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► To cite this version:

Hermann Pfitzner. Phillips' Lemma for L-embedded Banach spaces. Archiv der Mathematik, 2010, 94, pp.547-553. 10.1007/s00013-010-0120-6 . hal-00463770

HAL Id: hal-00463770

<https://hal.science/hal-00463770>

Submitted on 25 Mar 2010

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Phillips' Lemma for L-embedded Banach spaces

Hermann Pfitzner

Abstract. In this note the following version of Phillips' lemma is proved. The L-projection of an L-embedded space - that is of a Banach space which is complemented in its bidual such that the norm between the two complementary subspaces is additive - is weak*-weakly sequentially continuous.

Mathematics Subject Classification (2000). 46B20.

Keywords. Phillips' lemma, L-embedded Banach space, L-projection.

Phillips' classical lemma [9] refers to a sequence (μ_n) in $\text{ba}(\mathbb{N})$ (the Banach space of finitely bounded measures on the subsets of \mathbb{N}) and states that if $\mu_n(A) \rightarrow 0$ for all $A \subset \mathbb{N}$ then $\sum_k |\mu_n(\{i\})| \rightarrow 0$. It is routine to interpret this result as the weak*-weak-sequential continuity of the canonical projection from the second dual of l^1 onto l^1 because this continuity together with l^1 's Schur property gives exactly Phillips' lemma. (Cf., for example, [2, Ch. VII].) Therefore the following theorem generalizes Phillips' lemma (for the definitions see below):

Theorem 0.1. *The L-projection of an L-embedded Banach space is weak*-weakly sequentially continuous.*

The theorem will be proved at the end of the paper.

The theorem has been known in the two particular cases when the L-embedded space in question is the predual of a von Neumann algebra or the dual of an M-embedded Banach space Y . In the first case the result follows from [1, Th. III.1]; in the second case Y has Pełczyński's property (V) ([3] or [4, Th. III.3.4]) and has therefore, by [4, Prop. III.3.6], what in [6, p. 73] or in [10] is called the weak Phillips property whence the result by [4, Prop. III.2.4].

Preliminaries. By definition a Banach space X is *L-embedded* (or an *L-summand in its bidual*) if there is a linear projection P on its bidual X^{**} with range X such that $\|Px^{**}\| + \|x^{**} - Px^{**}\| = \|x^{**}\|$ for all $x^{**} \in X^{**}$. The projection P is called L-projection. Throughout this note X denotes an L-embedded Banach space with L-projection P . We have the decomposition $X^{**} = X \oplus_1 X_s$ where X_s denotes the kernel of P that is the range of the projection $Q = \text{id}_{X^{**}} - P$. We recall that a series $\sum z_j$ in a Banach space Z is called *weakly unconditionally Cauchy*

(wuC for short) if $\sum |z^*(z_j)|$ converges for each $z^* \in Z^*$ or, equivalently, if there is a number M such that $\|\sum_{j=1}^n \alpha_j z_j\| \leq M \max_{1 \leq j \leq n} |\alpha_j|$ for all $n \in \mathbb{N}$ and all scalars α_j . The presence of a non-trivial wuC-series in a dual Banach space is equivalent to the presence of an isomorphic copy of l^∞ . For general Banach space theory and undefined notation we refer to [5], [7], or [2]. The standard reference for L-embedded spaces is [4]; here we mention only that besides the Hardy space H^1 the preduals of von Neumann algebras - hence in particular $L^1(\mu)$ -spaces and l^1 - are L-embedded. Note in passing that in general an L-embedded Banach space, contrary to l^1 , need not be a dual Banach space.

The proof of the theorem consists of two halves. The first one states that the L-projection sends a weak*-convergent sequence to a relatively weakly sequentially compact set. This has already been proved in [8]. The second half asserts the existence of the 'right' limit and can be deduced from the corollary below which states that the singular part X_s of the bidual is weak*-sequentially closed. Note that X_s is weak*-closed if and only if X is the dual of an M-embedded Banach space [4, IV.1.9]. The following lemma contains the two main ingredients for the proof of the theorem namely two wuC-series $\sum x_k^*$ and $\sum u_k^*$ by means of which the theorem above will reduce to Phillips' original lemma. The first one has already been constructed in [8], the construction of the second one is (somewhat annoyingly) completely analogous, with the rôles of P and Q interchanged, cf. (0.20) and (0.21). (For the proof of the theorem it is not necessary to construct both wuC-series simultaneously but there is no extra effort in doing so and it might be useful elsewhere.)

Lemma 0.2. *Let X be L-embedded, let (x_n) be a sequence in X and let (t_n) be a sequence in X_s . Furthermore, suppose that $x + x_s$ is a weak*-cluster point of the x_n and that, along the same filter on \mathbb{N} , $u + u_s$ is a weak*-cluster point of the t_n (with $x, u \in X$, $x_s, u_s \in X_s$). Let finally $x^*, u^* \in X^*$ be normalized elements.*

Then there is a sequence (n_k) in \mathbb{N} and there are two wuC-series $\sum x_k^$ and $\sum u_k^*$ in X^* such that*

$$t_{n_k}(x_k^*) = 0 \quad \text{for all } k \in \mathbb{N}, \quad (0.1)$$

$$\lim_k x_k^*(x_{n_k}) = x_s(x^*), \quad (0.2)$$

$$\lim_k t_{n_k}(u_k^*) = u^*(u), \quad (0.3)$$

$$u_k^*(x_{n_k}) = 0 \quad \text{for all } k \in \mathbb{N}. \quad (0.4)$$

Proof. Let $1 > \varepsilon > 0$ and let (ε_j) be a sequence of numbers decreasing to zero such that $0 < \varepsilon_j < 1$ and $\prod_{j=1}^\infty (1 + \varepsilon_j) < 1 + \varepsilon$.

By induction over $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we shall construct four sequences $(x_k^*)_{k \in \mathbb{N}_0}$, $(y_k^*)_{k \in \mathbb{N}_0}$, $(u_k^*)_{k \in \mathbb{N}_0}$ and $(v_k^*)_{k \in \mathbb{N}_0}$ in X^* (of which the first members x_0^* , y_0^* , u_0^* , and v_0^* are auxiliary elements used only for the induction) and an increasing sequence (n_k) of indices such that, for all (real or complex) scalars α_j

and with $\beta = x_s(x^*)$, $\gamma = u^*(u)$, the following conditions hold for all $k \in \mathbb{N}_0$:

$$x_0^* = 0, \quad \|y_0^*\| = 1, \quad (0.5)$$

$$u_0^* = 0, \quad \|v_0^*\| = 1, \quad (0.6)$$

$$\left\| \alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^* \right\| \leq \left(\prod_{j=1}^k (1 + \varepsilon_j) \right) \max_{0 \leq j \leq k} |\alpha_j|, \quad \text{if } k \geq 1, \quad (0.7)$$

$$\left\| \alpha_0 v_k^* + \sum_{j=1}^k \alpha_j u_j^* \right\| \leq \left(\prod_{j=1}^k (1 + \varepsilon_j) \right) \max_{0 \leq j \leq k} |\alpha_j|, \quad \text{if } k \geq 1, \quad (0.8)$$

$$t_{n_k}(x_k^*) = 0, \quad (0.9)$$

$$u_k^*(x_{n_k}) = 0, \quad (0.10)$$

$$y_k^*(x) = 0, \quad \text{and} \quad x_s(y_k^*) = \beta, \quad (0.11)$$

$$u_s(v_k^*) = 0, \quad \text{and} \quad v_k^*(u) = \gamma, \quad (0.12)$$

$$|x_k^*(x_{n_k}) - \beta| < \varepsilon_k, \quad \text{if } k \geq 1, \quad (0.13)$$

$$|t_{n_k}(u_k^*) - \gamma| < \varepsilon_k, \quad \text{if } k \geq 1. \quad (0.14)$$

We set $n_0 = 1$, $x_0^* = 0$, $y_0^* = x^*$, $u_0^* = 0$ and $v_0^* = u^*$.

For the following it is useful to recall some properties of P : The restriction of P^* to X^* is an isometric isomorphism from X^* onto X_s^\perp with $(P^*y^*)_X = y^*$ for all $y^* \in X^*$, Q is a contractive projection and $X^{***} = X_s^\perp \oplus_\infty X^\perp$ (where X^\perp is the annihilator of X in X^{***}).

For the induction step suppose now that x_0^*, \dots, x_k^* , y_0^*, \dots, y_k^* , u_0^*, \dots, u_k^* , v_0^*, \dots, v_k^* and n_0, \dots, n_k have been constructed and satisfy conditions (0.5) - (0.14). Since $x + x_s$ is a weak*-cluster point of the x_n and $u + u_s$ is a weak*-cluster point of the t_n along the same filter there is an index n_{k+1} such that

$$|x_s(y_k^*) - y_k^*(x_{n_{k+1}} - x)| < \varepsilon_{k+1}, \quad (0.15)$$

$$|t_{n_{k+1}}(v_k^*) - (u + u_s)(v_k^*)| < \varepsilon_{k+1}, \quad (0.16)$$

Put

$$\begin{aligned} E &= \text{lin}(\{x^*, x_0^*, \dots, x_k^*, y_k^*, P^*x_0^*, \dots, P^*x_k^*, P^*y_k^*, \\ &\quad u^*, u_0^*, \dots, u_k^*, v_k^*, P^*u_0^*, \dots, P^*u_k^*, P^*v_k^*\}) \subset X^{***}, \\ F &= \text{lin}(\{x_{n_{k+1}}, t_{n_{k+1}}, x, x_s, u, u_s\}) \subset X^{**}. \end{aligned}$$

Clearly $Q^*x_j^*$, $Q^*y_k^*$, $Q^*u_j^*$, $Q^*v_k^* \in E$ for $0 \leq j \leq k$. By the principle of local reflexivity there is an operator $R : E \rightarrow X^*$ such that

$$\|Re^{***}\| \leq (1 + \varepsilon_{k+1})\|e^{***}\|, \quad (0.17)$$

$$f^{**}(Re^{***}) = e^{***}(f^{**}), \quad (0.18)$$

$$R|_{E \cap X^*} = \text{id}_{E \cap X^*} \quad (0.19)$$

for all $e^{***} \in E$ and $f^{**} \in F$.

We define

$$x_{k+1}^* = RP^*y_k^* \quad \text{and} \quad y_{k+1}^* = RQ^*y_k^*, \quad (0.20)$$

$$u_{k+1}^* = RQ^*v_k^* \quad \text{and} \quad v_{k+1}^* = RP^*v_k^*. \quad (0.21)$$

In the following we use the convention $\sum_{j=1}^0(\cdots) = 0$. Then we have that

$$\begin{aligned} \alpha_0 y_{k+1}^* + \sum_{j=1}^{k+1} \alpha_j x_j^* &= R \left(Q^* (\alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^*) + P^* (\alpha_{k+1} y_k^* + \sum_{j=1}^k \alpha_j x_j^*) \right), \\ \alpha_0 v_{k+1}^* + \sum_{j=1}^{k+1} \alpha_j u_j^* &= R \left(P^* (\alpha_0 v_k^* + \sum_{j=1}^k \alpha_j u_j^*) + Q^* (\alpha_{k+1} v_k^* + \sum_{j=1}^k \alpha_j u_j^*) \right). \end{aligned}$$

Now (0.7) (for $k+1$ instead of k) can be seen as follows:

$$\begin{aligned} &\left\| \alpha_0 y_{k+1}^* + \sum_{j=1}^{k+1} \alpha_j x_j^* \right\| \leq \\ &\stackrel{(0.17)}{\leq} (1 + \varepsilon_{k+1}) \left\| Q^* (\alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^*) + P^* (\alpha_{k+1} y_k^* + \sum_{j=1}^k \alpha_j x_j^*) \right\| \\ &= (1 + \varepsilon_{k+1}) \max \left\{ \left\| Q^* (\alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^*) \right\|, \left\| P^* (\alpha_{k+1} y_k^* + \sum_{j=1}^k \alpha_j x_j^*) \right\| \right\} \\ &\leq (1 + \varepsilon_{k+1}) \max \left\{ \left\| \alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^* \right\|, \left\| \alpha_{k+1} y_k^* + \sum_{j=1}^k \alpha_j x_j^* \right\| \right\} \\ &\leq \left(\prod_{j=1}^{k+1} (1 + \varepsilon_j) \right) \max \left\{ \max_{0 \leq j \leq k} |\alpha_j|, \max_{1 \leq j \leq k+1} |\alpha_j| \right\} \\ &= \left(\prod_{j=1}^{k+1} (1 + \varepsilon_j) \right) \max_{0 \leq j \leq k+1} |\alpha_j| \end{aligned}$$

where the last inequality comes from (0.5) if $k = 0$, and from (0.7), if $k \geq 1$.

Likewise, (0.8) (for $k+1$ instead of k) is proved.

The conditions (0.9) and (0.11) (for $k+1$ instead of k) are easy to verify because $Pt_{n_{k+1}} = 0$, $Qx = 0$ and $Qx_s = x_s$ thus, by (0.18)

$$\begin{aligned} t_{n_{k+1}}(x_{k+1}^*) &= Pt_{n_{k+1}}(y_k^*) = 0, \\ y_{k+1}^*(x) &= Qx(y_k^*) = 0 \quad \text{and} \quad x_s(y_{k+1}^*) = Q^*y_k^*(x_s) = x_s(y_k^*) = \beta. \end{aligned}$$

In a similar way we obtain (0.10) and (0.12) (for $k+1$ instead of k) by $u_{k+1}^*(x_{n_{k+1}}) = RQ^*v_k^*(x_{n_{k+1}}) = Qx_{n_{k+1}}(v_k^*) = 0$, $u_s(v_{k+1}^*) = Pu_s(v_k^*) = 0$ and $v_{k+1}^*(u) = v_k^*(u) = \gamma$.

Finally, we have

$$x_{k+1}^*(x_{n_{k+1}}) - \beta = y_k^*(x_{n_{k+1}}) - \beta = y_k^*(x_{n_{k+1}} - x) - x_s(y_k^*)$$

by (0.11) whence (0.13) for $k + 1$ by (0.15). Analogously, we get (0.14) for $k + 1$ via (0.16) and $t_{n_{k+1}}(u_{k+1}^*) = t_{n_{k+1}}(v_k^*)$ and $(u + u_s)(v_k^*) = \gamma$ by (0.12).

This ends the induction and the lemma follows immediately. \square

Corollary 0.3. *The complementary space X_s of an L-embedded Banach space X is weak*-sequentially closed.*

Proof. Suppose that (s_n) is a sequence in X_s that weak*-converges to $v + v_s$. Let $u^* \in X^*$ be normalized, set $t_n = s_n - v_s$. We apply the lemma to (t_n) with $u = v$, $u_s = 0$ and $x_n = u$ and define a sequence (μ_n) of finitely additive measures on the subsets of \mathbb{N} by $\mu_n(A) = (t_n - u)(\sum_{k \in A} u_k^*)$ for all $A \subset \mathbb{N}$ where $\sum_{k \in A} u_k^* \in X^*$ is to be understood in the weak*-topology of X^* and where the u_k^* are given by the lemma. Then $\mu_n(A) \rightarrow 0$ for all $A \subset \mathbb{N}$ and by Phillips' original lemma we get

$$|t_{n_k}(u_k^*)| \stackrel{(0.4)}{=} |(t_{n_k} - u)(u_k^*)| \leq \sum_j |(t_{n_k} - u)(u_j^*)| = \sum_j |\mu_{n_k}(\{j\})| \rightarrow 0.$$

Thus $u^*(u) = 0$ by (0.3) and $u = 0$ because u^* was arbitrary in the unit sphere of X^* . Hence (t_{n_k}) weak*-converges to 0 which is enough to see that (s_n) weak*-converges to v_s in X_s . \square

Proof. Proof of the theorem: Let X be an L-embedded Banach space with L-projection P . Suppose that the sequence (x_n^{**}) is weak*-null and that $x_n^{**} = x_n + t_n$ with $x_n = Px_n^{**}$. Let x^* be a normalized element of X . The sequence (x_n) is bounded and admits a weak*-cluster point $x + x_s$. We use the lemma, this time with the wuC-series $\sum x_k^*$, like in the proof of the corollary and define a sequence (μ_n) of finitely additive measures on the subsets of \mathbb{N} by $\mu_n(A) = x_n^{**}(\sum_{k \in A} x_k^*)$ for all $A \subset \mathbb{N}$. Then $\mu_n(A) \rightarrow 0$ for all $A \subset \mathbb{N}$ and by (0.1) and Phillips' original lemma we get

$$|x_k^*(x_{n_k})| = |x_{n_k}^{**}(x_k^*)| \leq \sum_j |x_{n_k}^{**}(x_j^*)| = \sum_j |\mu_{n_k}(\{j\})| \rightarrow 0.$$

Thus $x_s(x^*) = 0$ by (0.2) and $x_s = 0$ because x^* was arbitrary in the unit sphere of X^* . It follows that each weak*-cluster point of the set consisting of the x_n lies in X . Hence this set is relatively weakly sequentially compact by the theorem of Eberlein-Šmulian. If x is the limit of a weakly convergent sequence (x_{n_m}) then (t_{n_m}) weak*-converges to $-x$. Hence $x = 0$ by the corollary. This shows that the sequence (x_n) is weakly null and proves the theorem. \square

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ANR-06-BLAN-0015

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