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L-embedded Banach spaces and a weak version of Phillips lemma

Hermann Pfitzner

Abstract. After a short survey on L-embedded Banach spaces - for which Bochner spaces of the form $L^1(X)$ with reflexive X serve as examples - we prove that these spaces satisfy a weak form of Phillips' lemma: It is proved that the L-projection of an L-embedded Banach spaces sends relatively w^* -sequentially compact sets to relatively weakly sequentially compact sets.

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L-embedded Banach spaces. A projection P on a Banach space Z is called an *L-projection* if $\|Pz\| + \|z - Pz\| = \|z\|$ for all $z \in Z$. A Banach space X is called *L-embedded* (or *an L-summand in its bidual*) if it is the image of an L-projection on its bidual. In this case we write $X^{**} = X \oplus_1 X_s$. The standard reference for L-embedded Banach spaces is the monograph [3] which contains everything of this introduction except for [7]. For general Banach space theory and undefined notation we refer to [4], [6], or [1].

A special class of L-embedded spaces consists of the duals of M-embedded Banach spaces: A Banach space Y is called *M-embedded* (or *an M-Ideal in its bidual*) if its annihilator Y^\perp in Y^{***} is the range of an L-projection on Y^{***} in which case the dual Y^* identifies easily with the kernel of the L-projection and is therefore L-embedded. Examples of such L-embedded spaces include $l^1(\Gamma) = (c_0(\Gamma))^*$ (Γ any set), $N(H) = (K(H))^*$ (the nuclear operators on a Hilbert space H), $(K(l^p, l^q))^*$ where $1 < p \leq q < \infty$, the Hardy space $H_0^1 = (C(\mathbb{T})/A)^*$ where A is the disk algebra.

L^1 -spaces are L-embedded and so are, more generally, the preduals of von Neumann algebras, even the preduals of JBW*-triples. As to a generalization towards Bochner spaces of the form $L^1(X)$ not much seems to be known: From [3, p. 199] we resume the following cases. Since the projective tensor product of L^1 and the predual of von Neumann algebra is again the predual of a von Neumann algebra the Bochner space $L^1(X)$ is L-embedded if X is a von Neumann predual.

Further, if the underlying measure space is finite and if $X \subset L^1$ is (in Godefroy's terminology) nicely placed i.e. L-embedded, for example $H_0^1 \subset L^1(\mathbb{I})$, then $L^1(X)$ is L-embedded, too. Finally, if X is reflexive (hence trivially L-embedded) then $L^1(X)$ is L-embedded.

The class of L-embedded spaces is much larger than the one of duals of M-embedded spaces because among L^1 -spaces only $l^1(\Gamma)$ (Γ a set) is the dual of an M-embedded space, and, analogously, the only von Neumann predual which is the dual of an M-embedded space is the direct sum in the l^1 -sense of $N(H)$ -spaces which in turn is equivalent to the von Neumann predual having RNP [3, Prop. III.2.9, Prop. IV.2.9]. In particular, an L-embedded space need not have RNP whereas the dual of an M-embedded space does. There are dual L-embedded spaces with RNP that are not duals of M-embedded spaces [3, Ex. IV.4.12].

The interest of L-embeddedness lies in the fact that it unifies some Banach space properties which before were known only as special cases. A kind of breakthrough for this general point of view was Godefroy's discovery that L-embedded Banach spaces are weakly sequentially complete. Another example of this unifying point of view is the fact [7] that (at least) separable L-embedded spaces are unique preduals which comprises for example the corresponding result of Dixmier-Sakai for von Neumann preduals.

Talagrand [10] has shown that the Bochner space $L^1(X)$ is weakly sequentially complete if and only if X is and Randrianantoanina [9] has shown that $L^1(X)$ has Pełczyński's property (V^*) if and only if X does (see also [5]). Since L-embedded spaces are weakly sequentially complete and have (V^*) [3, IV.2.7] it seems plausible that $L^1(X)$ is L-embedded for more Banach spaces X than the ones mentioned above.

A weak version of Phillips' lemma. Phillips' lemma [8] states that the canonical projection from the second dual of l^1 onto l^1 is w^* -norm-sequentially continuous which if one takes l^1 's Schur property for granted is equivalent to the projection being w^* -weak-sequentially continuous. (Cf., for example, [1, Ch. VII].) Therefore the fact that the canonical projection from the third onto the first dual of a Banach space is w^* -weak sequentially continuous if the space in consideration has Pełczyński's property (V) (cf. [3, Prop. III.3.6]) can be considered a generalization of Phillips' lemma.

In this note we give a modest - see the remark after the proof - generalization in a similar direction by looking at l^1 as an L-embedded Banach space. Although the result is new the technique of its proof is not. In fact it is a modification of the proof in [7] but holds, contrary to the latter one, for all L-embedded Banach spaces, not only separable ones.

Theorem *The L-projection of an L-embedded Banach space sends relatively w^* -sequentially compact sets into relatively weakly sequentially compact sets.*

Proof. We recall that a series $\sum z_j$ in a Banach space Z is called *weakly unconditionally Cauchy* (wuC for short) if $\sum |z^*(z_j)|$ converges for each $z^* \in Z^*$ or,

equivalently, if there is a number M such that $\|\sum_{j=1}^n \alpha_j z_j\| \leq M \max_{1 \leq j \leq n} |\alpha_j|$ for all $n \in \mathbb{N}$ and all scalars α_j .

Let X be an L-embedded Banach space with L-projection P . We have the decomposition $X^{**} = X \oplus_1 X_s$ where X_s denotes the range of the projection $Q = \text{id}_{X^{**}} - P$. Let (x_n) be a sequence in X and let (t_n) be a sequence in X_s . Further, consider a cluster point $x + x_s$ of the x_n in the w^* -topology of X^{**} (with $x \in X$, $x_s \in X_s$).

CLAIM Given $x^* \in X^*$ of norm one there are a wuC -series $\sum x_k^*$ in X^* and a sequence (n_k) in \mathbb{N} such that, for all $k \in \mathbb{N}$,

$$t_{n_k}(x_k^*) = 0 \quad (0.1)$$

$$\lim_k x_k^*(x_{n_k}) = x_s(x^*). \quad (0.2)$$

Proof of the claim: Let $1 > \varepsilon > 0$ and let (ε_j) be a sequence of numbers decreasing to zero such that $0 < \varepsilon_j < 1$ and $\prod_{j=1}^{\infty} (1 + \varepsilon_j) < 1 + \varepsilon$.

By induction over $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we shall construct two sequences $(x_k^*)_{k \in \mathbb{N}_0}$ and $(y_k^*)_{k \in \mathbb{N}_0}$ in X^* (of which the first members x_0^* and y_0^* are auxiliary elements used only for the induction) and an increasing sequence (n_k) of indices such that, for all (real or complex) scalars α_j and with $\beta = x_s(x^*)$, the following hold:

$$x_0^* = 0, \quad \|y_0^*\| = 1, \quad (0.3)$$

$$\left\| \alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^* \right\| \leq \left(\prod_{j=1}^k (1 + \varepsilon_j) \right) \max_{0 \leq j \leq k} |\alpha_j|, \quad \text{if } k \geq 1, \quad (0.4)$$

$$t_{n_k}(x_k^*) = 0, \quad (0.5)$$

$$y_k^*(x) = 0, \quad \text{and} \quad x_s(y_k^*) = \beta, \quad (0.6)$$

$$|x_k^*(x_{n_k}) - \beta| < \varepsilon_k. \quad (0.7)$$

We set $n_0 = 1$, $x_0^* = 0$ and $y_0^* = x^*$.

For the following it is useful to recall some properties of P : The restriction of P^* to X^* is an isometric isomorphism from X^* onto X_s^\perp with $(P^* y^*)|_X = y^*$ for all $y^* \in X^*$, Q is a contractive projection and $X^{***} = X_s^\perp \oplus_\infty X^\perp$ (where X^\perp is the annihilator of X in X^{***}).

For the induction step suppose now that $x_0^*, \dots, x_k^*, y_0^*, \dots, y_k^*$ and n_0, \dots, n_k have been constructed and satisfy conditions (0.3) - (0.7). Since $x + x_s$ is a w^* -cluster point of the x_n there is an index n_{k+1} such that

$$|x_s(y_k^*) - y_k^*(x_{n_{k+1}} - x)| < \varepsilon_{k+1}. \quad (0.8)$$

Put

$$E = \text{lin}(\{x^*, x_0^*, \dots, x_k^*, y_k^*, P^* x_0^*, \dots, P^* x_k^*, P^* y_k^*\}) \subset X^{***},$$

$$F = \text{lin}(\{x_{n_{k+1}}, t_{n_{k+1}}, x, x_s\}) \subset X^{**}.$$

Clearly $Q^*x_j^*, Q^*y_k^* \in E$ for $0 \leq j \leq k$. By the principle of local reflexivity there is an operator $R: E \rightarrow X^*$ such that

$$(1 - \varepsilon_{k+1})\|e^{***}\| \leq \|Re^{***}\| \leq (1 + \varepsilon_{k+1})\|e^{***}\|, \quad (0.9)$$

$$f^{**}(Re^{***}) = e^{***}(f^{**}), \quad (0.10)$$

$$R|_{E \cap X^*} = \text{id}_{E \cap X^*} \quad (0.11)$$

for all $e^{***} \in E$ and $f^{**} \in F$.

We define

$$x_{k+1}^* = RP^*y_k^* \quad \text{and} \quad y_{k+1}^* = RQ^*y_k^*.$$

In the following we use the convention $\sum_{j=1}^0(\dots) = 0$. Then we have that

$$\alpha_0 y_{k+1}^* + \sum_{j=1}^{k+1} \alpha_j x_j^* = R \left(Q^* (\alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^*) + P^* (\alpha_{k+1} y_k^* + \sum_{j=1}^k \alpha_j x_j^*) \right).$$

Now (0.4) (for $k+1$ instead of k) can be seen as follows:

$$\begin{aligned} & \left\| \alpha_0 y_{k+1}^* + \sum_{j=1}^{k+1} \alpha_j x_j^* \right\| \leq \\ & \stackrel{(0.9)}{\leq} (1 + \varepsilon_{k+1}) \left\| Q^* (\alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^*) + P^* (\alpha_{k+1} y_k^* + \sum_{j=1}^k \alpha_j x_j^*) \right\| \\ & = (1 + \varepsilon_{k+1}) \max \left\{ \left\| Q^* (\alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^*) \right\|, \left\| P^* (\alpha_{k+1} y_k^* + \sum_{j=1}^k \alpha_j x_j^*) \right\| \right\} \\ & \leq (1 + \varepsilon_{k+1}) \max \left\{ \left\| \alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^* \right\|, \left\| \alpha_{k+1} y_k^* + \sum_{j=1}^k \alpha_j x_j^* \right\| \right\} \\ & \leq \left(\prod_{j=1}^{k+1} (1 + \varepsilon_j) \right) \max \left\{ \max_{0 \leq j \leq k} |\alpha_j|, \max_{1 \leq j \leq k+1} |\alpha_j| \right\} \\ & = \left(\prod_{j=1}^{k+1} (1 + \varepsilon_j) \right) \max_{0 \leq j \leq k+1} |\alpha_j| \end{aligned}$$

where the last inequality comes from (0.3) if $k=0$, and from (0.4), if $k \geq 1$.

The conditions (0.5) and (0.6) (for $k+1$ instead of k) are easy to verify because $Pt_{n_{k+1}} = 0$, $Qx = 0$ and $Qx_s = 0$ thus, by (0.10)

$$\begin{aligned} t_{n_{k+1}}(x_{k+1}^*) &= y_k^*(Pt_{n_{k+1}}) = 0, \\ y_{k+1}^*(x) &= y_k^*(Qx) = 0 \quad \text{and} \quad x_s(y_{k+1}^*) = Q^*y_k^*(x_s) = x_s(y_k^*) = \beta. \end{aligned}$$

Finally, we have

$$x_{k+1}^*(x_{n_{k+1}}) - \beta = y_k^*(x_{n_{k+1}}) - \beta = y_k^*(x_{n_{k+1}} - x) - x_s(y_k^*)$$

by (0.6) whence (0.7) for $k + 1$ by (0.8). This ends the induction and proves the CLAIM.

Put now $x_n^{**} = x_n + t_n$ and suppose that (x_n^{**}) is w^* -null. We define a sequence (μ_n) of finitely additive measures on the subsets of \mathbb{N} by $\mu_n(A) = x_n^{**}(\sum_{k \in A} x_k^*)$ for all $A \subset \mathbb{N}$ where $\sum_{k \in A} x_k^*$ is to be understood in the w^* -topology of X^* . Then $\mu_n(A) \rightarrow 0$ for all $A \subset \mathbb{N}$ and by (0.1) and Phillips' original lemma ([8], [1, p. 83]) we get

$$|x_k^*(x_{n_k})| = |x_{n_k}^{**}(x_k^*)| \leq \sum_j |x_{n_k}^{**}(x_j^*)| = \sum_j |\mu_{n_k}(\{j\})| \rightarrow 0.$$

Thus $x_s(x^*) = 0$ by (0.2). Since x^* was an arbitrary normalized element this means that $x_s = 0$. It follows that the w^* -closure in X^{**} of the set consisting of the x_n lies in X . Hence this set is relatively weakly compact (or, equivalently, relatively weakly sequentially compact). This proves the theorem. \square

It is not clear whether the L-projection is actually w^* -weak-continuous. In two cases this happens trivially: firstly if the dual X^* is a Grothendick space (which by definition means that in X^{**} w^* -convergent sequences converge weakly) and secondly if X is the dual of an M-embedded Banach space Y (see [3, Ch. III]) because in this case P is the canonical restriction projection from Y^{***} onto Y^* ([3, Prop. III.2.4]) and the x_n tend to 0 on the elements of Y and thus any weak cluster point x of (x_n) is necessarily zero. Alternatively, in the latter case the result [3, Prop. III.3.6], which was mentioned above, applies because M-embedded Banach spaces have property (V) ([2] or [3, Th. III.3.4]).

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