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# Bernstein-type inequalities for rational functions in weighted Bergman spaces and applications

RACHID ZAROUF

## Abstract

Generally speaking, given two Banach spaces  $X$  and  $Y$  of holomorphic functions on the unit disc  $\mathbb{D}$ , we are searching for the “best possible” constants  $\mathcal{C}_{n,r}(X, Y)$  and  $\mathcal{B}_{n,r}(X, Y)$  such that

$$\|f'\|_X \leq \mathcal{C}_{n,r}(X, Y) \|f\|_Y \quad \text{and} \quad \|f\|_X \leq \mathcal{B}_{n,r}(X, Y) \|f\|_Y,$$

for all rational functions  $f$  in  $\mathbb{D}$  having at most  $n$  poles all outside of  $\frac{1}{r}\mathbb{D}$ ,  $r \in [0, 1)$ . For the special case  $X = Y$ , we set  $\mathcal{C}_{n,r}(X) := \mathcal{C}_{n,r}(X, X)$ .

Throughout this paper the letter  $c$  denotes a positive constant that may change from one step to the next. For two positive functions  $a$  and  $b$ , we say that  $a$  is dominated by  $b$ , denoted by  $a = O(b)$ , if there is a constant  $c > 0$  such that  $a \leq cb$ ; and we say that  $a$  and  $b$  are equivalent, denoted by  $a \asymp b$ , if both  $a = O(b)$  and  $b = O(a)$  hold.

We show that  $\mathcal{C}_{n,r}(L_a^2, H^2) \asymp \mathcal{B}_{n,r}(B_{2,2}^{\frac{1}{2}}, H^2) \asymp \sqrt{\frac{n}{1-r}}$ , for all  $n \geq 1$  and  $r \in [0, 1)$ , where  $H^2$  is the classical Hardy space,  $L_a^2$  is the classical Bergman space and  $B_{2,2}^{\frac{1}{2}}$  is a Besov space also known as the Dirichlet space. Moreover, there exists a limit :  $\lim_{n \rightarrow \infty} \frac{\mathcal{B}_{n,r}(B_{2,2}^{\frac{1}{2}}, H^2)}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\mathcal{C}_{n,r}(L_a^2, H^2)}{\sqrt{n}} = \sqrt{\frac{1+r}{1-r}}$ .

Then, we apply our estimate of  $\mathcal{B}_{n,r}(B_{2,2}^{\frac{1}{2}}, H^2)$  to an effective Nevanlinna-Pick interpolation problem in the Dirichlet space.

Finally, if  $X = Y$  and if  $X$  is a radial-weighted Bergman space then we show that  $\mathcal{C}_{n,r}(X) \asymp \frac{n}{1-r}$  for all  $n \geq 1$  and  $r \in [0, 1)$ .

## Introduction

Surveys of inequalities for the derivatives of rational functions are given by Gonchar [Go] and Rusak [Ru]. Chapter 7 of [BoEr] is also devoted to this topic. Here, we present such inequalities for rational functions  $f$  of degree  $n$  with poles in  $\mathbb{D}_- = \{z : |z| > 1\}$ , involving Hardy norms, the Dirichlet-Besov norm and weighted-Bergman norms.

**1. Statement of the problem.** Let  $\mathcal{P}_n = \{p = \sum_{k=0}^n a_k z^k : a_k \in \mathbb{C}\}$ , be the complex space of polynomials of degree less or equal than  $n \geq 1$ . Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  and  $\overline{\mathbb{D}}$  its closure. Given  $r \in [0, 1)$ , we define

$$\mathcal{R}_{n,r} = \left\{ \frac{p}{q}, : p \in \mathcal{P}_{n-1}, q \in \mathcal{P}_n, q(\zeta) = 0 \implies \zeta \notin \frac{1}{r}\mathbb{D} \right\},$$

the set of all rational functions in  $\mathbb{D}$  of degree less or equal than  $n \geq 1$ , having at most  $n$  poles all outside of  $\frac{1}{r}\mathbb{D}$ . Notice that for  $r = 0$ , we get  $\mathcal{R}_{n,0} = \mathcal{P}_{n-1}$ .

**Proper definitions of constants  $\mathcal{C}_{n,r}(X, Y)$  and  $\mathcal{B}_{n,r}(X, Y)$ .**

Let  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  be a sequence in the unit disc  $\mathbb{D}$ , the finite Blaschke product  $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$ , where  $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$  is an elementary Blaschke factor for  $\lambda \in \mathbb{D}$ . Let also  $K_{B_\sigma}$  be the  $n$ -dimensional space defined by

$$K_{B_\sigma} = \mathcal{L}in(k_{\lambda_i} : i = 1..n),$$

where  $\sigma$  is a family of distincts elements of  $\mathbb{D}$ , and where  $k_\lambda = \frac{1}{1-\bar{\lambda}z}$  is the Szegő kernel associated to  $\lambda$ . An obvious modification allows to generalize the definition of  $K_{B_\sigma}$  in the case where the sequence  $\sigma$  admits multiplicities.

Notice that using the scalar product  $(\cdot, \cdot)_{H^2}$  on  $H^2$ , an equivalent description of this space is:

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2,$$

where  $H^2$  stand for the standard Hardy space of the unit disc  $\mathbb{D}$ ,

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{H^2}^2 =: \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) < \infty \right\},$$

$m$  being the Lebesgue normalized measure on  $\mathbb{T}$ . Let  $D$  be the operator of differentiation on  $(K_{B_\sigma}, \|\cdot\|_Y)$ :

$$\begin{aligned} D : (K_{B_\sigma}, \|\cdot\|_Y) &\rightarrow (X, \|\cdot\|_X) \\ f &\mapsto f'. \end{aligned}$$

For  $r \in [0, 1)$  and  $n \geq 1$ , we get

$$\mathcal{C}_{n,r}(X, Y) = \sup \left\{ \|D\|_{K_{B_\sigma} \rightarrow X} : 1 \leq \#\sigma \leq n, |\lambda| \leq r \forall \lambda \in \sigma \right\}.$$

In the same spirit, if  $Id$  is the identity operator

$$\begin{aligned} Id : (K_{B_\sigma}, \|\cdot\|_Y) &\rightarrow (X, \|\cdot\|_X) \\ f &\mapsto f, \end{aligned}$$

then for every  $r \in [0, 1)$  and  $n \geq 1$ , we get

$$\mathcal{B}_{n,r}(X, Y) = \sup \left\{ \|Id\|_{K_{B_\sigma} \rightarrow X} : 1 \leq \#\sigma \leq n, |\lambda| \leq r \forall \lambda \in \sigma \right\}.$$

**2. The spaces  $X$  and  $Y$  considered here.** Now let us define some Banach spaces  $X$  and  $Y$  of holomorphic functions in  $\mathbb{D}$  which we will consider throughout this paper.

**a.** The standard Hardy spaces  $H^p = H^p(\mathbb{D})$ ,  $1 \leq p \leq \infty$ ,

$$H^p = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{H^p}^p =: \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^p dm(z) < \infty \right\}.$$

An equivalent description of the space  $H^2$ , which has already be mentionned in 1, is the following

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{H^2} =: \left( \sum_{k \geq 0} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

**b.** The standard weighted Bergman spaces  $L_a^p(\beta) := L_a^p\left((1 - |z|^2)^\beta dx dy\right)$ ,  $1 \leq p \leq \infty$ ,  $\beta > -1$ ,

$$L_a^p(\beta) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{L_a^p(\beta)}^p =: \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\beta dx dy < \infty \right\}.$$

For  $\beta = 0$ , we shorten the notation and write  $L_a^p(0) = L_a^p$ . For  $p = 2$  and  $\beta = 0$  we get the standard Bergman space for which an equivalent description is,

$$L_a^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{L_a^2} =: \left( \sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{k+1} \right)^{\frac{1}{2}} < \infty \right\}.$$

**c.** The analytic Besov space  $B_{2,2}^{\frac{1}{2}}$  be (also known as the standard Dirichlet space) defined by

$$B_{2,2}^{\frac{1}{2}} = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{B_{2,2}^{\frac{1}{2}}} =: \left( \sum_{k \geq 0} (k+1) |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

Then if  $f \in B_{2,2}^{\frac{1}{2}}$ , we have the following equality

$$\|f\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|f'\|_{L_a^2}^2 + \|f\|_{H^2}^2, \quad (1)$$

which establish a link between the spaces  $B_{2,2}^{\frac{1}{2}}$  and  $L_a^2$ .

### 3. Known Results and motivations.

**a. An estimate for  $\mathcal{C}_{n,r}(H^p)$ ,  $1 \leq p \leq \infty$ .** The first known result belongs to K. Dyakonov [Dy2] and tells that

$$\mathcal{C}_{n,r}(H^p) \asymp \frac{n}{1-r}, \quad (2)$$

for every  $p \in [1, \infty]$ .

For the special case  $p = 2$ , it has been proved later in [Z1] that there exists a limit

$$\lim_{n \rightarrow \infty} \frac{\mathcal{C}_{n,r}(H^2)}{n} = \frac{1+r}{1-r}, \quad (3)$$

for every  $r$ ,  $0 \leq r < 1$ .

**b. An estimate for  $\mathcal{B}_{n,r}(B_{2,2}^{\frac{1}{2}}, H^2)$ .** The second result is also due to K. Dyakonov [Dy2] :

$$\mathcal{B}_{n,r} \left( B_{2,2}^{\frac{1}{2}}, H^2 \right) \leq c \left( \frac{n}{1-r} \right)^{\frac{1}{2}}. \quad (4)$$

Using (1) we get

$$\left( \mathcal{C}_{n,r} (L_a^2, H^2) \right)^2 = \left( \mathcal{B}_{n,r} \left( B_{2,2}^{\frac{1}{2}}, H^2 \right) \right)^2 - 1, \quad (5)$$

which establish a link between constants  $\mathcal{C}_{n,r}$  and  $\mathcal{B}_{n,r}$ .

### c. Motivations.

**c.1.** Now it is important to link and motivate such an inequality with one of Dolzhenko's inequality [Do] (see also [Pek], p.560 - inequality (12)) : let  $f$  be a rational function of degree  $n$  with poles in  $\mathbb{D}_-$  then

$$\|f\|_{B_{2,2}^{\frac{1}{2}}} \leq c\sqrt{n} \|f\|_{H^\infty}, \quad (6)$$

where  $\|f\|_{H^\infty} = \sup_{\zeta \in \mathbb{T}} |f(\zeta)|$ . In other words, Dolzhenko was interested in estimating the constant  $\mathcal{B}_{n,r} \left( B_{2,2}^{\frac{1}{2}}, H^\infty \right)$  and proved that  $\mathcal{B}_{n,r} \left( B_{2,2}^{\frac{1}{2}}, H^\infty \right) \leq c\sqrt{n}$  (this inequality is obviously sharp, taking a finite Blaschke product of order  $n$ ).

Notice that on one hand, inequality (6) was generalized by Danchenko [Da] for Hardy-Besov spaces. On the other hand, another generalization of (6) follows from a result of Peller [Pel] on best rational approximations for some class of Besov spaces, in BMOA : the space of all analytic functions of the bounded mean oscillation on  $\mathbb{D}$ . Later, Pekarskii [Pek] generalized and strengthened the results of Peller and Danchenko. An interesting motivation for such inequalities is to apply it to "inverse theorems of rational approximation" using the *classical Bernstein decomposition*.

**c.2.** A second motivation is to apply such inequalities to effective Nevanlinna-Pick interpolation problems. Indeed, one of the most important tool used in [Z2] (resp. in [Z3]) is the asymptotic sharp estimate of  $\mathcal{B}_{n,r} \left( B_{2,2}^1, H^2 \right)$  (resp.  $\mathcal{B}_{n,r} \left( B_{2,2}^s, H^2 \right)$ ,  $s > 0$ ), see paragraph 4 below for the definition of spaces  $B_{2,2}^s$ ,  $s > 0$ . In the same spirit, in Section II below we apply our estimate of  $\mathcal{B}_{n,r} \left( B_{2,2}^{\frac{1}{2}}, H^2 \right)$  to an effective Nevanlinna-Pick interpolation problem in the Dirichlet space.

**c.3.** Bernstein-type inequalities for rational functions are motivated by some applications in matrix analysis and in operator theory.

- Indeed, a result of Leveque and Trefethen [LeTr] improved by Spijker [Sp], shows that : if  $T$  is an  $n \times n$  matrix such that  $P(T) = \sup_{k \geq 0} \|T^k\|_{E_2 \rightarrow E_2} < \infty$ , where  $E_2$  stands for  $\mathbb{C}^n$  endowed with the Hilbert norm  $|\cdot|_2$  and setting  $\rho(T) = \sup_{|z| > 1} (|z| - 1) \|R(z, T)\|_{E_2 \rightarrow E_2}$ , where  $R(z, T)$  stands for the resolvent of  $T$  at point  $z$ , then

$$\rho(T) \leq P(T) \leq en \cdot \rho(T).$$

More precisely, the inequality  $\rho(T) \leq P(T)$  is obvious whereas the inequality  $P(T) \leq en \cdot \rho(T)$  is called Kreiss Matrix Theorem and is a straight consequence of the facts that  $C_{n,r}(H^1, H^\infty) \leq en$  and  $P(T) \leq C_{n,r}(H^1, H^\infty) \rho(T)$ .

- Moreover, replacing the algebra  $H^\infty$  by the Hardy space  $H^2$  in the constant  $C_{n,r}(H^1, H^\infty)$ , it is given in [Z4] an estimate for  $C_{n,r}(H^1, H^2)$  and an application to a resolvent estimate for operators with finite spectrum. The same Bernstein-type inequality between the Hardy spaces  $H^1$  and  $H^2$  is also applied in [Z5] to improve a resolvent estimate - from E.B. Davies and B. Simon [DS] - for power bounded operators.

In paragraph 4 below, we present our results : we first give an estimate for  $\mathcal{C}_{n,r}(L_a^2, H^2)$  and  $\mathcal{B}_{n,r}(B_{2,2}^{\frac{1}{2}}, H^2)$ , and then we generalize to radial-weighted Bergman spaces, the result of K. Dyakonov [Dy2] stated in 3.a for Hardy spaces : we prove that  $\mathcal{C}_{n,r}(L_a^p(\beta)) \asymp \frac{n}{1-r}$  for every  $\beta > -1$  and  $p \in [1, \infty]$ .

**4. Main Results.** Our first result is a partial case ( $p = q = 2, s = \frac{1}{2}, \gamma = +\infty$ ) of the following K. Dyakonov's result [Dy] : if  $p \in [1, \infty), s \in (0, +\infty), q \in [1, +\infty]$ , then there exists a constant  $c_{p,s} > 0$  such that

$$\mathcal{B}_{n,r}(B_{p,p}^s, H^q) \leq c_{p,s} \left\| B' \right\|_{H^\gamma}^s, \quad (7)$$

where  $B$  is a finite Blaschke product. Here  $B_{p,p}^s$  stands for the Hardy-Besov space which consist of analytic functions  $f$  on  $\mathbb{D}$  satisfying

$$\|f\|_{B_{p,p}^s} =: \sum_{k=0}^{n-1} |f^{(k)}(0)| + \int_{\mathbb{D}} (1-|w|)^{(n-s)p-1} |f^{(n)}(w)|^p dA(w) < \infty.$$

For the partial case considered here (Theorem A below), our proof is different and the constant  $c_{2,\frac{1}{2}}$  is slightly better.

**Theorem A.** *Let  $n \geq 1$  and  $r \in (0, 1)$ .*

(i) *If  $n = 1$  and  $\sigma = \{\lambda\}$ , we have*

$$\|D\|_{(K_{B_\sigma}, \|\cdot\|_{L_a^2}) \rightarrow (H^2, \|\cdot\|_{H^2})} = |\lambda|,$$

where  $D$  is the differentiation operator defined in 1. If  $n \geq 2$ ,

$$\tilde{a}(n,r) \sqrt{\frac{n}{1-r}} \leq \mathcal{C}_{n,r}(L_a^2, H^2) \leq \tilde{A}(n,r) \sqrt{\frac{n}{1-r}}, \quad (8)$$

where

$$\tilde{a}(n,r) \geq \left(1 - \frac{1-r}{n}\right)^{\frac{1}{2}} \quad \text{and} \quad \tilde{A}(n,r) \leq \left(1 + r + \frac{1}{\sqrt{n}}\right)^{\frac{1}{2}}.$$

In particular,

$$\sqrt{\frac{n}{1-r}} \leq \mathcal{B}_{n,r}(B_{2,2}^{\frac{1}{2}}, H^2) \leq \tilde{B}(n,r) \sqrt{\frac{n}{1-r}}, \quad (9)$$

where  $\tilde{B}(n,r) \leq \left(1 + r + \frac{1}{\sqrt{n}} + \frac{1-r}{n}\right)^{\frac{1}{2}}$ .

(ii) *Moreover, sequences*

$$\left(\frac{\mathcal{C}_{n,r}(L_a^2, H^2)}{\sqrt{n}}\right)_{n \geq 1} \quad \text{and} \quad \left(\frac{\mathcal{B}_{n,r}(B_{2,2}^{\frac{1}{2}}, H^2)}{\sqrt{n}}\right)_{n \geq 1}$$

are convergent and there exists a limit

$$\lim_{n \rightarrow \infty} \frac{\mathcal{B}_{n,r} \left( B_{2,2}^{\frac{1}{2}}, H^2 \right)}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\mathcal{C}_{n,r} (L_a^2, H^2)}{\sqrt{n}} = \sqrt{\frac{1+r}{1-r}}. \quad (10)$$

for all  $r \in [0, 1)$ .

Finally, we generalize Dyakonov's result (2) for Hardy spaces (see 3.a above) to radial-weighted Bergman spaces through the following Theorem B.

**Theorem B.** *Let  $p \in [1, \infty]$ ,  $\beta > -1$ . Then,*

$$\mathcal{C}_{n,r} (L_a^p(\beta)) \asymp \frac{n}{1-r}, \quad (11)$$

for all  $n > \beta + 2$  and  $r \in (0, 1)$ . Moreover

$$\mathcal{C}_{n,r} (L_a^p(\beta)) \leq 2^{\frac{1}{p}} \mathcal{C}_{n,r} (H^p), \quad (12)$$

for all  $p \in [1, \infty]$ ,  $\beta > -1$ .

Section I is devoted to the proofs of (8), (9) and (10). In Section II, we apply the right-hand side inequality from (8) to an effective Nevanlinna-Pick interpolation problem in the Dirichlet space, constrained by the  $H^2$ -norm. Finally in Section III, we prove (11) and (12).

## II. An estimate for $\mathcal{C}_{n,r} (L_a^2, H^2)$ and $\mathcal{B}_{n,r} \left( B_{2,2}^{\frac{1}{2}}, H^2 \right)$

The aim of this Section is to prove Theorem A.

**Proof of Theorem A.**

*Proof of (i). The case  $n = 1$ .* In this case,  $K_B = \mathbb{C}e_1$ , where

$$e_1 = \frac{(1 - |\lambda|^2)^{\frac{1}{2}}}{1 - \bar{\lambda}z}, \quad |\lambda| \leq r,$$

( $e_1$  being of norm 1 in  $H^2$ ). Calculating,

$$e'_1 = \frac{\bar{\lambda} (1 - |\lambda|^2)^{\frac{1}{2}}}{(1 - \bar{\lambda}z)^2},$$

and

$$\begin{aligned} \|e'_1\|_{L_a^2} &= |\lambda| (1 - |\lambda|^2)^{\frac{1}{2}} \left\| \frac{1}{(1 - \bar{\lambda}z)^2} \right\|_{L_a^2} = \\ &= |\lambda| (1 - |\lambda|^2)^{\frac{1}{2}} \left( \sum_{k \geq 0} \frac{k+1}{k+1} |\lambda|^{2k} \right)^{\frac{1}{2}} = |\lambda| (1 - |\lambda|^2)^{\frac{1}{2}} \frac{1}{(1 - |\lambda|^2)} = |\lambda|, \end{aligned}$$

we get

$$\|D|_{K_{B_\sigma}}\| = |\lambda|.$$

**The case  $n \geq 2$ .**

**Step 1.** First, we show the right hand side inequality of (8). Using both Cauchy-Schwarz inequality and the fact that  $\widehat{f}'(k) = (k+1)\widehat{f}(k+1)$  for all  $k \geq 0$ , we get

$$\begin{aligned} \|f'\|_{L_a^2}^2 &= \sum_{k \geq 0} \frac{|\widehat{f}'(k)|^2}{k+1} = \sum_{k \geq 0} \frac{(k+1)^2 |\widehat{f}(k+1)|^2}{k+1} = \\ &= \sum_{k \geq 1} k |\widehat{f}(k)|^2 \leq \left( \sum_{k \geq 1} k^2 |\widehat{f}(k)|^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} |\widehat{f}(k)|^2 \right)^{\frac{1}{2}} = \\ &= \|f'\|_{H^2} \|f\|_{H^2} \leq \mathcal{C}_{n,r}(H^2) \|f\|_{H^2}^2, \end{aligned}$$

and hence,

$$\|f'\|_{L_a^2} \leq \sqrt{\mathcal{C}_{n,r}(H^2)} \|f\|_{H^2},$$

which means

$$\mathcal{C}_{n,r}(L_a^2, H^2) \leq \sqrt{\mathcal{C}_{n,r}(H^2)}.$$

Then it remains to use [Z1] p.2 :

$$\mathcal{C}_{n,r}(H^2) \leq \left(1 + r + \frac{1}{\sqrt{n}}\right) \frac{n}{1-r},$$

for all  $n \geq 1$  and  $r \in [0, 1)$ .

**Step 2.** Now, we show the left-hand-side inequality of (8). Let  $e_n = \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-1}$ . Then  $e_n \in K_{b_r^n}$  and  $\|e_n\|_{H^2} = 1$ , (see [N1], Malmquist-Walsh Lemma, p.116). Moreover,

$$\begin{aligned} e'_n &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b'_r b_r^{n-2} = \\ &= -\frac{r}{(1-r^2)^{\frac{1}{2}}} b'_r b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b'_r b_r^{n-2}, \end{aligned}$$

since  $b'_r = \frac{r^2-1}{(1-rz)^2}$ . Then,

$$e'_n = b'_r \left[ -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-2} \right],$$



and

$$\begin{aligned}\|e'_n\|_{L_a^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{D}} \left| b'_r(w) \right|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} (b_r(w))^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} (b_r(w))^{n-2} \right|^2 dm(w) = \\ &= \frac{1}{2\pi} \int_{\mathbb{D}} \left| b'_r(w) \right|^2 |(b_r(w))^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r(w) + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} \right|^2 dm(w),\end{aligned}$$

which gives, using the variables  $u = b_r(w)$ ,

$$\|e'_n\|_{L_a^2}^2 = \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rb_r(u)} \right|^2 dm(u).$$

But  $1 - rb_r = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$  and  $b'_r \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2}$ . This implies

$$\begin{aligned}\|e'_n\|_{L_a^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-r^2} (1-ru) \right|^2 dm(u) = \\ &= \frac{1}{(1-r^2)} \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 |(-ru + (n-1)(1-ru))|^2 dm(u),\end{aligned}$$

which gives

$$\|e'_n\|_{L_a^2} = \frac{1}{(1-r^2)^{\frac{1}{2}}} \|\varphi_n\|_2,$$

where  $\varphi_n = z^{n-2}(-rz + (n-1)(1-rz))$ . Expanding, we get

$$\begin{aligned}\varphi_n &= z^{n-2}(-rz + n - 1 + rz - nrz) = \\ &= z^{n-2}(-nrz + n - 1) = (n-1)z^{n-2} - nrz^{n-1},\end{aligned}$$

and

$$\begin{aligned}\|e'_n\|_{L_a^2}^2 &= \frac{1}{(1-r^2)} \left( \frac{(n-1)^2}{n-1} + \frac{n^2}{n} r^2 \right) = \frac{1}{(1-r^2)} (n(1+r) - 1) \\ &= \frac{n}{(1-r)(1+r)} \left( (1+r) - \frac{1}{n} \right) = \frac{n}{1-r} \left( 1 - \frac{1-r}{n} \right),\end{aligned}$$

which gives

$$\mathcal{C}_{n,r}(L_a^2, H^2) \geq \sqrt{\frac{n}{1-r}} \left( 1 - \frac{1-r}{n} \right)^{\frac{1}{2}}.$$

*Proof of (ii).* **Step 1.** We first prove the right-hand-side inequality :

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \mathcal{C}_{n,r}(L_a^2, H^2) \leq \sqrt{\frac{1+r}{1-r}},$$

which becomes obvious since

$$\frac{1}{\sqrt{n}} \mathcal{C}_{n,r}(L_a^2, H^2) \leq \frac{1}{\sqrt{n}} \sqrt{\mathcal{C}_{n,r}(H^2)},$$

and

$$\frac{1}{\sqrt{n}} \sqrt{\mathcal{C}_{n,r}(H^2)} \rightarrow \sqrt{\frac{1+r}{1-r}},$$

as  $n$  tends to infinity, see [Z1] p. 2.

**Step 2.** We now prove the left-hand-side inequality :

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \mathcal{C}_{n,r} (L_a^2, H^2) \geq \sqrt{\frac{1+r}{1-r}}.$$

More precisely, we show that

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|D\|_{(K_{b_r^n}, \|\cdot\|_{L_a^2}) \rightarrow H^2} \geq \sqrt{\frac{1+r}{1-r}}.$$

Let  $f \in K_{b_r^n}$ . Then,

$$\begin{aligned} f' &= (f, e_1)_{H^2} \frac{r}{(1-rz)} e_1 + \sum_{k=2}^n (k-1) (f, e_k)_{H^2} \frac{b_r'}{b_r} e_k + r \sum_{k=2}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k = \\ &= r \sum_{k=1}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k + \frac{1-r^2}{(1-rz)(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} e_k = \\ &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{3}{2}}}{(1-rz)^2(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} = \\ &= -b_r' \left[ \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{z-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right]. \end{aligned}$$

Now using the change of variables  $v = b_r(u)$ , we get

$$\begin{aligned} \|f'\|_{L_a^2}^2 &= \int_{\mathbb{D}} |b_r'(u)|^2 \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{u-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right|^2 du = \\ &= \int_{\mathbb{D}} \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{b_r(v)-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv. \end{aligned}$$

But  $b_r - r = \frac{r-z-r(1-rz)}{1-rz} = \frac{z(r^2-1)}{1-rz}$ , which gives

$$\begin{aligned} \|f'\|_{L_a^2}^2 &= \int_{\mathbb{D}} \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{v(r^2-1)} (1-rv) \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv = \\ &= \frac{1}{1-r^2} \int_{\mathbb{D}} \left| r \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} - (1-rv) \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-2} \right|^2 dv = \\ &= \frac{1}{1-r^2} \int_{\mathbb{D}} \left| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k - (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right|^2 dv. \end{aligned}$$

But

$$\left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \leq r \left( \sum_{k=0}^{n-1} \frac{1}{k+1} |(f, e_{k+1})_{H^2}|^2 \right)^{1/2} \leq r \|f\|_{H^2},$$

and in particular

$$\lim_{n \rightarrow \infty} \frac{1}{n \|f\|_{H^2}} \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} = 0.$$

Now,

$$\begin{aligned} & (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k = \\ &= \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k - r \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^{k+1} = \\ &= \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k - r \sum_{k=1}^{n-1} k (f, e_{k+1})_{H^2} v^k = \\ &= (f, e_2)_{H^2} + 2(f, e_3)_{H^2} v + \sum_{k=2}^{n-2} [(k+1) (f, e_{k+2})_{H^2} - rk (f, e_{k+1})_{H^2}] v^k + \\ &\quad -r [(f, e_2)_{H^2} v + (n-1) (f, e_n)_{H^2} v^{n-1}] = \\ &= (f, e_2)_{H^2} + [(f, e_3)_{H^2} - r(f, e_2)_{H^2}] v + \sum_{k=2}^{n-2} [(k+1) (f, e_{k+2})_{H^2} - rk (f, e_{k+1})_{H^2}] v^k + \\ &\quad -r(n-1) (f, e_n)_{H^2} v^{n-1}. \end{aligned}$$

Since

$$\begin{aligned} & \frac{1}{\|f\|_{H^2} \sqrt{n(1-r^2)}} \left[ \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} + \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \right] \geq \\ & \geq \frac{1}{\sqrt{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq \\ & \geq \frac{1}{\|f\|_{H^2} \sqrt{n(1-r^2)}} \left[ \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} - \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \right], \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} = 0,$$

we get that

$$\begin{aligned} & \frac{1}{\sqrt{1+r}} \lim_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} \geq \\ & \geq \lim_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq \\ & \geq \frac{1}{\sqrt{1+r}} \lim_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}. \end{aligned}$$

This gives

$$\underline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}}} = \frac{1}{\sqrt{1+r}} \underline{\lim}_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}.$$

But

$$\begin{aligned} &= (f, e_2)_{H^2} + [(f, e_3)_{H^2} - r(f, e_2)_{H^2}]v + \sum_{k=2}^{n-2} [(k+1)(f, e_{k+2})_{H^2} - rk(f, e_{k+1})_{H^2}]v^k + \\ &\quad -r(n-1)(f, e_n)_{H^2}v^{n-1}. \\ &\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = |(f, e_2)_{H^2}|^2 + \\ &+ \frac{1}{2} |(f, e_3)_{H^2} - r(f, e_2)_{H^2}|^2 + \frac{1}{n} r^4 (n-1)^2 |(f, e_n)_{H^2}|^2 + \\ &+ \sum_{k=2}^{n-2} \left| (f, e_{k+2})_{H^2} - \frac{rk}{k+1} (f, e_{k+1})_{H^2} \right|^2. \end{aligned}$$

Now, let  $s = s_n$  be a sequence of even integers such that

- $\lim_{n \rightarrow \infty} s_n = \infty$  and
- $s_n = o(n)$  as  $n \rightarrow \infty$ .

Then we consider the following function  $f$  in  $K_{b^2}$ :

$$f = \sum_{k=0}^{s+2} (-1)^k e_{n-k}.$$

With such an  $f$ , we get

$$\begin{aligned} &\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = |(f, e_2)_{H^2}|^2 + \\ &+ \frac{1}{2} |(f, e_3)_{H^2} - r(f, e_2)_{H^2}|^2 + \frac{1}{n} r^4 (n-1)^2 |(f, e_n)_{H^2}|^2 + \\ &+ \sum_{k=2}^{n-2} (k+1) \left| (f, e_{k+2})_{H^2} - \frac{rk}{k+1} (f, e_{k+1})_{H^2} \right|^2. \\ &\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = \\ &= r^4 \frac{(n-1)^2}{n} + \end{aligned}$$

$$+ \sum_{l=2}^{n-2} (n-l+1) \left| (f, e_{n-l+2})_{H^2} - \frac{r(n-l)}{n-l+1} (f, e_{n-l+1})_{H^2} \right|^2,$$

setting the change of index  $l = n - k$  in the last sum. This finally gives

$$\begin{aligned} & \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = \\ & = r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left| 1 + \frac{r(n-l)}{n-l+1} \right|^2 = \\ & = r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left[ 1 + r \left( 1 - \frac{1}{n-l+1} \right) \right]^2 \end{aligned}$$

And

$$\begin{aligned} & \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 \geq \\ & \geq r^4 \frac{(n-1)^2}{n} + (s+1-2+1)(n-(s+1)+1) \left[ 1 + r \left( 1 - \frac{1}{n-(s+1)+1} \right) \right]^2 = \\ & = r^4 \frac{(n-1)^2}{n} + s(n-s) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2. \end{aligned}$$

In particular,

$$\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 \geq s(n-s) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2.$$

Now, since  $\|f\|_{H^2}^2 = s+3 = s_n+3$ , we get

$$\begin{aligned} & \underline{\lim}_{n \rightarrow \infty} \frac{1}{n \|f\|_{H^2}^2} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 \geq \\ & \geq \underline{\lim}_{n \rightarrow \infty} \frac{1}{n \|f\|_{H^2}^2} \|f\|_{H^2}^2 (n - \|f\|_{H^2}^2) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2 = \\ & = \lim_{n \rightarrow \infty} \left( 1 - \frac{s_n}{n} \right) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2 = (1+r)^2. \end{aligned}$$

We can now conclude that

$$\begin{aligned} \underline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} &= \frac{1}{\sqrt{1+r}} \underline{\lim}_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} \geq \\ &\geq \frac{1+r}{\sqrt{1+r}} = \sqrt{1+r}, \end{aligned}$$

and

$$\underline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \|D\|_{K_{b_r^n} \rightarrow H^2} \geq \underline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq \sqrt{1+r}.$$

**Step 3. Conclusion.** Using both **Step 1** and **Step 2**, we get

$$\overline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{C}_{n,r}(L_a^2, H^2) = \underline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{C}_{n,r}(L_a^2, H^2) = 1+r,$$

which means that the sequence  $\left(\frac{1}{\sqrt{n}} \mathcal{C}_{n,r}(L_a^2, H^2)\right)_{n \geq 1}$  is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \mathcal{C}_{n,r}(L_a^2, H^2) = \sqrt{\frac{1+r}{1-r}}.$$

Now using equality (5), we get inequalities (9) from Theorem A and prove in particular that the sequence  $\left(\frac{1}{\sqrt{n}} \mathcal{B}_{n,r}(B_{2,2}^{\frac{1}{2}}, H^2)\right)_{n \geq 1}$  is also convergent with

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \mathcal{B}_{n,r}(B_{2,2}^{\frac{1}{2}}, H^2) = \sqrt{\frac{1+r}{1-r}}. \quad \square$$

## II. Interpolation in the Dirichlet space

**a. Statement of the problem.** Let  $X$  and  $Y$  two Banach spaces of holomorphic functions on the unit disc  $\mathbb{D}$  such that  $X \supset Y$ , and a finite set  $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$ . We consider the following interpolation constant

$$I(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \},$$

and

$$\mathcal{I}_{n,r}(X, Y) = \sup \{ I(\sigma, X, Y) : \#\sigma \leq n, \forall j = 1..n, |\lambda_j| \leq r \},$$

for all  $n \geq 1$  and  $r \in [0, 1)$ .

**b. Motivations.** We can give three main motivations for this problem.

1. It is explained in [Z3] why the classical interpolation problems, those of Nevanlinna-Pick (1908) and Carathéodory-Schur (1916) (see [N2] p.231 for these two problems), on the one hand and Carleson's free interpolation problem (1958) (see [N1] p.158) on the other hand, are of the nature of our interpolation problem.

2. It is also explained in [Z3] why this constrained interpolation is motivated by some applications in matrix analysis and in operator theory.

3. It has already been proved in [Z2] that for  $X = H^2$  and  $Y = H^\infty$ ,

$$\frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq I(\sigma_{n,r}, H^2, H^\infty) \leq \mathcal{I}_{n,r}(H^2, H^\infty) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}},$$

a question especially stimulated by L. Baratchart (a part of a more complicated question arising in an applied situation in [BL1] and [BL2]) : given a set  $\sigma \subset \mathbb{D}$ , how to estimate  $I(\sigma, H^2, H^\infty)$  in terms of  $n = \text{card}(\sigma)$  and  $\max_{\lambda \in \sigma} |\lambda| = r$  only?

**c. The result.** Here, looking at motivation 3, we replace the algebra  $H^\infty$  by the Dirichlet space  $B_{2,2}^{\frac{1}{2}}$ . We show that the “gap” between  $X = H^2$  and  $Y = H^\infty$  is asymptotically the same as the one which exists between  $X = H^2$  and  $Y = B_{2,2}^{\frac{1}{2}}$ . In other words,

$$\mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \asymp \mathcal{I}_{n,r} \left( H^2, H^\infty \right) \asymp \sqrt{\frac{n}{1-r}}. \quad (13)$$

Our aim in this Section is to prove the following Theorem in which the implicit constants of (13) are precised.

**Theorem 3.1.** *Let  $n \geq 1$ , and  $r \in [0, 1)$ . Then,*

$$\mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \leq \mathcal{B}_{n,r} \left( B_{2,2}^{\frac{1}{2}}, H^2 \right). \quad (14)$$

We consider  $\lambda \in \mathbb{D}$  and  $\sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}$ , a one-point set of multiplicity  $n \geq 2$ . Then,

$$I \left( \sigma_{n,\lambda}, H^2, B_{2,2}^{\frac{1}{2}} \right) \geq \sqrt{\frac{n}{1-|\lambda|}} \left[ \frac{(1+|\lambda|)^2 - \frac{2}{n} - \frac{2|\lambda|}{n}}{2(1+|\lambda|)} \right]^{\frac{1}{2}}. \quad (15)$$

In particular,

$$\sqrt{\frac{n}{1-r}} \left[ \frac{1+r}{2} \left( 1 - \frac{1}{n} \right) \right]^{\frac{1}{2}} \leq \mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \leq \mathcal{B}_{n,r} \left( B_{2,2}^{\frac{1}{2}}, H^2 \right), \quad (16)$$

$$\sqrt{\frac{\frac{1+r}{2}}{1-r}} \leq \underline{\lim}_{n \rightarrow \infty} \frac{\mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right)}{\sqrt{n}} \leq \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right)}{\sqrt{n}} \leq \sqrt{\frac{1+r}{1-r}}, \quad (17)$$

and

$$\frac{\sqrt{2}}{2} \leq \underline{\lim}_{r \rightarrow 1^-} \underline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \leq \overline{\lim}_{r \rightarrow 1^-} \overline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \leq \sqrt{2}. \quad (18)$$

**Proof of Theorem 3.1.**

*Proof of inequality (14).* Let  $\sigma$  be a sequence in  $\mathbb{D}$ ,  $H = B_{2,2}^{\frac{1}{2}}$  and  $B = B_\sigma$  the finite Blaschke product corresponding to  $\sigma$ . Let  $P_B$  be the orthogonal projection of  $(H, (\cdot, \cdot)_{H^2})$  onto  $K_B$  (see [Z2], p.8 - (1.1.4) and Lemma 1.1.5 for details). If  $f \in H^2$ , let  $g = P_B f \in K_B$ . Then  $g - f \in BH^2$  and

$$\|g\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|g\|_{H^2}^2 + \sum_{k \geq 1} k |\hat{g}(k)|^2 = \|g'\|_{L_a^2}^2 + \|g\|_{H^2}^2 .$$

Now, using Theorem 2.1,

$$\|g'\|_{L_a^2}^2 \leq (\mathcal{C}_{n,r}(L_a^2, H^2))^2 \|g\|_{H^2}^2 ,$$

which gives

$$\|g\|_{B_{2,2}^{\frac{1}{2}}}^2 \leq (\mathcal{C}_{n,r}(L_a^2, H^2) + 1)^2 \|g\|_{H^2}^2 .$$

Using the fact that  $\|g\|_{H^2} = \|P_B f\|_{H^2} \leq \|f\|_{H^2}$ , we finally get

$$\|g\|_{B_{2,2}^{\frac{1}{2}}} \leq (\mathcal{C}_{n,r}(L_a^2, H^2) + 1) \|f\|_{H^2} ,$$

and as a result,

$$I\left(\sigma, H^2, B_{2,2}^{\frac{1}{2}}\right) \leq \mathcal{C}_{n,r}(L_a^2, H^2) + 1 = \mathcal{B}_{n,r}\left(B_{2,2}^{\frac{1}{2}}, H^2\right) .$$

*Proof of inequality (15).* 1) We set

$$f = \sum_{k=0}^{n-1} (1 - |\lambda|^2)^{\frac{1}{2}} b_\lambda^k (1 - \bar{\lambda}z)^{-1} .$$

Then  $\|f\|_2^2 = n$ ,

2) Since the spaces  $H^2$  and  $\mathcal{D}$  are rotation invariant, we have  $I\left(\sigma_{n,\lambda}, H^2, B_{2,2}^{\frac{1}{2}}\right) = I\left(\sigma_{n,\mu}, H^2, B_{2,2}^{\frac{1}{2}}\right)$  for every  $\lambda, \mu$  with  $|\lambda| = |\mu| = r$ . Let  $\lambda = -r$ . To get a lower estimate for  $\|f\|_{H_\varphi/b_\lambda^n H_\varphi}$  consider  $g$  such that  $f - g \in b_\lambda^n \text{Hol}(\mathbb{D})$ , i.e. such that  $f \circ b_\lambda - g \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$ .

3) First, we notice that

$$\begin{aligned} \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 &= \left\| (g \circ b_\lambda)' \right\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \|b_\lambda \cdot (g' \circ b_\lambda)\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \\ &= \int_{\mathbb{D}} |b_\lambda(u)|^2 |g'(b_\lambda(u))|^2 du + \|g \circ b_\lambda\|_{H^2}^2 = \int_{\mathbb{D}} |g'(w)|^2 dw + \|g \circ b_\lambda\|_{H^2}^2 , \end{aligned}$$

using the changing of variables  $w = b_\lambda(u)$ . We get

$$\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|g'\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \|g\|_{B_{2,2}^{\frac{1}{2}}}^2 + \|g \circ b_\lambda\|_{H^2}^2 - \|g\|_{H^2}^2 ,$$

and

$$\begin{aligned} \|g\|_{B_{2,2}^{\frac{1}{2}}}^2 &= \|g\|_{H^2}^2 + \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 = \\ &\geq \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 . \end{aligned}$$

Now, we notice that



$$\begin{aligned}
f \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1 - |\lambda|^2)^{\frac{1}{2}}}{1 - \bar{\lambda} b_\lambda(z)} = (1 - |\lambda|^2)^{-\frac{1}{2}} \left( 1 + (1 - \bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) = \\
&= (1 - r^2)^{-\frac{1}{2}} \left( 1 + (1 + r) \sum_{k=1}^{n-1} z^k + r z^n \right).
\end{aligned}$$

4) Next,

$$\begin{aligned}
\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 &= \sum_{k \geq 1} k \left| \widehat{g \circ b_\lambda}(k) \right|^2 \geq \\
&\geq \sum_{k=1}^{n-1} k \left| \widehat{g \circ b_\lambda}(k) \right|^2 = \sum_{k=1}^{n-1} k \left| \widehat{f \circ b_\lambda}(k) \right|^2,
\end{aligned}$$

since  $\widehat{g \circ b_\lambda}(k) = \widehat{f \circ b_\lambda}(k)$ ,  $\forall k \in [0, n-1]$ . This gives

$$\begin{aligned}
\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 &\geq \frac{1}{1 - r^2} \left( (1 + r)^2 \sum_{k=1}^{n-1} k \right) = \\
&= \frac{(1 + r)^2 n(n-1)}{1 - r^2} \frac{1}{2} = \frac{1 + r}{1 - r} \frac{n(n-1)}{2} = \frac{1 + r}{1 - r} \frac{(n-1)}{2} \|f\|_{H^2}^2,
\end{aligned}$$

for all  $n \geq 2$  since  $\|f\|_{H^2}^2 = n$  ( $f$  is indeed the sum of  $n$  elements of  $H^2$  which are an orthonormal family known as Malmquist's basis, see see [N1] p. 117). Finally,

$$\|g\|_{B_{2,2}^{\frac{1}{2}}}^2 \geq \frac{n}{1 - r} \frac{1 + r}{2} \left( 1 - \frac{1}{n} \right) \|f\|_{H^2}^2.$$

In particular,

$$\mathcal{I} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \geq \sqrt{\frac{n}{1 - r} \left[ \frac{1 + r}{2} \left( 1 - \frac{1}{n} \right) \right]^{\frac{1}{2}}}. \quad \square$$

### III. Bernstein-type inequalities in weighted Bergman spaces

The aim of this Section is to prove Theorem B.

#### Proof of Theorem B.

**Step 1.** We first prove inequality (12) from Theorem B. Let  $f \in \mathcal{R}_{n,r}$  with  $r \in (0, 1)$  and  $n \geq 1$ . Let also  $\rho \in (0, 1)$  and  $f_\rho : w \mapsto f(\rho w)$ . Then,

$$\|f'\|_{L_a^p(\beta)}^p = \frac{1}{\pi} \int_{\mathbb{D}} |f'(w)|^p (1 - |w|^2)^\beta dx dy =$$

$$\begin{aligned}
&= 2 \int_0^1 \int_{\mathbb{T}} |f'_\rho(\zeta)|^p (1 - \rho^2)^\beta \rho dm(\zeta) d\rho = \\
&= 2 \int_0^1 (1 - \rho^2)^\beta \rho \left( \int_{\mathbb{T}} |f'_\rho(\zeta)|^p dm(\zeta) \right) d\rho = \\
&= 2 \int_0^1 (1 - \rho^2)^\beta \rho \left( \|f'_\rho\|_{H^p}^p \right) d\rho.
\end{aligned}$$

Now using the fact  $f_\rho \in \mathcal{R}_{n,\rho r} \subset \mathcal{R}_{n,r}$  for every  $\rho \in (0, 1)$ , we get

$$\begin{aligned}
\|f'\|_{L_a^p(\beta)}^p &\leq 2 \int_0^1 (1 - \rho^2)^\beta \rho (\mathcal{C}_{n,r}(H^p) \|f_\rho\|_{H^p})^p d\rho = \\
&= 2 (\mathcal{C}_{n,r}(H^p))^p \int_0^1 (1 - \rho^2)^\beta \rho \|f_\rho\|_{H^p}^p d\rho = \\
&= 2 (\mathcal{C}_{n,r}(H^p))^p \int_0^1 (1 - \rho^2)^\beta \rho \int_{\mathbb{T}} |f_\rho(\zeta)|^p dm(\zeta) d\rho = \\
&= 2 (\mathcal{C}_{n,r}(H^p))^p \int_0^1 \int_{\mathbb{T}} |f_\rho(\zeta)|^p (1 - \rho^2)^\beta \rho dm(\zeta) d\rho = 2 (\mathcal{C}_{n,r}(H^p))^p \|f\|_{L_a^p(\beta)}^p,
\end{aligned}$$

which proves (12). In particular, using the right-hand inequality from K. Dyakonov's result [Dy2], we get

$$\mathcal{C}_{n,r}(L_a^p(\beta)) \leq c_p \frac{n}{1-r},$$

for all  $p \in [1, \infty]$ , and  $\beta \in (-1, \infty)$ , where  $c_p$  is a constant depending on  $p$  only.

**Step 2.** We now prove statement (11) of Theorem B and more precisely the left-hand side inequality from (11), since the right-hand side one has already been proved in Step 1. Let  $\beta \in (-1, \infty)$ ,  $r \in [0, 1)$ ,  $n \in \mathbb{N}$  such that  $n > \beta + 2$  and  $f_n = \frac{1}{(1-rz)^n}$ . Then

$$\|f_n\|_{L_a^p(\beta)}^p = \frac{1}{\pi} \int_{\mathbb{D}} \frac{(1 - |z|^2)^\beta}{|1 - rz|^{pn}} dx dy,$$

and since  $f'_n = \frac{rn}{(1-rz)^{n+1}}$ , we get

$$\|f'_n\|_{L_a^p(\beta)}^p = \frac{1}{\pi} r^p n^p \int_{\mathbb{D}} \frac{(1 - |z|^2)^\beta}{|1 - rz|^{p(n+1)}} dx dy.$$

Now according to Z. Wu [Wu] we set

$$I_{t,\beta}(r) = \int_{\mathbb{D}} \frac{(1 - |z|^2)^\beta}{|1 - rz|^t} dx dy,$$

and recall that (see [Wu] Section 3, p.155)

$$I_{t,\beta}(r) \asymp \frac{1}{(1 - r^2)^{t-\beta-2}},$$

for every  $t$  such that  $t > \beta + 2$ . Now,

$$\frac{\|f'_n\|_{L_a^p(\beta)}^p}{\|f_n\|_{L_a^p(\beta)}^p} = r^p n^p \frac{I_{p(n+1),\beta}(r)}{I_{pn,\beta}(r)} \asymp r^p n^p \frac{\frac{1}{(1-r^2)^{pn+p-\beta-2}}}{\frac{1}{(1-r^2)^{pn-\beta-2}}} = r^p \left( \frac{n}{1-r^2} \right)^p.$$

In particular there exists a positive constant  $c_{p,\beta}$  such that,

$$\mathcal{C}_{n,r}(L_a^p(\beta)) \geq c_{p,\beta} \frac{r}{1+r} \frac{n}{1-r}.$$

Therefore, if  $r \in [\frac{1}{2}, 1)$ ,

$$\mathcal{C}_{n,r}(L_a^p(\beta)) \geq \frac{1}{3} c_{p,\beta} \frac{n}{1-r}.$$

Now if  $r \in [0, \frac{1}{2})$ , then we first notice that  $n \leq \frac{n}{1-r} \leq 2n$ , we simply take  $g_n = z^{n-1} \in K_{z^n} = \mathcal{P}_{n-1} = \mathcal{R}_{n,0}$  and recalling that for every real numbers  $t, l$

$$\int_0^1 \rho^t (1-\rho^2)^l d\rho = \frac{1}{2} \frac{\Gamma(l+1)\Gamma(\frac{1}{2}t + \frac{1}{2})}{\Gamma(\frac{1}{2}t + \frac{3}{2} + l)},$$

where  $\Gamma$  stands for the usual Gamma function  $\Gamma(z) = \int_0^{+\infty} e^{-s} s^{z-1} ds$ , we get

$$\begin{aligned} \|g_n\|_{L_a^p(\beta)}^p &= 2 \int_0^1 \int_{\mathbb{T}} |\rho\zeta|^{p(n-1)} (1-\rho^2)^\beta \rho dm(\zeta) d\rho = \\ &= 2 \int_0^1 \rho^{p(n-1)+1} (1-\rho^2)^\beta d\rho = \frac{\Gamma(\beta+1)\Gamma(\frac{p(n-1)+1}{2} + \frac{1}{2})}{\Gamma(\frac{p(n-1)+1}{2} + \frac{3}{2} + \beta)}, \end{aligned}$$

on one hand.

And on the other hand,

$$\begin{aligned} \|g'_n\|_{L_a^p(\beta)}^p &= 2(n-1) \int_0^1 \int_{\mathbb{T}} |\rho\zeta|^{p(n-2)} (1-\rho^2)^\beta \rho dm(\zeta) d\rho = \\ &= 2(n-1) \int_0^1 \rho^{p(n-2)+1} (1-\rho^2)^\beta d\rho = \frac{\Gamma(\beta+1)\Gamma(\frac{p(n-2)+1}{2} + \frac{1}{2})}{\Gamma(\frac{p(n-2)+1}{2} + \frac{3}{2} + \beta)}. \end{aligned}$$

Now,

$$\frac{\|g'_n\|_{L_a^p(\beta)}^p}{\|g_n\|_{L_a^p(\beta)}^p} = (n-1) \frac{\Gamma(\frac{p(n-1)+1}{2} + \frac{3}{2} + \beta)}{\Gamma(\frac{p(n-1)+1}{2} + \frac{1}{2})} \frac{\Gamma(\frac{p(n-2)+1}{2} + \frac{1}{2})}{\Gamma(\frac{p(n-2)+1}{2} + \frac{3}{2} + \beta)} \geq c'_{p,\beta} n,$$

for every  $n \geq 2$ , where  $c'_{p,\beta}$  is a positive constant depending on  $p$  and  $\beta$  only. (The case  $n = 1$  is obvious). Then if  $r \in [0, \frac{1}{2})$ ,

$$\mathcal{C}_{n,r}(L_a^p(\beta)) \geq c'_{p,\beta} n \geq \frac{c'_{p,\beta}}{2} 2n \geq \frac{c'_{p,\beta}}{2} \frac{n}{1-r},$$

since  $\frac{n}{1-r} \leq 2n$ . Finally,

$$\mathcal{C}_{n,r}(L_a^p(\beta)) \geq \begin{cases} \frac{c_{p,\beta}}{3} \frac{n}{1-r} & \text{if } r \in [\frac{1}{2}, 1) \\ \frac{c'_{p,\beta}}{2} \frac{n}{1-r} & \text{if } r \in [0, \frac{1}{2}) \end{cases} \geq \min\left(\frac{c_{p,\beta}}{3}, \frac{c'_{p,\beta}}{2}\right) \frac{n}{1-r},$$

for all  $r \in [0, 1)$ , which completes the proof of Step 2.

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## References

- [BL1] L. Baratchart, *Rational and meromorphic approximation in  $L_p$  of the circle : system-theoretic motivations, critical points and error rates*. In N. Papamichael, S. Ruscheweyh, and E. Saff, editors, *Computational Methods and Function Theory*, pages 45–78. World Scientific Publish. Co, (1999).
- [BL2] L. Baratchart, F. Wielonsky, *Rational approximation problem in the real Hardy space  $H_2$  and Stieltjes integrals : a uniqueness theorem*, *Constr. Approx.* 9 (1993), 1-21.
- [BoEr] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer, New York, (1995).
- [Go] A. A. Gonchar, *Degree of approximation by rational fractions and properties of functions*, *Proc. Internat. Congr. Math. (Moscow, 1966 (“Mir”, Moscow), 1968, 329–356 ; English transl. Amer. Math. Soc. Transl. (2), 91 (1970)*
- [Da] V. I. Danchenko, *An integral estimate for the derivative of a rational function*, *Izv. Akad. Nauk SSSR Ser. Mat.*, 43 (1979), 277–293; English transl. *Math. USSR Izv.*, 14 (1980)
- [Do] E. P. Dolzhenko, *Bounds for derivatives of rational functions*, *Izv. Akad. Nauk SSSR Ser. Mat.*, 27 (1963), 9-28, (Russian)
- [Dy1] K. M. Dyakonov, *Smooth functions in the range of a Hankel operator*, *Indiana Univ. Math. J.* 43 (1994), 805-838.
- [Dy2] K. M. Dyakonov, *Differentiation in Star-Invariant Subspaces I. Boundedness and Compactness*, *J.Funct.Analysis*, 192 (2002), 364-386.
- [HeKoZh] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, *Grad. Texts in Math.* 199, Springer, New York, (2000).
- [N1] N.Nikolski, *Treatise on the shift operator*, Springer-Verlag, Berlin etc., 1986 (Transl. from Russian, *Lekzii ob opereore sdviga*, “Nauja”, Moskva, 1980).
- [N2] N.Nikolski, *Operators, Function, and Systems : an easy reading*, Vol.1, Amer. Math. Soc. Monographs and Surveys, 2002.
- [Pel] V. V. Peller, *Hankel operators of class  $S_p$  and their applications (rational approximations, Gaussian processes, the problem of majorizing operators)*, *Mat. Sb.*, 113(155) (1980), 538–581; English transl. *Math. USSR Sb.*, 41 (1982)

- [Pek] A. A. Pekarskii, *Inequalities of Bernstein type for derivatives of rational functions, and inverse theorems of rational approximation*, Math. USSR-Sb.52 (1985), 557-574.
- [Ru] V. N. Rusak, *Rational functions as approximation apparatus*, (Izdat. Beloruss. Gos. Univ., Minsk.), (1979), (Russian)
- [Wu] Z. Wu, *Carleson measures and multipliers for Dirichlet spaces*, J. Funct. Anal. 169 , no. 1, 148-163, (1999)
- [Z1] R. Zarouf, *Asymptotic sharpness of a Bernstein-type inequality for rational functions in  $H^2$* , to appear in St. Petersburg. Math. Journal (2009).
- [Z2] R. Zarouf, *Effective  $H^\infty$  interpolation constrained by Hardy and Bergman norms*, submitted.
- [Z3] R. Zarouf, *Effective  $H^\infty$  interpolation constrained by Hardy and Bergman weighted norms*, submitted.
- [Z4] R. Zarouf, *A resolvent estimate for operators with finite spectrum*, submitted.
- [Z5] R. Zarouf, *Sharpening a result by E.B. Davies and B. Simon*, C. R. Acad. Sci. Paris, Ser. I 347 (2009).
- [Zhu] K. H. Zhu, *Operator theory in function spaces*, Monographs and Textbooks in Pure and Applied Mathematics 139, Marcel Dekker Inc., New York, (1990).

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