is exactly the binomial coefficient $\binom{nS}{n}$. We denote $H : [0, 1] \rightarrow \mathbb{R}$ the map defined by $H(t) = -t \log t - (1-t) \log(1-t)$. It is easily seen that

$$(3) \quad \log \binom{nS}{n} \leq nSH(S^{-1}) + 1$$

For all $\gamma > 0$ we fix $S_\gamma \in \mathbb{Z}^+$ so that $H(S^{-1}) < \gamma$ for all $S \geq S_\gamma$.

We show now our Ruelle's inequality. First let us explain shortly the outline of the proof. We will consider some iterate T^N of T such that the negative Lyapunov exponents of ν are almost given by the average of the norm of $(DT^N)^{-1}$ along the orbits of typical points. Then, given a nN -invertible branch A_{nN} , we bound the cardinality of any (n, δ) separated (for T^N) set E in A_{nN} by shadowing the orbits of $T^{nN}E$ under the action of T^{-N} . In the usual Ruelle inequality the orbits under forwards iterates are shadowed. Our situation is more difficult because the derivative of T^{-N} is not bounded near the critical values of T^N . However we

show that the integrability assumption, $\int \log \text{Jac}_x T d\nu(x) > -\infty$, and the Hölder property of the differential allows us to neglect the growth of orbits near the critical values. The proof goes as follows :

- we first define N and exploit our integrability assumption;
- then we bound the volume T^{-N} -growth of balls by distinguishing two cases depending on whether we are far or close to the critical values of T^N ;
- we prove a combinatorial estimate which allows us to consider (n, δ) separated sets E such that the size of $\|DT^{-1}\|$ along the T -orbit of E is fixed;
- finally we detail our shadowing construction.

PROOF OF THEOREM 5 : Fix an ergodic measure ν with $\int \log \text{Jac}_x T d\nu(x) > -\infty$. Let $\gamma > 0$. By Lemma 2 there exists an integer N and a Borel set G with $\nu(G) > 1 - \frac{\gamma}{\max(-\sum_{i=1}^d \chi_i^-(\nu), 1)}$ such that for all $x \in G$:

$$(4) \quad -\sum_{i=1}^d \chi_i^-(\nu) - \gamma < \frac{1}{N} \log^+ \|(D_x T^N)^{-1}\|^\wedge < -\sum_{i=1}^d \chi_i^-(\nu) + \gamma$$

and

$$\frac{1}{N} \int \log^+ \|(D_x T^N)^{-1}\|^\wedge d\nu(x) < -\sum_{i=1}^d \chi_i^-(\nu) + \gamma$$

From the above inequalities one deduces easily that

$$\frac{1}{N} \int_{M \setminus G} (\log^+ \|(D_x T^N)^{-1}\|^\wedge + 1) d\nu(x) < 4\gamma$$

Observe also that the set $\text{Crit}(T)$ of critical points has zero ν -measure. Let us denote by $\text{Crit}(T)^\theta = \{y \in M, \partial(y, \text{Crit}(T)) < \theta\}$ the θ -neighborhood of the set of critical points. We also put $\text{Crit}(T)_N^\theta = \bigcup_{k=0}^{N-1} T^{-k} \text{Crit}(T)^\theta$. Since $x \mapsto \sum_{0 \leq j < N} \log^+ \|(D_{T^j x} T)^{-1}\|$ is a ν -integrable function there exists $\theta > 0$ such that :

$$\int_{\text{Crit}(T)_N^\theta} \sum_{0 \leq j < N} (\log^+ \|(D_{T^j x} T)^{-1}\| + 1) d\nu(x) < \gamma$$

Let $\sigma \in]0, 1[$. By Birkhoff's ergodic theorem and the two previous equations there exists a set F with $\nu(F) > \sigma$ and an integer n_0 such that for all $x \in F$ and for all $n \geq n_0$:

$$(5) \quad \frac{1}{nN} \sum_{0 \leq l < nN} \mathbf{1}_{\text{Crit}(T)_N^\theta}(T^l x) \sum_{0 \leq j < N} (\log^+ \|(D_{T^{l+j} x} T)^{-1}\| + 1) < \gamma$$

and

$$\frac{1}{nN^2} \sum_{0 \leq l < nN} \mathbb{1}_{M \setminus G}(T^l x) (\log^+ \|(D_{T^l x} T^N)^{-1}\|^\wedge + 1) < 4\gamma,$$

in particular there exists some $0 \leq i(x) < N$ such that

$$(6) \quad \frac{1}{nN} \sum_{0 \leq k < N} \mathbb{1}_{M \setminus G}(T^{i(x)+kN} x) (\log^+ \|(D_{T^{i(x)+kN} x} T^N)^{-1}\|^\wedge + 1) < 4\gamma$$

Let n be an integer larger than n_0 and let A_{nN} be an nN -invertible branch. We first control the growth of balls under T^{-1} for pieces of orbits far from the set of critical values. As T is a \mathcal{C}^1 map there exists, by an easy argument of continuity, a number $0 < \delta < \theta$ such that for all $0 < r < \delta$, for all $y \in T^{i+kN} A_{nN} \cap (M \setminus \text{Crit}(T)_N^\theta)$ with $i + (k+1)N < nN$ and for all $z \in B(T^N y, r)$:

$$(7) \quad (T^N|_{T^{i+kN} A_{nN}})^{-1} B(z, r) \subset \exp_y ((D_y T^N)^{-1} B_{T^N y}(0, 3r))$$

Observe that δ can be chosen regardless of the choice of the invertible branch A_{nN} .

Now we give satisfactory estimates for pieces of orbits close to the set of critical points. Choose $R < R' < R_{inj}$ such that $T(B(x, R)) \subset B(Tx, R')$ for all $x \in M$. We consider the local dynamic $\mathcal{T}_x : B_x(0, R) \rightarrow B_{Tx}(0, R')$ at x in the local charts defined by the exponential map, i.e. $\mathcal{T}_x : \exp_{Tx}^{-1} \circ T \circ \exp_x$. Fix $y \in T^t A_{nN}$ with $0 \leq t < nN$. Let $0 < Q < R$ be a constant depending only on T such that $\|(D\mathcal{T}_y)|_{B_y(0, Q)}\|_\eta \leq 2\|DT\|_\eta$ and let $h \in T_y M$ with $\|h\| \leq Q$. First notice that :

$$\begin{aligned} \mathcal{T}_y(h) - D_y T(h) &= \mathcal{T}_y(h) - D_0 \mathcal{T}_y(h) \\ &= \int_0^1 (D_{th} \mathcal{T}_y(h) - D_0 \mathcal{T}_y(h)) dt \end{aligned}$$

Then by using the Hölder property of the differential we have :

$$\begin{aligned} \|\mathcal{T}_y(h) - D_y T(h)\| &\leq \|(D\mathcal{T}_y)|_{B_y(0, Q)}\|_\eta \|h\|^{1+\eta} \\ &\leq 2\|DT\|_\eta \|h\|^{1+\eta} \end{aligned}$$

By assuming $r < a \max(\|(D_y T)^{-1}\|, 1)^{-\frac{1+\eta}{\eta}}$ with a constant a we have for all $h \in B_y(0, 2 \max(\|(D_y T)^{-1}\|, 1)r)$:

$$\begin{aligned} \|\mathcal{T}_y(h) - D_y T(h)\| &< 2\|DT\|_\eta (2 \max(\|(D_y T)^{-1}\|, 1))^{1+\eta} r^{1+\eta} \\ &< 2\|DT\|_\eta (2 \max(\|(D_y T)^{-1}\|, 1))^{1+\eta} a^\eta \max(\|(D_y T)^{-1}\|, 1)^{-1-\eta} r \\ &< r \end{aligned}$$

where the last inequality follows from an appropriate choice of the constant a .

Moreover we have obviously that $B_{T_y}(0, 2r) \subset D_y T(B_y(0, 2 \max(\|(D_y T)^{-1}\|, 1)r))$. Therefore we get that $B_{T_y}(0, r) \subset \mathcal{T}_y(B_y(0, 2 \max(\|(D_y T)^{-1}\|, 1)r))$. Finally by

taking the exponential map \exp_{Ty} we have for all $0 \leq r < a \max(\|(D_y T)^{-1}\|, 1)^{-\frac{n+1}{\eta}}$:

$$(8) \quad (T|_{T^t A_{nN}})^{-1} B(Ty, r) \subset B(y, 2 \max(\|(D_y T)^{-1}\|, 1)r)$$

We are going to bound $\max\{\#E : E \subset F \cap A_{nN} \text{ and } E \text{ is a } (nN, \delta) \text{ separated set}\}$. There exists $\delta' < \delta$ such that $\partial(y, z) < \delta' \Rightarrow \partial(T^k y, T^k z) < \delta, \forall 0 \leq k \leq N$. We also choose $\delta' < a$. Notice that δ' (as δ) does not depend on the invertible branch A_{nN} .

In order to estimate the growth of orbits in $F \cap A_{nN}$ near the critical points, for each $z \in M \setminus \text{Crit}(T)$ we let

$$\psi(z) = [\log_2^+ \|(D_z T)^{-1}\|] + 1$$

We also consider the following sequences for all $y \in F \cap A_{nN}$ (we write $\mathbb{1}_E$ for the characteristic function of a subset E of M) :

$$J_n(y) = \left(\mathbb{1}_{\text{Crit}(T)_N^c} (T^{i(y)+kN} y) \right)_{0 \leq k \leq n-2}$$

$$\widetilde{J}_n(y) = \left(\mathbb{1}_{\text{Crit}(T)_N^c} (T^{i(y)+kN} y) \psi(T^{i(y)+kN+l} y) \right)_{\substack{0 \leq k \leq n-2, \\ 0 \leq l \leq N-1}}$$

In order to control the dynamics far from the critical points, for each $z \in M \setminus \text{Crit}(T^N)$ we let

$$\xi(z) = [\log_2^+ \|(D_z T^N)^{-1}\|] + 1$$

and we consider the following sequences for all $y \in F \cap A_{nN}$:

$$H_n(y) = \left(\mathbb{1}_{M \setminus (G \cup \text{Crit}(T)_N^c)} (T^{i(y)+kN} y) \right)_{0 \leq k \leq n-2}$$

$$\widetilde{H}_n(y) = \left(\mathbb{1}_{M \setminus (G \cup \text{Crit}(T)_N^c)} (T^{i(y)+kN} y) \xi(T^{i(y)+kN} y) \right)_{0 \leq k \leq n-2}$$

Now we estimate the cardinality of $\{(\widetilde{H}_n(y), \widetilde{J}_n(y)) : y \in F \cap A_{nN} \text{ and } i(y) = i, H_n(y) = \underline{H}, J_n(y) = \underline{J}\}$ for some fixed $n \geq n_0, i \in \{0, \dots, N-1\}, \underline{H} = (\underline{H}_0, \dots, \underline{H}_{n-2}) \in \{0, 1\}^{n-1}$ and $\underline{J} = (\underline{J}_0, \dots, \underline{J}_{n-2}) \in \{0, 1\}^{n-1}$. We consider the sequences $\{h_1, \dots, h_{M_{\underline{H}}}\} = \{0 \leq j \leq n-2, \underline{H}_j = 1\}$ and $\{j_1, \dots, j_{M_{\underline{J}}}\} = \{0 \leq j \leq n-2, \underline{J}_j = 1\}$. For all $y \in F \cap A_{nN}$ with $i(y) = i, H_n(y) = \underline{H}$ and $J_n(y) = \underline{J}$ we put :

$$\mathcal{H}_n(y) = (\xi(T^{i+h_m N} y))_{1 \leq m \leq M_{\underline{H}}}$$

$$\mathcal{J}_n(y) = (\psi(T^{i+j_m N+l} y))_{\substack{1 \leq m \leq M_{\underline{J}}, \\ 0 \leq l \leq N-1}}$$

The above sequences $\mathcal{H}_n(y)$ and $\mathcal{J}_n(y)$ coincide respectively with $\widetilde{H}_n(y)$ and $\widetilde{J}_n(y)$ once the zeros are taken off. Observe that these sequences $\mathcal{H}_n(y)$ and $\mathcal{J}_n(y)$ miss respectively the values $S_{\underline{H}}$ and $S_{\underline{J}}$ with

$$S_{\underline{H}} := \frac{1}{M_{\underline{H}}} \sup_{y \in F} \sum_{1 \leq m \leq M_{\underline{H}}} (\log_2^+ \|(D_{T^{i+hmNy}} T^N)^{-1}\|^\wedge + 1)$$

$$S_{\underline{J}} := \frac{1}{M_{\underline{J}}N} \sup_{y \in F} \sum_{1 \leq m \leq M_{\underline{J}}} \sum_{0 \leq l < N} (\log_2^+ \|(D_{T^{i+jmN+l}y} T)^{-1}\| + 1)$$

We have then by Equation (6) and Equation (5) :

$$M_{\underline{H}} S_{\underline{H}} = \sup_{y \in F} \sum_{1 \leq m \leq M_{\underline{H}}} (\log_2^+ \|(D_{T^{i+hmNy}} T^N)^{-1}\|^\wedge + 1) < \frac{4\gamma nN}{\log 2}$$

$$M_{\underline{J}} N S_{\underline{J}} = \sup_{y \in F} \sum_{1 \leq m \leq M_{\underline{J}}} \sum_{0 \leq l < N} (\log_2^+ \|(D_{T^{i+jmN+l}y} T)^{-1}\| + 1) < \frac{\gamma nN}{\log 2}$$

By Inequality (3) the logarithm of $\#\{(\widetilde{H}_n(y), \widetilde{J}_n(y)), y \in F \cap A_{nN} \text{ and } i(y) = i, H_n(y) = \underline{H}, J_n(y) = \underline{J}\}$ is bounded above by $5\gamma nN + 2$ and thus we get finally :

$$(9) \quad \log \#\{(i(y), \widetilde{H}_n(y), \widetilde{J}_n(y)), y \in F \cap A_{nN}\} \leq (2 \log 2 + 5\gamma N)n + \log N + 2$$

Fix $0 \leq i < N$, $\widehat{H} = (\widehat{H}_0, \dots, \widehat{H}_{n-2}) \in \mathbb{Z}^{+(n-1)}$ and $\widehat{J} = (\widehat{J}_0, \dots, \widehat{J}_{(n-1)N-1}) \in \mathbb{Z}^{+(n-1)N}$. By the combinatorial estimate (9) we only need to bound the cardinality of (n, δ) separated set $\widehat{\tau}$ in $F(i, \widehat{H}, \widehat{J}) := F \cap A_{nN} \cap \{y : i(y) = i, \widetilde{H}_n(y) = \widehat{H}, \widetilde{J}_n(y) = \widehat{J}\}$.

To this end we would like to δ' -shadow the orbits of $T^{nN} F(i, \widehat{H}, \widehat{J})$ under the action of T^{-N} by sequences of points in $M_{\delta'}$, i.e. associate to each $y \in F(i, \widehat{H}, \widehat{J})$ a sequence $y_1, \dots, y_N \in M_{\delta'}$ such that $d(y_k, T^{kN} y) < \delta'$ for all $k = 1, \dots, N$. But the volume T^{-1} -growth of balls near the critical values of T is not uniformly bounded and is controled for balls with radius small compared to the inverse of the norm of DT^{-1} according to Equation (7). Therefore we will also shadow the orbit under the action of T^{-1} when we are close to the critical value of T and the shadowing scales at this times will vary with respect to the norm of DT^{-1} .

We define the sequence $(\delta_j)_{i \leq j \leq i+(n-1)N}$ of shadowing scales by $\delta_i = \delta'$, $\delta_{i+t} = \min(\delta', a2^{-\frac{n+1}{n}\widehat{J}_{t-1}})$ for $t = 1, \dots, (n-1)N - 1$ and $\delta_{i+(n-1)N} = \delta'$ again. Observe that $\frac{\max(\delta_{i+t}, \delta_{i+t+1})}{\delta_{i+t+1}} \leq 2^{\frac{n+1}{n}\widehat{J}_t}$ for all $0 \leq t < (n-1)N$. We shadow the orbits of the set $F(i, \widehat{H}, \widehat{J})$ by associating to any y in this set a point $d(y)$ in $M_{\delta'} \cap B(y, \frac{\delta'}{2})$ and

⁷We use the slightly different notation $\widetilde{J}_n(x) := \left(\mathbf{1}_{\text{Crit}(T)_N^0} (T^{i(x)+[\frac{x}{N}]} x) \psi(T^{i(x)+s} x) \right)_{0 \leq s < (n-1)N}$.

a sequence $C_n(y) = (c(T^{i+kN}y))_{0 \leq k \leq n-1}$ where $c(T^{i+kN}y)$ is a point in $M_{\delta_{i+kN}} \cap B\left(T^{i+kN}y, \frac{\delta_{i+kN}}{2}\right)$ if $k = n-1$ or $T^{i+kN}y \notin \text{Crit}(T)_N^\theta$ and $c(T^{i+kN}y)$ is a N -tuple $c(T^{i+kN}y) = (C_0, \dots, C_{N-1})$ with $C_l \in M_{\delta_{i+kN+l}} \cap B\left(T^{i+kN+l}y, \frac{\delta_{i+kN+l}}{2}\right)$ otherwise.

If $d(y) = d(z)$ and $C_n(y) = C_n(z)$ then z belongs to the Bowen ball $B(y, nN, \delta)$ and therefore y and z are not (nN, δ) separated. Indeed we have firstly $\partial(y, z) \leq \partial(y, d(y)) + \partial(d(y), z) < \delta'$ and then $\partial(T^l y, T^l z) < \delta$ for all $0 \leq l < N$. Secondly if m is an integer with $i < N \leq m < nN$ there exist $0 \leq k < n$ and $0 \leq l < N$ satisfying $m = i + kN + l$. But $c(T^{i+kN}y) = c(T^{i+kN}z)$ implies that

$$\begin{aligned} \partial(T^{i+kN}y, T^{i+kN}z) &\leq \partial(T^{i+kN}y, c(T^{i+kN}y)) + \partial(c(T^{i+kN}z), T^{i+kN}z) \\ &< \frac{\delta_{i+kN} + \delta_{i+kN}}{2} \leq \delta' \end{aligned}$$

and then $\partial(T^m y, T^m z) < \delta$.

Now we build the sequences $C_n(y)$ for $y \in F(i, \widehat{H}, \widehat{J})$ and we estimate their cardinality. We will use the following claim which follows easily from Equations (7) and (8) and Lemma 1 :

Claim . *There exists a constant P depending only on d such that we have for all $0 \leq k < n-1$ and $0 \leq l < N$ and for all $y \in F(i, \widehat{H}, \widehat{J})$:*

- $\# \text{Cov}\left(\delta_{i+kN}, (T^N|_{T^{i+kN}A_{nN}})^{-1} B\left(z, \frac{\delta_{i+(k+1)N}}{2}\right)\right) \leq P \left(\frac{\delta_{i+(k+1)N}}{\delta_{i+kN}}\right)^d e^{N(-\sum_{l=1}^d \chi_l^-(\nu) + \gamma)}$
for all $z \in B\left(T^{i+(k+1)N}y, \frac{\delta_{i+(k+1)N}}{2}\right)$ with $T^{i+kN}y \in G \cap (M \setminus \text{Crit}(T)_N^\theta)$;
- $\# \text{Cov}\left(\delta_{i+kN}, (T^N|_{T^{i+kN}A_{nN}})^{-1} B\left(z, \frac{\delta_{i+(k+1)N}}{2}\right)\right) \leq P \left(\frac{\delta_{i+(k+1)N}}{\delta_{i+kN}}\right)^d 2^{\widehat{H}_k}$ for all
 $z \in B\left(T^{i+(k+1)N}y, \frac{\delta_{i+(k+1)N}}{2}\right)$ with $T^{i+kN}y \in M \setminus (G \cup \text{Crit}(T)_N^\theta)$;
- $\# \text{Cov}\left(\delta_{i+kN+l}, (T|_{T^{i+kN+l}A_{nN}})^{-1} B\left(z, \frac{\delta_{i+kN+l+1}}{2}\right)\right) \leq P \left(\frac{\max(\delta_{i+kN+l}, \delta_{i+kN+l+1})}{\delta_{i+kN+l}} 2^{\widehat{J}_{kN+l}}\right)^d$
for all $z \in B\left(T^{i+kN+l+1}y, \frac{\delta_{i+kN+l+1}}{2}\right)$ with $T^{i+kN}y \in \text{Crit}(T)_N^\theta$.

Notice that we have $\delta_{i+(k+1)N} = \delta' \geq \delta_{i+kN}$ in the two first cases of the Claim since $\widehat{J}_{(k+1)N-1} = 0$ when $T^{i+kN}y \notin \text{Crit}(T)_N^\theta$ for some $y \in F(i, \widehat{H}, \widehat{J})$.

By the decreasing induction on k we define $c(T^{i+kN}y)$ for all $y \in F(i, \widehat{H}, \widehat{J})$ and show that

(10)

$$\# \left\{ (c(T^{i+lN}y))_{k \leq l \leq n-1}, y \in F(i, \widehat{H}, \widehat{J}) \right\} \leq \frac{C(M)}{\delta_{i+kN}^d} \times P^{n-k} e^{(n-k)N(-\sum_{l=1}^d \chi_l^-(\nu) + \gamma)} \\ \times 2^{\sum_{t=k}^{n-2} \widehat{H}_t} \times 2^{\left(\sum_{t=kN}^{(n-1)N-1} \widehat{J}_t\right) \left(d \frac{2\eta+1}{\eta} + \log_2 P\right)}$$

First for all $y \in F(i, \widehat{H}, \widehat{J})$ we put $c(T^{i+(n-1)N}y) = z$ where z is chosen in $\text{Cov}(\delta_{i+(n-1)N}, M) \cap B\left(T^{i+(n-1)N}y, \frac{\delta_{i+(n-1)N}}{2}\right)$. Then inequality (10) for $k = n - 1$ follows from the Claim. Assume we have already defined $c(T^{i+(k+1)N}y)$ and that Inequality (10) holds for $k + 1$. We distinguish two cases :

- $\widehat{J}_{kN} = 0$, i.e. $T^{i+kN}y$ is far from the critical set of T^N , then we choose

$$c(T^{i+kN}y) \in \text{Cov}\left(\delta_{i+kN}, (T^N|_{T^{i+kN}A_{nN}})^{-1} B\left(c(T^{i+(k+1)N}y), \frac{\delta_{i+(k+1)N}}{2}\right)\right) \cap B\left(T^{i+kN}y, \frac{\delta_{i+kN}}{2}\right)$$

- $\widehat{J}_{kN} \neq 0$, i.e. $T^{i+kN}y$ is close to the critical set of T^N , then we define $c(T^{i+kN}y) = (C_0, \dots, C_{N-1})$ with for all $0 \leq l \leq N - 1$

$$C_l \in \text{Cov}\left(\delta_{i+kN+l}, (T|_{T^{i+kN+l}A_{nN}})^{-1} B\left(C_{l+1}, \frac{\delta_{i+kN+l+1}}{2}\right)\right) \cap B\left(T^{i+kN+l}y, \frac{\delta_{i+kN+l}}{2}\right)$$

and with the convention $C_N = c(T^{i+(k+1)N}y)$.

Notice that we have

$$\frac{1}{\delta_{i+(k+1)N}} \times \prod_{l=0}^{N-1} \frac{\max(\delta_{i+kN+l}, \delta_{i+kN+l+1})}{\delta_{i+kN+l}} \leq \frac{1}{\delta_{i+kN}} \times \prod_{l=0}^{N-1} \frac{\max(\delta_{i+kN+l}, \delta_{i+kN+l+1})}{\delta_{i+kN+l+1}} \\ (11) \qquad \qquad \qquad \leq \frac{2^{\frac{\eta+1}{\eta} \sum_{l=0}^{N-1} \widehat{J}_{kN+l}}}{\delta_{i+kN}}$$

By using the Claim and the above inequality (11) we easily check by the decreasing induction on k that the estimate (10) holds for all $k = 0, \dots, n - 1$. Then according to the inequalities (6) and (5) we get for all $n \geq n_0$

$$\log \# \left\{ \left(d(y), (c(T^{i+lN}y))_{k \leq l \leq n-1} \right), y \in F(i, \widehat{H}, \widehat{J}) \right\} \leq -2d \log \delta' + 2 \log C(M) + n \log P + \\ nN \left(-\sum_{l=1}^d \chi_l^-(\nu) + \gamma \right) + 4\gamma nN + \gamma nN \left(d \frac{2\eta+1}{\eta} + \log_2 P \right)$$

then by using the combinatorial estimate (9) we have

$$\begin{aligned} & \log \max\{\#E : E \subset F \cap A_{nN} \text{ and } E \text{ is a } (nN, \delta) \text{ separated set}\} \\ & \leq \log \#\{(i(y), \tilde{H}_n(y), \tilde{J}_n(y)), y \in F \cap A_{nN}\} + \sup_{\substack{i, \hat{H}, \hat{J} \\ \text{with } F(i, \hat{H}, \hat{J}) \neq \emptyset}} \log \#\{(d(y), C_n(y)), y \in F(i, \hat{H}, \hat{J})\} \\ & \leq -2d \log \delta' + 2 \log C(M) + n(2 \log 2 + \log P) + \log N + 2 \\ & \quad + nN \left(- \sum_{l=1}^d \chi_l^-(\nu) \right) + \gamma nN \left(d \frac{2\eta + 1}{\eta} + \log_2 P + 10 \right) \end{aligned}$$

By taking N and then n_0 large enough, we get for $n \geq n_0$:

$$\begin{aligned} \log \max\{\#E : E \subset F \cap A_{nN} \text{ and } E \text{ is a } (nN, \delta) \text{ separated set}\} & \leq nN \left(- \sum_{l=1}^d \chi_l^-(\nu) \right) \\ & \quad + \gamma nN \left(d \frac{2\eta + 1}{\eta} + \log_2 P + 11 \right) \end{aligned}$$

Finally for general $m \in \mathbb{Z}^+$ observe that if $\sigma > \rho > 0$ satisfies $d(x, y) < \rho \Rightarrow d(T^k x, T^k y) < \sigma$ for all $0 \leq k < N$, then any (m, σ) separated set in A_m is $(\lfloor \frac{m}{N} \rfloor N, \rho)$ separated in $A_{\lfloor \frac{m}{N} \rfloor N}$. This last remark concludes easily the proof of Theorem 5. \square

5. COUNTING LEMMA

The following lemma is a generalization in any dimension of Lemma 4.1 of [14]. The provided proof is independent and based on a semi-algebraic approach.

Lemma 3. *Let $f :]-1, 1[^d \rightarrow \mathbb{R}$ be a C^r map with $r > 0$. Then there exists a constant c depending only on r and d such that for every $0 < s < 1$ the number of connected components of the open set $\{x : f(x) \neq 0\}$ on which $|f|$ reaches or exceeds the value s is at most $c \max(\|f\|_r, 1)^{\frac{d}{r}} s^{-\frac{d}{r}}$ where $\|f\|_r$ is the supremum norm $\|D^r f\|_\infty$ of the r derivative if $r \in \mathbb{Z}^+$ and the $r - [r]$ Hölder norm $\|D^{[r]} f\|_{r-[r]}$ of the $[r]$ derivative⁸ if $r \notin \mathbb{Z}^+$.*

PROOF : We cover the unit square $] - 1, 1[^d$ by $\left(2 \left[\left(\frac{as}{\max(\|f\|_r, 1)} \right)^{-\frac{1}{r}} \right] + 1 \right)^d$ subsquares of size $< \left(\frac{as}{\max(\|f\|_r, 1)} \right)^{\frac{1}{r}}$ where $a = a(r, d)$ is a constant depending only on r and d which we specify later. Consider one such subsquare S and let P_S the Lagrange polynomial of order $[r - 1]$ at the center x_0 of S . By the Taylor-Lagrange formula we have for all $x \in S$:

$$f(x) = P_S(x) + \frac{1}{[r - 1]!} \int_0^1 (1 - t)^{[r] - 1} D_{x_0 + t(x - x_0)}^{[r]} f(x - x_0)^{[r]} dt$$

⁸By convention the 0 derivative of T is the map T itself.

where $(x - x_0)^{[r]}$ denotes the vector $\underbrace{(x - x_0, \dots, x - x_0)}_{[r] \text{ times}} \in (\mathbb{R}^d)^{[r]}$.

Therefore if $r \in \mathbb{Z}^+$:

$$|f(x) - P_S(x)| \leq \frac{\|D^r f\|_\infty}{r!} \|x - x_0\|^r$$

and if $r \notin \mathbb{Z}^+$ we have :

$$f(x) - P_S(x) - \frac{D_{x_0}^{[r]} f(x - x_0)^{[r]}}{[r]!} = \frac{1}{[r-1]!} \int_0^1 (1-t)^{[r]-1} \left(D_{x_0+t(x-x_0)}^{[r]} f - D_{x_0}^{[r]} f \right) (x - x_0)^{[r]} dt$$

and thus

$$\left| f(x) - P_S(x) - \frac{D_{x_0}^{[r]} f(x - x_0)^{[r]}}{[r]!} \right| \leq \frac{\|D^{[r]} f\|_{r-[r]}}{[r]!} \|x - x_0\|^r$$

Put $Q_S = P_S$ if $r \in \mathbb{Z}^+$ and $Q_S := P_S + \frac{D_{x_0}^{[r]} f(x_0)(\cdot - x_0)^{[r]}}{[r]!}$ if $r \notin \mathbb{Z}^+$, then

$$\|f - Q_S\|_\infty \leq \frac{\text{diam}(S)^r \|f\|_r}{[r]!}$$

Then the constant $a = a(r, d)$ can be chosen so that

$$\|f - Q_S\|_\infty < \frac{s}{2}$$

By the above inequality any connected component of $\{x : f(x) \neq 0\}$ meeting S and on which $|f|$ reaches or exceeds the value s contains at least one connected component of $\{|Q_S| > \frac{s}{2}\}$. In particular the number of such connected components is bounded by the number of connected components of $\{|Q_S| > \frac{s}{2}\}$. But the set $\{|Q_S| > \frac{s}{2}\}$ is a semi-algebraic set of \mathbb{R}^d and it is well known [11] that the number of connected components of such sets is bounded by a constant $b = b(r, d)$ depending only on r and d and not on the coefficients of the polynomial Q_S and not on s (it is obvious for $d = 1$ because this number is bounded from above by the number of roots of the polynomial $Q_S^2 - \frac{s^2}{4}$ which is less than $2r$). We conclude that the number of connected components of the open set $\{x : f(x) \neq 0\}$ on which $|f|$ reaches or exceeds the value s is at most $b \left(2 \left[\left(\frac{as}{\max(\|f\|_r, 1)} \right)^{-\frac{1}{r}} + 1 \right]^d \right)$.

□

6. PROOF OF THE MAIN THEOREM

Let $\gamma > 0$ and $\mu \in \mathcal{M}(M, T)$. If $\int \log \text{Jac}_x(T) d\mu(x) < 0$ then by upper semicontinuity of $g : \xi \mapsto \int \log \text{Jac}_x(T) d\xi(x)$ we have $\int \log \text{Jac}_x(T) d\nu(x) = \sum_{i=1}^d \chi_i(\nu) < 0$ for ergodic measures ν close to μ . The map T being nonuniformly entropy expanding it implies that $h(\nu) = 0$ and then Equation (2) is checked.

We assume now that $g(\mu) = \int \log \text{Jac}_x(T) d\mu(x) \geq 0$. In particular the set $\text{Crit}(T)$ of critical points of T has zero μ -measure. Let U be an open neighborhood of $\text{Crit}(T)$ satisfying the following properties :

- $\log \text{Jac}_x(T) < -S_\gamma$ for $x \in U$ (recall $S_\gamma \in \mathbb{Z}^+$ was fixed such that $H(S^{-1}) < \gamma$ for all $S \geq S_\gamma$);
- $\mu(\partial U) = 0$ and $\mu(U) < \gamma$;
- $\int \log \text{Jac}_x(T) d\mu(x) \leq \int_{M \setminus \bar{U}} \log \text{Jac}_x(T) d\mu(x) \leq \int \log \text{Jac}_x(T) d\mu(x) + \gamma$.

We fix a riemanian structure $\|\cdot\|$ on the manifold M . We denote by R_{inj} the radius of injectivity and by $\exp_x : T_x M \rightarrow M$ the exponential map at $x \in M$. There exist $R < R' < \frac{R_{inj}}{\sqrt{d}}$ such that $T(B(x, R)) \subset B(Tx, R')$ for all $x \in M$. Let $\mathcal{V}_\mu = (W_1, \dots, W_p, U_1, \dots, U_q)$ be a finite open cover of M such that :

- $\text{diam}(W_i) < R, \text{diam}(U_i) < R$;
- $\bigcup_{i=1, \dots, q} U_i = U$;
- $T|_{W_i}$ is a diffeomorphism onto its image.

It is a well-known fact that the function $\xi \mapsto \int f(x)\xi(x)$ is upper semicontinuous on $\mathcal{M}(M, T)$ when f is an upper semicontinuous function on M . In particular $\xi \mapsto \xi(\bar{U})$ is upper semicontinuous on $\mathcal{M}(M, T)$. The function $\xi \mapsto \int_{M \setminus \bar{U}} \log \text{Jac}_x(T) d\xi(x)$ is also upper semicontinuous on $\mathcal{M}(M, T)$ since $x \mapsto \mathbb{1}_{M \setminus \bar{U}}(x) \log \text{Jac}_x(T)$ is upper semicontinuous on M : the function $x \mapsto \log \text{Jac}_x(T)$ is continuous on the closure of $M \setminus \bar{U}$ and negative on its boundary. We choose a parameter $\tau_\mu > 0$ such that for all ergodic measures ν with $\text{dist}(\nu, \mu) < \tau_\mu$ we have

$$(12) \quad \nu(\bar{U}) < \gamma$$

$$\int_{M \setminus \bar{U}} \log \text{Jac}_x(T) d\nu(x) < \int_{M \setminus \bar{U}} \log \text{Jac}_x(T) d\mu(x) + \gamma$$

We fix an ergodic measure ν with $\text{dist}(\mu, \nu) < \tau_\mu$. One can assume $g(\nu) = \int \log \text{Jac}_x(T) d\nu(x) = \sum_{i=1}^d \chi_i(\nu) \geq 0$ (if not then $h(\nu) = 0$ as we have already noticed it) and thus $g^+(\nu) = g(\nu)$.

We break the integral $\int \log \text{Jac}_x(T) d\nu(x)$ as the sum of three integrals : over U , over $M \setminus \bar{U}$ and over ∂U . Since $\log \text{Jac}_x(T)$ is negative on ∂U , by dropping the last term we can only increase the right hand side. Moving the terms around we get :

$$(13) \quad \begin{aligned} - \int_U \log \text{Jac}_x(T) d\nu(x) &\leq \int_{M \setminus \bar{U}} \log \text{Jac}_x(T) d\nu(x) - \int \log \text{Jac}_x(T) d\nu(x) \\ &\leq \int_{M \setminus \bar{U}} \log \text{Jac}_x(T) d\mu(x) - \int \log \text{Jac}_x(T) d\nu(x) + \gamma \\ &\leq \int \log \text{Jac}_x(T) d\mu(x) - \int \log \text{Jac}_x(T) d\nu(x) + 2\gamma \end{aligned}$$

Let $\sigma \in]0, 1[$. Let F be a Borel set of ν -measure larger than σ such that

$$(14) \quad h^{inv}(M|F) < -\sum_{i=1}^d \chi_i^-(\nu) + \gamma$$

One can also assume by Birkhoff's ergodic theorem that the sequences $(\frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_U(T^k x))_n$ and $(\frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_U(T^k x) \log \text{Jac}_{T^k x} T)_n$ converge uniformly in $x \in F$ to $\nu(U)$ and $\int_U \log \text{Jac}_x(T) d\nu(x)$, respectively.

Let $V^n = \bigcap_{0 \leq k < n} T^{-k} V_k \in \mathcal{V}_\mu^n$. Consider the sequence $\{i_1, \dots, i_N\} = \{0 \leq k < n, \exists 0 \leq l \leq q \text{ such that } V_k = U_l\}$. To any maximal n -invertible branch A_n intersecting V^n we associate the sequence $\mathcal{K}(A_n) = (k_1(A_n), \dots, k_N(A_n))$ defined by

$$\forall j = 1, \dots, N, \quad k_j(A_n) = \left[\inf_{x \in A_n \cap V^n} -\log \text{Jac}_{T^{i_j x}}(T) \right] + 1$$

With these notations note that the \mathcal{C}^{r-1} function defined on M by $x \mapsto \text{Jac}_x(T)$ reaches or exceeds the value $e^{-k_j(A_n)}$ on $T^{i_j} A_n \cap V_{i_j}$. We consider a sequence $\mathcal{K} = (k_1, \dots, k_N)$ of N positive integers. By Lemma 3 applied for $1 \leq j \leq N$ to the Jacobian of $\exp_{T^{x_j}}^{-1} \circ T \circ \exp_{x_j}(R \cdot) :]-1, 1[^d \subset T_{x_j} M \rightarrow T_{T^{x_j} M}$ with some fixed $x_j \in V_{i_j}$, the number of maximal n -invertible branches A_n meeting V^n with $\mathcal{K}(A_n) = \mathcal{K}$ is bounded above by $c^N e^{\sum_{j=1}^N \frac{dk_j}{r-1}}$ where c depends only on r, d, M and $\max_{s=1, \dots, [r], r} \|T\|_s$. If we assume moreover that A_n meets $F \cap V^n$ then $\mathcal{K}(A_n)$ misses the value $S := \sup_{x \in F} \frac{1}{N} \sum_{j=1}^N (-\log \text{Jac}_{T^{i_j x}} T + 1) \geq S_\gamma$, i.e. $\frac{1}{N} \sum_{j=1}^N k_j(A_n) \leq S$. Since the number of sequences of N positive integers missing the value S is $\binom{NS}{N}$ we get by inequality (3) and since we have arranged that $H(S^{-1}) < \gamma$, that the logarithm of the number of maximal n -invertible branches meeting $F \cap V^n$ is bounded above by $N \log c + NS \left(\frac{d}{r-1} + \gamma \right) + 1$. Observe now that $N \leq \sup_{x \in F} \sum_{k=0}^{n-1} \mathbf{1}_U(T^k x)$ and $NS = \sup_{x \in F} \sum_{j=1}^N (-\log \text{Jac}_{T^{i_j x}} T + 1) \leq \sup_{x \in F} \sum_{k=0}^{n-1} \mathbf{1}_U(T^k x) (-\log \text{Jac}_{T^k x} T + 1)$. Therefore we get for each $V^n \in \mathcal{V}_\mu^n$ (by A_n we will always denote a maximal n -invertible branch) :

$$(15) \quad \begin{aligned} & \log \#\{A_n : A_n \cap F \cap V^n \neq \emptyset\} \\ & \leq N \log c + NS \left(\frac{d}{r-1} + \gamma \right) + 1 \\ & \leq \left(\sup_{x \in F} \sum_{k=0}^{n-1} \mathbf{1}_U(T^k x) (-\log \text{Jac}_{T^k x} T + 1) \right) \left(\frac{d}{r-1} + \gamma \right) \\ & \quad + \left(\sup_{x \in F} \sum_{k=0}^{n-1} \mathbf{1}_U(T^k x) \right) \log c + 1 \end{aligned}$$

Fix $\delta > 0$. Clearly $E \cap A_n$ is a (n, δ) separated set in $F \cap A_n$ for any maximal n -invertible branch A_n when E is a (n, δ) separated set in F . It follows that

$$\begin{aligned} \max\{\#\!E : E \text{ is a } (n, \delta) \text{ separated set in } F \cap V^n \text{ with } V^n \in \mathcal{V}_\mu^n\} \\ \leq \max_{V^n \in \mathcal{V}_\mu^n} \#\!A_n : A_n \cap F \cap V^n \neq \emptyset \\ \times \sup_{A_n} (\max\{\#\!E : E \text{ is a } (n, \delta) \text{ separated in } F \cap A_n\}) \end{aligned}$$

By taking the logarithmic limit in n and then by letting δ go to zero, we get

$$h(M|F, \mathcal{V}_\mu) \leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \max_{V^n \in \mathcal{V}_\mu^n} \#\!A_n : A_n \cap F \cap V^n \neq \emptyset + h^{inv}(M|F)$$

Finally we obtain by the inequalities (14) and (15) and by the uniform convergence on F of the Birkhoff sums

$$\begin{aligned} h(M|F, \mathcal{V}_\mu) &\leq \lim_{n \rightarrow +\infty} \left(\sup_{x \in F} -\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_U(T^k x) \log \text{Jac}_{T^k x} T \right) \left(\frac{d}{r-1} + \gamma \right) \\ &\quad + \lim_{n \rightarrow +\infty} \left(\sup_{x \in F} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_U(T^k x) \right) \left(\frac{d}{r-1} + \gamma + \log c \right) - \sum_{i=1}^d \chi_i^-(\nu) + \gamma \\ &\leq - \int_U \log \text{Jac}_x(T) d\nu(x) \left(\frac{d}{r-1} + \gamma \right) + \nu(U) \left(\frac{d}{r-1} + \gamma + \log c \right) \\ &\quad - \sum_{i=1}^d \chi_i^-(\nu) + \gamma \end{aligned}$$

Since ν has been chosen close to μ , according to the equations (12) and (13) we have (denoting by C the constant $\frac{d}{r-1} + \gamma + \log c + 1$):

$$\begin{aligned} h(M|F, \mathcal{V}_\mu) &\leq \left(\int \log \text{Jac}_x(T) d\mu(x) - \int \log \text{Jac}_x(T) d\nu(x) + 2\gamma \right) \left(\frac{d}{r-1} + \gamma \right) \\ &\quad - \sum_{i=1}^d \chi_i^-(\nu) + C\gamma \\ &\leq (g^+(\mu) - g^+(\nu) + 2\gamma) \left(\frac{d}{r-1} + \gamma \right) - \sum_{i=1}^d \chi_i^-(\nu) + C\gamma \end{aligned}$$

Then by taking the infimum over the Borel sets F of ν -measure larger than σ and by letting σ go to one we get

$$h^{New}(M|\nu, \mathcal{V}_\mu) \leq (g^+(\mu) - g^+(\nu) + 2\gamma) \left(\frac{d}{r-1} + \gamma \right) - \sum_{i=1}^d \chi_i^-(\nu) + C\gamma$$

This concludes the proof since γ can be chosen arbitrarily small.

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