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► **To cite this version:**

G. Leugering, J. Sokolowski. Topological derivatives for networks of elastic strings. *Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 2011, 91 (12), pp.926-943. 10.1002/zamm.201000067 . hal-00463693

**HAL Id: hal-00463693**

**<https://hal.science/hal-00463693>**

Submitted on 14 Mar 2010

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# TOPOLOGICAL DERIVATIVES FOR NETWORKS OF ELASTIC STRINGS

G. LEUGERING AND J. SOKOLOWSKI

ABSTRACT. We consider second order problems on metric graphs under given boundary and nodal conditions. We consider the problem of changing the topology of the underlying graph in that we replace a multiple node by an imported subgraph, or, in reverse, concentrate a subgraph to a single node or delete or add edges, respectively. We wish to do so in some optimal fashion. More precisely, given a cost function we may look at such operations in order to find an optimal topology of the graph. Thus, finally we are looking into the topological gradient of an elliptic problem on a graph.

## 1. INTRODUCTION

Ordinary or time-dependent partial differential equations on metric graphs are a subject of great importance in various applications. In contrast to such objects on discrete graphs, see e.g. [21], where only node-to-node relations and consequently discrete linear and nonlinear partial difference equations are considered, on a metric graph we consider a material variable  $x$  along each individual edge, such that we can introduce local equations, i.e. differential equations, possibly with varying coefficients along edges, which are then to be coupled at the inner vertices. Such couplings, as seen below, depend on the local equations, and more importantly, can be parametrized by generalized Kirchhoff conditions. To this end, one can introduce Sturm-Liouville type operators on metric graphs and discuss the problem of characterizing self-adjoint nodal conditions. It is interesting to observe that the development of differential equations on metric graphs has taken place in parallel, for mechanical structures (see e.g. [15]) and, more recently, for quantum graphs (see e.g. [23]). While in mechanical networks typically vectorial quantities are to be considered, quantum graph problems are genuinely scalar. The problem for scalar equations, in particular motivated by quantum graphs, has been solved by Kostykin and Schrader [23]. For mechanical systems, typically the classical Kirchhoff condition has been used (see Lagnese et.al. [15], Lagnese and Leugering [16]). In this paper we extend the theory to vectorial equations, that, in the context of mechanics, account for longitudinal, vertical and lateral motion along an edge - even twist and shearing can be included when dealing with Timoshenko-beams along edges.

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1991 *Mathematics Subject Classification.* 34B20, 34E05, 34L25, 34E40, 47B20, 81Q10.

*Key words and phrases.* Differential equations on metric graphs, Steklov-Poincaé operators on graphs, self-adjoint nodal conditions, topological sensitivities.

We gratefully acknowledge the support of the DAAD PROCOPE program and the DFG-cluster of excellence 'Engineering of advanced materials'.

Moreover, the optimization of the topology of the metric graph 'carrying a process' is of paramount interest in many applications.

Edgewise linear problems on graphs in that context reduce to truss-like structures and the topology optimization thereof has been of vital interest in structural mechanics (cf. e.g. Rozvany and Kocvara and Zowe [26, 11] as well as Mroz and Bojczuk [19] where only longitudinal strains are transmitted). Also in the context of flow problems in branching vascular and irrigational systems ( cf. e.g. Durand [8] for flow in branching vascular systems, de Wolf [7] for flow in gas pipe-networks, and Brenot et.al. [5] for irrigational networks ), problems of optimal topology have been analyzed in recent years. See Klarbring et.al. [10]

Locally varying problems, as discussed here, relate to elasticity problems on metric graphs. We leave the corresponding treatment of beams and, hence, frame-structures to a forthcoming publication. Such elastic 3-D grid-structures, where a thickness parameter and local stiffness is involved, can be taken as being representative of material structures like ceramic, polymeric or metallic foams, but also they can be taken in the context of carbon nano-tubes as the mechanical structure carrying other processes, like light-transmission. It is well-known, see e.g. the work of Kuchment [12, 13], Post [25], Exner and Post [9] and many others, that the topology of such metric graphs plays an important role in the study of their physical properties. In particular, the spectrum of such an operator on a metric graph is largely determined by the underlying discrete graph structure. See the work of von Below [28], see also Nicaise [22]. The occurrence of so-called band-gaps is thus crucially related to the spectrum of the incidence matrix of the graph (see definition below). Thus, changing the incidence matrix at a given multiple node results in changing the spectrum. Also the inverse problem of determining the physical properties of an underlined metric graph problem including its topology has been of much interest recently. See Belishev [3], Belishev and Vakulenko [4], Avdonin and Kurasov [1] and Avdonin, Leugering and Mykhailov [2]. In these articles the Steklov-Poincaré map plays a key role.

In material sciences one asks the reverse question: how can we achieve a certain desired optimal band-gap structure in changing the topology of the graph. Moreover, in the context of meta-materials one asks for auxetic structures, such that the global bulk material, a homogenized one, has, say, a negative Poisson ratio. Questions of this sort necessitate a variational theory of topology changes, a topological sensitivity analysis of differential operators on metric graphs. A first paper in this direction is Leugering and Sokolowski [18].

We need to introduce some notation. We consider a simple graph  $(V, E) = G$  in  $\mathbf{R}^d$ ,  $d = 2, 3$ , with vertices  $V = \{v_J | J \in \mathcal{J}\}$  and edges  $E = \{e_i | i \in \mathcal{I}\}$ . Let  $m = |\mathcal{J}|$ ,  $n = |\mathcal{I}|$  be the numbers of vertices and edges, respectively.

In general the edge-set may be a collection of smooth curves in  $\mathbf{R}^2$ , parametrized by their arc lengths. The restriction to *straight edges* is for the sake of simplicity only. The more general case, which is of course also interesting in the combination of shape and topology optimization, can also be handled. However this is beyond these notes.

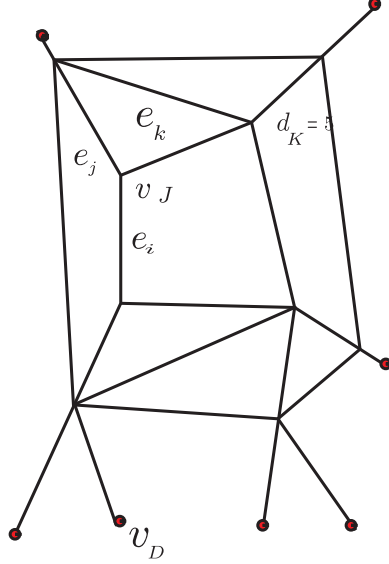


FIGURE 1. A typical graph

We associate to the edge  $e_i$  the unit vector  $\mathbf{e}_i$  aligned along the edge.  $(\mathbf{e}_i^\perp)^1, (\mathbf{e}_i^\perp)^2$  denote the orthogonal unit vectors. In the planar case we only have  $\mathbf{e}_i^\perp$ . Given a node  $v_J$  we define

$$\mathcal{I}_J := \{i \in \mathcal{I} | e_i \text{ is incident at } v_J\}$$

the incidence set, and  $d_J = |\mathcal{I}_J|$  the edge degree of  $v_J$ . The set of nodes splits into simple nodes  $\mathcal{J}_S$  and multiple nodes  $\mathcal{J}_M$  according to  $d_J = 1$  and  $d_J > 1$ , respectively. On  $G$  we consider a vector-valued function  $r$  representative of the displacement of the network (see Figure 2)

$$(1.1) \quad r : G \rightarrow \mathbf{R}^{np} := \prod_{i \in \mathcal{I}} \mathbf{R}^{p_i}, \quad p_i \geq 1 \forall i \in \mathcal{I}.$$

The numbers  $p_i$  represent the degrees of freedom of the physical model used to describe the behavior of the edge with number  $i$ . For instance,  $p = 1$  is representative of a heat problem, whereas  $p = 2, 3$  is used in an elasticity context on graphs in 2 or 3 dimensions. The  $p_i$ 's may change in the network in principle. However, in this paper we insist on  $p_i = p = 2, \forall i$  in the planar case. See Lagnese, Leugering and Schmidt [15] and Lagnese and Leugering [16] for details on the modeling.

Once the function  $r$  is understood as being representative of, say, a deformation of the graph, we may localize it to the edges

$$(1.2) \quad r_i := r|_{e_i} : [\alpha_i, \beta_i] \rightarrow \mathbf{R}^p, \quad i \in \mathcal{I},$$

where  $e_i$  is parametrized by  $x \in [\alpha_i, \beta_i] =: I_i, 0 \leq \alpha_i < \beta_i, \ell_i := \beta_i - \alpha_i$ . See Figure 2

We introduce the incidence relation

$$d_{iJ} := \begin{cases} 1 & \text{if } e_i \text{ ends at } v_J \\ -1 & \text{if } e_i \text{ starts at } v_J. \end{cases}$$

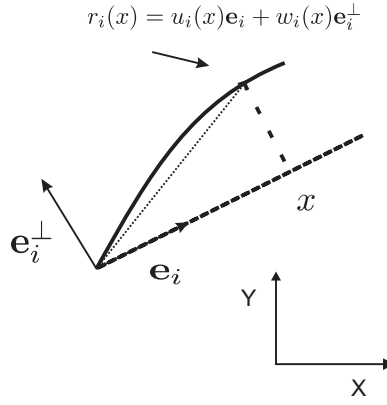


FIGURE 2. Representation of planar displacement

Accordingly, we define

$$x_{iJ} := \begin{cases} 0 & \text{if } d_{iJ} = -1 \\ \ell_i & \text{if } d_{iJ} = 1. \end{cases}$$

We will use the notation  $r_i(v_J)$  instead of  $r_i(x_{iJ})$ . In order to represent the material considered on the graph, we introduce stiffness matrices

$$(1.3) \quad K_i = E_i \text{diag}(\kappa_i, \kappa_i^{\perp 1}, \kappa_i^{\perp 2}) E_i^T,$$

$$(1.4) \quad E_i = (\mathbf{e}_i, \mathbf{e}_i^{\perp 1}, \mathbf{e}_i^{\perp 2}) \in \mathbf{R}^{d,d}.$$

Obviously, the longitudinal stiffness is given by  $\kappa_i$ , whereas the transverse and lateral stiffness is given by  $\kappa_i^{\perp 1}, \kappa_i^{\perp 2}$ . This can be related to 1-d analogs of the Lamé parameters. In general, the stiffness parameters can vary along the edge. We introduce Dirichlet and Neumann simple nodes as follows. As the displacements and, consequently, the forces are vectorial quantities, we may consider nodes, where the longitudinal (or tangential) displacement or forces are kept zero, while the transverse displacements or forces are not constrained, and the other way round. We thus define

$$\mathcal{J}_D^t := \{J \in \mathcal{J}_S \mid r_i(v_J) \cdot \mathbf{e}_i = 0\},$$

$$\mathcal{J}_D^n := \{J \in \mathcal{J}_S \mid r_i(v_J) \cdot \mathbf{e}_i^{\perp} = 0\},$$

$$\mathcal{J}_N^t := \{J \in \mathcal{J}_S \mid d_{iJ} K_i r_i'(v_J) \cdot \mathbf{e}_i = 0\},$$

$$\mathcal{J}_N^n := \{J \in \mathcal{J}_S \mid d_{iJ} K_i r_i'(v_J) \cdot \mathbf{e}_i^{\perp} = 0\},$$

for the planar situation, the 3-d case being completely analogous. Notice that these sets are not necessarily disjoint. Obviously, the set of completely clamped vertices can be expressed as

$$(1.5) \quad \mathcal{J}_D^0 := \mathcal{J}_D^t \cap \mathcal{J}_D^n$$

Similarly, a vertex with completely homogenous Neumann conditions is expressed as  $\mathcal{J}_N^n \cap \mathcal{J}_N^t$ . At tangential Dirichlet nodes in  $\mathcal{J}_D^t$  we may, however,

consider normal Neumann-conditions as in  $\mathcal{J}_N^n$  and so on. The system of equations governing the full transient motion is then given by

$$(1.6) \quad \left\{ \begin{array}{l} \rho \ddot{r}^i - (K_i(r^i))' + c_i \dot{r}^i = f^i \quad \in (0, \ell_i) \\ r^i(v_D) = u_D, \quad i \in \mathcal{I}_D, D \in \mathcal{J}_D \\ d_{iJ}(r^i)'(v_N) = g_J \quad i \in \mathcal{I}_N, N \in \mathcal{J}_N \\ r^i(v_J) = r^j(v_J) \quad i, j \in \mathcal{I}_J, J \in \mathcal{J}_M \\ \sum_{i \in \mathcal{I}_J} d_{iJ}(K_i(r^i))'(v_J) = 0 \quad J \in \mathcal{J}_M \\ r^i(\cdot, 0) = r_0^i, \quad r_t^i(\cdot, 0) = r_1^i, \end{array} \right. ,$$

where the dot signifies a time-derivative and the prime a spatial derivative. In this representation we used capital letters for vertices in order to improve the readability of the formulae. It is important to understand the coupling conditions (1.6)<sub>4,5</sub>. Indeed, the first of these conditions simply expresses the continuity of displacements across the vertex  $v_J$ . Without this condition the network falls apart. The second condition reflects the physical law that the forces at the vertex  $v_J$ , in the absence of additional external forces acting on  $v_J$ , should add up to zero. Notice that the coupling at multiple nodes  $v_J$ , those where  $|I_J| > 1$ , is a vectorial equation. This is in contrast to the out-of-the-plane model, where no such vectorial couplings occur which, in turn, makes the problem then independent of the particular geometry. In the case treated here the geometry, represented by the triple  $(e_i, e_i^{\perp 1}, e^{\perp 2}$  does play a crucial role. See Figure 3

In this paper we consider the time-invariant case with constant coefficients, obtained from (1.6) using time-harmonics. We will also put  $c_i = 0$ , thus we do not consider an elastic coupling to the environment. Then we obtain the classical Helmholtz problem locally on each edge together with nodal conditions as above.

## 2. SELF-ADJOINT OPERATORS ON METRIC GRAPHS IN $\mathbf{R}^d$

Using time-harmonics, i.e.  $r^i(t, x) = e^{i\omega t} r_i(x)$ ,  $i \in \mathcal{I}$ , we can transform (1.6) into the following Helmholtz-problem on the metric graph  $G$ .

$$(2.7) \quad \left\{ \begin{array}{l} K_i r_i'' + \omega^2 r_i = f_i \quad \text{on } (0, \ell_i) \\ A_J^D r_i(v_J) + B_J^N r_i'(v_J) = g_J \quad i \in I_J, J \in \mathcal{J}_S \\ \sum_{i \in I_J} d_{iJ} K_i r_i'(v_J) = g_J \quad J \in \mathcal{J}_M \\ r_i(v_J) = r_j(v_J) \quad \forall i, j \in I_J. \end{array} \right. .$$

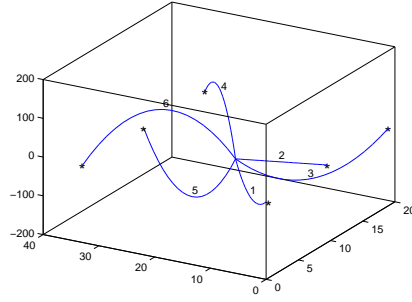
We define

$$A_J^D, B_J^N \in \mathbf{R}^{d,d}, \quad \text{rank} [A_J^D, B_J^N] = d, \\ \text{rg}(A_J^D) \perp \text{rg}(B_J^N).$$

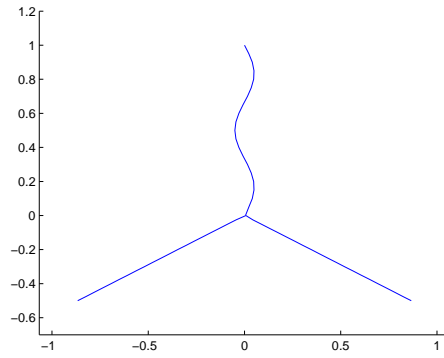
At a multiple node  $v_J$ ,  $J \in \mathcal{J}_M$  we may introduce matrices

$$(2.8) \quad A_J, B_J \in \mathbf{R}^{d d_J \times d d_J}$$

such that



(a) 3-d displacement



(b) Planar displacement

FIGURE 3. Star-graph

$$(2.9) \quad \text{rank} [A_J, B_J] = d d_J.$$

If we introduce  $r^J, Kr^{J'}$  as

$$(2.10) \quad r^J = \left( r_{i_1}(v_J), \dots, r_{i_{d_J}}(v_J) \right)^T,$$

$$(2.11) \quad Kr^{J'} = \left( d_{i_1} K_{i_1} r'_{i_1}(v_J), \dots, d_{i_{d_J}} K_{i_{d_J}} r'_{i_{d_J}}(v_J) \right)^T.$$

We may express the multiple-node conditions (2.7)<sub>3,4</sub> as

$$(2.12) \quad A_J r^J + B_J Kr^{J'} = G_J, \quad J \in J_M.$$

In particular,

$$(2.13) \quad A_J = \begin{pmatrix} I_d & -I_d & & & \\ I_d & 0 & -I_d & & \\ \vdots & \vdots & \ddots & \ddots & \\ I_d & 0 & \cdots & 0 & -I_d \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$(2.14) \quad B_J = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ I_d & I_d & \cdots & I_d & I_d \end{pmatrix},$$

$$(2.15) \quad G_J = (0 \dots 0, g_J)^T.$$

At a simple node one may likewise introduce

$$(2.16) \quad \begin{cases} r^S & = r_i(v_S) & S \in J_S & i \in I_S \\ Kr^{S'} & = d_{iS} K_i r'_i(v_S) & S \in J_S & i \in I_S, \end{cases},$$

where, however,  $|I_S| = d_S = 1$ , and  $B_J^D =: B_S, A_J^D =: A_S, g_J = g_S$ . Then, also the boundary condition (2.7)<sub>2</sub> of simple nodes can be expressed as

$$(2.17) \quad A_S r^S + B_S K r^{S'} = g_S \quad s \in J_S.$$

The rationale behind this notation becomes clear, if one considers the corresponding Lagrange-identities for the operator

$$(2.18) \quad \mathcal{A}^{\max} r := (K_i r_i'' + \omega^2 r_i)_{i \in I} =: \mathcal{A}_0 r$$

on

$$(2.19) \quad \mathcal{D}^{\max} = \mathcal{H}^0 = \prod_{i=1}^N L_2(0, \ell_i)^d, \quad N = |N|.$$

We take  $\mathcal{A}_0$  as the differential expression, rather than the operator. We introduce the bilinear form

$$(r, w)_1 := \sum_{i=1}^N \int_0^{\ell_i} K_i r'_i \cdot w'_i \, dx$$

for all  $r, w \in \mathcal{H}^1 := \prod_{i=1}^N H^1(0, \ell_i)^d$ . In order to perform integration by parts,

we consider  $r, w \in \mathcal{H}^2 := \prod_{i=1}^N H^2(0, \ell_i)^d$ . Then

$$(r, w)_1 = \sum_{i=1}^N \int_0^{\ell_i} K_i r'_i \cdot w'_i \, dx = \sum_{J \in \mathcal{J}} \sum_{i \in I_J} d_{iJ} K_i'(v_J) w_i(v_J) - \sum_{i=1}^N \int_0^{\ell_i} K_i r_i'' w_i \, dx$$



and

$$\begin{aligned} \sum_{i=1}^N \int_0^{\ell_i} -K_i r_i'' w_i dx &= - \sum_{J \in \mathcal{J}} \sum_{i \in I_J} d_{iJ} \cdot K_i r_i'(v_J) w_i(v_J) \\ &+ \sum_{J \in \mathcal{J}} \sum_{i \in I_J} r_i(v_J) d_{iJ} K_i w_i'(v_J) \\ &+ \sum_{i=1}^N \int_0^{\ell_i} r_i(-K_i w_i'') dx. \end{aligned}$$

We define the symplectic form for each node  $v_J$

$$(2.20) \quad \langle r, w \rangle_J := \sum_{i \in I_J} r_i(v_J) d_{iJ} K_i w_i'(v_J) - \sum_{i \in I_J} d_{iJ} K_i r_i'(v_J) w_i(v_J).$$

This can be represented as

$$\begin{aligned} \mathcal{I}_J &= \begin{pmatrix} 0 & I_{dd_J} \\ -I_{dd_J} & 0 \end{pmatrix} \\ \langle r, w \rangle_J &= \langle (r^J, K r^{J'})^T, J_J(w^J, K w^{J'}) \rangle \\ &= \left\langle \begin{pmatrix} r^J \\ K r^{J'} \end{pmatrix}, \begin{pmatrix} 0 & I_{3d_J} \\ -I_{3d_J} & 0 \end{pmatrix} \begin{pmatrix} w^J \\ K w^{J'} \end{pmatrix} \right\rangle \\ &= \langle r^J, K w^{J'} \rangle - \langle K r^{J'}, w^J \rangle \\ &= \sum_{i \in I_J} r_i(v_J) d_{iJ} K_i w_i'(v_J) - \sum_{i \in I_J} d_{iJ} K_i r_i'(v_J) w_i(v_J). \end{aligned}$$

We, thus, have the identity

$$(2.21) \quad (\mathcal{A}^{\max} r, w)_0 = \langle r, w \rangle_J + (r, \mathcal{A}^{\max} w)_0,$$

where

$$(r, w)_0 = \sum_{i=1}^N \int_0^{\ell_i} r w dx \quad \forall r, w \in \mathcal{H}^0.$$

If we define  $\mathcal{D}_\infty = \prod_{i=1}^N C_0^\infty(0, \ell_i)$ , then obviously the *minimal operator*

$$(2.22) \quad \mathcal{A}^0, \mathcal{D}(\mathcal{A}^0) := \mathcal{D}_\infty$$

is symmetric i.e.

$$(2.23) \quad (\mathcal{A}^0 r, w)_0 = (r, \mathcal{A}^0 w)_0, \quad \forall r \in \mathcal{D}_\infty$$

**Definition 2.1.** We define the maximal and the minimal operator on the metric graph  $G$  by (2.18), (2.22), respectively. Let the operator  $\mathcal{A}_{A,B}$  between the minimal and the maximal operator on  $G$  be defined as follows.

$$(2.24) \quad \mathcal{A}_{A,B} r = (-K_i r_i'')_{i=1}^N, \quad \mathcal{D}(\mathcal{A}_{A,B}) = \mathcal{D}_{A,B}$$

$$(2.25) \quad \mathcal{D}_{A,B} := \left\{ r \in \mathcal{H}^2 \mid \begin{array}{l} A_S r_S + B_S K r'_S = 0, \quad S \in J_S \\ A_J r_J + B_J K r'_J = 0, \quad J \in J_M \end{array} \right\}.$$

**Theorem 2.1.** *Let the operator  $\mathcal{A}_{A,B}$  be given by Definition 2.1. Let  $A_J B_J^*$  be self-adjoint and  $\text{rank}[A_J, B_J] = dd_J$ . Then  $\mathcal{A}_{A,B}$  is a self-adjoint extension of  $\mathcal{A}^0$  and all self-adjoint extension of  $\mathcal{A}^0$  are parametrized by such matrices.*

*Proof.* Let

$$(2.26) \quad \begin{cases} \begin{pmatrix} r_J \\ K_J r'_J \end{pmatrix} = \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \phi \\ \begin{pmatrix} w_J \\ K_J w'_J \end{pmatrix} = \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \psi \end{cases}$$

with  $\phi, \psi \in \mathbf{R}^{dd_J}$ . Then

$$\begin{aligned} \langle (r^J, K_J r^{J'})^T, J_J (w^J, K_J w^{J'})^T \rangle &= \left\langle \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \phi, \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \psi \right\rangle \\ &= \langle \phi, (A_J B_J^* - B_J A_J^*) \psi \rangle \\ &= 0 \end{aligned}$$

$$(2.27) \quad \text{iff } (A_J B_J^*)^* = A_J B_J^*.$$

Moreover, if

$$\begin{pmatrix} r^J \\ K r^{J'} \end{pmatrix} = J_J \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \phi$$

and

$$\begin{pmatrix} w^J \\ K w^{J'} \end{pmatrix} = J_J \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \psi,$$

then, according to  $J_J^2 = -I$ , we have

$$(2.28) \quad \begin{aligned} \left\langle \begin{pmatrix} r^J \\ K r^{J'} \end{pmatrix}, J_J \begin{pmatrix} w^J \\ K w^{J'} \end{pmatrix} \right\rangle &= \left\langle J_J \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \phi, J_J \left( J_J \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \psi \right) \right\rangle \\ &= - \langle J_J \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \phi, \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \psi \rangle \\ &= 0 \end{aligned}$$

according to (2.27). However,

$$(2.29) \quad (A_J, B_J) J_J \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \phi = (A_J B_J^* - B_J A_J^*) \phi = 0,$$

again according to (2.27).

We have a parametrization of elements of  $\ker(A_J, B_J)$ , namely:

$$(2.30) \quad z \in \ker(A_J, B_J) \text{ iff } z \in \text{rg}\left(J_J \begin{pmatrix} A_J^* \\ B_J \end{pmatrix}\right)$$

and (2.28) shows that

$$(2.31) \quad \langle r, w \rangle_J = 0 \quad \forall r, w \in \ker(A_J, B_J).$$

□

We assume in the sequel (2.27).

As  $[A_J, B_J]$  is of maximal rank and  $A_J B_J^*$  is selfadjoint, the right-inverse

$$(2.32) \quad R_{AB} := \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \left[ (A_J, B_J) \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} \right]^{-1}$$

exists and  $[A_J, B_J] R_{A,B} z = z \quad \forall z \in \mathbf{R}^{dd_J}$ .

$$\text{But } [A_J B_J] \begin{pmatrix} A_J^* \\ B_J^* \end{pmatrix} = A_J A_J^* + B_J B_J^*.$$

Let  $P_J, \tilde{P}_J$  be the orthoprojectors onto  $\ker B_J$  and  $\ker B_J^*$ , respectively. Let  $Q_J, \tilde{Q}_J$  be the corresponding complementary operators, that map onto  $R = \text{rg}(B_J^*)$  and  $\tilde{R} = \text{rg}(B_J)$ , respectively.

$$(2.33) \quad \begin{aligned} I &= P_J \oplus Q_J &= \tilde{P}_J \oplus \tilde{Q}_J \\ \mathbf{R}^{dd_J} &= \ker B_J \oplus \text{rg}(B_J^*) &= \ker B_J^* \oplus \text{rg}(B_J). \end{aligned}$$

We consider the mappings

$$(2.34) \quad \tilde{B}_J := \tilde{Q}_J B_J Q_J : R \longrightarrow \tilde{R}, \quad C_J := \tilde{B}_J^{-1} A_J.$$

Then  $C_J$  is self-adjoint. Assume

$$(2.35) \quad P_J r_J = 0, \quad C_J Q_J r_J + Q_J K r_J' = 0$$

holds, then

$$\begin{aligned} \tilde{B}_J^{-1} (A_J r^J + B_J K r^{J'}) &= \\ \tilde{B}_J^{-1} A_J (Q_J + P_J) r^J + (Q_J + P_J) \tilde{B}_J^{-1} B_J K r^{J'} &= \\ C_J Q_J r^J + C_J P_J r^J + Q_J K r^{J'} &= 0. \end{aligned}$$

Hence

$$(2.36) \quad A_J r^J + B_J K r^{J'} = 0.$$

The reverse direction is also true.

**Corollary 2.1.** *Let  $\mathcal{A}_{A,B}$  be self-adjoint according to Theorem 2.1. Then there are operators  $P, Q, C$  given by (2.33), (2.34) such that (2.36) is equivalent to (2.35).*

We may then introduce the space

$$(2.37) \quad \mathcal{V} := \{r \in \mathcal{H}^1 \mid P_J r^J = 0 \ \forall J \in \mathcal{J}_M\}$$

and compute

$$\begin{aligned} (r, w)_1 &= \sum_{i=1}^N \int_0^{\ell_i} K_i r'_i w'_i \, dx \\ &= - \sum_{J \in \mathcal{J}} \langle C_J r^J, w^J \rangle + (\mathcal{A}_0 r, w)_0, \quad \forall w \in \mathcal{V}, \, r \text{ with (2.35)} \end{aligned}$$

We therefore define the following bilinear form on  $\mathcal{V} \times \mathcal{V}$ :

$$(2.38) \quad a(r, w) = \sum_{i=1}^N \int_0^{\ell_i} K_i r'_i \cdot w'_i \, dx + \sum_{J \in \mathcal{J}} \langle C_J r^J, w^J \rangle.$$

Then we have

$$a(r, w) = (\mathcal{A}^0 r, w) \quad \forall r \in \mathcal{D}_0, \, w \in \mathcal{V}.$$

**Example 2.1.**

$$A_J = \begin{pmatrix} I & -I & & & \\ \vdots & 0 & -I & & \\ I & \dots & \dots & -I & \\ 0 & & & & 0 \end{pmatrix} \quad B_J = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \\ I & & I \end{pmatrix}$$

$$\begin{aligned} \ker B_J &= \{\phi \mid \sum \phi_i = 0\} \\ \ker B_J^* &= \{\phi \mid \phi_{d_J} = 0\} \\ \text{rg } B_J^* &= \{cE \mid E_i = I, \, i = 1, \dots, d_J\} \\ \text{rg } B_J &= \{c(0, \dots, I)^T\} \end{aligned}$$

$$A_J A_J^* = \begin{pmatrix} 2I & I & \dots & I & 0 \\ I & 2I & \ddots & I & \\ I & \dots & & 2I & \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

$$B_J B_J^* = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & & 0 \\ 0 & \dots & 0 & d_J I \end{pmatrix}$$

$$A_J A_J^* + B_J B_J^* = \begin{pmatrix} I & & & & \\ & \ddots & & & \\ & & & & \\ & & & d_J I & \end{pmatrix} + \begin{pmatrix} I \\ \vdots \\ I \\ 0 \end{pmatrix} (I, I, \dots, I, 0)$$

Using the Sherman-Morrison-Woodbury formula we obtain

$$\begin{aligned}
(A_J A *_J + B_J B_* )^{-1} &= \begin{pmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & \frac{1}{d_J} I \end{pmatrix} - \frac{1}{d_J} \begin{pmatrix} I & \dots & I & 0 \\ \vdots & & & \\ I & \dots & I & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix} \\
\mathbf{R}_{AB}\phi &= \begin{pmatrix} I & \dots & I & 0 \\ -I & & & \\ & \ddots & -I & 0 \\ 0 & & 0 & I \\ & & \vdots & \vdots \\ 0 & & 0 & I \end{pmatrix} \begin{pmatrix} \phi_1 - \frac{1}{d_J} \sum_{i=1}^{d_J-1} \phi_i \\ \vdots \\ \phi_{d_J-1} - \frac{1}{d_J} \sum_{i=1}^{d_J-1} \phi_i \\ \frac{1}{d_J} \phi_{d_J} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^{d_J-1} \phi_i - \frac{d_J}{d_J} \sum_{i=1}^{d_J-1} \phi_i \\ - \left( \phi_2 - \frac{1}{d_J} \sum_{i=1}^{d_J-1} \phi_i \right) \\ - \left( \phi_{d_J-i} - \frac{1}{d_J} \sum_{i=1}^{d_J-1} \phi_i \right) \\ \frac{1}{d_J} \phi_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{d_J} \sum_{i=1}^{d_J-1} \phi_i \\ - \left( \phi_1 - \frac{1}{d_J} \sum_{i=1}^{d_J-1} \phi_i \right) \\ \vdots \\ - \left( \phi_{d_J-i} - \frac{1}{d_J} \sum_{i=1}^{d_J-1} \phi_i \right) \\ \frac{1}{d_J} \phi_{d_J} \\ \vdots \\ \frac{1}{d_J} \phi_{d_J} \end{pmatrix}
\end{aligned}$$

Now,  $P r^J = 0 \iff r^J \in \text{rg}(B_J^*)$

$$\iff r_{i_1}(v_J) = r_{i_2}(v_J) = \dots = r_{i_{d_J}}(v_J)$$

$$\implies A_J r^J = 0 \implies C_J r^J = 0$$

$$\implies a(r, w) = \sum_{i=1}^N \int_0^{\ell_i} K_i r'_i \cdot w'_i dx$$

As it ist well-known for this case

**Example 2.2.**

$$A_J = \begin{pmatrix} I & -I & & & \\ I & 0 & -I & & \\ \vdots & \vdots & \ddots & \ddots & \\ I & \vdots & 0 & \ddots & -I \\ -\alpha I & 0 & \dots & \dots & 0 \end{pmatrix} B_J = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \\ I & \dots & I \end{pmatrix}$$

$$P_J r^J = 0 \iff r_{i_1}(v_J) = \dots = r_{i_{d_J}}(v_J)$$

$$\begin{aligned}
 A_J r^J &= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\alpha r_{i_1}(v_J) \end{pmatrix} \\
 \tilde{Q}_J B_J Q_J x &= A_J r = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\alpha r_{i_1}(v_J) \end{pmatrix} \\
 B_J \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{d_J} \end{pmatrix} &= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{i=1}^{d_y} \phi_i \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\alpha r_{i_1}(v_J) \end{pmatrix} \\
 &= \sum_{i=1}^N \int_0^{\ell_i} K_i r'_i w'_i dx - \sum_{J \in \mathcal{J}} \frac{\alpha}{d_J} r_{i_1}(v_J) w_{i_1}(v_J) \\
 &= \sum_{J \in \mathcal{J}} \left( \sum_{J \in \mathcal{J}} d_{i_J} K_i r'_i(v_J) - (\alpha r_{i_1}(v_J)) \right) w_{i_1}(v_J) + (A^0 r, w)_0.
 \end{aligned}$$

### 3. SPECTRAL PROBLEM AND STEKLOV-POINCARÉ-MAPS FOR SUBGRAPHS

It is well-known that scalar self-adjoint problems on finite metric graphs with certain connectivity properties have a discrete spectrum, see e.g. von Below [28], Nicaise [22], Kuchment [13] and Post [25]. For vectorial self-adjoint problems, in particular for the classical continuity- and Kirchhoff-condition at multiple nodes, as in (2.7), see e.g. Lagnese, Leugering and Schmidt [15]. In particular, the spectrum consists of positive eigenvalues, if the graph satisfies a path-constraint. Essentially that constraint says that from each node there is a path to a Dirichlet node. As a result, the spectrum of a self-adjoint operator on a metric graph is strongly related to the spectrum of the adjacency matrix of the graph itself. This correspondence is most evident in the case of an equilateral homogenous metric graph, where equilateral indicates that all edges have the same length. From this it is clear that the spectrum is largely determined by the topology of the underlined discrete graph.

In reverse, it is therefore reasonable to ask in which way the topology of a graph should be changed in order to maximize (or minimize) some given merit (or cost) function that may or may not depend on the eigenvalues. In this section we provide a first sketch of the possibilities of inserting a subgraph into a given graph. For scalar problems on graphs this problem has been considered in a PhD thesis by Ong [24], where the procedure to insert a given graph at every node of the original graph has been termed graph-decoration. The idea there is to delete eigenvalues from the original graph which are related to those eigenvalues of the subgraphs 'decorating'

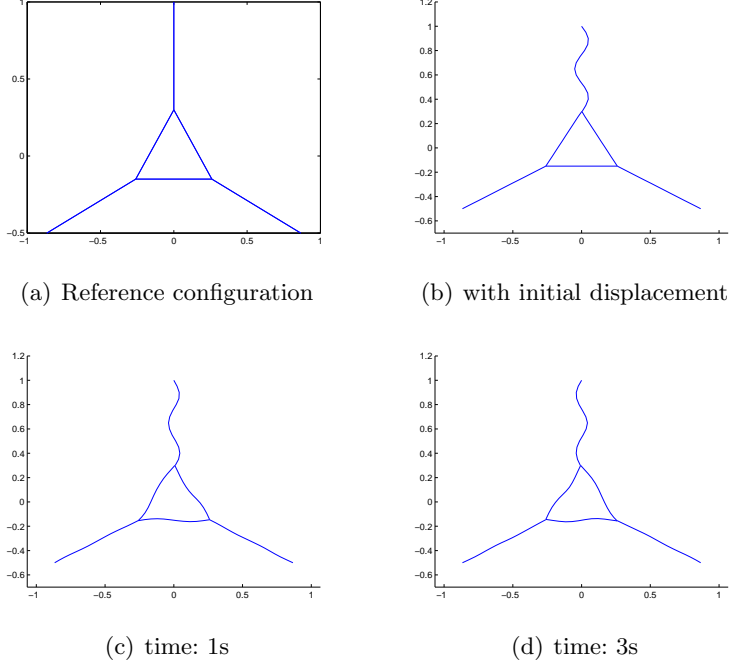


FIGURE 4. Star-graph with a 'hole' at its center

each multiple node. As always, we concentrate on a star-graph, as all other situations can be composed out of such 'cells'.

**Example 3.1.** *As for a homogenous situation, where we may also introduce a potential, we consider*

$$\begin{aligned}
 (3.39) \quad & -r_j'' + \gamma_j r_j = \lambda_j r_j \quad x \in (0, \ell_j) \\
 & r_j(v_D) = 0 \quad \forall j \in \mathcal{I}_D, \\
 & r_j(v_J) = r_k(v_J) \quad j, k \in \mathcal{I}_J, \\
 & \sum_{j \in \mathcal{I}_J} d_j r_j'(v_J) = 0 \quad J \in \mathcal{J}_M,
 \end{aligned}$$

See Figure 4

Even more specific, we consider a simple three-star with scalar displacement (for the sake of convenience only) and  $\gamma = 0$  we obtain the eigen-pairs  $\lambda^{k,\gamma}, \phi_j^{k,\gamma}$ :

$$\begin{aligned}
 1.) \quad & \lambda^{k,0} = \left(\frac{k\pi}{2}\right)^2, \quad \phi_j^{k,0} = \sqrt{\frac{2}{3}} \cos\left(\frac{k}{2}\pi x\right), \quad k \text{ odd}, j = 1 : 3 \\
 2.) \quad & \lambda^{k,0} = (k\pi)^2, \quad \phi_1^{k,0}(x) = \phi_2^{k,0}(x) = \frac{1}{\sqrt{3}} \sin(k\pi x), \\
 & \phi_3^{k,0}(x) = -\frac{2}{\sqrt{3}} \sin(k\pi x)
 \end{aligned}$$

Notice that 'individual' eigenvalues correspond to Dirichlet conditions on a given edge while 'structural' eigenvalues correspond to eigenvalues of the

incidence matrix. Spectral gaps according the structural eigenvalues may appear in general. See e.g. von Below[28], Nicaise [22] or Post[25] for a general treatment of (3.39) in the scalar case.

If  $\omega^2$  is not in the spectrum  $\sigma(\mathcal{A})$  of  $\mathcal{A}$ , then problem (2.7) admits a unique solution  $r$ . The proof is standard and, hence, is omitted. As the main concern in this paper is the investigation of a metric graph under perturbations, we proceed to consider a star-graph with the central node being inflated by a subgraph. In other words, let  $G_s = (V_s, E_s)$  be a subgraph of  $G$ . Furthermore, let  $\partial G_s$  denote the set of vertices in  $V_s$  adjacent to the nodes in  $G \setminus G_s$  in the remaining graph. For simplicity we assume that each vertex  $v_J \in \partial G_s$  has only one edge from  $G \setminus G_s$  attached to it, i.e.

$$(3.40) \quad \forall v_J \in \partial G_s \exists! i \in \mathcal{I}_{G \setminus G_s} : d_{iJ} \neq 0.$$

We consider the problem

$$(3.41) \quad \begin{cases} K_i \cdot r_i'' + \omega^2 r_i = f_i & (0, \ell_i) & i \in I_{G_s} \\ r_i(v_J) = \phi_J & i \in \mathcal{I}_J^s & v_J \in \partial G_s \\ A_J r^J + B_J K r^J = 0 & v_J \in \dot{V}_s, \end{cases}$$

**Definition 3.1.** Let  $r_i$  be the solution of (3.41)  $i \in \mathcal{I}_{G_s}$ . For  $v_J \in \partial G_s, i \in \mathcal{I}_J$  and for  $\phi = (\phi_J)_{\{J; v_J \in \partial G_s\}}$  we define the Steklov-Poincaré map by

$$(3.42) \quad \Lambda(\omega, G_s)\phi = \left( \sum_{i \in \mathcal{I}_J^s} d_{iJ} K_i r_i'(v_J) \right).$$

If we set  $n_0 := |\partial G_s|$  then we have exactly  $n_0$  edges from  $E \setminus E_s$  connected to  $v_J \in \partial G_s$ . Let these indices be relabeled as  $e_1, \dots, e_{n_0}$ . Then

$$(3.43) \quad \begin{cases} r_i(v_j) = \phi_J, & v_J \in \partial G_s, i \in \mathcal{I}_J^{G \setminus G_s} \\ d_{iJ} K_i r_i'(v_J) + (\Lambda(\omega, G_s)\phi)_J = 0 \end{cases}$$

or in short

$$(3.44) \quad d_{kJ_k} K_k r_k'(v_{J_k}) + \Lambda(\omega, G_s)(r_1(v_{J_1}) \dots r_{n_0}(v_{J_{n_0}})^T)_{J_k} = 0$$

$k = 1, \dots$ , is equivalent to

$$A_{J_k} r^{J_k} + B_{J_k} K r^{J_k} = 0 \quad k = 1, \dots, n_0$$

in the original formulation. Thus, loosely speaking, solving the problem with the subgraph included, as described above, is equivalent to solving the problem on the graph without the subgraph but with nodal condition (2.7)<sub>3</sub> replaced with boundary conditions (3.44). More precisely, we have:

**Theorem 3.1.** Let  $r$  be a solution of (2.7) and let  $v_J, J \in \mathcal{J}^M$  be a multiple node with edge degree  $n_0 = d_J$ . Resolve the node  $v_J$  into  $n_0$  simple nodes  $V_J := \{v_{J_1}, \dots, v_{J_{n_0}}\}$  such that the remaining graph  $G_r$  is given



by  $V(G_r) := V(G) \setminus \{v_J\} \cup V_J$ ,  $E_r = E$ , where we identify the  $n_0$  edges connecting to  $V_J$  with those connecting to  $v_J$  in  $G$ . Consider a graph  $G_s$  such that  $V(G_s) = \partial G_s \cup \overset{\circ}{G}$ , where  $|\partial G_s| = n_0$ . Connect  $V_J$  with the  $n_0$  nodes from  $\partial G_s$ . Then the problem (2.7) on the new graph  $G^*$  with  $V(G^*) = V(G_r) \cup V(G_s)$ ,  $E(G^*) = E(G_r) \cup E(G_s)$ , is equivalent to the problem (2.7) on  $G_r$  with nodal conditions (3.44).

**Remark 3.1.** *The methodology behind Theorem 3.1 is based on the idea of domain decomposition using the Steklov-Poincaré map. Iterative domain decomposition for problems on metric graphs have been considered in Lagnese and Leugering [16] even for networked 2-d partial differential equations. This technique will be used in a forthcoming study for the numerical realization of complex graph problems.*

*In the context of inverse problems on metric graphs see Avdonin, Leugering and Mykhailov [2]*

*Results analogous to Theorem 3.1 have been provided in the scalar case of quantum graphs by Ong [24], Kuchment [13], Post [25] and others. Kuchment and Ong denote the procedure described in Theorem 3.1 as decoration. They consider in addition the problem of possible band-gaps resulting from this decoration. A similar analysis for the vectorial problems discussed here is under way, and beyond the scope of this paper.*

#### 4. A STAR WITH A SUBGRAPH INCLUDED AT IST CENTER

As described in the last section, using the Steklov-Poincaré map defined in Definition 3.1 we can decompose the graph  $G$  into star-graphs. We thus confine ourselves with just a single star-graph with a single multiple node  $v_J^0 \in V$ . Assume that  $d_{J0} = n$ . We may again relabel the edge indices  $i = 1, \dots, n$  of the edges incident at  $v_J^0$ .

To simplify notation we assume w.l.o.g. that the edges  $e_i$  start at  $v_J^0$  i.e.  $d_{iJ0} = -1$   $i = 1, \dots, n$ . We introduce  $n$  new vertices and label those  $v_1, \dots, v_n$  at the points  $\delta \rho_i e_i$ ,  $\rho_i > 0$  sufficiently small such that  $\delta \rho_i < \ell_i$ . We then cut out the partial edges  $e_i^\delta := [0, \delta \rho_i]$  from the star graph, and connect the newly created vertices through a finite graph  $G_s = (V_s, E_s)$ . We confine ourselves with a subgraph  $G_s$  with  $n = |V_s|$ , thus,  $\partial G_s = G_s$  and all nodes of  $G_s$  are connected to  $V_J = \{v_1, \dots, v_n\}$ . See Figure 5 for an inclusion with internal multiple node. Notice that we confine ourselves with subgraphs without internal nodes. This is, however, just for the sake of simplicity. No additional mathematical difficulty occurs in the more general case.

$$(4.45) \quad e_{ij} := \begin{cases} \frac{\rho_i e_i - \rho_j e_j}{\|\rho_i e_i - \rho_j e_j\|} := \tau_{ij} & \text{if } a_{ij}^s \neq 0 \\ 0 & \text{if } a_{ij}^s = 0 \end{cases}$$

where  $a_{iJ}^s$  is the adjacency matrix of  $G_s$ . The length of the edge with unit vector  $e_{ij}$  is  $\ell_{ij} = \delta \tau_{ij}$ .

At each vertex  $v_1, \dots, v_n$  we introduce Dirichlet data  $\phi_i$   $i = 1, \dots, n$ . We consider the Steklov - Poincaré map for the subgraph  $G_s$  with respect to the vertices  $\{v_1, \dots, v_n\} =: \partial G_s$ .

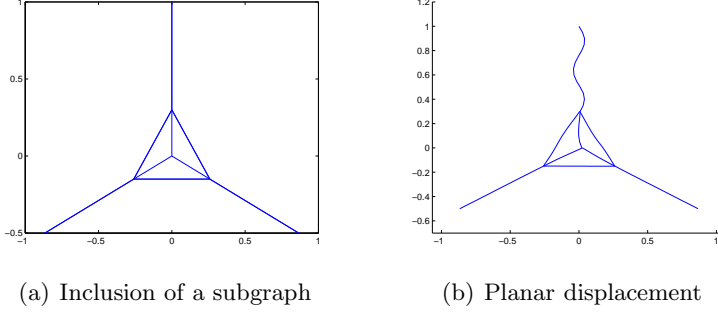


FIGURE 5. Star-graph with inclusion at the center

$$(4.46) \quad \begin{cases} K_{ij} r_{ij}'' + w^2 r_{ij} &= f_{ij} \quad (0, \ell_{ij}) \\ r_{ij}(0) &= \Phi_i \\ r_{ij}(\ell_{ij}) &= \Phi_j. \end{cases}$$

We then compute  $d_{ij}K_{ij}r'_{ij}(v_j)$  and compose for  $v_j$

$$(4.47) \quad \sum_{i \in \mathcal{I}_j^s} d_{ij}K_{ij}r'_{ij}(v_j)$$

In particular, we have

$$\begin{aligned} r_{ij}(x) &= \sin \omega K_{ij}^{-\frac{1}{2}} x a_{ij} + \cos \omega K_{ij}^{-\frac{1}{2}} x b_{ij} \\ r_{ij}(0) &= b_{ij} = \Phi_i, \quad \forall j \in \mathcal{I}_i \\ r_{ij}(\ell_{ij}) &= \sin \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} a_{ij} + \cos \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \Phi_i = \Phi_j. \end{aligned}$$

Hence,

$$a_{ij} = \left( \sin \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \right)^{-1} \left( \Phi_j - \cos \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \Phi_i \right)$$

and therefore

$$r'_{ij}(0) = \omega K_{ij}^{-\frac{1}{2}} \left( \sin \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \right)^{-1} \left( \Phi_j - \cos \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \Phi_i \right)$$

$r'_{ij}(0)$  starting at  $v_j$  into  $v_i$  and

$$\begin{aligned} r'_{ij}(\ell_{ij}) &= \omega K_{ij}^{-\frac{1}{2}} \cot \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \left( \Phi_j - \cos \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \Phi_i \right) \\ &\quad - \omega K_{ij}^{-\frac{1}{2}} \sin \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \Phi_i. \end{aligned}$$

Inserting these expressions into (4.47) we obtain

$$(4.48) \quad \begin{aligned} \sum_{i \in \mathcal{I}_j^s} d_{ij}K_{ij}r'_{ij}(v_j) &= \omega \left( \sum_{i \in \mathcal{I}_j^s} K_{ij}^{\frac{1}{2}} \cot \omega K_{ij}^{-\frac{1}{2}} \right) \Phi_j \\ &\quad - \omega \sum_{i \in \mathcal{I}_j^s} K_{ij}^{\frac{1}{2}} \left( \sin \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \right)^{-1} \Phi_i, \quad j = 1, \dots, n \end{aligned}$$

We now use  $r_i(\delta\rho_i) = \Phi_i$ ,  $i = 1, \dots, n$  in (4.48) and recall that  $d_{ii} = -1$ ,  $i = 1, \dots, n$ . Thus, the nodal condition for  $r_j$  at  $v_j$  is given by

$$(4.49) \quad \begin{aligned} K_j r'_j(\delta\rho_j) &= \omega \left( \sum_{i \in \mathcal{I}_j^s} K_{ij}^{\frac{1}{2}} \cot \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \right) r_j(\delta\rho_j) \\ &\quad - \omega \sum_{i \in \mathcal{I}_j^s} K_{ij}^{\frac{1}{2}} (\sin \omega K_{ij}^{-\frac{1}{2}} \ell_{ij})^{-1} r_i(\delta\rho_i), \quad i = 1, \dots, n. \end{aligned}$$

**Example 4.1.** *We assume a planar graph  $G$  and  $G^s$  a cycle connecting the vertices  $v_1, \dots, v_n$ . Assume further that the local stiffness matrices are all the same, e.g.  $K_{ij} = \kappa^2 I$ , and the graph  $G^s$  is symmetric and 2-regular with  $\ell_{ij} = \delta\tau$ . Then (4.49) reads*

$$(4.50) \quad K_j r'_j(\delta\rho_j) = \frac{\omega\kappa}{\sin(\omega\kappa^{-1}\delta\tau)} \left\{ 2 \cos(\omega\kappa^{-1}\delta\tau) r_j(\delta\rho_j) - \sum_{i \in \mathcal{I}_j} r_i(\delta\rho_i) \right\}, \quad j = 1, \dots, n.$$

## 5. ASYMPTOTIC ANALYSIS

We derive an auxiliary system from the problem on star graph with a hole:

$$(5.51) \quad \left\{ \begin{array}{l} K_i r_i'' + \omega^2 r_i = 0 \quad \text{in } (\delta\rho_i, \ell_i) \\ r_i(\ell_i) = u_i \\ K_i r'_i(\delta\rho_i) = \omega \left( \sum_{j \in \mathcal{I}_i^s} K_{ij}^{\frac{1}{2}} \cot \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} \right) r_i(\delta\rho_i) \\ \quad - \omega \sum_{j \in \mathcal{I}_i^s} K_{ij}^{\frac{1}{2}} (\sin \omega K_{ij}^{-\frac{1}{2}} \ell_{ij})^{-1} r_j(\delta\rho_j), \end{array} \right.$$

First of all we derive an asymptotic expansion for the Steklov-Poincaré map in the right hand side of (5.51)<sub>1</sub>. To this end we notice

$$(5.52) \quad (\sin \omega K_{ij}^{-\frac{1}{2}} \ell_{ij})^{-1} = (\omega K_{ij}^{-\frac{1}{2}} \delta\tau_{ij})^{-1} + \frac{1}{6} (\omega K_{ij}^{-\frac{1}{2}} \delta\tau_{ij}) + O(\delta^3)$$

and

$$(5.53) \quad \cos \omega K_{ij}^{-\frac{1}{2}} \ell_{ij} = I - \frac{1}{2} \omega^2 K_{ij}^{-\frac{1}{2}} \delta^2 \tau_{ij}^2 + O(\delta^4).$$

Then (4.48) has the expansion  
(5.54)

$$\begin{aligned}
 & \sum_{j \in \mathcal{I}_i^s} d_{ji} K_{ij} r'_{ji}(v_i) = \\
 & \frac{1}{\delta} \left\{ \left[ \sum_{j \in \mathcal{I}_i^s} \frac{1}{\tau_{ij}} K_{ij} - \frac{1}{3} \omega^2 \delta^2 \tau_{ij} \right] \Phi_i - \sum_{j \in \mathcal{I}_i^s} \left[ \frac{1}{\tau_{ij}} K_{ij} + \frac{1}{6} \omega^2 \delta^2 \tau_{ij} \right] \Phi_j + O(\delta^3) \right\} \\
 & = \frac{1}{\delta} \left\{ \left[ \sum_{j \in \mathcal{I}_i^s} \frac{1}{\tau_{ij}} K_{ij} \right] \Phi_i - \sum_{j \in \mathcal{I}_i^s} \frac{1}{\tau_{ij}} K_{ij} \Phi_j \right\} \\
 & \quad - \delta \frac{\omega^2}{6} \left\{ \left[ \sum_{j \in \mathcal{I}_i^s} 2\tau_{ij} \right] \Phi_i + \omega^2 \tau_{ij} \Phi_j \right\} + O(\delta^2).
 \end{aligned}$$

In matrix notation this reads as  
(5.55)

$$\begin{aligned}
 & \left( \sum_{j \in \mathcal{I}_i^s} d_{ji} K_{ij} r'_{ji}(v_j) \right)_{i=1}^n = \\
 & \frac{1}{\delta} \begin{pmatrix} \sum_{j \in \mathcal{I}_1^s} \frac{1}{\tau_{1j}} K_{1j} & -\frac{1}{\tau_{12}} K_{12} & \dots & \dots & -\frac{1}{\tau_{1n}} K_{1n} \\ -\frac{1}{\tau_{12}} K_{12} & \sum_{j \in \mathcal{I}_2^s} \frac{1}{\tau_{2j}} K_{2j} & \dots & \dots & \dots \\ \dots & \dots & \ddots & \dots & \dots \\ \dots & \dots & \dots & \ddots & \dots \\ -\frac{1}{\tau_{1n}} K_{1n} & -\frac{1}{\tau_{2n}} K_{2n} & \dots & \dots & \sum_{j \in \mathcal{I}_n^s} \frac{1}{\tau_{jn}} K_{jn} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \vdots \\ \vdots \\ \Phi_n \end{pmatrix} \\
 & - \frac{\omega^2 \delta}{6} \begin{pmatrix} 2 \sum_{j \in \mathcal{I}_1^s} \tau_{1j} & \tau_{12} & \dots & \dots & \tau_{1n} \\ \tau_{12} & 2 \sum_{j \in \mathcal{I}_2^s} \tau_{2j} & \dots & \dots & \dots \\ \dots & \dots & \ddots & \dots & \dots \\ \dots & \dots & \dots & \ddots & \dots \\ \tau_{1n} & \tau_{2n} & \dots & \dots & 2 \sum_{j \in \mathcal{I}_n^s} \tau_{jn} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \vdots \\ \vdots \\ \Phi_n \end{pmatrix} + O(\delta^2) I \\
 & =: \frac{1}{\delta} \mathcal{A}_0 \Phi - \delta \mathcal{A}_1 \Phi + O(\delta^2) I.
 \end{aligned}$$

We introduce the asymptotic expansions

$$(5.56) \quad r_i^\delta(x) = r_i^0(x) + \delta \tilde{r}_i(x) + \delta^2 \tilde{\tilde{r}}_i(x) + O(\delta^2)$$

and

$$\begin{aligned}
 (5.57) \quad r_i^0(\delta \rho_i) &= r_i^0(0) + \delta \rho_i (r_i^0)'(0) + O(\delta^2) \\
 r_i^0(\delta \rho_i) &= r_i^0(0) + \delta \rho_i (r_i^0)'(0) + O(\delta^2).
 \end{aligned}$$

Comparing coefficients, after differentiating (5.56) twice with respect to  $x$  and pre-multiplying by  $K_i$ , implies that  $r_i^0, \tilde{r}_i, \tilde{\tilde{r}}_i$  solve the Helmholtz equation  $K_i r_i'' + \omega^2 r_i = 0$ . Therefore (5.57) implies

$$(5.58) \quad \begin{aligned} K_i(r_i^0)'(\delta\rho_i) &= K_i(r_i^0)'(0) - \omega^2 \delta\rho_i r_i^0(0) + O(\delta^2) \\ K_i(\tilde{r}_i^0)'(\delta\rho_i) &= K_i(\tilde{r}_i^0)'(0) - \omega^2 \delta\rho_i \tilde{r}_i^0(0) + O(\delta^2). \end{aligned}$$

We obtain

$$(5.59) \quad \begin{aligned} K_i(r_i^\delta)'(\delta\rho_i) &= K_i(r_i^0)'(0) + \delta (K_i \tilde{r}_i'(0) - \rho_i \omega^2 r_i^0(0)) \\ &\quad + \delta^2 (K_i \tilde{\tilde{r}}_i''(0) - \rho_i \omega^2 \tilde{r}_i(0)) + O(\delta^2) \\ &= \frac{1}{\delta} \left\{ \left( \sum_j \frac{K_{ij}}{\tau_{ij}} \right) \Phi_i - \sum_j \frac{K_{ij}}{\tau_{ij}} \Phi_j \right\} \\ &\quad - \delta \omega^2 \left\{ \frac{1}{3} \left( \sum_j \tau_{ij} \right) \Phi_i + \frac{1}{6} \sum_j \tau_{ij} \Phi_j \right\} + O(\delta^2). \end{aligned}$$

Now,  $\Phi_i = r_i^\delta(\delta\rho_i)$  and hence

$$\Phi_i = r_i^0(0) + \delta(\rho_i(r_i^0)'(0) + \tilde{r}_i(0)) + \delta^2(\rho_i^2(r_i^0)''(0) + \rho_i \tilde{r}_i'(0)) + O(\delta^3).$$

Therefore, (5.59) leads, after some calculus and by comparing powers of  $\delta$ , to the following identities  $\forall i \in \mathcal{I}$ :

$$(5.60) \quad \sum_j \frac{K_{ij}}{\tau_{ij}}(r_i^0)(0) = \sum_j \frac{K_{ij}}{\tau_{ij}} r_j^0(0),$$

$$(5.61) \quad \begin{aligned} K_i(r_i^0)'(0) &= \left( \sum_j \frac{K_{ij}}{\tau_{ij}} \right) \rho_i(r_i^0)'(0) \\ &\quad - \sum_j \frac{K_{ij}}{\tau_{ij}} \rho_j(r_j^0)'(0) \\ &\quad + \left\{ \left( \sum_j \frac{K_{ij}}{\tau_{ij}} \right) \tilde{r}_i(0) - \sum_j \frac{K_{ij}}{\tau_{ij}} \tilde{r}_j(0) \right\}, \end{aligned}$$

$$(5.62) \quad \begin{aligned} K_i(\tilde{r}_i)'(0) - \omega^2 \rho_i r_i^0 &= \\ &\left( \sum_j \frac{K_{ij}}{\tau_{ij}} \right) (\rho_i \tilde{r}_i'(0) + \tilde{r}_i(0) - \rho_i^2 \omega^2 K_i^{-1} r_i^0) \\ &\quad - \sum_j \frac{K_{ij}}{\tau_{ij}} (\rho_j \tilde{r}_j'(0) + \tilde{r}_j(0) - \rho_j^2 \omega^2 K_j^{-1} r_j^0) \\ &\quad - \omega^2 \left( \frac{1}{3} \left( \sum_j \frac{K_{ij}}{\tau_{ij}} \right) r_i^0(0) + \frac{1}{6} \sum_j \frac{K_{ij}}{\tau_{ij}} r_j^0(0) \right). \end{aligned}$$

Now, (5.60) implies

$$(5.63) \quad r_i^0(0) = r_j^0(0) =: r^0(0),$$

while (5.60),(5.61) imply

$$(5.64) \quad \sum_i K_i(r_i^0)'(0) = 0.$$

Indeed, conditions (5.63) and (5.64) are precisely the nodal conditions for a star graph with  $n$  edges coonected at  $x = 0$ . Thus,  $(r_i^0)_{i=1,\dots,n}$  solves the unperturbed problem, as it should. Adding up (5.62) we obtain

$$(5.65) \quad \sum_j K_i(\tilde{r}_i)'(0) = \omega^2 \left( \sum_i \rho_j - \frac{1}{2} \sum_{ij} \tau_{ij} \right) r^0(0) =: G^1(r^0(0))$$

Still, rewriting (5.64) we obtain

$$(5.66) \quad \begin{aligned} & \left( \sum_j \frac{K_{ij}}{\tau_{ij}} \right) \tilde{r}_i(0) - \sum_j \frac{K_{ij}}{\tau_{ij}} \tilde{r}_j(0) \\ &= K_i(r_i^0)'(0) - \left( \sum_j \frac{K_{ij}}{\tau_{ij}} \right) \rho_i(r_i^0)'(0) + \sum_j \frac{K_{ij}}{\tau_{ij}} \rho_j(r_j^0)'(0) \\ &:= G^0((r^0)'(0))_i. \end{aligned}$$

We have the following boundary condition

$$u_i = r_i^\delta(\ell_i) = r_i^0(\ell_i) + \delta \tilde{r}_i(\ell_i) + O(\delta^2)$$

which implies

$$(5.67) \quad \tilde{r}_i(\ell_i) = 0.$$

Collecting (5.65),(5.66) and (5.67) together with the fact, that  $\tilde{r}_i$  solves the Helmolzt equation locally on the edge  $i$ , we obtain the following system for  $\tilde{r}$ :

$$(5.68) \quad \left\{ \begin{array}{l} K_i \tilde{r}'' + \omega^2 \tilde{r}_i = 0 \\ \tilde{r}_i(\ell_i) = 0 \\ \left( \sum_j \frac{K_{ij}}{\tau_{ij}} \right) \tilde{r}_i(0) - \sum_j \frac{K_{ij}}{\tau_{ij}} \tilde{r}_j(0) = G^0((r^0)'(0))_i \\ \sum_j K_i(\tilde{r}_i)'(0) = G^1(r^0(0)) \end{array} \right.$$

We proceed to show that this system (5.68) is actually self-adjoint. Indeed, if we define

$$(5.69) \quad \mathcal{B} = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ I & I & \dots & I & I \end{pmatrix}$$

and  $G^0 = (G^0((r^0)'(0))_1, \dots, G^0((r^0)'(0))_n)^T$  as well as  $G^1 = (0, 0, \dots, G^1(r^0(0)))^T$ ,  $\tilde{r}(0) = (\tilde{r}_1(0), \dots, \tilde{r}_n(0))^T$ ,  $\tilde{r}'(0) = (\tilde{r}'_1(0), \dots, \tilde{r}'_n(0))^T$ , we obtain

$$(5.70) \quad \mathcal{A}_0 \tilde{r}(0) = G^0, \quad \mathcal{B} K \tilde{r}'(0) = G^1.$$

It can be shown that the LU-decomposition  $\mathcal{A}_0 = \mathcal{L}\mathcal{U}$  is such that  $\mathcal{L}^{-1}\mathcal{B} = \mathcal{B}$  and, hence,  $[\mathcal{U}, \mathcal{B}]$  is equivalent to  $[\mathcal{A}_0, \mathcal{B}]$  in the sense that the nodal condition  $\mathcal{A}_0\tilde{r}(0) = G^0$ ,  $\mathcal{B}K\tilde{r}'(0) = G^1\mathbf{1}$  can be rewritten as  $\mathcal{U}\tilde{r}(0) + \mathcal{B}K\tilde{r}'(0) = G$ , where  $G = \mathcal{L}^{-1}K(r^0)'(0) - \mathcal{U}\rho(r^0)'(0) + \mathcal{B}G^1$ . The matrix  $\mathcal{U}\mathcal{B}^*$  is indeed self-adjoint and  $[\mathcal{U}, \mathcal{B}]$  has full rank, such that that (5.68) is a self-adjoint problem. Similarly, one can derive the following self-adjoint system for the second order term  $\tilde{r}$  in (5.56).

$$(5.71) \quad \begin{cases} K_i\tilde{r}'' + \omega^2\tilde{r}_i = 0 \\ \tilde{r}_i(\ell_i) = 0 \\ K_i\tilde{r}'(0) = \rho_i\omega^2\tilde{r}_i(0) \end{cases}.$$

**Remark 5.1.** *If we introduce*

$$(5.72) \quad \hat{r}_i(x) := \tilde{r}_i(x) + \rho_i(r_i^0)'(x), \quad i = 1, \dots, n,$$

then  $\hat{r}$  satisfies

$$(5.73) \quad \begin{cases} K_i\hat{r}'' + \omega^2\hat{r}_i = 0 \\ \hat{r}_i(\ell_i) = \rho_i(r_i^0)'(\ell_i) \\ \left( \sum_j \frac{K_{ij}}{\tau_{ij}} \right) \hat{r}_i(0) - \sum_j \frac{K_{ij}}{\tau_{ij}} \hat{r}_j(0) = K_i(r_i^0)'(0) \cdot \\ \sum_j K_i(\hat{r}_i)'(0) = -\omega^2 \frac{1}{2} \sum_{ij} \tau_{ij} r_i^0(0) \end{cases}.$$

**Theorem 5.1.** *Consider the solution  $r^\delta$  of the perturbed problem (5.51). Let the asymptotic expansion (5.56) be given. Then, the zeroth order term  $r^0$  satisfies the self-adjoint equations of the unperturbed  $n$ -star-graph, while the first order term  $\tilde{r}$  and the second order term  $\tilde{\tilde{r}}$  satisfies the self-adjoint problems (5.68), (5.71), respectively.*

**Remark 5.2.** *Theorem 5.1 is in clear analogy to similar problems in 2-D or 3-D. However, it is also clear that the perturbation in 'digging a hole' into a graph is regular, while similar perturbations for 2-D or 3-D problems are singular. This reflects the fact that the Green function for the locally 1-D problem is proportional to the absolute value function, where in 2-D or 3-D it is proportional to  $\ln$  or  $1/(\cdot)$ , respectively.*

## 6. TOPOLOGICAL SENSITIVITY FOR THE TOTAL ENERGY

In this section we consider the sensitivity of the energy with respect to changes in the topology. We mainly focus on graph inclusions as in the last section. For the sake of simplicity we consider a star-graph with  $n$  edges as before. We recall the energy of this system:

$$(6.74) \quad \mathcal{E}_0(r^0) = \frac{1}{2} \sum_{i=1}^n \int_0^{\ell_i} K_i r_i' \cdot r_i' - \omega^2 r_i^2 dx.$$

Using the boundary and nodal conditions (2.7) we have after integrating by parts

$$(6.75) \quad \begin{aligned} \mathcal{E}_0(r^0) &= \frac{1}{2} \sum_{i=1}^n \int_0^{\ell_i} K_i r_i' \cdot r_i' - \omega^2 r_i^2 dx \\ &= \frac{1}{2} \langle S(r^0), r^0 \rangle, \end{aligned}$$

where

$$(6.76) \quad S(r^0) := (K_i(r_i^0)'(0))_{i=1}^n$$

is the Steklov-Poincaré operator at the central node. Notice that the center is located at  $x = 0$  for all participating edges, and hence  $d_{i,J} = -1, \forall i \in \mathcal{I}_J$ . We now consider the energy in the  $n$  principal edges of the perturbed system, i.e. the edges of the star that has been perturbed by inserting the subgraph (hole).

$$(6.77) \quad \mathcal{E}_0(r^\delta) = \frac{1}{2} \sum_{i=1}^n \int_0^{\ell_i^\delta} K_i(r_i^\delta)' \cdot (r_i^\delta)' - \omega^2 (r_i^\delta)^2 dx.$$

Again, integrating by parts we obtain

$$(6.78) \quad \begin{aligned} \mathcal{E}_0(r^\delta) &= \frac{1}{2} \sum_{i=1}^n \int_0^{\ell_i^\delta} K_i(r_i^\delta)' \cdot (r_i^\delta)' - \omega^2 (r_i^\delta)^2 dx \\ &= \frac{1}{2} \sum_{i=1}^n K_i(r_i^\delta)'(\ell_i) \cdot (r_i^\delta)'(\ell_i) - \frac{1}{2} \sum_{i=1}^n K_i(r_i^\delta)'(\delta\rho_i) \cdot (r_i^\delta)'(\delta\rho_i) \\ &= \frac{1}{2} \sum_{i=1}^n K_i((r_i^0)'(\ell_i) + \delta\tilde{r}'_i(\ell_i) + \delta^2\tilde{r}''_i(\ell_i) + O(\delta^3)) \cdot (r_i^0(\ell_i) + \delta\tilde{r}(\ell_i) + O(\delta^2)) \\ &\quad - \frac{1}{2} \sum_{i=1}^n K_i(r_i^\delta)'(\delta\rho_i) r_i^\delta(\delta\rho_i), \end{aligned}$$

where we have used (5.56). We recall (3.42), (3.43), (3.44) and (4.48), leading to (5.51) and (5.55). If we insert the asymptotic expansion, we obtain

$$(6.79) \quad \begin{aligned} &\sum_{i=1}^n K_i(r_i^\delta)'(\delta\rho_i) \cdot r_i^\delta(\delta\rho_i) \\ &= \delta \left\{ \langle \mathcal{A}_0(\tilde{r}(0) + \rho(r^0)'(0)), \tilde{r}(0) + \rho(r^0)'(0) \rangle - \langle \mathcal{A}_1 r^0(0), r^0(0) \rangle \right\} \\ &= \sum_{i=1}^n K_i(r_i^0)'(0) r_i^0(0) \\ &+ \delta \left\{ \sum_{i=1}^n K_i(r_i^0)'(0) \cdot (\tilde{r}(0) + \rho_i r_i^0(0)) - \frac{\omega^2}{2} \left( \sum_{ij} \tau_{ij} \right) r^0(0) \cdot r^0(0) \right\} + O(\delta^2). \end{aligned}$$



We go back to (6.78). We have

$$\begin{aligned}
(6.80) \quad & \sum_{i=1}^n K_i(r_i^\delta)'(\ell_i) \cdot (r_i^\delta)(\ell_i) \\
& = \sum_{i=1}^n K_i(r_i^0)'(\ell_i) \cdot r_i^0(\ell_i) \\
& + \delta \left\{ \sum_{i=1}^n K_i \tilde{r}'(\ell_i) \cdot r_i^0(\ell_i) + K_i(r_i^0)'(\ell_i) \cdot \tilde{r}_i(\ell_i) \right\} + O(\delta^2).
\end{aligned}$$

But, according to (5.68)  $\tilde{r}(\ell_i) = 0$ , and  $r_i^0(\ell_i) = u_i$ ,  $i = 1, \dots, n$ . We thus obtain

$$\begin{aligned}
(6.81) \quad \mathcal{E}_0(r^\delta) & = \sum_{i=1}^n K_i(r_i^\delta)'(\ell_i) \cdot (r_i^\delta)(\ell_i) - \sum_{i=1}^n K_i(r_i^\delta)'(\delta\rho_i) \cdot r_i^\delta(\delta\rho_i) \\
& = \mathcal{E}_0(r^0) + \delta \left\{ \sum_{i=1}^n K_i \tilde{r}'(\ell_i) \cdot r_i^0(\ell_i) \right. \\
& \left. - \sum_{i=1}^n K_i(r_i^0)'(0) \cdot (\tilde{r}(0) + \rho_i(r_i^0)'(0)) + \frac{\omega^2}{2} \left( \sum_{ij} \tau_{ij} \right) r^0(0) \cdot r^0(0) \right\} \\
& \quad + O(\delta^2).
\end{aligned}$$

We need to add the energy stored in the subgraph. To this end we introduce expansions of  $r_{ij}^\delta$ . However, as the lengths  $\ell_{ij} = \delta\tau_{ij}$  are already first order, we only need first order expansions here.

$$\begin{aligned}
(6.82) \quad & r_{ij}^\delta(x) =: r_{ij}^0(x) + O(\delta) \\
& r_{ij}(\delta\tau_{ij}) = r_{ij}^0(0) + \delta\tau_{ij}(r_{ij}^0)'(0) \\
& (r_{ij}^\delta)'(\delta\tau_{ij}) = (r_{ij}^0)'(0) - \delta\tau_{ij}\omega^2 K_{ij}^{-1} r_{ij}^0(0).
\end{aligned}$$

We use (6.82) in order to derive the following expansion

$$\begin{aligned}
(6.83) \quad & \sum_{ij} K_{ij}(r_{ij}^\delta)'(\delta\tau_{ij}) r_{ij}^\delta(\delta\tau_{ij}) - K_{ij}(r_{ij}^\delta(0)) r_{ij}^\delta(0) \\
& = -\delta \frac{1}{2} \sum_{ij} (\omega^2 \tau_{ij} r_{ij}^0(0)^2 - \tau_{ij} K_{ij}(r_{ij}^0)'(0) (r_{ij}^0)'(0)).
\end{aligned}$$

Actually, the second part in the expansion (6.87) is equal to the Steklov-Poincaré operator times displacements of the subgraph. Indeed, by (3.44), (4.48) and (6.79), the energy stored in the subgraph has the expansion

$$(6.84) \quad \mathcal{E}_\delta(r^\delta) = -\delta \frac{1}{2} \sum_{ij} (\omega^2 \tau_{ij} r_{ij}^0(0)^2 - \tau_{ij} K_{ij}(r_{ij}^0)'(0) (r_{ij}^0)'(0)).$$

Therefore, the total energy of the perturbed system has the asymptotic expansion

$$\begin{aligned}
 \mathcal{E}(r^\delta) &= \mathcal{E}_0(r^\delta) + \mathcal{E}_\delta(r^\delta) \\
 &= \mathcal{E}_0(r^0) + \delta \left\{ \sum_{i=1}^n K_i \hat{r}'(\ell_i) \cdot r_i^0(\ell_i) \right. \\
 (6.85) \quad &- \sum_{i=1}^n K_i (r_i^0)'(0) \cdot (\hat{r}(0) + \rho_i (r_i^0)'(0)) + \frac{\omega^2}{2} \left( \sum_{ij} \tau_{ij} r^0(0) \cdot r^0(0) \right. \\
 &\quad \left. \left. - \frac{1}{2} \sum_{ij} (\omega^2 \tau_{ij} r_{ij}^0(0)^2 - \tau_{ij} K_{ij} (r_{ij}^0)'(0) (r_{ij}^0)'(0)) \right) \right\}.
 \end{aligned}$$

We may use  $\hat{r}$  to simplify (6.84). We notice that

$$(r_{ij}^0)'(0) = \frac{1}{\tau_{ij}} (\hat{r}_j(0) - \hat{r}_i(0))$$

and therefore

$$\begin{aligned}
 \mathcal{E}^\delta(r^\delta) &= \mathcal{E}_0(r^0) \\
 (6.86) \quad &+ \delta \frac{1}{2} \sum_i \left\{ \rho_i K_i (r_i^0)'(\ell_i) (r_i^0)'(\ell_i) - 2K_i \hat{r}(0) (r_i^0)'(0) + \omega^2 \rho_i r_i^0(\ell_i)^2 \right. \\
 &\quad \left. - \omega^2 \frac{1}{2} \sum_{j \leftrightarrow i} \tau_{ij} r_i^0(0)^2 + \frac{1}{2} \sum_{j \leftrightarrow i} \frac{K_{ij}}{\tau_{ij}} (\hat{r}_j(0) - \hat{r}_i(0))^2 \right\}.
 \end{aligned}$$

We define the topological derivative of the energy.

**Definition 6.1.** *Let  $\mathcal{E}^\delta(r^\delta)$  and  $\mathcal{E}_0(r^0)$  be the energies corresponding to the perturbed and the unperturbed graph, respectively. Then the limits, if it exists,*

$$(6.87) \quad \frac{1}{\delta} \left( \mathcal{E}^\delta(r^\delta) - \mathcal{E}_0(r^0) \right) =: \mathcal{T}(r^0(0), (r^0)'(0))$$

*is called the topological derivative of  $\mathcal{E}$  at the center node of the star-graph.*

The asymptotic expansion of the energy given by (6.86) is still not explicit, as it involves the solution  $\hat{r}$  of the auxiliary system (5.73). This solution, however is completely determined by the solution and its derivative at zero,  $r^0(0), (r^0)'(0)$ , of the original unperturbed problem. Also,  $r^0(\ell_i), (r_i^0)'(\ell_i)$  can be expressed by  $r^0(0), (r^0)'(0)$  by solving on the individual edges once the corresponding data (Dirichlet or Neumann) are given. Thus, solving the auxiliary problem in terms of these values gives a quadratic form

$$\begin{aligned}
 &\langle P(r^0(0), (r^0)'(0)), (r^0(0), (r^0)'(0)) \rangle \\
 (6.88) \quad &= \frac{1}{2} \sum_i \left\{ \rho_i K_i (r_i^0)'(\ell_i) (r_i^0)'(\ell_i) - 2K_i \hat{r}(0) (r_i^0)'(0) + \omega^2 \rho_i r_i^0(\ell_i)^2 \right. \\
 &\quad \left. - \omega^2 \frac{1}{2} \sum_{j \leftrightarrow i} \tau_{ij} r_i^0(0)^2 + \frac{1}{2} \sum_{j \leftrightarrow i} \frac{K_{ij}}{\tau_{ij}} (\hat{r}_j(0) - \hat{r}_i(0))^2 \right\}.
 \end{aligned}$$

The matrix  $P$  in (6.88) can be viewed as an analogue of the *polarization matrix* in 2-D and 3-D elasticity problems. See [20].

**Theorem 6.1.** *Under the assumption above, the sensitivity of the energy associated with the star-graph containing a cyclic subgraph (hole) with respect to letting  $\delta \rightarrow 0$  is given by*

$$(6.89) \quad \mathcal{T}(r^0(0), (r^0)'(0)) = P(r^0(0), (r^0)'(0)), (r^0(0), (r^0)'(0)),$$

where  $P$  according to (6.88) is given by the solution of (5.73).

**Example 6.1.** *We consider as a simple case the situation where the material is completely homogenous and the geometry of the hole is symmetric i.e.  $K_i = \kappa I$ ,  $\rho_i = \rho$ ,  $\ell_i = \ell$   $i = 1, \dots, n$ ,  $K_{ij} = \kappa I$ ,  $\tau_{ij} = \tau \forall i, j$ . The latter assumption implies that we have a subgraph that is complete (fully connected). In this situation we can explicitly compute  $\tilde{r}$ , the solution of the auxiliary problem (5.68). Indeed, after some elementary calculus we arrive at*

(6.90)

$$\hat{r}_i(0) = -\frac{\omega}{\kappa^{\frac{1}{2}}} \tan(\omega \kappa^{-\frac{1}{2}} \ell) \left( \rho - \frac{n\tau}{n} \right) r^0(0) + \left( \frac{\tau}{n} - \rho \right) \frac{\omega \kappa^{-\frac{1}{2}}}{\sin \omega \kappa^{-\frac{1}{2}} \ell} u_i + (\rho r_i^0)'(0).$$

The solution of the unperturbed system  $r^0$  can also be given explicitly. In order to actually compute the sensitivity of the total energy with respect to inserting a symmetric fully connected subgraph, we use the Dirichlet boundary condition  $r_i(\ell) = u_i$ ,  $i = 1 \dots, n$ . In particular,

$$(6.91) \quad r_i^0(\ell) = u_i = \sin \omega \kappa^{-\frac{1}{2}} \ell \frac{\kappa^{\frac{1}{2}}}{\omega} (r_i^0)'(0) + \cos \omega \kappa^{-\frac{1}{2}} \ell r^0(0),$$

$$(6.92) \quad \begin{aligned} r_i^0(0) &= \frac{1}{n} \frac{1}{\cos \omega \kappa^{-\frac{1}{2}} \ell} \sum_j u_j =: r^0(0) \\ (r_i^0)'(0) &= \frac{\omega \kappa^{-\frac{1}{2}}}{\sin \omega \kappa^{-\frac{1}{2}} \ell} \left( u_i - \frac{1}{n} \sum_j u_j \right). \end{aligned}$$

This gives the topological gradient in terms of the inputs  $u$  as follows

(6.93)

$$\begin{aligned} & \mathcal{T}(r^0(0), (r^0)'(0)) \\ &= \frac{\omega^2}{2} \left\{ \left( \rho (\tan^2(\omega \kappa^{-\frac{1}{2}} \ell) - \cot^2(\omega \kappa^{-\frac{1}{2}} \ell)) - \frac{\tau n}{2 \cos^2(\omega \kappa^{-\frac{1}{2}} \ell)} \right) \frac{1}{n} \left| \sum_i u_i \right|^2 \right. \\ & \quad \left. + \frac{\rho}{\sin^2(\omega \kappa^{-\frac{1}{2}} \ell)} \sum_i |u_i|^2 \right\} \\ & \quad - \frac{\kappa^{\frac{1}{2}} \omega}{\sin(\omega \kappa^{-\frac{1}{2}} \ell)} \sum_i \hat{r}(0) \left( u_i - \frac{1}{n} \sum_j u_j \right) + \frac{\kappa n}{2\tau} \sum_{i \in \mathcal{I}} \sum_{i \leftrightarrow j} |\hat{r}_i(0) - \hat{r}_j(0)|^2. \end{aligned}$$

**Example 6.2.** *In this example we are even more specific, in that we assume that  $r^0(\ell) = u_i = \mu e_i$ ,  $i = 1, \dots, n$ , where  $\sum_i e_i = 0$  which corresponds to a symmetric displacement at all simple nodes of the star-graph. In this situation we conclude with*

$$(6.94) \quad \mathcal{T}(0, u) = \omega^2 \frac{1}{\sin^2 \omega \kappa^{-\frac{1}{2}} \ell} (\rho n - \tau) \mu^2.$$

It follows in particular that for a three-star, where  $\rho = \frac{\sqrt{3}}{3}, \tau = 1$

$$(6.95) \quad \mathcal{T}(0, u) = \omega^2 \frac{1}{\sin^2 \omega \kappa^{-\frac{1}{2}} \ell} (\sqrt{3} - 1) \mu^2 > 0.$$

Thus, the three-star under  $120^\circ$  degree angles is energetically optimal with respect to the edge-degree. The analogous result has been shown in [18] for a string with Wrinkler support.

**Example 6.3.** The situation is similar for 4 strings in  $\mathbf{R}^3$  where  $\rho = \frac{1}{4}\sqrt{6}$  and

$$(6.96) \quad \mathcal{T}(0, u) = \omega^2 \frac{1}{\sin^2 \omega \kappa^{-\frac{1}{2}} \ell} (\sqrt{6} - 1) \mu^2 > 0,$$

which shows that also that configuration is energetically optimal under symmetric boundary conditions. Indeed, for six edges  $\rho = 1$  and this is exactly the bounding case where cutting out a hole (i.e. including a cyclic subgraph) compensates the total loss of the edges that had been shortened. In this case

$$(6.97) \quad \mathcal{T}(0, u) = -\omega^2 \frac{3}{\sin^2 \omega \kappa^{-\frac{1}{2}} \ell} \mu^2 < 0,$$

which implies that releasing the edge-degree 6 of the central node to a series of 6 nodes with degree 3 is favored. This result strongly suggests that hexagonal structures are energetically optimal. This is common place in material sciences where one deals with foam-structures.

## 7. COMPLIANCE OPTIMIZATION

Energy, as a functional to optimize is strongly related to the widely used compliance-functional. This amounts to minimizing

$$(7.98) \quad \mathcal{J}(r, f, g) := \sum_{i \in \mathcal{I}} \int_0^{\ell_i} f_i \cdot r_i dx + \sum_{J \in \mathcal{J}} \sum_{i \in \mathcal{I}_J} g_J \cdot r_j(v_J),$$

where  $r$  solves (2.7). In order to stay in the context of this presentation, we allow for nodal forces only, keeping the more general case including spatial varying coefficients for a future publication. In this case the energy is given by

$$(7.99) \quad \mathcal{E}(r) := \frac{1}{2} \sum_{i \in \mathcal{I}} \int_0^{\ell_i} K_i r'_i \cdot r'_i - \omega^2 r_i \cdot r_i dx - \sum_{J \in \mathcal{J}} \sum_{i \in \mathcal{I}_J} g_J \cdot r_i(v_J)$$

and as  $r$  solves the variational problem

$$(7.100) \quad \sum_{i \in \mathcal{I}} \int_0^{\ell_i} K_i r'_i \cdot w'_i - \omega^2 r_i \cdot w_i dx = \sum_{J \in \mathcal{J}} \sum_{i \in \mathcal{I}_J} g_J \cdot w_i(v_J),$$

the compliance functional can be expressed for a solution of (7.100) as

$$(7.101) \quad \sum_{J \in \mathcal{J}} \sum_{i \in \mathcal{I}_J} g_j \cdot r_j(v_J) = -2\mathcal{E}(r).$$

Therefore, one may use (6.86), where, however, the solution  $r^0$  has to be evaluated for the problem with nodal forces at the simple nodes. Then the quantities of interest are

$$\begin{aligned}
r_i^0(0) &= -\frac{1}{\omega} \left( \sum_j K_j^{\frac{1}{2}} \tan \omega K_j^{-\frac{1}{2}} \ell_j \right)^{-1} \sum_i \left( \cos \omega K_i^{-\frac{1}{2}} \ell_i \right)^{-1} g_i, \\
(r_i^0)'(0) &= K_i^{-\frac{1}{2}} \frac{1}{\omega} \left( K_i^{-\frac{1}{2}} \omega \left( \cos \omega K_i^{-\frac{1}{2}} \ell_i \right)^{-1} g_i \right. \\
(7.102) \quad &\left. - \frac{1}{\omega} \tan \omega K_i^{-\frac{1}{2}} \ell_i \left\{ \sum_j \left( K_j^{-\frac{1}{2}} \ell_j \right)^{-\frac{1}{2}} \sum_i \left( \cos \omega K_i^{-\frac{1}{2}} \ell_i \right)^{-1} g_i \right\} \right), \\
r_i(\ell_i) &= \sin \omega K_i^{-\frac{1}{2}} \ell_i \frac{K_i^{\frac{1}{2}}}{\omega} (r_i^0)'(0) + \sin \omega K_i^{-\frac{1}{2}} \ell_i r_i^0(0).
\end{aligned}$$

We may now introduce (7.102) into (6.86) in order to evaluate the topological gradient for the energy functional in this case. We refrain from doing this here in the full complexity, as no new mathematical insight is to be expected from this. Now, minimizing the compliance is typically accompanied with penalizing the total volume. In the context of our elastic network we come to penalize the total length. Indeed, the stiffness operators  $K_i$  involve, in the context of linear elasticity, the cross section and the Young modulus. Therefore, we would have to separate the cross-sectional part and multiply with the length. As in our example we will be dealing with homogenous material, we just take the total length as a measure.

$$(7.103) \quad \mathcal{P}(\ell^\delta) := \alpha \left( \sum_i (\ell_i - \delta \rho_i) + \frac{1}{2} \sum_i \sum_{j \in \mathcal{I}_i^s} (\delta \tau_{ij}) \right),$$

where obviously  $\mathcal{P}(\ell^0) = \alpha \sum_i \ell_i$  is the total length of the unperturbed star-graph. We then obtain

$$(7.104) \quad \frac{1}{\delta} \left\{ \mathcal{P}(\ell^\delta) - \mathcal{P}(\ell^0) \right\} = \alpha \left( \frac{1}{2} \sum_i \sum_{j \in \mathcal{I}_i^s} \tau_{ij} - \sum_i \rho_i \right).$$

As to be expected, the penalized compliance minimization turns out to be a balance between energy- and 'perimeter'-sensitivity. We give a canonical example.

**Example 7.1.** *We assume that the material is homogenous, as in the other examples. We pull at each end of the elastic star-network with a force  $g_i$ . Moreover, we assume that the sum of forces is zero  $\sum_i g_i = 0$ . We also consider a planar situation with a 2-regular cycle as subgraph. In this case*

some calculus shows that (7.102) reduces to

$$(7.105) \quad r_i^0(0) = -\frac{1}{\omega} \cot(\omega \kappa^{-\frac{1}{2}} \ell) \frac{1}{n} \sum_i g_i = 0$$

$$(7.106) \quad \begin{aligned} (r_i^0)'(0) &= \frac{1}{\kappa \cos \omega \kappa^{-\frac{1}{2}} \ell} \left( g_i - \frac{1}{n} \sum_j g_j \right) \\ &= \frac{1}{\kappa \cos \omega \kappa^{-\frac{1}{2}} \ell} g_i, \end{aligned}$$

$$(7.107) \quad \begin{aligned} r_i^0(\ell) &= \sin \omega \kappa^{-\frac{1}{2}} \ell \frac{\kappa^{\frac{1}{2}}}{\omega} (r_i^0)'(0) + \cos \omega \kappa^{-\frac{1}{2}} \ell r_i^0(0) \\ &= \frac{1}{\kappa^{\frac{1}{2}} \omega} \tan(\omega \kappa^{\frac{1}{2}} \ell) g_i. \end{aligned}$$

Moreover, the first order variation  $\hat{r}$  of (5.73) gives

$$(7.108) \quad \begin{aligned} \hat{r}_i(\ell) &= \rho (r_i^0)'(\ell) = \rho \frac{1}{\kappa \cos \omega \kappa^{-\frac{1}{2}} \ell} g_i \\ \hat{r}_i(0) &= \frac{1}{n \kappa \cos \omega \kappa^{-\frac{1}{2}} \ell} g_i. \end{aligned}$$

This gives the total sensitivity

$$(7.109) \quad \mathcal{T}(r) = -\frac{1}{n \kappa \cos \omega \kappa^{-\frac{1}{2}} \ell} (\rho n - 1) \sum_i g_i^2 + \alpha n (1 - \rho).$$

In case of a symmetric three-star  $n = 3$ ,  $\tau_{ij} = 1$ ,  $\rho = \frac{\sqrt{3}}{3}$  where one pulls at each end such that the system is in equilibrium  $\sum_i g_i = 0$ , then with we obtain

$$(7.110) \quad \mathcal{T}(r) = -\frac{1}{3 \kappa \cos \omega \kappa^{-\frac{1}{2}} \ell} (\sqrt{3} - 1) \sum_i g_i^2 + \alpha 3 \left( 1 - \frac{\sqrt{3}}{3} \right).$$

This shows that if one takes  $\alpha$  sufficiently large, thereby insisting on the volume constraint, then no hole is favored depending on how close  $\omega$  is to  $\frac{\pi k \kappa^{\frac{1}{2}}}{\ell}$ , whereas otherwise a hole may be favorable.

## 8. OTHER FUNCTIONALS AND GRAPH OPERATIONS

Besides the energy and the compliance we can also consider other cost functionals, such as tracking-type functions.

$$(8.111) \quad \mathcal{J}(r) := \frac{1}{2} \sum_{i \in \mathcal{I}} \int_0^{\ell_i} |r_i - r_i^d|^2 dx,$$

where  $r_i^d \in C^0(0, \ell_i)$  is a profile to be tracked by the solutions  $r_i$  of (2.7). Now, let  $r^\delta$  be the solution of the perturbed problem and set

$$(8.112) \quad \mathcal{J}^\delta(r^\delta) := \frac{1}{2} \sum_{i \in \mathcal{I}_{\delta \rho_i}} \int_0^{\ell_i} |r_i^\delta - r_i^d|^2 dx + \frac{1}{2} \sum_{i \in \mathcal{I}, j \in \mathcal{I}_j^s} \int_0^{\delta \tau_{ij}} |r_{ij} - r_{ij}^d|^2 dx.$$

In this case the topological derivative can be calculated as follows.

(8.113)

$$\begin{aligned} \frac{1}{\delta} \left\{ \mathcal{J}^\delta(r^\delta) - \mathcal{J}(r^0) \right\} &= \mathcal{T}(r) \\ &= \sum_i \int_0^{\ell_i} \tilde{r}_i \cdot (r_i^0 - r_i^d) dx \\ &= \sum_i |r_i^0(0) - r_i^d(0)|^2 - \sum_{i \in \mathcal{I}, j \in \mathcal{I}_j^s} \tau_{ij} |\tilde{r}_i(0) + \rho_i (r_i^0)'(0)|^2. \end{aligned}$$

We may introduce the adjoint  $p = (p_i)_i$  corresponding to  $r^0$  as follows.

$$(8.114) \quad \begin{cases} K_i p_i'' + \omega^2 p_i = r_i^0 - r_i^d & \text{in } (0, \ell_i) \\ p_i(\ell_i) = 0 \\ p_i(0) = p_j(0), \quad i \neq j = 1, \dots, n \quad . \\ \sum_{i=1}^n K_i p_i'(0) = 0 \end{cases}$$

Then integrating by parts and using the boundary and nodal conditions for  $\tilde{r}$ ,

$$\sum_i \int_0^{\ell_i} \tilde{r}_i (K_i p_i'' + \omega^2 p_i) dx = - \sum_i K_i p_i'(0) \tilde{r}_i(0) + \omega^2 \left( \sum_i \rho_i - \frac{1}{2} \sum_{ij} \tau_{ij} \right) r^0(0) p(0).$$

Indeed, the expression  $\sum_i K_i p_i'(0) \tilde{r}_i(0)$  can be expressed in terms of a quadratic form using the pseudoinverse  $R_{A,B}$  (2.32). Thereby, one can define an analogue of the polarization matrix. Notice that  $r_i^0(0)$ ,  $p_i(0)$  are independent of  $i$ . Indeed, it can be shown, that the following explicit representation holds.

(8.115)

$$\begin{aligned} p(0) &= - \left( \omega \sum_i K_i^{\frac{1}{2}} \cot \omega K_i^{-\frac{1}{2}} \ell_i \right)^{-1} \\ &\cdot \sum_i \int_0^{\ell_i} \left( \sin \omega K_i^{-\frac{1}{2}} \ell_i \right)^{-1} \sin(\omega K_i^{-\frac{1}{2}} (\ell_i - s)) \{r_i^0 - r_i^d\}(s) ds \\ p_i'(0) &= -\omega K_i^{-\frac{1}{2}} \cot(\omega K_i^{-\frac{1}{2}} \ell_i) p(0) \\ &+ K_i^{-1} \int_0^{\ell_i} \left( \sin \omega K_i^{-\frac{1}{2}} \ell_i \right)^{-1} \sin(\omega K_i^{-\frac{1}{2}} (\ell_i - s)) \{r_i^0 - r_i^d\}(s) ds. \end{aligned}$$

We put this together with (8.113) and obtain.

$$(8.116) \quad \begin{aligned} \mathcal{T}(r) = & - \sum_i K_i p'_i(0) \tilde{r}_i(0) \\ & + \omega^2 \left( \sum_i \rho_i - \frac{1}{2} \sum_{ij} \right) r^0(0) p(0) - \sum_i |r_i^0(0) - r_i^d(0)|^2 \\ & - \sum_{i \in \mathcal{I}, j \in \mathcal{I}_j^s} \tau_{ij} |\tilde{r}_i(0) + \rho_i (r_i^0)'(0)|^2. \end{aligned}$$

**Example 8.1.** *If we assume homogenous material and Dirichlet conditions  $r_i^0(\ell) = u_i$  where we may consider the standard symmetric scenario  $\sum e_i = 0$  and  $u_i = \mu e_i$  then*

$$(8.117) \quad \mathcal{T}(r) = - \left( \sum_i p'_i(0) \cdot e_i \right) \mu - \omega^2 (\rho n - 1)^2 \frac{1}{\sin \omega \ell} \mu^2 - \sum_i |r_i^d(0)|^2.$$

*Obviously, depending on the load at 0, the frequency  $\omega$ , the topology given by the unite vectors  $e_i$  and the adjoint  $p$ , (8.1) provides a positive or a negative sensitivity.*

**Remark 8.1.** *We remark that more complicated functionals can be investigated with respect to topological sensitivities along the same line. It is clear that as the load and topological scenario becomes asymmetric, it is more likely that the topological gradient is negative for a range of frequencies. Robustness issues with respect to frequencies, however, are beyond this paper and will be investigated in a forthcoming publication.*

**Remark 8.2.** *As has been demonstrated in Leugering and Sokolowski [18], on can also compute the sensitivities of the energy, the compliance and other functionals with respect to introducing a single edge or with respect to releasing a node of degree, say  $n$ , into one of degree  $n - 1$  and one with degree 3. We do not want to overload this paper with these additional possibilities. It is, however, clear that everything can be done in the framework developed here.*

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