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► **To cite this version:**

Maxim N. Chernodub. Conformal Anomaly in Yang-Mills Theory and Thermodynamics of Open Confining Strings. Universe, 2020, 6 (11), pp.202. 10.3390/universe6110202 . hal-00464529

**HAL Id: hal-00464529**

**<https://hal.science/hal-00464529v1>**

Submitted on 9 Nov 2020

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Article

# Conformal Anomaly in Yang-Mills Theory and Thermodynamics of Open Confining Strings

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Received: 7 October 2020; Accepted: 29 October 2020; Published: 31 October 2020

**Abstract:** We discuss thermodynamic properties of open confining strings introduced via static sources in the vacuum of Yang-Mills theory. We derive new sum rules for the chromoelectric and chromomagnetic condensates and use them to show that the presence of the confining string lowers the gluonic pressure in the bulk of the system. The pressure deficit of the gluon plasma is related to the potential energy in the system of heavy quarks and anti-quarks in the plasma.

**Keywords:** conformal anomaly; quark-gluon plasma; thermodynamics; heavy quarks

## 1. Introduction

Quantum Chromodynamics (QCD) features many nonperturbative properties thanks to the dynamics of the gluons which mediate the strong fundamental forces. The pure gluonic sector of the theory is described by Yang-Mills theory which respects the conformal invariance at the classical level because the classical Yang-Mills Lagrangian possesses no dimensionful parameters. As a result, the classical processes look identically the same under a rescaling of all coordinates, fields, energies, and momenta according to their canonical dimensions.

The conformal symmetry is, however, broken at quantum level. This phenomenon is called quantum conformal anomaly. The appearance of the conformal anomaly is a natural feature of almost any interacting theory because quantum corrections affect differently the physical phenomena that develop at different energy (or spatial) scales. The interaction phenomena are determined by coupling constants of the theory that are called “constants” in the classical version of the theory. The presence of the quantum conformal anomaly reveals itself in the fact of the “running” of the coupling(s): The running implies the dependence of the coupling with respect to the energy scale or momentum scale which is involved in the interaction process that is described by this coupling. Thus, in an interacting quantum (field) theory, the coupling constants are not, strictly speaking, constants: they are functions of the energy involved in the interaction itself.

The most exciting outcome of experiments on the relativistic heavy-ion collisions is the creation of the quark-gluon plasma [1–3]. The thermodynamic and transport phenomena of this deconfined state of strongly interacting matter attract the increasing attention of the scientific community [4–6].

Long time ago it was suggested that an enhanced dissociation of heavy quark-antiquark bound states could be a good signal of the deconfinement phase [7]. Theoretical efforts aimed to study properties of heavy quarkonia require a nonperturbative input in a form of quark-antiquark potentials at finite temperature [8,9]. A powerful numerical tool to study such potentials is based on first-principle calculations of Polyakov loop correlators in lattice simulations of QCD [9]. The renormalized correlator of Polyakov loops provides us with the free energy, and—via thermodynamic relations—with internal energy

and entropy of the heavy quarks [10]. The Polyakov loops introduce the sources of the chromoelectric field in the vacuum. Since the sources of the chromoelectric field are static they cannot be excited thermally, so that the term “free energy of heavy quarks” is usually understood as an excess in free energy of thermal gluons that appears due to presence of the strong chromoelectric fields of the heavy quarks [10].

It is clear that presence of heavy quarks—considered as sources and sinks of the chromoelectric fields—affects thermodynamics of the quark-gluon plasma. Despite the fact that the thermodynamic effect of one quark-antiquark pair is not an extensive quantity, a multiple production of quarkonia may provide a contribution to the bulk properties of the plasma.

The presence of the heavy quarks and the emergence of the associated stringy effects were linked in the literature to certain interesting observable effects. The dilute admixture of heavy quarks was argued in Reference [11] to lower the speed of sound in QGP, while the open string dynamics in a simple model was shown to lead for a thermal-like distribution of transverse mass of particles created in the string decay [12,13].

In our paper, we show that, in addition to the free energy, internal energy and entropy of the heavy quarks, we find that the gluons around the quarks are also able to make contribution to the pressure of the system. The effect of pressure is not seen in a standard approach because the pressure is usually associated with variations of the volume of the system, which does not enter the excess in the free energy of a finite-sized quark pair. However, below we show that the variations in size of a quark system do couple to the pressure.

In the thermodynamics, the spatial extent of the quark pair can be treated as a external variable while the pressure enters a quantity that plays a role of the corresponding generalized force. From this point of view the renormalized quark potential can be associated with an excess in a generalized Helmholtz free energy.

We use a sum rule approach that is generally known as powerful analytical tool for investigation of certain nonperturbative properties of QCD physics [14,15]. For example, in absence of the external sources, the sum rules may help constrain the transport coefficients [16], while the susceptibilities of components of the energy-momentum tensor determine (derivatives of) thermodynamic potentials [17].

We derive exact finite-temperature sum rules for excesses in internal energy and in (volume-integrated) pressure in the presence of external heavy quarks. In the limit of zero temperature certain combinations of these sum rules reduce to the well-known action and energy sum rules derived by Michael and Rothe in a lattice formulation of the theory. The Michael-Rothe equations associate the quark-antiquark potential to an excess in chromoelectric and chromomagnetic condensates. Historically, the first attempt to derive such sum rules was done in Reference [18]. The rules were subsequently corrected [19] and extended [20], and an important role of the conformal anomaly was stressed [21]. A lattice check of the sum rules was done in Reference [22]. Below we use the new finite-temperature sum rules to study the gluon thermodynamics around color sources further.

It is important to notice that the “sum rules” in the present paper are closer to the finite-temperature generalizations of the “low-energy QCD theorems” (similar to the ones developed in References [23,24]) as compared to the original sum rules defined in Reference [14,15]. Here, we follow the lattice terminology, originally used in References [18–21].

The structure of this paper is as follows. In Section 2 we give a short introduction to the conformal anomaly in Yang-Mills theory and its relation to the thermodynamics of gluons. In Section 3 we derive the new finite-temperature sum rules that relate the inter-quark-potential to the excesses in expectations values of the chromoelectric and chromomagnetic condensates, that arise due to the presence of the external quark sources. Our conclusions are summarized in the last section.

## 2. The Conformal Anomaly and Thermodynamics of Gluons

### 2.1. Conformal Anomaly with Isotropic Renormalization Scale: Equation of State of Gluon Plasma

While our paper focuses on the thermodynamics of gluon plasma in the presence of heavy quarks, it is illuminating to discuss first the thermodynamics of Yang-Mills theory without any external quark sources. Although the derivation is well known, we repeat it for the sake of further convenience.

Because of further applications in the lattice gauge theory, we consider a Euclidean formulation of the Yang-Mills theory. The Euclidean Lagrangian of the theory with  $N$  colors,

$$\mathcal{L} = \frac{1}{2} \text{Tr} G_{\mu\nu}^2 \equiv \frac{1}{2} (\vec{E}^2 + \vec{B}^2), \tag{1}$$

is expressed via the chromoelectric ( $E_i^a = G_{i0}^a$ ) and chromomagnetic ( $B_i^a = \frac{1}{2} \epsilon_{ijk} G_{jk}^a$ ) fields, where

$$G_{\mu\nu} \equiv G_{\mu\nu}^a t^a = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \tag{2}$$

is the field strength tensor of the gluon field  $A_\mu \equiv t^a A_\mu^a$ . The generators of the gauge group  $t^a$  (with the adjoint index  $a = 1, \dots, N^2 - 1$ ) are normalized in the standard way,  $\text{Tr} t^a t^b = \frac{1}{2} \delta^{ab}$ . The bold symbols are used for the vectors in coordinate space while the arrow over a symbol denotes a vector in the color space.

At the classical level, the energy momentum tensor of the Yang-Mills theory Equation (1)

$$\mathcal{T}_{\mu\nu} = 2\text{Tr} \left[ G_{\mu\sigma} G_{\nu\sigma} - \frac{1}{4} \delta_{\mu\nu} G_{\sigma\rho} G_{\sigma\rho} \right], \tag{3}$$

is traceless,  $(\mathcal{T}^\mu_\mu)_{\text{cl}} = 0$ . This property is an inevitable consequence of the fact that the classical Yang-Mills theory is a conformally-invariant theory that contains no mass or length scales in its classical Lagrangian.

The conformal invariance is broken at the quantum level. This phenomenon makes the trace of the energy-momentum tensor nonzero:

$$\langle \mathcal{T}^\mu_\mu(x) \rangle = \left\langle \frac{\beta(g)}{2g} G_{\mu\nu}^a(x) G_{\mu\nu}^a(x) \right\rangle \equiv \left\langle \frac{2\beta(g)}{g} \mathcal{L}(x) \right\rangle, \tag{4}$$

where

$$\beta(g) = \frac{\partial g(\mu)}{\partial \ln \mu} = -g^3(b_0 + b_1 g^2 + \dots), \quad b_0 = \frac{11N}{3(4\pi)^2}, \quad b_1 = \frac{34N^2}{3(4\pi)^4}. \tag{5}$$

is the beta function of Yang-Mills theory expressed via the perturbative coefficients  $b_i$ . The beta function determines how the Yang-Mills coupling  $g$  changes with the renormalization scale  $\mu$  which corresponds to the energy of a given process that involves the gluon interaction via  $g = g(\mu)$ . This quantum phenomenon makes the vacuum expectation value of the energy-momentum tensor non-vanishing.

Consider the gluonic system in a finite but large volume  $V$  (we will send it to infinity at the end of calculation). In the thermodynamic limit, the total energy of the system is  $E = \varepsilon V$ , where  $\varepsilon$  is the energy density. The total energy  $E$  and the pressure  $P$  are determined via the derivatives of the partition function  $\mathcal{Z}$  with respect to the temperature  $T$  and volume, respectively,

$$E = T \frac{\partial \ln \mathcal{Z}}{\partial \ln T}, \quad PV = T \frac{\partial \ln \mathcal{Z}}{\partial \ln V}, \quad \ln \mathcal{Z} = -\frac{F}{T} \equiv -f \frac{V}{T}, \tag{6}$$

and the partition function  $\mathcal{Z}$  is expressed via the free energy  $F \equiv fV$ .

Since the partition is a dimensionless function of the dimensional quantities  $V$ ,  $T$ , and  $\mu$ , on the dimensional grounds one gets the following relation:

$$\left(3\frac{\partial}{\partial \ln V} - \frac{\partial}{\partial \ln T} - \frac{\partial}{\partial \ln \mu}\right) \ln Z = 0. \tag{7}$$

The free energy is a physical quantity which characterizes the system as a whole. Therefore, in the thermodynamic limit, it does not depend on the interaction energy scale [25]:

$$\frac{d \ln Z}{d \ln \mu} \equiv \left(\frac{\partial}{\partial \ln \mu} + \beta(g)\frac{\partial}{\partial g}\right) \ln Z = 0, \tag{8}$$

where the beta function is given in Equation (5). Then we use Equations (1), (4), (6), (7) and (8) to identify

$$\frac{\partial \ln Z}{\partial g} = \frac{2V}{T} \left\langle \frac{1}{g} \mathcal{L} \right\rangle, \tag{9}$$

and to derive the following conformal anomaly relation:

$$\Theta = \left\langle \mathcal{T}^\mu_\mu \right\rangle \equiv E - 3PV = \int d^3x \left\langle \left\langle \frac{\beta(g)}{g} \left[ \vec{E}^2(x) + \vec{B}^2(x) \right] \right\rangle \right\rangle_T. \tag{10}$$

The left-hand side of this equation is related to the equation of state of the gluon plasma while its right-hand side represents the spatial integral of the anomalous trace Equation (4) of the energy-momentum tensor Equation (3).

To pick up the thermal effects, Equation (10) is regularized via the subtraction of the zero-temperature contribution, with the notations  $\langle \dots \rangle_T = \langle \dots \rangle_T - \langle \dots \rangle_{T=0}$ . Here  $\langle \dots \rangle_T$  indicates the expectation value at temperature  $T$  while  $\langle \dots \rangle_{T=0}$  denotes the expectation value of the same operator at zero temperature. The reason for this subtraction is that in the perturbation theory, the vacuum expectation value of the gluon condensate Equation (4) diverges quartically with the ultraviolet cutoff. This hazardous divergence is related to the zero-point quantum fluctuations rather than the system’s thermodynamics. Since the divergence is a temperature-independent property, it is customary to renormalize the right-hand side of Equation (10) by subtracting the divergent  $T = 0$  part.

The conformal anomaly Equation (10) is used in numerical calculations of the equation of state of Yang-Mills theory implemented in first-principle lattice simulations [26,27]. Equation (6) give us the total energy and pressure of the gluon plasma, and determines the quantity on the right-hand side of Equation (10):

$$E = F - T \frac{\partial F}{\partial T}, \quad P = -\frac{\partial F}{\partial V} \equiv -\frac{F}{V} \equiv -f, \quad \Theta = VT^5 \frac{\partial}{\partial T} \frac{P(T)}{T^4}. \tag{11}$$

Thus, the left hand side of the anomaly Equation (10) is expressed via pressure  $P$  while the right-hand side of this equation can be evaluated using numerical calculations in lattice gauge theory. The full information on the thermodynamics of the theory,  $E = E(T)$  and  $P = P(T)$  may be obtained via a numerical integration of this simple equation.

In the thermodynamic limit of a uniform system—such as the vacuum of  $SU(N)$  gauge theory—the energy density and pressure are extensive variables related to each other via the thermodynamical identities. The relation of Equation (10) provides us with a relationship between the system’s energy and pressure to have the complete system of two equations for two unknown quantities. However, the contributions to energy density and pressure from external sources (from static quark-antiquark

systems) are not extensive variables. Thus, the standard relation between energy and pressure should not work in these systems by default, and we need to define these quantities independently. We will perform this task below.

2.2. Conformal Anomaly with Anisotropic Renormalization Scales: Energy and Pressure via Condensates

Let us generalize the anomaly relation Equation (10) by introducing separate renormalization scales in temporal ( $\mu = \mu_t$ ) and spatial ( $\mu = \mu_s$ ) directions. This procedure has been implemented already in the lattice gauge theory [28,29], and here we will reformulate it in continuum terms for the future use. The essential point is that the spatial scale  $\mu_s$  controls the three-dimensional volume  $V$  of the system but does not affect the temperature while the (Euclidean) temporal scale  $\mu_t$  controls the temperature  $T$  and does not affect the volume. The usefulness of this separation is evident in view of the thermodynamic relations Equation (11). Then, the analogues of the dimensional identity Equation (7) are as follows:

$$3 \frac{\partial \ln \mathcal{Z}}{\partial \ln V} - \frac{\partial \ln \mathcal{Z}}{\partial \ln \mu_s} = 0, \quad \frac{\partial \ln \mathcal{Z}}{\partial \ln T} + \frac{\partial \ln \mathcal{Z}}{\partial \ln \mu_t} = 0, \tag{12}$$

while the requirement of the scale independence of the partition function provides us with following counterparts of Equation (8):

$$\frac{d \ln Z}{d \ln \mu_a} \equiv \frac{\partial \ln Z}{\partial \ln \mu_a} + \sum_{b=s,t} \frac{\partial g_b}{\partial \ln \mu_a} \frac{\partial \ln Z}{\partial g_b} = 0, \tag{13}$$

for spatial ( $a = s$ ) and temporal ( $a = t$ ) components. The spatial and temporal couplings in Equation (13) are defined, respectively, via the relations:

$$\frac{1}{g_s^2(\mu_s, \mu_t)} = \frac{1}{g^2(\mu_s, \mu_t)} \frac{\mu_t}{\mu_s}, \quad \frac{1}{g_t^2(\mu_s, \mu_t)} = \frac{1}{g^2(\mu_s, \mu_t)} \frac{\mu_s}{\mu_t}. \tag{14}$$

The couplings  $g_s$  and  $g_t$  enter, respectively, the chromomagnetic  $\vec{B}$  and chromoelectric  $\vec{E}$  components of the gluon strength tensor Equation (2).

While the relations similar to Equation (14) appear naturally in the lattice gauge theory on asymmetric lattices [29], these definitions of the couplings are not obvious in the continuum context which we use in the paper. To this end, let us briefly adapt our notations to the lattice ones and follow the logic of Reference [29] but in terms of the continuum variables. We rescale the gauge fields  $A_\mu \rightarrow g^{-1} A_\mu$  so that the gluon strength tensor Equation (2) loses the dependence on the coupling  $g$  similarly to the lattice gauge theory. These formulations are equivalent in the continuum limit. The action of the isotropic theory—with the same scales in the spatial and temporal directions—is as follows:

$$S = \int d^4x \left[ \frac{1}{2g^2} \vec{B}^2 + \frac{1}{2g^2} \vec{E}^2 \right] \quad (\text{isotropic}). \tag{15}$$

We would like to make the theory invariant under independent rescaling transformations of spatial,  $x \rightarrow \lambda_s x$ , and temporal,  $x_4 \rightarrow \lambda_t x_4$ , coordinates where  $\lambda_s$  and  $\lambda_t$  are real-valued factors. The corresponding energy (renormalization) scales transform as  $\mu_s \rightarrow \lambda_s^{-1} \mu_s$  and  $\mu_t \rightarrow \lambda_t^{-1} \mu_t$ . The chromoelectric and chromomagnetic fields rescale, respectively, as follows:  $\vec{E} \rightarrow \lambda_s^{-1} \lambda_t^{-1} \vec{E}$  and  $\vec{B} \rightarrow \lambda_s^{-2} \vec{B}$ . Since the magnetic

and electric parts of the action scale in different manner, the couplings in front of these terms should not, logically, be the same. In the theory with anisotropic scales  $\mu_s \neq \mu_t$ , one gets:

$$S = \int d^4x \left[ \frac{1}{2g_s^2} \vec{B}^2 + \frac{1}{2g_t^2} \vec{E}^2 \right] \quad (\text{anisotropic}), \tag{16}$$

where the couplings  $g_s$  and  $g_t$  are given in Equation (14), respectively.

The classical theory Equation (16) is invariant under the scaling transformations with independent spatial  $\lambda_s$  and temporal  $\lambda_t$  factors. Since the measure of the integration transforms as  $d^4x \rightarrow \lambda_s^3 \lambda_t d^4x$  we arrive to:

$$d^4x \frac{1}{g_s^2} \vec{B}^2 \rightarrow (\lambda_t \lambda_s^3 d^4x) \left( \frac{\lambda_s}{\lambda_t} \frac{1}{g_s^2} \right) \left( \frac{1}{\lambda_s^4} \vec{B}^2 \right) \rightarrow d^4x \frac{1}{g_s^2} \vec{B}^2, \tag{17}$$

$$d^4x \frac{1}{g_t^2} \vec{E}^2 \rightarrow (\lambda_t \lambda_s^3 d^4x) \left( \frac{\lambda_t}{\lambda_s} \frac{1}{g_t^2} \right) \left( \frac{1}{\lambda_s^2 \lambda_t^2} \vec{E}^2 \right) \rightarrow d^4x \frac{1}{g_t^2} \vec{E}^2. \tag{18}$$

These relations imply that the classical action does not change under the scaling transformation,  $S \rightarrow S$ . The derivation in lattice terms also uses the naive continuum limit. We refer the interested reader to Reference [29] for the details.

In Equation (14), the quantity  $g^2 = g^2(\mu_s, \mu_t)$  is the isotropic Yang-Mills coupling which depends on both scales  $\mu_s$  and  $\mu_t$ . At the end of our derivations, after performing all the differentiations, we always set  $\mu_t = \mu_s = \mu$  to achieve a locally isotropic theory at the single renormalization scale  $\mu$ .

Then Equations (12)–(14) provide us with following expressions for the energy and pressure:

$$\int d^3x \varepsilon = \int d^3x \left\{ \left\langle \left\langle \frac{1}{2} [\vec{B}^2(x) - \vec{E}^2(x)] \right\rangle \right\rangle_T + \left\langle \left\langle \frac{\beta_t(g)}{g} [\vec{E}^2(x) + \vec{B}^2(x)] \right\rangle \right\rangle_T \right\}, \tag{19}$$

$$3 \int d^3x P = \int d^3x \left\{ \left\langle \left\langle \frac{1}{2} [\vec{B}^2(x) - \vec{E}^2(x)] \right\rangle \right\rangle_T - \left\langle \left\langle \frac{\beta_s(g)}{g} [\vec{E}^2(x) + \vec{B}^2(x)] \right\rangle \right\rangle_T \right\}, \tag{20}$$

where the generalizations of the beta function Equation (5) are expressed via the isotropic coupling  $g^2$ :

$$\beta_a(g(\mu)) = \left. \frac{\partial g(\mu_t, \mu_s)}{\partial \ln \mu_a} \right|_{\mu_s = \mu_t = \mu}, \quad a = s, t. \tag{21}$$

The relations Equations (19) and (20) are the continuum analogues of the discretized lattice expressions derived in Reference [28]. Notice also that  $\langle \langle \vec{B}^2(x) - \vec{E}^2(x) \rangle \rangle_T \equiv \langle \vec{B}^2(x) - \vec{E}^2(x) \rangle_T$  because the zero-temperature contributions disappear in this term.

The first term in the curly brackets in each of Equations (19) and (20) is a classical contribution coming from the classical energy-momentum tensor Equation (3). The second term in these expressions corresponds to the quantum correction related to the conformal trace anomaly Equation (4). Thus, it is natural that the first terms cancel each other in the trace of the energy-momentum tensor,  $\mathcal{T}^\mu_\mu = \varepsilon - 3P$ , while the second terms add up giving us Equation (10). At the last step, we used the equality  $\beta_t + \beta_s = \beta$ , which follows immediately from the definitions Equation (14). The spatial and temporal beta functions Equation (21) were calculated numerically in Reference [29].



### 3. Gluon Energy and Pressure via Gluon Condensates in Presence of Confining String

#### 3.1. Thermodynamics from the Conformal Anomaly

Now imagine that we have inserted an infinitely heavy quark-antiquark pair in the a thermal vacuum of Yang-Mills theory. Since the quark and the antiquark are so heavy, they cannot be excited by the thermal fluctuations. Nevertheless, the gluon field around the heavy-quark pair affects the thermal fluctuations of gluons in the thermal bath, and, consequently, contributes to energy density and pressure around these static sources of the chromoelectric fields. In this section, we generalize the thermodynamic relations Equations (19) and (20) taking into account the presence of the static quarks.

How to take into account the influence of the heavy quarks on thermodynamics of gluons? An obvious way is to evaluate the gluon condensates in Equations (19) and (20) with insertions of quark creation operators.

A static quark is created at the position  $\vec{x}$  by the gauge-invariant Polyakov loop operator

$$L(\vec{x}) = \frac{1}{N} \text{Tr} \mathcal{P} \exp \left\{ ig \int_0^{1/T} dt A_4(\vec{x}, t) \right\}, \tag{22}$$

where the operator  $\mathcal{P}$  implies the path ordering of the integration which goes along the straight line parallel to compactified (temperature) direction of the Euclidean space-time. The length of the compactified direction is  $1/T$ . The path of the integration in Equation (22) is closed via the periodic boundary condition imposed on the gluonic fields in the compactified direction. The conjugated operator  $L^\dagger(\vec{x})$  creates an antiquark at the spatial point  $\vec{x}$ .

A quantum state corresponding to a system  $\mathcal{Q} = Q_1 \dots Q_{N_Q} \bar{Q}_1 \dots \bar{Q}_{N_{\bar{Q}}}$ , consisting of  $N_Q$  quarks located at spatial positions  $\vec{x}_i$  and  $N_{\bar{Q}}$  antiquarks at points  $\vec{y}_k$ , is created by the operator

$$L_{\mathcal{Q}}(\{\vec{R}_i\}) = L(\vec{x}_1) \dots L(\vec{x}_{N_Q}) \cdot L^\dagger(\vec{y}_1) \dots L^\dagger(\vec{y}_{N_{\bar{Q}}}), \tag{23}$$

where  $\{\vec{R}_i\} \equiv \{\vec{x}_1, \dots, \vec{x}_{N_Q}, \vec{y}_1, \dots, \vec{y}_{N_{\bar{Q}}}\}$ . Since generalizations to various representations is straightforward, we consider here the color-averaged operators Equation (23) only.

According to Equations (19) and (20), in order to estimate the effect of the heavy (anti)quark system on the thermodynamics of gluons one should evaluate the chromoelectric and chromomagnetic condensates in the presence of the static (heavy) quarks. Since Equations (19) and (20) are linear in the condensates, it is convenient to consider the excess, respectively, in the energy and in the (integrated-over-volume) pressure caused by the presence of the heavy quarks:

$$E_{\mathcal{Q}}(T) = \int d^3x [\varepsilon_{\mathcal{Q}}(x) - \varepsilon(x)], \quad \Pi_{\mathcal{Q}}(T) = \int d^3x [P_{\mathcal{Q}}(x) - P(x)]. \tag{24}$$

Notice, that both these equations do not depend on the spatial size of the system. These quantities are finite but, in general, nonzero.

The contribution of the quark system  $\mathcal{Q}$  to both chromoelectric and chromomagnetic gluon condensates  $\mathcal{O}$  is given by:

$$\langle \mathcal{O} \rangle_{\mathcal{Q},T} = \frac{\langle \mathcal{O} L_{\mathcal{Q}} \rangle_T}{\langle L_{\mathcal{Q}} \rangle_T} - \langle \mathcal{O} \rangle_T, \tag{25}$$



where the product  $L_Q$  of the Polyakov loops is defined in Equation (23). This definition is valid at any nonzero temperature.

Using Equations (19) and (20) along with the definition in Equation (24), we get the excesses in the energy and the pressure of the system due to the presence of the heavy (anti)quarks

$$E_Q(T) = \int d^3x \left\{ \left\langle \frac{1}{2} [\vec{B}^2(x) - \vec{E}^2(x)] \right\rangle_{Q,T} + \left\langle \frac{\beta_t(g)}{g} [\vec{E}^2(x) + \vec{B}^2(x)] \right\rangle_{Q,T} \right\}, \quad (26)$$

$$3\Pi_Q(T) = \int d^3x \left\{ \left\langle \frac{1}{2} [\vec{B}^2(x) - \vec{E}^2(x)] \right\rangle_{Q,T} - \left\langle \frac{\beta_s(g)}{g} [\vec{E}^2(x) + \vec{B}^2(x)] \right\rangle_{Q,T} \right\}. \quad (27)$$

Notice that the expectation value  $\langle\langle \mathcal{O} \rangle\rangle_{Q,T}$  in Equations (19) and (20) gets automatically replaced by  $\langle \mathcal{O} \rangle_{Q,T}$  in Equations (26) and (27) because of cancelations of zero-temperature contributions.

Equations (26) and (27) can already be used for a numerical estimation of the contribution of the heavy (anti)quark systems to the thermodynamics of gluons. Yet, there is a way to proceed further analytically.

### 3.2. Generalization of Michael-Rothe Sum Rules

Consider now a quark  $Q$  and an antiquark,  $\bar{Q}$ , separated by the distance  $R$ . The color-averaged  $Q\bar{Q}$  potential  $V_{Q\bar{Q}}$  is given by the following expectation value:

$$e^{-V_{Q\bar{Q}}/T} = \langle L(\vec{0})L^\dagger(\vec{R}) \rangle, \quad (28)$$

where we use the symbol  $V$  both for the volume (which always appears without a subscript) and for the multi-quark potential  $V_Q$  (which always comes with a subscript).

Notice that the physical interaction potential of heavy quarks comes from the renormalized Polyakov loops  $L^{\text{ren}}(\vec{R})$  that are multiplied by a perimeter/length-dependent factor  $Z$  in order to remove the divergent perturbative contribution:

$$L(\vec{R}) \rightarrow L^{\text{ren}}(\vec{R}) = [Z(g)]^{\mu/T} \cdot L(\vec{R}). \quad (29)$$

The renormalization of the quark potential is, however, irrelevant for the determination of both the energy Equation (26) and the pressure Equation (27), because Equation (25)—used both in Equation (26) and in Equation (27)—is invariant under the renormalization shift Equation (29). This property is well understood on physical grounds as the energy Equation (26) and pressure Equation (27) reflect the thermodynamics of gluons around the static quarks and not the quarks themselves.

The contribution of the gluons around the finite-size heavy-quark system to the thermodynamics is not an extensive quantity. Thus, the heavy quark potential Equation (28) is a function of four dimensionful parameters: the temperature  $T$ , the distance between the quark and antiquark  $R$ , and the spatial and temporal renormalization scales  $\mu_s$  and  $\mu_t$ . In the presence of the heavy quarks, we rewrite Equation (13) and the scale-independence requirements Equation (12) by using the quantity  $\ln \langle L(\vec{0})L^\dagger(\vec{R}) \rangle \equiv -V_{Q\bar{Q}}/T$  instead of  $\ln \mathcal{Z}$ . Equation (13) reduces to as follows:

$$\left( \frac{\partial}{\partial \ln R} - \frac{\partial}{\partial \ln \mu_s} \right) \frac{V_{Q\bar{Q}}(R, T; \mu_s, \mu_t)}{T} = 0, \quad \left( \frac{\partial}{\partial \ln T} + \frac{\partial}{\partial \ln \mu_t} \right) \frac{V_{Q\bar{Q}}(R, T; \mu_s, \mu_t)}{T} = 0, \quad (30)$$

while the second pair of equations in Equation (13) gives us the following identities:

$$T^2 \frac{\partial}{\partial T} \frac{V_{Q\bar{Q}}}{T} = - \int d^3x \left\{ \left\langle \frac{1}{2} [\vec{B}^2(x) - \vec{E}^2(x)] \right\rangle_{Q\bar{Q},T} + \left\langle \frac{\beta_t(g)}{g} [\vec{E}^2(x) + \vec{B}^2(x)] \right\rangle_{Q\bar{Q},T} \right\}, \quad (31)$$

$$R \frac{\partial V_{Q\bar{Q}}}{\partial R} = - \int d^3x \left\{ \left\langle \frac{1}{2} [\vec{B}^2(x) - \vec{E}^2(x)] \right\rangle_{Q\bar{Q},T} - \left\langle \frac{\beta_s(g)}{g} [\vec{E}^2(x) + \vec{B}^2(x)] \right\rangle_{Q\bar{Q},T} \right\}. \quad (32)$$

Equations (31) and (32) are new finite-temperature sum rules that relate the  $Q\bar{Q}$  potential to the excesses in expectations values of the chromoelectric and chromomagnetic condensates, that arise due to the presence of the external quark sources. These sum rules represent a generalization of already known low-energy relations that were derived in  $T = 0$  lattice gauge theory in References [18–21,25]. For example, subtracting Equation (31) from Equation (32) and using Equation (22) one gets:

$$\left(1 + R \frac{\partial}{\partial R} - T \frac{\partial}{\partial T}\right) V_{Q\bar{Q}} = \left\langle \frac{2\beta(g)}{g} \int d^3x \mathcal{L}(x) \right\rangle_{Q\bar{Q},T} \quad (33)$$

This relation is nothing but a natural finite-temperature extension of a well-known “action sum rule” first found in the lattice formulation in Reference [19]:

$$V_{Q\bar{Q}} + R \frac{\partial V_{Q\bar{Q}}}{\partial R} = \left\langle \frac{2\beta(g)}{g} \int d^3x \mathcal{L}(x) \right\rangle_{Q\bar{Q}(T=0)}. \quad (34)$$

Moreover, the zero-temperature limit of Equation (31) gives us back a continuum version of the known “energy sum rule” [19]:

$$V_{Q\bar{Q}} = \int d^3x \left\{ \left\langle \frac{1}{2} [\vec{B}^2(x) - \vec{E}^2(x)] \right\rangle_{Q\bar{Q}(T=0)} + \left\langle \frac{\beta_t(g)}{g} [\vec{E}^2(x) + \vec{B}^2(x)] \right\rangle_{Q\bar{Q}(T=0)} \right\}. \quad (35)$$

Let us mention that we have three pairs of equations that look similar but have rather different physical meanings:

- (i) The pair Equations (19) and (20) is an *equality*, that relates the energy and pressure of the gluonic fields to the expectation values of the gluon condensates. This pair is a continuum version of the corresponding lattice formulae derived in Reference [28].
- (ii) The pair Equations (26) and (27) is a natural *definition* of the contribution of heavy quarks to the energy Equation (19) and pressure Equation (20) of the gluons.
- (iii) Finally, the pair Equations (31) and (32) describes the new finite-temperature *sum rules*.

### 3.3. New Sum Rules and Exact Relations for Gluon Thermodynamics around Static Sources

A comparison of the exact relations Equation (26), Equation (27) with new sum rules Equation (31), Equation (32) shows that the excess in the energy Equation (24) and the (volume-integrated) excess in the pressure Equation (24) due to the presence of the heavy quark-antiquark pair are, respectively:

$$E_{Q\bar{Q}}(R, T) = V_{Q\bar{Q}} - T \frac{\partial V_{Q\bar{Q}}}{\partial T} \equiv -T^2 \frac{\partial}{\partial T} \frac{V_{Q\bar{Q}}}{T}, \quad \Pi_{Q\bar{Q}}(R, T) = -\frac{R}{3} \frac{\partial V_{Q\bar{Q}}}{\partial R}. \quad (36)$$

The important property of these relations is that they are exact.

From Equation (36) we get a Maxwell-type relation between the excesses in the energy and pressure:

$$R \left( \frac{\partial E_{Q\bar{Q}}}{\partial R} \right)_T = 3T^2 \left( \frac{\partial \Pi_{Q\bar{Q}}}{\partial T} \frac{1}{T} \right)_R, \tag{37}$$

where the subscripts mean that the derivatives are taken at the fixed temperature  $T$  and the fixed inter-quark distance  $R$ , respectively. Equation (37) is an exact equation of state for heavy quarks.

A comparison of the energy excess in Equation (36) with the corresponding thermodynamical relation [the first formula in Equation (11)] would lead us to the conclusion that the heavy quark potential  $V_{Q\bar{Q}}$  may be interpreted—up to a renormalization constant Equation (29)—as an excess in the Helmholtz free energy  $F_{Q\bar{Q}}$  due to the presence of the heavy quarks. However, the excess in the pressure Equation (36) has nothing to do with its analogue in the thermodynamical limit [the second formula in Equation (11)], making the free energy interpretation of the heavy quark potential  $V_{Q\bar{Q}}$  obscure.

Nevertheless, one can still interpret  $V_{Q\bar{Q}}$  as an excess in the Helmholtz free energy of gluons due to the presence of the heavy (anti)quarks. Indeed, besides the temperature  $T$ , the heavy quark thermodynamics has the additional external variable  $R$  but lacks the usual volume variable  $V$ . In this case the usual fundamental thermodynamic relation,  $dE = TdS - PdV$ , should be written as follows

$$dE_{Q\bar{Q}} = TdS_{Q\bar{Q}} - X_{Q\bar{Q}}dR, \tag{38}$$

where  $S_{Q\bar{Q}}$  is the excess in the entropy and  $X_{Q\bar{Q}}$  is the generalized force associated with the distance  $R$  between the quark and antiquark. From Equation (36) one gets:

$$S_{Q\bar{Q}} = - \left( \frac{\partial V_{Q\bar{Q}}}{\partial T} \right)_R, \quad X_{Q\bar{Q}} = - \left( \frac{\partial V_{Q\bar{Q}}}{\partial R} \right)_T \equiv - \frac{3}{R} \Pi_{Q\bar{Q}}, \tag{39}$$

so that the generalized force  $X_{Q\bar{Q}}$  is related to the (integrated) pressure in Equation (36). Then the differential of the excess in the Helmholtz free energy  $F_{Q\bar{Q}} \equiv E_{Q\bar{Q}} - TS_{Q\bar{Q}}$  is:

$$dF_{Q\bar{Q}} = -S_{Q\bar{Q}}dT - X_{Q\bar{Q}}dR. \tag{40}$$

Notice that the excess in any thermodynamical quantity coming as a result of the presence of a finite number of finitely-separated quarks is not an extensive property of the system. Due to the effect of screening (i.e., due to a finite mass gap in the spectrum), the finite-volume corrections to these quantities should vanish with the increase in the volume of the system. Thus, the volume is a thermodynamically irrelevant variable for large enough systems.

The excess in the free energy  $F_{Q\bar{Q}}$  can be easily obtained by an integration of Equation (40):

$$F_{Q\bar{Q}}(R, T) = V_{Q\bar{Q}}(R, T). \tag{41}$$

A free integration constant—which inevitably appears in any integration—is to be interpreted as the renormalization quark-antiquark potential Equation (29). We have just shown that Equation (41)—despite its standard appearance—has somewhat nonstandard interpretation because it involves the generalized fundamental thermodynamic relation Equation (38) that includes the generalized force Equation (39).

### 3.4. Negative Pressure Excess

The excess in the pressure due to the presence of the heavy quark pairs is always negative. This property is seen from the second formula in Equation (36) taking into account that the  $V_{Q\bar{Q}}$  potential

is a concave function of the distance [30]. Thus, the presence of a heavy quarkonium state should decrease the pressure in the gluon plasma. A gas of weakly-interacting heavy-quark bound states with the density  $\rho$  should produce the total pressure deficit in the gluonic plasma (for the sake of simplicity, here we assume that all  $Q\bar{Q}$ -states have the same size  $R$ ):

$$\Delta P_{Q\bar{Q}}(R, T) = -\frac{\rho(R, T)R}{3} \frac{\partial V_{Q\bar{Q}}(R, T)}{\partial R} < 0. \tag{42}$$

It is interesting to notice that even at zero temperature the integrated pressure in Equation (36) is not a positive-definite quantity. This fact does not contradict the common wisdom. For example, a similar feature characterizes the conventional Casimir effect in the vacuum of electrodynamics: the vacuum between two perfectly conducting metallic plates has a negative pressure, negative free energy and negative internal energy [31,32]. Less exotic example of a similar system at nonzero temperature is a solution of ethanol in water, that demonstrates in normal conditions an effect of volume contraction (i.e., negative excess of volume). The maximal volume contraction in this binary mixture is reached at approximately 20% of mole fraction of ethanol [33], thus mimicking, in very loose terms, a noticeable (excess of) negative pressure.

Our calculations are in line with the observation that the contribution of the heavy quarks to the specific heat of the gluon plasma becomes negative at high enough temperatures [34]. The identification Equation (41) represents the excess in the free due to heavy quarks which is equal to the (renormalized) heavy-quark potential (see, e.g., Reference [10]). Moreover, there is no ab initio restriction on the sign of the excess in any thermodynamic quantity due to presence of external particles.

### 3.5. Multiquarks and a Single Quark

A generalization of our results to multiquark systems is rather straightforward. Instead of the  $Q\bar{Q}$  potential  $V_{Q\bar{Q}}$  one should use its multiquark counterpart  $V_Q$  defined via the multiquark loop Equation (23) as follows:

$$e^{-V_Q(\{\vec{R}_i\}, T)/T} = \langle L_Q(\{\vec{R}_i\}) \rangle. \tag{43}$$

In addition, the derivatives with respect to the  $Q\bar{Q}$  distance should be replaced by the following scale derivative:

$$R \frac{\partial}{\partial R} V_{Q\bar{Q}}(R, T) \rightarrow \frac{\partial}{\partial \lambda} V_Q(\{\lambda \vec{R}_i\}, T) \Big|_{\lambda=1}. \tag{44}$$

According to Equations (36), (43) and (44), a single quark  $Q$  cannot affect the pressure of the system even in the deconfinement phase because the distance variable  $R$  is evidently absent. Therefore, the contribution to the vacuum gluonic pressure from the infinite string open at one end is zero. Then the single-quark contribution depends only on temperature, and we get the following relation, which is a direct analogue of Equation (32) for  $Q = Q$ :

$$\left\langle \left( \frac{1}{2} - \frac{\beta_s(g)}{g} \right) \int d^3x \vec{B}^2(x) \right\rangle_{Q,T} = \left\langle \left( \frac{1}{2} + \frac{\beta_s(g)}{g} \right) \int d^3x \vec{E}^2(x) \right\rangle_{Q,T}. \tag{45}$$

This relation leads to an analogue of Equation (33) for a single quark:

$$V_Q - T \frac{\partial V_Q}{\partial T} = \left\langle \frac{2\beta(g)}{g} \int d^3x \mathcal{L}(x) \right\rangle_{Q,T}. \tag{46}$$

The above equation represents yet another new sum rule, that reduces to the known relation Equation (34) in the limit  $T \rightarrow 0$  (to verify the correspondence, one should notice that  $\partial V_Q / \partial R \equiv 0$ ).

### 3.6. A Few Phenomenological Examples

The excesses in free energy Equation (41), entropy Equation (39) and internal energy Equation (36) were thoroughly studied in lattice simulations of  $SU(3)$  Yang-Mills theory [10,35]. Here we discuss briefly the new quantity, the (volume-integrated) excess in the pressure Equation (36) caused by the presence of the heavy quark-antiquark pair.

#### 3.6.1. Deconfinement Phase

The deconfinement phase, which is realized at temperatures higher than the critical temperature  $T_c$ , the real part of the heavy-quark potential can quantitatively be described by the following formula [36]:

$$V_{Q\bar{Q}}(R, T) = -\frac{e(T)T}{(RT)^{d(T)}} e^{-\mu(T)R}, \quad [T \geq T_c], \tag{47}$$

where  $e(T)$  and  $d(T)$  are dimensionless temperature-dependent parameters, and  $\mu(T)$  is the screening mass. For a reference, the behaviour of the screening factor is  $\mu(T) \simeq 2.5T$  at  $T \gtrsim 1.5T_c$ ; it decreases drastically as temperature approaches its critical value  $T_c$  from above. The exponent  $d$  increases from  $d \simeq 1$  at  $T = T_c$  to a stable value  $d \simeq 1.5$  at  $T \gtrsim 1.5T_c$  up to temperatures  $T \approx 3T_c$  [36]. The heavy-quark potential develops also an imaginary part [37], which may become important for the dynamics of the bottomonium bound state at temperatures around 250 MeV and higher. While the imaginary part of the potential does not enter our analysis below, we mention that its presence reflects the fact that heavy quarkonia are unstable states with finite widths. We refer the Reader to References [38,39] for recent reviews on the subject.

Relation Equation (36) implies that in the deconfinement phase the absolute value of the  $Q\bar{Q}$ -induced pressure is a decreasing function of the distance  $R$ :

$$\Pi_{Q\bar{Q}}(R, T \geq T_c) = \frac{d + \mu(T)R}{3} V_{Q\bar{Q}}(R, T) < 0. \tag{48}$$

At large separations between the sources, the pressure excess vanishes due to the color screening.

#### 3.6.2. Confinement Phase at Finite Temperature

In the confinement phase at finite-temperature  $0 < T < T_c$ , the interaction potential of quarks in the quarkonium state can be described, at large distances, by the following formula [36]:

$$V_{Q\bar{Q}} = V_0 + \sigma(T)R + CT \ln(2RT), \quad RT \gg 1, \quad [T < T_c], \tag{49}$$

where  $V_0$  and  $C$  are parameters obtained by a fit of the lattice numerical data. We get from Equation (36):

$$\Pi_{Q\bar{Q}}(R, T < T_c) = -\frac{CT + \sigma R}{3}. \tag{50}$$

At large distances, the pressure deficit is a linearly decreasing function of the separation between the quark and the antiquark.

### 3.6.3. Zero Temperature

At zero temperature, the heavy-quark interactions at short and long distances can be described by the single Cornell potential:

$$V_{Q\bar{Q}}(R, T = 0) = -\frac{\alpha}{R} + \sigma R, \quad (51)$$

where  $\alpha \simeq 0.356 \pm 0.015$  and  $\sigma \simeq (0.207 \pm 0.088) \text{ GeV}^2$  are the phenomenological constants which may be obtained, for example, from the heavy quarkonia (bottomonia) fits [40,41]). The integrated pressure deficit,

$$\Pi_{Q\bar{Q}}(R, T = 0) = -\frac{\alpha}{3R} - \frac{\sigma R}{3}, \quad (52)$$

takes its maximum  $\Pi_{Q\bar{Q}}^{\max} = -2\sqrt{\alpha\sigma}/3 \simeq -(178 \pm 34) \text{ MeV}$  at the distance between the heavy quarks  $R^{\max} = \sqrt{\alpha/\sigma} \simeq (0.28 \pm 0.06) \text{ fm}$ .

In the confinement phase the integrated excess in the gluon pressure, Equations (50) and (52), is always a negative quantity, as expected. At small distances  $R$ , the effect appears due to perturbative gluons in the chromoelectric flux which spans between the sources.

The pressure deficit Equation (52) decreases linearly with the increase in the  $Q\bar{Q}$  separation  $R$  at large enough  $R$ . Physically, this effect emerges due to the presence of the QCD string that is spanned between the quark and antiquark. The presence of the string causes the backreaction of the gluons in a close proximity.

## 4. Conclusions

We derived the finite-temperature sum rules Equations (31) and (32) of gluons in heavy quarks systems. From our sum rules, the Michael-Rothe sum rules for the action Equation (34) and for the energy Equation (35) are recovered automatically in the zero-temperature limit. In addition, the new sum rules provide us with the new expression Equation (36) for the internal energy  $E$  of the quarks Equation (24).

At the same time, these sum rules lead to the new expression Equation (36) for the spatial integral Equation (24) of the excess in the gluonic pressure,  $\Pi_Q$  around quarks. The excess corresponds to the difference between the pressure in the gluonic system in the presence of the quarks and without quarks. In the confining phase, this excess may be interpreted as the effect of the confining string, which is open at the (anti-)quarks that serve as sources and sinks of the chromoelectric field. We found that the pressure excess of the gluons around a heavy quark-antiquark state is always negative, independent of the phase of the gluonic system.

We also derived the exact equation of state Equation (37) that relates the excess in the gluonic energy to the excess in the gluonic pressure around the heavy-quark systems. The generalization of our results to multi-quark systems is straightforward and given in Equations (43) and (44).

**Funding:** The work was partially supported by Grant No. 0657-2020-0015 of the Ministry of Science and Higher Education of Russia.

**Acknowledgments:** The author is grateful to Stam Nicolis for useful discussions.

**Conflicts of Interest:** The author declares no conflict of interest.

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