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Solving the noncommutative Batalin-Vilkovisky equation.

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Abstract

I show that summation over ribbon graphs give the construction of the solutions to the noncommutative Batalin-Vilkovisky equation, including the equivariant version, introduced in my previous papers. This generalizes the known construction of A_∞ -algebra via summation over ribbon trees. In particular, the quadratic Maurer-Cartan equation is replaced by the linear equation $\Delta(\Psi) = 0$. Such solutions are naturally the matrix action functionals giving the equivariantly closed differential forms on the matrix spaces, described in my previous paper.

I explain that the summation over ribbon graphs with legs produces solutions to the noncommutative Batalin-Vilkovisky equation, introduced in [B06a]. This generalizes the known construction of A_∞ -structure via summation over trees, see [M].

Notations. I work in the tensor category of super vector spaces, over an algebraically closed field k , $\text{char}(k) = 0$. Let $V = V_0 \oplus V_1$ be a super vector space. I denote by $\bar{\alpha}$ the parity of an element α and by ΠV the super vector space with inversed parity. Element $(a_1 \otimes a_2 \otimes \dots \otimes a_n)$ of $A^{\otimes n}$ is denoted by (a_1, a_2, \dots, a_n) . I denote by V^\vee the dual vector space $\text{Hom}(V, k)$.

1 Algebra with odd differentiation.

I consider a $\mathbb{Z}/2\mathbb{Z}$ -graded algebra A , $\dim_k A < \infty$, with multiplication denoted by $m_2 : A^{\otimes 2} \rightarrow A$ and an *odd differentiation* $I : A \rightarrow \Pi A$

$$Im_2(a, b) = m_2(Ia, b) + (-1)^{\bar{a}}m_2(a, Ib),$$

in particular, if $I^2 = 0$ then this is a $d(\mathbb{Z}/2\mathbb{Z})$ -algebra. I assume, at the beginning, that the algebra is cyclic with odd scalar product

$$\beta : A^{\otimes 2} \rightarrow \Pi A,$$

so that the three tensor

$$m \in ((\Pi A)^{\otimes 3})^\vee$$

$$m(\pi a, \pi b, \pi c) = (-1)^{\bar{b}} \beta(m_2(a, b), c)$$

is cyclically invariant

$$m(\pi a, \pi b, \pi c) = (-1)^{(c+1)(a+b)} m(\pi c, \pi a, \pi b)$$

and that β is preserved by I :

$$\beta(Ia, b) + (-1)^{\bar{a}} \beta(a, Ib) = 0.$$

The modification for the variant with an even scalar product are described below.

Below I consider also the variant for general $d(\mathbb{Z}/2\mathbb{Z})$ -algebra without scalar product. It is reduced to the case with even/odd scalar product by putting $\tilde{A} = A \oplus A^\vee$, or $\tilde{A} = A \oplus \Pi A^\vee$ with their natural scalar products.

I have

$$m(Ia, b, c) + (-1)^{\bar{a}} m(a, Ib, c) + (-1)^{\bar{a}+\bar{b}} m(a, b, Ic) = 0 \quad (1)$$

which reflect the Leibnitz rule for the differentiation I . Denote by $\beta^\vee \in (\Pi A)^{\otimes 2}$ the tensor of the scalar product on the dual vector space, then for any $a, b, c, d \in A$,

$$\langle m(\pi a, \pi b, \cdot) m(\cdot, \pi c, \pi d), \beta^\vee \rangle = (-1)^\varepsilon \langle m(\pi d, \pi a, \cdot) m(\cdot, \pi b, \pi c), \beta^\vee \rangle \quad (2)$$

which is the associativity of the multiplication m .

Let H be an odd selfadjoint operator

$$H : A \rightarrow \Pi A, \quad H^\vee = H$$

such that

$$Id - [I, H] = P \quad (3)$$

is an idempotent operator $P : A \rightarrow A$,

$$P^2 = P.$$

I assume also that H commutes with I^2 , this is automatic if $I^2 = 0$. I denote by B the subspace which is the image of the idempotent P .

Let Γ be a tri-valent ribbon graph, i.e. the trivalent graph with fixed cyclic orders on the sets of the three flags attached to every vertex. Let Σ_Γ be the corresponding two-dimensional surface. Then I put:

- the three-tensors

$$m^v \in ((\Pi A)^{\otimes \text{Flag}(v)})^\vee$$

on every vertex v and define

$$\alpha_\Gamma = \bigotimes_{v \in \text{Vert}(\Gamma)} m^v$$

- the two tensors

$$\begin{aligned} \beta_H^{\vee, e} &\in (\Pi A)^{\otimes \{f, f'\}}, \\ \beta_H^{\vee, e} &= \beta^\vee(H^\vee u_f, v_{f'}) = (-1)^{\overline{u_f} \overline{v_{f'}}} \beta^\vee(H^\vee v_{f'}, u_f) \end{aligned}$$

for any interior edge $e = (ff')$

- element $a_l \in \Pi B$, for any exterior leg l , this gives a partition of the set of elements $\{a_l\}_{l \in \text{Leg}(\Gamma)}$ to the subsets corresponding to the components of the boundary $\partial \Sigma_\Gamma$ and cyclic orders on these subsets.

Notice that both m^v and $\beta_H^{\vee, e}$ are even elements, so that the products $\bigotimes_{v \in \text{Vert}(\Gamma)} m^v$ and $\bigotimes_{e \in \text{Edge}(\Gamma)} \beta_H^{\vee, e}$ are canonically defined.

Definition 1 I define the tensor W_Γ as the contraction

$$W_\Gamma(\bigotimes_{l \in \text{Leg}(\Gamma)} a_l) = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left(\bigotimes_{e \in \text{Edge}(\Gamma)} \beta_H^{\vee, e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right\rangle \quad (4)$$

Notice that W_Γ is cyclically invariant on every subset of $\{a_l\}_{l \in \text{Leg}(\Gamma)}$ corresponding to a component of the boundary of Σ_Γ . Moreover the cyclic orders on flags at vertices induce the orientation on the ribbon graph Γ , whose more detailed analysis, see e.g.[B09b], shows that W_Γ lies in

$$W_\Gamma \in \text{Symm}(\bigoplus_{j=1}^\infty (\Pi B^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})^\vee$$

Let $\chi(\Sigma_\Gamma)$ denotes the genus of Σ_Γ ,

$$\chi(\Sigma_\Gamma) = 2 - 2g(\Sigma_\Gamma) - i(\Sigma_\Gamma),$$

where $g(\Sigma_\Gamma)$, $i(\Sigma_\Gamma)$ are the genus and the number of boundary components of Σ_Γ . I put

$$S = \sum_{\{\Gamma\}} \hbar^{-\chi(\Sigma_\Gamma)} W_\Gamma \quad (5)$$

where the sum is over isomorphism classes of connected trivalent graphs with nonempty subsets of legs on every boundary component of Σ_Γ . One can include the graphs with empty subsets of legs on boundary components by adding the constant extra term to the Batalin-Vilkovisky operator Δ below, I leave the details to the interested reader.

I've described in [B06a],[B06b] the Batalin-Vilkovisky formalism associated with a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space with odd scalar product and in particular I've defined the second order Batalin-Vilkovisky operator Δ and the odd Poisson bracket $\{\cdot, \cdot\}$ on $\text{Symm}(\bigoplus_{j=1}^\infty (\Pi B^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})^\vee$.

Theorem 2 *The sum over graphs S defined in (5) satisfy the equivariant non-commutative Batalin-Vilkovisky equation:*

$$\hbar\Delta S + \frac{1}{2}\{S, S\} + I^\vee S = 0,$$

in particular if $I|_B$ is zero then S is the solution of the non-commutative Batalin-Vilkovisky equation from [B06a],[B06b]

$$\hbar\Delta S + \frac{1}{2}\{S, S\} = 0$$

Remark 3 *If $I|_B \neq 0$, but $I^2|_B = 0$, then $S + S_{0,2}$ is also a solution to the non-commutative Batalin-Vilkovisky equation, where $S_{0,2} = \beta(I, \cdot)|_B$ is the quadratic term corresponding to the differential $I|_B$.*

Proof. The proof is straightforward, see e.g. [B09b]. For a trivalent graph Γ and an internal edge $e \in \text{Edge}(\Gamma)$ consider the three tensors

$$W_{\Gamma,e}^{[I,H]}, W_{\Gamma,e}^{Id}, W_{\Gamma,e}^P \in \text{Symm}(\bigoplus_{j=1}^{\infty} ((\Pi B)^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})^\vee$$

which are defined by the same contraction as W_Γ except that at the edge $e \in \text{Edge}(\Gamma)$ I put the tensors

$$\beta^\vee([I^\vee, H^\vee]u_f, v_{f'}), \beta^\vee(u_f, v_{f'}), \beta^\vee(P^\vee u_f, v_{f'})$$

correspondingly instead of $\beta_H^{\vee,e}$. Then, from (3)

$$W_{\Gamma,e}^P = W_{\Gamma,e}^{Id} - W_{\Gamma,e}^{[I,H]}.$$

By summing over $v \in \text{Vert}(\Gamma)$ of the Leibnitz rule (1) and noticing that

$$\beta^\vee(H^\vee I^\vee u_f, v_{f'}) + \beta^\vee(u_f, H^\vee I^\vee v_{f'}) = -\beta^\vee([I^\vee, H^\vee]u_f, v_{f'})$$

I get

$$I^\vee W_\Gamma - \sum_e W_{\Gamma,e}^{[I,H]} = 0.$$

Next I use (2) to substitute in $W_{\Gamma,e}^{Id}$ the contraction

$$\langle m(\pi a, \pi b, \cdot)m(\cdot, \pi c, \pi d), \beta^\vee \rangle$$

corresponding to the internal edge $e \in \text{Edge}(\Gamma)$ by

$$(-1)^\varepsilon \langle m(\pi d, \pi a, \cdot)m(\cdot, \pi b, \pi c), \beta^\vee \rangle.$$

This corresponds to passing from the trivalent ribbon graph Γ to the trivalent ribbon graph Γ' obtained by the standard transformation on the edge e , preserving the overall cyclic order of the flags corresponding to $\pi a, \pi b, \pi c, \pi d$. This transformation preserves the surface Σ_Γ and the distribution of elements of $\text{Leg}(\Gamma)$ over the boundary components of Σ_Γ . Therefore the sum of $W_{\Gamma,e}^{Id}$

over all internal edges and over the set of trivalent graphs, having the same Σ_Γ with same distribution of $Leg(\Gamma)$ over the boundary components, is zero:

$$\sum_{\{\Gamma\}, \Sigma_\Gamma = \Sigma, e \in Edge(\Gamma)} W_{\Gamma, e}^{Id} = 0.$$

Notice that $P^2 = P$ implies that

$$\beta^\vee(P^\vee u_f, v_{f'}) = \beta^\vee(P^\vee u_f, P^\vee v_{f'}).$$

Then, from the definition of the Batalin-Vilkovisky operator and the odd Poisson bracket on B it follows that

$$\begin{aligned} \sum_{\{\Gamma\}} \hbar^{-\chi(\Sigma_\Gamma)+1} \Delta W_\Gamma + \frac{1}{2} \{ \sum_{\{\Gamma\}} \hbar^{-\chi(\Sigma_\Gamma)} W_\Gamma, \sum_{\{\Gamma'\}} \hbar^{-\chi(\Sigma_{\Gamma'})} W_{\Gamma'} \} = \\ = \sum_{\{\tilde{\Gamma}\}, e \in Edge(\tilde{\Gamma})} \hbar^{-\chi(\Sigma_\Gamma)} W_{\tilde{\Gamma}, e}^P \end{aligned}$$

where each term on left hand side corresponds precisely to the right hand side term $\hbar^{-\chi(\Sigma_\Gamma)} W_{\tilde{\Gamma}, e}^P$, where $\tilde{\Gamma}$ is obtained by gluing two legs to form the edge e from either the single surface or the two surfaces. Notice that the condition, that Δ does not get contributions from the neighboring points on the same circle, corresponds precisely to the fact that the resulting surface $\Sigma_{\tilde{\Gamma}}$ has always nonempty subsets of $Legs(\tilde{\Gamma})$ on the boundary components. ■

Remark 4 *In general, for infinite dimensional cyclic $\mathbb{Z}/2\mathbb{Z}$ -graded algebra A the tensor $\beta^\vee(H^\vee \cdot, \cdot)$ belongs to some completion of $A \otimes_k A$, and extra work is needed in the infinite dimensional situation in order to verify that the tensors W_Γ are well-defined. One can either apply the technique of analysis and construct the propagators as forms with singularity on the diagonal and then verify that the resulting integrals defining W_Γ converge, or the technique of algebraic geometry by working in the \mathbb{Z} -graded category and with dg-algebras whose resolution of the diagonal is constructed from modules with finite-dimensional graded pieces, for example, for smooth dg-algebras, and this implies that the tensors W_Γ are well-defined, see ??.*

Corollary 5 *Given a $\mathbb{Z}/2\mathbb{Z}$ -graded cyclic A_∞ -algebra B , if B has a cyclic $d(\mathbb{Z}/2\mathbb{Z})g$ associative model A , such that for A the contractions (4) over trivalent ribbon graphs are well-defined, then the summation over such graphs gives an extension of the cyclic A_∞ -algebra on B to the quantum A_∞ -algebra, i.e. to the solution of the non-commutative Batalin-Vilkovisky equation.*

1.1

1.2 Even scalar product.

Assume now that the scalar product on A is even:

$$\beta : A^{\otimes 2} \rightarrow A.$$

Then given an *odd differentiation* $I : A \rightarrow \Pi A$ and an odd selfadjoint operator $H : A \rightarrow \Pi A$, satisfying (3), I construct the tensors W_Γ for any ribbon trivalent graph by the same contraction (4). The only difference is that in this case, both the three-tensors m^v attached to the vertices and the two-tensors $\beta_H^{\vee,e}$ are odd and an analysis of the corresponding orientation on Γ , similar to [B09b], shows that W_Γ belongs to the *exterieur* power of the space of cyclic tensors

$$W_\Gamma \in \text{Symm}(\oplus_{j=1}^{\infty} \Pi(\Pi B^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})^{\vee}$$

For the case of even scalar product on $\mathbb{Z}/2\mathbb{Z}$ -graded vector space V I've also described a variant of noncommutative Batalin-Vilkovisky formalism in [B06a].

Theorem 6 *The sum over trivalent ribbon graphs S satisfy the equivariant non-commutative Batalin-Vilkovisky equation:*

$$\hbar \Delta S + \frac{1}{2} \{S, S\} + I^\vee S = 0,$$

in particular if $I|_B = 0$ then S is the solution of the non-commutative Batalin-Vilkovisky equation from [B06a],[B06b]

$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0$$

1.3 General algebras.

Let now A be an arbitrary $\mathbb{Z}/2\mathbb{Z}$ -graded algebra, $\dim_k A < \infty$, with an *odd differentiation* $I : A \rightarrow \Pi A$. This case is reduced to the two previous cases by putting $\tilde{A} = A \oplus (\Pi A)^\vee$ with odd scalar product β given by natural odd pairing between A and $(\Pi A)^\vee$, or by putting $\tilde{A} = A \oplus A^\vee$ with even scalar product β given by natural even pairing between A and A^\vee . Then \tilde{A} is naturally an associative algebra with odd, respectively even, scalar product and the *odd differentiation* I , whose action extends naturally to \tilde{A} .

Consider the case of odd scalar product. Suppose that H is an odd operator

$$H : A \rightarrow \Pi A,$$

such that

$$Id - [I, H] = P \tag{6}$$

is an idempotent operator $P : A \rightarrow A$,

$$P^2 = P.$$

Then both H and P act naturally on \tilde{A} as self-adjoint operators and I apply to this situation the construction of tensors W_Γ for ribbon graphs described above. The tensors

$$W_\Gamma^B \in \text{Symm}(\oplus_{j=1}^{\infty} ((\Pi B \oplus B^\vee)^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})^{\vee}$$

defined by the contraction (4) are given by the sum over markings

$$Flag(\Gamma) \rightarrow \{\Pi A, A^\vee\}$$

such that for any edge, its two flags are marked differently, and for any vertex there is exactly one flag which is marked by ΠA , with no other extra restrictions. In particular such marking gives an orientation for every edge, from A^\vee to ΠA , and there must be exactly one edge exiting every vertex. There are no other restrictions, in particular the edges can in principle form cycles. The legs of Γ , which correspond to the points sitting on the boundary of the surface Σ_Γ , are also marked as either entries (B^\vee) or exits (ΠB). And I define S^B by the summation as above

$$S^B = \sum_{\{\Gamma\}} \hbar^{-\chi(\Sigma_\Gamma)} W_\Gamma^B$$

where the sum is over isomorphism classes of connected trivalent graphs with such markings and with nonempty subsets of legs on every boundary component of Σ_Γ .

Similarly I define the tensors W_Γ in the case of $\tilde{A} = A \oplus A^\vee$ with its even pairing . These tensors are from the space of *exterieur* powers of linear functionals on cyclic words consisting of elements of ΠB and ΠB^\vee

$$W_\Gamma^{\Pi B} \in \text{Symm}(\oplus_{j=1}^{\infty} \Pi((\Pi B \oplus \Pi B^\vee)^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})^\vee$$

and I define $S^{\Pi B}$ as their sum over ribbon graphs as above.

Theorem 7 *Let A an arbitrary $\mathbb{Z}/2\mathbb{Z}$ -graded algebra, $\dim_k A < \infty$, with an odd differentiation $I : A \rightarrow \Pi A$ and a homotopy H , such that the operator (6) is idempotent. The sums over ribbon graphs S^B and $S^{\Pi B}$ give the solutions to the two variants of the equivariant noncommutative Batalin-Vilkovisky equation in the spaces of symmetric, respectfully exterieur powers, of cyclic words, consisting of elements from ΠB^\vee and B , respectfully from ΠB^\vee and ΠB :*

$$\begin{aligned} \hbar \Delta S^B + \frac{1}{2} \{S^B, S^B\} + I^\vee S^B &= 0 \\ \hbar \Delta S^{\Pi B} + \frac{1}{2} \{S^{\Pi B}, S^{\Pi B}\} + I^\vee S^{\Pi B} &= 0. \end{aligned}$$

2 Graphs with the insertion of A_∞ -tensors.

Assume now that A is a $\mathbb{Z}/2\mathbb{Z}$ -graded A_∞ -algebra, $\dim_k A < \infty$, as above I relax the condition $d^2 = 0$, and assume that it is simply an *odd operator* $I : A \rightarrow \Pi A$, which together with other structure maps $m_n \in ((\Pi A)^{\otimes n})^\vee \otimes A$, $n \geq 2$, satisfy the standard A_∞ -constraints, except the very first, so that I^2 is in general nonzero: for any $n \geq 2$

$$\begin{aligned} Im_n(v_1, \dots, v_n) - \sum_l (-1)^\epsilon m_n(v_1, \dots, Iv_l, \dots, v_n) &= \\ = \sum_{i+j=n+1} (-1)^\epsilon m_i(v_1, \dots, m_j(\dots), \dots, v_n) & \end{aligned}$$

I assume first that A has also an invariant odd scalar product β so that all tensors

$$m_n \in ((\Pi A)^{\otimes n+1})^\vee, \beta(m_n(v_1, \dots, v_n), v_{n+1})$$

are cyclic invariant, the variant without scalar product is reduced as above to this case by taking $\tilde{A} = A \oplus (\Pi A)^\vee$.

Let as above H be an odd selfadjoint operator

$$H : A \rightarrow \Pi A, \quad H^\vee = H$$

such that

$$Id - [I, H] = P$$

is an idempotent operator $P : A \rightarrow A$, whose image I denote by B .

Now I define the tensors W_Γ , by inserting the cyclic tensors $m_{n(v)} \in ((\Pi A)^{\otimes Flag(v)})^\vee$ at vertices, as above, where Γ is now a ribbon graph, with valency $n(v)$ for any vertice at least three:

$$W_\Gamma(\otimes_{l \in Leg(\Gamma)} a_l) = \left\langle \otimes_{v \in Vert(\Gamma)} m_{n(v)}, \left(\otimes_{e \in Edge(\Gamma)} \beta_H^{\vee, e} \right) \otimes_{l \in Leg(\Gamma)} a_l \right\rangle$$

and

$$W_\Gamma \in Symm(\oplus_{j=1}^\infty (\Pi B^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})^\vee$$

Next however, looking carefully at the proof of the equation for S above, one sees that one immediately runs into a problem because of tadpoles, unless the important condition

$$\Delta m_n = 0$$

is imposed, which I assume from now on.

I define now, similarly to above,

$$S = \sum_{\{\Gamma\}} \hbar^{-\chi(\Sigma_\Gamma)} W_\Gamma \quad (7)$$

where the sum is over isomorphism classes of connected ribbon graphs with vertices of valency $n(v) \geq 3$, and with nonempty subsets of legs on every boundary component of Σ_Γ .

Theorem 8 *Let the odd operator I and the cyclically invariant tensors $m_n \in ((\Pi A)^{\otimes n+1})^\vee$, $n \geq 2$, satisfy*

$$\begin{aligned} I^\vee m + \{m, m\} &= 0 \\ \Delta m &= 0 \end{aligned}$$

Then the sum over ribbon graphs S satisfy the equivariant noncommutative Batalin-Vilkovisky equation associated with $(B, \beta|_B)$:

$$\hbar \Delta S + \frac{1}{2} \{S, S\} + I^\vee S = 0,$$

in particular if $I|_B = 0$ then S is the solution of the non-commutative Batalin-Vilkovisky equation from [B06a],[B06b]

$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0$$

Proof. The proof is parallel to the above. ■

The same result holds starting from the arbitrary solution to the non-commutative Batalin-Vilkovisky equation and the tensors W_Γ constructed similarly for stable ribbon graphs. Details will appear elsewhere.

Analogous result holds for arbitrary modular operad.

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