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Bernstein-type inequalities for rational functions in weighted Bergman spaces and applications

RACHID ZAROUF

Abstract

Two Bernstein-type inequalities in the standard Bergman space L_a^2 of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, for rational functions in \mathbb{D} having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, $0 < r < 1$, are considered. The asymptotic sharpness is shown as $n \rightarrow \infty$ and $r \rightarrow 1$. We apply the first Bernstein-type inequality to an effective estimate of a Nevanlinna-Pick type interpolation constant in the Dirichlet space, constrained by the H^2 norm, H^2 being the classical Hardy space. We also extend our second Bernstein-type inequality to radial-weighted Bergman spaces.

I. Introduction

Surveys of inequalities for the derivatives of rational functions are given by Gonchar [Go] and Rusak [Ru]. Chapter 7 of [BoEr] is also devoted to this topic. Here, we present such inequalities for rational functions f of degree n with poles in $\mathbb{D}_- = \{z : |z| > 1\}$, involving weighted-Bergman norms.

Let $L_a^2 = l_a^2\left(\frac{1}{\sqrt{k+1}}\right)$ be the Bergman space of all $f(z) = \sum_{k \geq 0} \hat{f}(k)z^k$ satisfying

$$\|f\|_{L_a^2} =: \left(\sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{k+1} \right)^{\frac{1}{2}} < \infty.$$

An equivalent description of this space is the space of holomorphic functions f in \mathbb{D} such that

$$\|f\|_{L_a^2} = \left(\int_{\mathbb{D}} |f(z)|^2 dA \right)^{\frac{1}{2}} < \infty,$$

where dA stands for the area measure. Our first result (see Section 1) is that if f is a rational function in \mathbb{D} having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, then

$$\|f'\|_{L_a^2} \leq \tilde{A}(n, r) \sqrt{\frac{n}{1-r}} \|f\|_{H^2}, \tag{1}$$

where $\tilde{A}(n, r) \leq \left(1 + r + \frac{1}{\sqrt{n}}\right)^{\frac{1}{2}}$ and H^2 stands for the standard Hardy space of the unit disc \mathbb{D} ,

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) < \infty \right\},$$

m being the Lebesgue normalized measure on \mathbb{T} .

Now let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional space defined by

$$K_{B_\sigma} = \mathcal{L}in(k_{\lambda_i} : i = 1 \dots n),$$

where σ is a family of distincts elements of \mathbb{D} , and where $k_\lambda = \frac{1}{1-\bar{\lambda}z}$ is the Szegő kernel associated to λ . An obvious modification allows to generalize the definition of K_{B_σ} in the case where the sequence σ admits multiplicities.

Notice that using the scalar product $(\cdot, \cdot)_{H^2}$ on H^2 , an equivalent description of this space is:

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Notice also that every rational functions with poles in \mathbb{D}_- lies in a space K_{B_σ} . More precisely, if f is a rational function in \mathbb{D} having n poles $\{\mu_1, \dots, \mu_n\}$ all outside of $\frac{1}{r}\mathbb{D}$, then $f \in K_{B_\sigma}$ where $\sigma = \{\lambda_1, \dots, \lambda_n\}$ and $\lambda_i = \frac{1}{\bar{\mu}_i}$ for all $i \in [1, n]$. It has already been proved in [Z1] that setting $r = \max_j |\lambda_j|$, and $f \in K_{B_\sigma}$, then

$$\|f'\|_{H^2} \leq C_{n,r} \|f\|_{H^2}, \quad (2)$$

and

$$a(n, r) \frac{n}{1-r} \leq C_{n,r} \leq A(n, r) \frac{n}{1-r},$$

for all $n \geq 2$, where

$$a(n, r) \geq \frac{1}{1+r} \left(1 + 5r^4 - \frac{4r^4}{n} - \min\left(\frac{3}{4}, \frac{2}{n}\right) \right)^{\frac{1}{2}},$$

and

$$A(n, r) \leq 1 + r + \frac{1}{\sqrt{n}}.$$

Moreover, it is shown in [Z1] that the sequence

$$\left(\frac{1}{n} C_{n,r} \right)_{n \geq 1},$$

is convergent for each $r \in [0, 1)$, and

$$\lim_{n \rightarrow \infty} \frac{1}{n} C_{n,r} = \frac{1+r}{1-r},$$

for all $r \in [0, 1)$. In the same spirit, we prove in Section 1, the asymptotic sharpness - in inequality (1) - of the ratio $\sqrt{\frac{n}{1-r}}$ as $n \rightarrow \infty$, for all $r \in [0, 1)$, in the following sense : we find an inequality for

$$\sup \|D\|_{(K_B, \|\cdot\|_{L_a^2}) \rightarrow H^2} = \tilde{C}_{n,r},$$

(\sup is over all B with given $n = \deg B$ and $r = \max_{\lambda \in \sigma} |\lambda|$), which is asymptotically sharp as $n \rightarrow \infty$, for all for all $r \in [0, 1)$. Moreover, there exists a limit

$$\lim_{n \rightarrow \infty} \frac{\tilde{C}_{n,r}}{\sqrt{n}} = \sqrt{\frac{1+r}{1-r}},$$

for every r , $0 \leq r < 1$.

Now it is important to link and motivate such an inequality with one of Dolzhenko's inequality [Do] : let f be a rational function of degree n with poles in \mathbb{D}_- and let $B_{2,2}^{\frac{1}{2}}$ be the analytic Besov space of all $f(z) = \sum_{k \geq 0} \hat{f}(k)z^k$ satisfying

$$\|f\|_{B_{2,2}^{\frac{1}{2}}} =: \left(\sum_{k \geq 0} (k+1) |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty,$$

then

$$\|f\|_{B_{2,2}^{\frac{1}{2}}} \leq c\sqrt{n} \|f\|_{H^\infty}, \quad (3)$$

where $\|f\|_{H^\infty} = \sup_{\zeta \in \mathbb{T}} |f(\zeta)|$ and c is a numerical constant. Notice that on one hand, inequality (3) was generalized by Danchenko [Da] for Hardy-Besov spaces. On the other hand, another generalization of (3) follows from a result of Peller [Pel] on best rational approximations for some class of Besov spaces, in BMOA : the space of all analytic functions of the bounded mean oscillation on \mathbb{D} . Later, Pekarskii [Pek] generalized and strengthened the results of Peller and Danchenko. An interesting motivation for such inequalities is to apply it to “ inverse theorems of rational approximation ”.

Here, the link between inequalities (1) and (3) is that $\|f'\|_{L_a^2}$ is comparable with $\|f\|_{B_{2,2}^{\frac{1}{2}}}$. Indeed,

$$\|f\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|f'\|_{L_a^2}^2 + \|f\|_{H^2}^2.$$

In particular, a corollary of our result is that if f is a rational functions in \mathbb{D} having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, then

$$\|f\|_{B_{2,2}^{\frac{1}{2}}} \leq \tilde{B}(n, r) \sqrt{\frac{n}{1-r}} \|f\|_{H^2}, \quad (4)$$

where $\sup_n \tilde{B}(n, r) \leq \sqrt{1+r}$ for all $r \in [0, 1)$, which is a Dolzhenko-type inequality and could be compared with (3) : the H^∞ -norm is replaced by the H^2 -norm which is smaller, but another parameter “ r ” appears with a factor $\frac{1}{\sqrt{1-r}}$. In the same spirit, we find an inequality for

$$\sup \|Id\|_{(K_B, \|\cdot\|_{L_a^2}) \rightarrow H^2} = \tilde{B}_{n,r},$$

(Id is the identity operator and \sup is over all B with given $n = \deg B$ and $r = \max_{\lambda \in \sigma} |\lambda|$), which is asymptotically sharp as $n \rightarrow \infty$ for all $r \in [0, 1)$. Indeed,

$$\tilde{B}_{n,r} = \tilde{C}_{n,r} + 1.$$

In particular, there exists a limit

$$\lim_{n \rightarrow \infty} \frac{\tilde{B}_{n,r}}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\tilde{C}_{n,r}}{\sqrt{n}} = \sqrt{\frac{1+r}{1-r}},$$

for every r , $0 \leq r < 1$.

In Section 2, we apply inequality (1) to an effective estimate of a Nevanlinna-Pick type interpolation constant in the Dirichlet space, constrained by the H^2 -norm.

In Section 3, we give a Bernstein-type inequality for the same class of rational functions f in \mathbb{D} having at most n poles all outside of $\frac{1}{r}\mathbb{D}$,

$$\|f'\|_{L_a^2} \leq B(n, r) \frac{n}{1-r} \|f\|_{L_a^2}, \quad (5)$$

where $B(n, r) = 20^{\frac{1}{4}} \left(1 + r + \frac{1}{\sqrt{n}}\right)$. We prove that inequality (5) is asymptotically sharp as $n \rightarrow \infty$ and $r \rightarrow 1$ in the following sense : if $n \geq 2$, there exists $f_n \in H_\alpha$ such that

$$\frac{\|f_n'\|_{L_a^2}}{\|f_n\|_{L_a^2}} \sim \frac{1}{2} \frac{n}{1-r},$$

as r tends to 1. Finally, in Section 4, we generalize inequality (5) replacing the Bergman norm $\|\cdot\|_{L_a^2}$ by weighted-Bergman norms $\|\cdot\|_{H_\alpha}$ for positive α . We recall that the space $H_\alpha = l_a^2 \left(\frac{1}{(k+1)^\alpha}\right)$, $\alpha \geq 0$ is defined to be the Hardy weighted space of all $f(z) = \sum_{k \geq 0} \hat{f}(k) z^k$ satisfying

$$\|f\|_{H_\alpha} =: \left(\sum_{k \geq 0} \frac{1}{(k+1)^{2\alpha}} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty.$$

It is also important to recall that

$$H_\alpha = L_a^2 \left((1 - |z|^2)^{2\alpha-1} dA \right), \quad \alpha > 0,$$

where $L_a^2 \left((1 - |z|^2)^\beta dA \right)$, $\beta > -1$, stand for the Bergman weighted spaces of all holomorphic functions f such that

$$\|f\|_{H_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\beta dA \right)^{\frac{1}{2}} < \infty.$$

Notice also that $H^2 = H_0$ and $L_a^2 = H_{-\frac{1}{2}}$. We get

$$\|f'\|_{H_\alpha} \leq K_\alpha \frac{n}{1-r} \|f\|_{H_\alpha}, \quad (6)$$

where K_α is a constant depending on α only. We prove that inequality (6) is asymptotically sharp as $n \rightarrow \infty$ and $r \rightarrow 1$ in the following sense : if $n > \alpha + 1$, there exists $f_n \in H_\alpha$ such that

$$\frac{\|f_n'\|_{H_\alpha}}{\|f_n\|_{H_\alpha}} \sim \frac{1}{2} \frac{n}{1-r},$$

as r tends to 1.

II. A first Bernstein-type inequality in L_a^2

Theorem 2.1

Let $n \geq 1$, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , and B_σ the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional subspace of H^2 defined by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Let D be the operator of differentiation on $(K_{B_\sigma}, \|\cdot\|_{L_a^2})$:

$$D : (K_{B_\sigma}, \|\cdot\|_{L_a^2}) \rightarrow (H^2, \|\cdot\|_{H^2})$$

$$f \mapsto f'.$$

For $r \in [0, 1)$ and $n \geq 1$, we set

$$\tilde{C}_{n,r} = \sup \left\{ \|D\|_{K_{B_\sigma} \rightarrow H^2} : 1 \leq \#\sigma \leq n, |\lambda| \leq r \forall \lambda \in \sigma \right\}.$$

Then, if $n = 1$ and $\sigma = \{\lambda\}$, we have

$$\|D\|_{K_{B_\sigma} \rightarrow H^2} = |\lambda|.$$

If $n \geq 2$, we have

$$\tilde{C}_{n,r} \leq \sqrt{C_{n,r}},$$

and

$$\tilde{a}(n, r) \sqrt{\frac{n}{1-r}} \leq \tilde{C}_{n,r} \leq \tilde{A}(n, r) \sqrt{\frac{n}{1-r}},$$

where

$$\tilde{a}(n, r) \geq \left(1 - \frac{1-r}{n}\right)^{\frac{1}{2}},$$

and

$$\tilde{A}(n, r) \leq \left(1 + r + \frac{1}{\sqrt{n}}\right)^{\frac{1}{2}}.$$

(ii) Moreover, the sequence

$$\left(\frac{1}{\sqrt{n}} \tilde{C}_{n,r}\right)_{n \geq 1},$$

is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \tilde{C}_{n,r} = \sqrt{\frac{1+r}{1-r}},$$

for all $r \in [0, 1)$.

Proof.

Proof of (i). **The case $n = 1$.** In this case, $K_B = \mathbb{C}e_1$, where

$$e_1 = \frac{(1 - |\lambda|^2)^{\frac{1}{2}}}{1 - \bar{\lambda}z}, \quad |\lambda| \leq r,$$

(e_1 being of norm 1 in H^2). Calculating,

$$e_1' = \frac{\bar{\lambda}(1 - |\lambda|^2)^{\frac{1}{2}}}{(1 - \bar{\lambda}z)^2},$$

and

$$\begin{aligned} \|e_1'\|_{L_a^2} &= |\lambda|(1 - |\lambda|^2)^{\frac{1}{2}} \left\| \frac{1}{(1 - \bar{\lambda}z)^2} \right\|_{L_a^2} = \\ &= |\lambda|(1 - |\lambda|^2)^{\frac{1}{2}} \left(\sum_{k \geq 0} \frac{k+1}{k+1} |\lambda|^{2k} \right)^{\frac{1}{2}} = |\lambda|(1 - |\lambda|^2)^{\frac{1}{2}} \frac{1}{(1 - |\lambda|^2)} = |\lambda|, \end{aligned}$$

we get

$$\|D|_{K_{B\sigma}}\| = |\lambda|.$$

The case $n \geq 2$. First, we show the right hand side inequality. Using both Cauchy-Schwarz inequality and the fact that $\widehat{f}'(k) = (k+1)\widehat{f}(k+1)$ for all $k \geq 0$, we get

$$\begin{aligned} \|f'\|_{L_a^2}^2 &= \sum_{k \geq 0} \frac{|\widehat{f}'(k)|^2}{k+1} = \sum_{k \geq 0} \frac{(k+1)^2 |\widehat{f}(k+1)|^2}{k+1} = \\ &= \sum_{k \geq 1} k |\widehat{f}(k)|^2 \leq \left(\sum_{k \geq 1} k^2 |\widehat{f}(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k \geq 1} |\widehat{f}(k)|^2 \right)^{\frac{1}{2}} = \\ &= \|f'\|_{H^2} \|f\|_{H^2} \leq C_{n,r} \|f\|_{H^2}^2, \end{aligned}$$

and hence,

$$\|f'\|_{L_a^2} \leq \sqrt{C_{n,r}} \|f\|_{H^2},$$

which means

$$\tilde{C}_{n,r} \leq \sqrt{C_{n,r}}.$$

Now, we show the left-hand-side one. Let

$$e_n = \frac{(1 - r^2)^{\frac{1}{2}}}{1 - rz} b_r^{n-1}.$$

Then $e_n \in K_{b_r^n}$ and $\|e_n\|_2 = 1$, (see [N1], Malmquist-Walsh Lemma, p.116). Moreover,

$$\begin{aligned} e_n' &= \frac{r(1 - r^2)^{\frac{1}{2}}}{(1 - rz)^2} b_r^{n-1} + (n-1) \frac{(1 - r^2)^{\frac{1}{2}}}{1 - rz} b_r' b_r^{n-2} = \\ &= -\frac{r}{(1 - r^2)^{\frac{1}{2}}} b_r' b_r^{n-1} + (n-1) \frac{(1 - r^2)^{\frac{1}{2}}}{1 - rz} b_r' b_r^{n-2}, \end{aligned}$$

since $b'_r = \frac{r^2-1}{(1-rz)^2}$. Then,

$$e'_n = b'_r \left[-\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-2} \right],$$

and

$$\begin{aligned} \|e'_n\|_{L_a^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{D}} \left| b'_r(w) \right| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} (b_r(w))^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} (b_r(w))^{n-2} \right|^2 dm(w) = \\ &= \frac{1}{2\pi} \int_{\mathbb{D}} \left| b'_r(w) \right|^2 |(b_r(w))^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r(w) + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} \right|^2 dm(w), \end{aligned}$$

which gives, using the variables $u = b_r(w)$,

$$\|e'_n\|_{L_a^2}^2 = \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rb_r(u)} \right|^2 dm(u).$$

But $1 - rb_r = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$ and $b'_r \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2}$. This implies

$$\begin{aligned} \|e'_n\|_{L_a^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-r^2} (1-ru) \right|^2 dm(u) = \\ &= \frac{1}{(1-r^2)} \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 |(-ru + (n-1)(1-ru))|^2 dm(u), \end{aligned}$$

which gives

$$\|e'_n\|_{L_a^2} = \frac{1}{(1-r^2)^{\frac{1}{2}}} \|\varphi_n\|_2,$$

where $\varphi_n = z^{n-2}(-rz + (n-1)(1-rz))$. Expanding, we get

$$\begin{aligned} \varphi_n &= z^{n-2}(-rz + n - 1 + rz - nrz) = \\ &= z^{n-2}(-nrz + n - 1) = (n-1)z^{n-2} - nrz^{n-1}, \end{aligned}$$

and

$$\begin{aligned} \|e'_n\|_{L_a^2}^2 &= \frac{1}{(1-r^2)} \left(\frac{(n-1)^2}{n-1} + \frac{n^2}{n} r^2 \right) = \\ &= \frac{1}{(1-r^2)} (n(1+r) - 1) = \frac{n}{(1-r)(1+r)} \left((1+r) - \frac{1}{n} \right) = \\ &= \frac{n}{1-r} \left(1 - \frac{1-r}{n} \right), \end{aligned}$$

which gives

$$\widetilde{C}_{n,r} \geq \sqrt{\frac{n}{1-r}} \left(1 - \frac{1-r}{n} \right)^{\frac{1}{2}}.$$

Proof of (ii). **Step 1.** We first prove the right-hand-side inequality:

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \widetilde{C}_{n,r} \leq \sqrt{\frac{1+r}{1-r}},$$

which becomes obvious since

$$\frac{1}{\sqrt{n}} \widetilde{C}_{n,r} \leq \frac{1}{\sqrt{n}} \sqrt{C_{n,r}},$$

and

$$\frac{1}{\sqrt{n}} \sqrt{C_{n,r}} \rightarrow \sqrt{\frac{1+r}{1-r}},$$

as n tends to infinity.

Step 2. We now prove the left-hand-side inequality :

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \widetilde{C}_{n,r} \geq \sqrt{\frac{1+r}{1-r}}.$$

More precisely, we show that

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|D\|_{(K_{b_r^n}, \|\cdot\|_{L_a^2}) \rightarrow H^2} \geq \sqrt{\frac{1+r}{1-r}}.$$

Let $f \in K_{b_r^n}$. Then,

$$\begin{aligned} f' &= (f, e_1)_{H^2} \frac{r}{(1-rz)} e_1 + \sum_{k=2}^n (k-1) (f, e_k)_{H^2} \frac{b_r'}{b_r} e_k + r \sum_{k=2}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k = \\ &= r \sum_{k=1}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k + \frac{1-r^2}{(1-rz)(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} e_k = \\ &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{3}{2}}}{(1-rz)^2(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} = \\ &= -b_r' \left[\frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{z-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right]. \end{aligned}$$

Now using the change of variables $v = b_r(u)$, we get

$$\begin{aligned} \|f'\|_{L_a^2}^2 &= \int_{\mathbb{D}} \left| b_r'(u) \right|^2 \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{u-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right|^2 du = \\ &= \int_{\mathbb{D}} \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{b_r(v)-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv. \end{aligned}$$

But

$$b_r - r = \frac{r - z - r(1 - rz)}{1 - rz} = \frac{z(r^2 - 1)}{1 - rz},$$

which gives

$$\begin{aligned}
\|f'\|_{L_a^2}^2 &= \int_{\mathbb{D}} \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{v(r^2-1)} (1-rv) \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv = \\
&= \frac{1}{1-r^2} \int_{\mathbb{D}} \left| r \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} - (1-rv) \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-2} \right|^2 dv = \\
&= \frac{1}{1-r^2} \int_{\mathbb{D}} \left| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k - (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right|^2 dv.
\end{aligned}$$

But

$$\begin{aligned}
\left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} &\leq r \left(\sum_{k=0}^{n-1} \frac{1}{k+1} |(f, e_{k+1})_{H^2}|^2 \right)^{1/2} \leq \\
&\leq r \|f\|_{H^2},
\end{aligned}$$

and in particular

$$\lim_{n \rightarrow \infty} \frac{1}{n \|f\|_{H^2}} \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} = 0.$$

Now,

$$\begin{aligned}
&(1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k = \\
&= \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k - r \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^{k+1} = \\
&= \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k - r \sum_{k=1}^{n-1} k (f, e_{k+1})_{H^2} v^k = \\
&= (f, e_2)_{H^2} + 2(f, e_3)_{H^2} v + \sum_{k=2}^{n-2} [(k+1) (f, e_{k+2})_{H^2} - rk (f, e_{k+1})_{H^2}] v^k + \\
&\quad -r [(f, e_2)_{H^2} v + (n-1) (f, e_n)_{H^2} v^{n-1}] = \\
&= (f, e_2)_{H^2} + [(f, e_3)_{H^2} - r(f, e_2)_{H^2}] v + \sum_{k=2}^{n-2} [(k+1) (f, e_{k+2})_{H^2} - rk (f, e_{k+1})_{H^2}] v^k + \\
&\quad -r(n-1) (f, e_n)_{H^2} v^{n-1}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{\|f\|_{H^2} \sqrt{n(1-r^2)}} \left[\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} + \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \right] &\geq \\
&\geq \frac{1}{\sqrt{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq
\end{aligned}$$

$$\geq \frac{1}{\|f\|_{H^2} \sqrt{n(1-r^2)}} \left[\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} - \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \right],$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} = 0,$$

we get that

$$\begin{aligned} \frac{1}{\sqrt{1+r}} \lim_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} &\geq \\ &\geq \lim_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq \\ &\geq \frac{1}{\sqrt{1+r}} \lim_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}. \end{aligned}$$

This gives

$$\lim_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} = \frac{1}{\sqrt{1+r}} \lim_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}.$$

But

$$\begin{aligned} &= (f, e_2)_{H^2} + [(f, e_3)_{H^2} - r(f, e_2)_{H^2}]v + \sum_{k=2}^{n-2} [(k+1)(f, e_{k+2})_{H^2} - rk(f, e_{k+1})_{H^2}]v^k + \\ &\quad -r(n-1)(f, e_n)_{H^2}v^{n-1}. \\ &\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = |(f, e_2)_{H^2}|^2 + \\ &+ \frac{1}{2} |(f, e_3)_{H^2} - r(f, e_2)_{H^2}|^2 + \frac{1}{n} r^4 (n-1)^2 |(f, e_n)_{H^2}|^2 + \\ &+ \sum_{k=2}^{n-2} \left| (f, e_{k+2})_{H^2} - \frac{rk}{k+1} (f, e_{k+1})_{H^2} \right|^2. \end{aligned}$$

Now, let $s = s_n$ be a sequence of even integers such that

- $\lim_{n \rightarrow \infty} s_n = \infty$ and
- $s_n = o(n)$ as $n \rightarrow \infty$.

Then we consider the following function f in K_{b_r} :

$$\begin{aligned} f &= e_n - e_{n-1} + e_{n-2} - e_{n-3} + \dots + (-1)^k e_{n-k} + \dots + e_{n-s} - e_{n-s-1} + e_{n-s-2} = \\ &= \sum_{k=0}^{s+2} (-1)^k e_{n-k}. \end{aligned}$$

With such an f , we get

$$\begin{aligned}
& \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = |(f, e_2)_{H^2}|^2 + \\
& + \frac{1}{2} |(f, e_3)_{H^2} - r(f, e_2)_{H^2}|^2 + \frac{1}{n} r^4 (n-1)^2 |(f, e_n)_{H^2}|^2 + \\
& + \sum_{k=2}^{n-2} (k+1) \left| (f, e_{k+2})_{H^2} - \frac{rk}{k+1} (f, e_{k+1})_{H^2} \right|^2. \\
& \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = \\
& = r^4 \frac{(n-1)^2}{n} + \\
& + \sum_{l=2}^{n-2} (n-l+1) \left| (f, e_{n-l+2})_{H^2} - \frac{r(n-l)}{n-l+1} (f, e_{n-l+1})_{H^2} \right|^2,
\end{aligned}$$

setting the change of index $l = n - k$ in the last sum. This finally gives

$$\begin{aligned}
& \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = \\
& = r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left| 1 + \frac{r(n-l)}{n-l+1} \right|^2 = \\
& = r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left[1 + r \left(1 - \frac{1}{n-l+1} \right) \right]^2
\end{aligned}$$

And

$$\begin{aligned}
& \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 \geq \\
& \geq r^4 \frac{(n-1)^2}{n} + (s+1-2+1)(n-(s+1)+1) \left[1 + r \left(1 - \frac{1}{n-(s+1)+1} \right) \right]^2 = \\
& = r^4 \frac{(n-1)^2}{n} + s(n-s) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2.
\end{aligned}$$

In particular,

$$\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 \geq s(n-s) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2.$$

Now, since

$$\|f\|_{H^2}^2 = s+3 = s_n+3,$$

we get

$$\begin{aligned}
& \underline{\lim}_{n \rightarrow \infty} \frac{1}{n \|f\|_{H^2}^2} \left\| (1 - rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq \\
& \geq \underline{\lim}_{n \rightarrow \infty} \frac{1}{n \|f\|_{H^2}^2} \|f\|_{H^2}^2 (n - \|f\|_{H^2}^2) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2 = \\
& = \underline{\lim}_{n \rightarrow \infty} \left(1 - \frac{s_n}{n} \right) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2 = (1+r)^2.
\end{aligned}$$

We can now conclude that

$$\begin{aligned}
\underline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} &= \frac{1}{\sqrt{1+r}} \underline{\lim}_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1 - rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} \geq \\
&\geq \frac{1+r}{\sqrt{1+r}} = \sqrt{1+r},
\end{aligned}$$

and

$$\underline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \|D\|_{K_{b^2} \rightarrow H^2} \geq \underline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq \sqrt{1+r}.$$

Step 3. Conclusion. Using both **Step 1** and **Step 2**, we get

$$\overline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \widetilde{C}_{n,r} = \underline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \widetilde{C}_{n,r} = 1+r,$$

which means that the sequence $(\frac{1}{n} C_{n,r})_{n \geq 1}$ is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \widetilde{C}_{n,r} = \sqrt{\frac{1+r}{1-r}}. \quad \square$$

Now, in order to “compare” our result to Dolzhenko’s inequality [Do], mentioned in the introduction, we give the following corollary.

Corollary 2.2

Let $n \geq 1$, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , and B_σ the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional subspace of H^2 defined by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Let Id be the identity operator on $(K_{B_\sigma}, \|\cdot\|_{B_{2,2}^{\frac{1}{2}}})$:

$$\begin{aligned}
Id : (K_{B_\sigma}, \|\cdot\|_{B_{2,2}^{\frac{1}{2}}}) &\rightarrow (H^2, \|\cdot\|_{H^2}) \\
f &\mapsto f.
\end{aligned}$$

For $r \in [0, 1)$ and $n \geq 1$, we set

$$\tilde{B}_{n,r} = \sup \left\{ \|D\|_{K_{B_\sigma} \rightarrow H^2} : 1 \leq \#\sigma \leq n, |\lambda| \leq r \forall \lambda \in \sigma \right\}.$$

Then

$$\tilde{B}_{n,r} = \tilde{C}_{n,r} + 1,$$

for all $n \geq 1$ and $r \in [0, 1)$. In particular, the sequence

$$\left(\frac{1}{\sqrt{n}} \tilde{B}_{n,r} \right)_{n \geq 1},$$

is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \tilde{B}_{n,r} = \sqrt{\frac{1+r}{1-r}},$$

for all $r \in [0, 1)$.

Proof. Indeed,

$$\begin{aligned} \|f'\|_{L_a^2}^2 &= \sum_{k \geq 1} k |\hat{f}(k)|^2 = \sum_{k \geq 1} (k+1) |\hat{f}(k)|^2 - \sum_{k \geq 1} |\hat{f}(k)|^2 = \\ &= \left(\|f\|_{B_{2,2}^{\frac{1}{2}}}^2 - |\hat{f}(0)|^2 \right) - \left(\|f\|_{H^2}^2 - |\hat{f}(0)|^2 \right) = \|f\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|f\|_{H^2}^2, \end{aligned}$$

for every $f \in K_B$. Then,

$$\frac{\|f\|_{B_{2,2}^{\frac{1}{2}}}^2}{\|f\|_{H^2}^2} = 1 + \frac{\|f'\|_{L_a^2}^2}{\|f\|_{H^2}^2}.$$

Taking the supremum over all $f \in K_B$ of norm 1 in H^2 , this gives

$$\tilde{B}_{n,r} = \tilde{C}_{n,r} + 1,$$

which completes the proof, applying Theorem 2.1. \square

III. Interpolation in the Dirichlet space

Let X and Y two Banach spaces of holomorphic functions on the unit disc \mathbb{D} such that $X \supset Y$, and a finite set $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$. Following [Z2] and [Z3], we consider the following interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \},$$

and

$$C_{n,r}(X, Y) = \sup \{ c(\sigma, X, Y) : \#\sigma \leq n, \forall j = 1..n, |\lambda_j| \leq r \}.$$

It has already been proved in [Z2] that for $X = H^2$ and $Y = H^\infty$,

$$\frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq c(\sigma_{n,r}, H^2, H^\infty) \leq C_{n,r}(H^2, H^\infty) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}},$$

a question especially stimulated by L. Baratchart (a part of a more complicated question arising in an applied situation in [BL1] and [BL2]) : given a set $\sigma \subset \mathbb{D}$, how to estimate $c(\sigma, H^2, H^\infty)$ in terms of $n = \text{card}(\sigma)$ and $\max_{\lambda \in \sigma} |\lambda| = r$ only?

Here, we replace the algebra H^∞ by the Dirichlet space \mathcal{D} all $f(z) = \sum_{k \geq 0} \hat{f}(k) z^k$ satisfying

$$\|f\|_{\mathcal{D}}^2 := \left| \hat{f}(0) \right|^2 + \sum_{k \geq 1} k \left| \hat{f}(k) \right|^2 < \infty.$$

Notice that \mathcal{D} coincides with the Besov space $B_{2,2}^{\frac{1}{2}}$: the spaces are the same and the norms are equivalent. We show that the ‘‘gap’’ between $X = H^2$ and $Y = H^\infty$ is asymptotically the same as the one which exists between $X = H^2$ and $Y = \mathcal{D}$.

Corollary 3.1. *Let σ be a sequence in \mathbb{D} . Then,*

$$c(\sigma, H^2, \mathcal{D}) \leq \left(\tilde{C}_{n,r}^2 + 1 \right)^{\frac{1}{2}}.$$

Indeed, let $H = \mathcal{D}$ and $B = B_\sigma$ the finite Blaschke product corresponding to σ . We recall that P_B is the orthogonal projection of H onto $K_B = K_B(H^2)$. If $f \in H^2$, let $g = P_B f \in K_B$. Then $g - f \in BH^2$ and

$$\begin{aligned} \|g\|_{\mathcal{D}}^2 &= |\hat{g}(0)|^2 + \sum_{k \geq 1} k |\hat{g}(k)|^2 = \\ &= \left\| g' \right\|_{L_a^2}^2 + |\hat{g}(0)|^2. \end{aligned}$$

Now, using Theorem 2.1,

$$\left\| g' \right\|_{L_a^2}^2 \leq C_{n,r} \|g\|_{H^2}^2,$$

which gives

$$\|g\|_{\mathcal{D}}^2 \leq (C_{n,r} + 1) \|g\|_{H^2}^2.$$

Using the fact that $\|g\|_{H^2} = \|P_B f\|_{H^2} \leq \|f\|_{H^2}$, we finally get

$$\|g\|_{\mathcal{D}} \leq (C_{n,r} + 1)^{\frac{1}{2}} \|f\|_{H^2},$$

and as a result,

$$c(\sigma, H^2, \mathcal{D}) \leq (C_{n,r} + 1)^{\frac{1}{2}}. \quad \square$$

Theorem 3.2. *Let $n \geq 1$, $\lambda \in \mathbb{D}$ and $r \in [0, 1)$. We consider $\sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}$, a one-point set of multiplicity n . Then,*

$$c(\sigma_{n,\lambda}, H^2, \mathcal{D}) \geq \sqrt{\frac{n}{1-|\lambda|}} \left[\frac{(1+|\lambda|)^2 - \frac{2}{n} - \frac{2|\lambda|}{n}}{2(1+|\lambda|)} \right]^{\frac{1}{2}}.$$

In particular,

$$\sqrt{\frac{n}{1-r}} \left[\frac{(1+r)^2 - \frac{2}{n} - \frac{2r}{n}}{2(1+r)} \right]^{\frac{1}{2}} \leq \mathcal{C}_{n,r}(H^2, \mathcal{D}) \leq (C_{n,r} + 1)^{\frac{1}{2}},$$

$$\sqrt{\frac{\frac{1+r}{2}}{1-r}} \leq \liminf_{n \rightarrow \infty} \frac{\mathcal{C}_{n,r}(H^2, \mathcal{D})}{\sqrt{n}} \leq \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{C}_{n,r}(H^2, \mathcal{D})}{\sqrt{n}} \leq \sqrt{\frac{1+r}{1-r}}$$

and

$$\frac{\sqrt{2}}{2} \leq \lim_{r \rightarrow 1} \liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{C}_{n,r}(H^2, \mathcal{D}) \leq \overline{\lim}_{r \rightarrow 1} \overline{\lim}_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{C}_{n,r}(H^2, \mathcal{D}) \leq \sqrt{2}.$$

Proof. 1) We set

$$f = \sum_{k=0}^{n-1} (1-|\lambda|^2)^{\frac{1}{2}} b_\lambda^k (1-\bar{\lambda}z)^{-1}.$$

Then $\|f\|_2^2 = n$,

2) Since the spaces H^2 and \mathcal{D} are rotation invariant, we have $c(\sigma_{n,\lambda}, H^2, \mathcal{D}) = c(\sigma_{n,\mu}, H^2, \mathcal{D})$ for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|f\|_{H_\varphi/b_\lambda^n H_\varphi}$ consider g such that $f - g \in b_\lambda^n \text{Hol}(\mathbb{D})$, i.e. such that $f \circ b_\lambda - g \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$.

3) First, we notice that

$$\begin{aligned} \|g \circ b_\lambda\|_{\mathcal{D}}^2 &= \left\| (g \circ b_\lambda)' \right\|_{L_a^2}^2 + |g(b_\lambda(0))|^2 = \|b_\lambda \cdot (g' \circ b_\lambda)\|_{L_a^2}^2 + |g(-r)|^2 = \\ &= \int_{\mathbb{D}} |b_\lambda(u)|^2 |g'(b_\lambda(u))|^2 du + |g(-r)|^2 = \int_{\mathbb{D}} |g'(w)|^2 dw + |g(-r)|^2, \end{aligned}$$

using the changing of variables $w = b_\lambda(u)$. We get

$$\|g \circ b_\lambda\|_{\mathcal{D}}^2 = \|g'\|_{L_a^2}^2 + |g(-r)|^2 = \|g\|_{\mathcal{D}}^2 + |g(-r)|^2 - |g(0)|^2,$$

and

$$\begin{aligned} \|g\|_{\mathcal{D}}^2 &= \|g \circ b_\lambda\|_{\mathcal{D}}^2 + |g(0)|^2 - |g(-r)|^2 = \\ &= \|g \circ b_\lambda\|_{\mathcal{D}}^2 + |g(0)|^2 - |f(-r)|^2 \geq \\ &\geq \|g \circ b_\lambda\|_{\mathcal{D}}^2 - |f(-r)|^2, \end{aligned}$$

but

$$|f(-r)|^2 = |\langle f, k_{-r} \rangle|^2 \leq \|f\|_{H^2}^2 \|k_{-r}^2\|_{H^2} = \|f\|_{H^2}^2 \frac{1}{1-r^2},$$

which gives

$$\|g\|_{\mathcal{D}}^2 \geq \|g \circ b_\lambda\|_{\mathcal{D}}^2 - \|f\|_{H^2}^2 \frac{1}{1-r^2}.$$

Now, we notice that

$$\begin{aligned} f \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1 - |\lambda|^2)^{\frac{1}{2}}}{1 - \bar{\lambda} b_\lambda(z)} = (1 - |\lambda|^2)^{-\frac{1}{2}} \left(1 + (1 - \bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) = \\ &= (1 - r^2)^{-\frac{1}{2}} \left(1 + (1 + r) \sum_{k=1}^{n-1} z^k + r z^n \right). \end{aligned}$$

4) Next,

$$\begin{aligned} \|g \circ b_\lambda\|_{\mathcal{D}}^2 &= \sum_{k \geq 0} (k+1) \left| \widehat{g \circ b_\lambda}(k) \right|^2 \geq \\ &\geq \sum_{k=0}^{n-1} (k+1) \left| \widehat{g \circ b_\lambda}(k) \right|^2 = \sum_{k=0}^{n-1} (k+1) \left| \widehat{f \circ b_\lambda}(k) \right|^2, \end{aligned}$$

since $\widehat{g \circ b_\lambda}(k) = \widehat{f \circ b_\lambda}(k)$, $\forall k \in [0, n-1]$. This gives

$$\begin{aligned} \|g \circ b_\lambda\|_{\mathcal{D}}^2 &\geq \frac{1}{1-r^2} \left(1 + (1+r)^2 \sum_{k=1}^{n-1} (k+1) + (n+1)r^2 \right) = \\ &= \frac{1}{1-r^2} \left(1 + (1+r)^2 \left(n-1 + \frac{n(n-1)}{2} \right) + (n+1)r^2 \right) = \\ &= \frac{1}{1-r^2} \left(1 + (1+r)^2 \left(\frac{n^2+n-2}{2} \right) + (n+1)r^2 \right) \geq \frac{(1+r)^2}{1-r^2} \left(\frac{n^2+n-2}{2} \right) \geq \\ &\geq \frac{n^2}{1-r^2} \frac{(1+r)^2}{2} = \frac{n}{1-r} \frac{1+r}{2} \|f\|_{H^2}^2, \end{aligned}$$

for all $n \geq 2$, since $\|f\|_{H^2} = n$.

$$\begin{aligned} \|g\|_{\mathcal{D}}^2 &\geq \frac{n}{1-r} \frac{1+r}{2} \|f\|_{H^2}^2 - \frac{1}{1-r^2} \|f\|_{H^2}^2 = \\ &= \frac{1}{1-r} \left[\frac{1+r}{2} n - \frac{1}{1+r} \right] \|f\|_{H^2}^2 = \\ &= \frac{n}{1-r} \left[\frac{1+r}{2} - \frac{1}{1+r} \right] \|f\|_{H^2}^2 = \frac{n}{1-r} \left[\frac{(1+r)^2 - \frac{2}{n} - \frac{2r}{n}}{2(1+r)} \right] \|f\|_{H^2}^2. \end{aligned}$$

In particular,

$$C_{n,r}(H^2, \mathcal{D}) \geq \sqrt{\frac{n}{1-r}} \left[\frac{(1+r)^2 - \frac{2}{n} - \frac{2r}{n}}{2(1+r)} \right]^{\frac{1}{2}}. \quad \square$$

IV. A second Bernstein-type inequality in L_a^2

Theorem 4.1.

Let $n \geq 1$, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , and B_σ the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional subspace of H^2 defined by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Let D be the operator of differentiation on $(K_{B_\sigma}, \|\cdot\|_{L_a^2})$:

$$D : (K_{B_\sigma}, \|\cdot\|_{L_a^2}) \rightarrow (L_a^2, \|\cdot\|_{L_a^2})$$

$$f \mapsto f'.$$

For $r \in [0, 1)$ and $n \geq 1$, we set

$$B_{n,r} = \sup \left\{ \|D\|_{K_{B_\sigma} \rightarrow L_a^2} : 1 \leq \#\sigma \leq n, |\lambda| \leq r \forall \lambda \in \sigma \right\}.$$

Then

$$B_{n,r} \leq 20^{\frac{1}{4}} C_{n,r}.$$

The inequality is asymptotically sharp as $n \rightarrow \infty$ and $r \rightarrow 1$ in the following sense : if $n \geq 2$, there exists $f_n \in L_a^2$ such that

$$\frac{\|f_n'\|_{L_a^2}}{\|f_n\|_{L_a^2}} \sim \frac{1}{2} \frac{n}{1-r},$$

as r tends to 1.

Proof. We already know (Theorem 2.1) that

$$\|f'\|_{L_a^2} \leq \sqrt{C_{n,r}} \|f\|_{H^2}.$$

Using Hölder's inequality setting $p = 3$, $p' = \frac{3}{2}$, $x = \frac{2}{3}$ so as $\frac{1}{p} + \frac{1}{p'} = 1$ and,

$$\begin{cases} px = 2 \\ p'x = 1 \\ p'(2-x) = 2 \end{cases},$$

we get

$$\begin{aligned} \|f\|_{H^2}^2 &= \sum_{k \geq 0} |\widehat{f}(k)|^2 = \sum_{k \geq 0} (k+1)^x |\widehat{f}(k)|^x \frac{|\widehat{f}(k)|^{2-x}}{(k+1)^x} \leq \\ &\leq \left(\sum_{k \geq 0} (k+1)^{px} |\widehat{f}(k)|^{px} \right)^{\frac{1}{p}} \left(\sum_{k \geq 0} \frac{|\widehat{f}(k)|^{p'(2-x)}}{(k+1)^{p'x}} \right)^{\frac{1}{p'}} = \\ &= \left(\sum_{k \geq 0} (k+1)^2 |\widehat{f}(k)|^2 \right)^{\frac{1}{3}} \left(\sum_{k \geq 0} \frac{|\widehat{f}(k)|^2}{k+1} \right)^{\frac{2}{3}} = \end{aligned}$$

$$\begin{aligned}
&= \left(\left| \widehat{f}(0) \right|^2 + \sum_{k \geq 1} (k+1)^2 \left| \widehat{f}(k) \right|^2 \right)^{\frac{1}{3}} \|f\|_{L_a^2}^{\frac{4}{3}} \leq \\
&\leq \left(\left| \widehat{f}(0) \right|^2 + \sup_{k \geq 1} \left\{ \left(\frac{k+1}{k} \right)^2 \right\} \sum_{k \geq 1} k^2 \left| \widehat{f}(k) \right|^2 \right)^{\frac{1}{3}} \|f\|_{L_a^2}^{\frac{4}{3}} = \\
&= \left(\left| \widehat{f}(0) \right|^2 + 2^2 \|f'\|_{H^2}^2 \right)^{\frac{1}{3}} \|f\|_{L_a^2} \leq \left(\|f\|_{H^2}^2 + 4 \|f'\|_{H^2}^2 \right)^{\frac{1}{3}} \|f\|_{L_a^2}^{\frac{4}{3}} \leq \\
&\leq \left(\|f\|_{H^2}^2 + 4C_{n,r}^2 \|f\|_{H^2}^2 \right)^{\frac{1}{3}} \|f\|_{L_a^2}^{\frac{4}{3}} = (1 + 4C_{n,r}^2)^{\frac{1}{3}} \|f\|_{H^2}^{\frac{2}{3}} \|f\|_{L_a^2}^{\frac{4}{3}},
\end{aligned}$$

using the definition of $C_{n,r}$. This gives

$$\|f\|_{H^2}^2 \leq (1 + 4C_{n,r}^2)^{\frac{1}{3}} \|f\|_{H^2}^{\frac{2}{3}} \|f\|_{L_a^2}^{\frac{4}{3}},$$

which means

$$\|f\|_{H^2}^{\frac{4}{3}} \leq (1 + 4C_{n,r}^2)^{\frac{1}{3}} \|f\|_{L_a^2}^{\frac{4}{3}},$$

or equivalently

$$\|f\|_{H^2} \leq (1 + 4C_{n,r}^2)^{\frac{1}{4}} \|f\|_{L_a^2}.$$

Finally,

$$\|f'\|_{L_a^2} \leq \sqrt{C_{n,r}} (1 + 4C_{n,r}^2)^{\frac{1}{4}} \|f\|_{L_a^2} = C_{n,r} \left(4 + \frac{1}{C_{n,r}^2} \right)^{\frac{1}{4}} \|f\|_{L_a^2}.$$

But

$$\begin{aligned}
\frac{1}{C_{n,r}} &\leq \frac{1}{a(n,r)} \frac{1-r}{n} \leq \\
&\leq \frac{1+r}{\left(1 + 5r^4 - \frac{4r^4}{n} - \min\left(\frac{3}{4}, \frac{2}{n}\right)\right)^{\frac{1}{2}}} \frac{1-r}{n} \leq \frac{2}{\left(1 - \frac{3}{4}\right)^{\frac{1}{2}}} \frac{1-r}{n} = \\
&= 4 \frac{1-r}{n} \leq 4.
\end{aligned}$$

As a result,

$$\|f'\|_{L_a^2} \leq 20^{\frac{1}{4}} C_{n,r} \|f\|_{L_a^2}.$$

In order to prove that this inequality is asymptotically sharp as $n \rightarrow \infty$ and $r \rightarrow 1$, it is sufficient to consider $f_n = \frac{1}{(1-rz)^n}$. Then

$$\frac{\|f_n'\|_{L_a^2}}{\|f_n\|_{L_a^2}} \sim \frac{1}{2} \frac{n}{1-r},$$

as $r \rightarrow 1$. Indeed, recalling that

$$\|g\|_{H_\alpha}^2 = \int_{\mathbb{D}} |g(z)|^2 dA,$$

for all $g \in L_a^2$, and setting

$$I_{c,t}(\lambda) = \int_{\mathbb{D}} \frac{(1-|z|^2)^t}{|1-\bar{\lambda}z|^{2+t+c}} dA,$$

we get

$$\frac{\|f'_n\|_{L_a^2}^2}{\|f_n\|_{L_a^2}^2} = r^2 n^2 \frac{I_{2n,0}(r)}{I_{2n-2,0}(r)} \sim n^2 \frac{\frac{1}{(1-r^2)^{2n}}}{\frac{1}{(1-r^2)^{2n-2}}} \sim \left(\frac{n}{1-r^2}\right)^2,$$

as r tends to 1, (see [Zhu] Chapter 4, or equivalently [HeKoZh]). This gives

$$\frac{\|f'_n\|_{L_a^2}}{\|f_n\|_{L_a^2}} \sim \frac{1}{1+r} \frac{n}{1-r} \sim \frac{1}{2} \frac{n}{1-r},$$

as r tends to 1.

V. Bernstein-type inequalities in weighted Bergman spaces

In this Section, we generalize Theorem 4.1 to the case of spaces $H_\alpha = l_a^2\left(\frac{1}{(k+1)^\alpha}\right)$, $\alpha \geq 0$.

Theorem 5.1.

Let $n \geq 1$, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , and B_σ the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional subspace of H^2 defined by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Let D be the operator of differentiation on $(K_{B_\sigma}, \|\cdot\|_{H_\alpha})$:

$$D : (K_{B_\sigma}, \|\cdot\|_{H_\alpha}) \rightarrow (H_\alpha, \|\cdot\|_{H_\alpha})$$

$$f \mapsto f'.$$

For $r \in [0, 1)$ and $n \geq 1$, we set

$$B_{n,r}^\alpha = \sup \left\{ \|D\|_{K_{B_\sigma} \rightarrow H_\alpha} : 1 \leq \#\sigma \leq n, |\lambda| \leq r \forall \lambda \in \sigma \right\}.$$

Then

$$B_{n,r}^\alpha \leq K_\alpha C_{n,r},$$

where $K_\alpha = 2^{\frac{\alpha^2}{2(\alpha+1)}} 20^{\frac{\alpha}{2(\alpha+1)}}$, for all $\alpha \in \mathbb{R}^+$. The inequality is asymptotically sharp as $n \rightarrow \infty$ and $r \rightarrow 1$ in the following sense: if $n > \alpha + 1$, there exists $f_n \in H_\alpha$ such that

$$\frac{\|f'_n\|_{H_\alpha}}{\|f_n\|_{H_\alpha}} \sim \frac{1}{2} \frac{n}{1-r},$$

as r tends to 1.

Proof . Let $f \in H_\alpha$ and $p = \alpha + 1$, $p' = \frac{1+\alpha}{\alpha} \geq 1$, $t = \frac{2}{\alpha+1}$, so as $\frac{1}{p} + \frac{1}{p'} = 1$ and

$$\begin{cases} pt = 2 \\ p'(2(1-\alpha) - t) = -2\alpha \\ p'(2-t) = 2 \end{cases} .$$

Then

$$\begin{aligned} \|f'\|_{H_\alpha}^2 &= \sum_{k \geq 0} \frac{(k+1)^2 |\hat{f}(k+1)|^2}{(k+1)^{2\alpha}} = \sum_{k \geq 0} (k+1)^{2(1-\alpha)} |\hat{f}(k+1)|^2 = \\ &= \sum_{k \geq 0} (k+1)^t |\hat{f}(k+1)|^t (k+1)^{2(1-\alpha)-t} |\hat{f}(k+1)|^{2-t} . \end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned} \|f'\|_{H_\alpha}^2 &\leq \left(\sum_{k \geq 0} (k+1)^2 |\hat{f}(k+1)|^2 \right)^{\frac{1}{p}} \left(\sum_{k \geq 0} \frac{|\hat{f}(k+1)|^2}{(k+1)^{2\alpha}} \right)^{\frac{1}{p'}} = \\ &= \|f'\|_{H^2}^{\frac{2}{p}} \left(\sum_{k \geq 0} \left(\frac{k+2}{k+1} \right)^\alpha \frac{|\hat{f}(k+1)|^2}{(k+2)^{2\alpha}} \right)^{\frac{1}{p'}} \leq \\ &\leq C_{n,r}^{\frac{2}{p}} \|f\|_{H^2}^{\frac{2}{p}} \sup_{k \geq 0} \left\{ \left(\frac{k+2}{k+1} \right)^{\frac{\alpha}{p'}} \right\} \left(\sum_{k \geq 0} \frac{|\hat{f}(k+1)|^2}{(k+2)^{2\alpha}} \right)^{\frac{1}{p'}} \leq \\ &\leq 2^{\frac{\alpha}{p'}} C_{n,r}^{\frac{2}{p}} \|f\|_{H^2}^{\frac{2}{p}} \left(\|f\|_{H_\alpha}^2 - |\hat{f}(0)|^2 \right)^{\frac{1}{p'}} \leq \\ &\leq 2^{\frac{\alpha}{p'}} C_{n,r}^{\frac{2}{p}} \|f\|_{H^2}^{\frac{2}{p}} \|f\|_{H_\alpha}^{\frac{2}{p'}} . \end{aligned}$$

Now let $x = \frac{2\alpha}{\alpha+1}$, so as

$$\begin{cases} p'x = 2 \\ px = 2\alpha \\ p(2-x) = 2 \end{cases} .$$

Using again Hölder's inequality, we get

$$\begin{aligned} \|f\|_{H^2}^2 &= \sum_{k \geq 0} |\hat{f}(k)|^2 = \sum_{k \geq 0} (k+1)^x |\hat{f}(k)|^x \frac{|\hat{f}(k)|^{2-x}}{(k+1)^x} \leq \\ &\leq \left(\sum_{k \geq 0} (k+1)^{p'x} |\hat{f}(k)|^{p'x} \right)^{\frac{1}{p'}} \left(\sum_{k \geq 0} \frac{|\hat{f}(k)|^{p(2-x)}}{(k+1)^{px}} \right)^{\frac{1}{p}} = \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{k \geq 0} (k+1)^2 |\widehat{f}(k)|^2 \right)^{\frac{1}{p'}} \left(\sum_{k \geq 0} \frac{|\widehat{f}(k)|^2}{(k+1)^{2\alpha}} \right)^{\frac{2}{p}} = \\
&= \left(|\widehat{f}(0)|^2 + \sum_{k \geq 1} (k+1)^2 |\widehat{f}(k)|^2 \right)^{\frac{1}{p'}} \|f\|_{H_\alpha}^{\frac{2}{p}} \leq \\
&\leq \left(|\widehat{f}(0)|^2 + \sup_{k \geq 1} \left\{ \left(\frac{k+1}{k} \right)^2 \right\} \sum_{k \geq 1} k^2 |\widehat{f}(k)|^2 \right)^{\frac{1}{p'}} \|f\|_{H_\alpha}^{\frac{2}{p}} = \\
&= \left(|\widehat{f}(0)|^2 + 2^2 \|f'\|_{H^2}^2 \right)^{\frac{1}{p'}} \|f\|_{H_\alpha}^{\frac{2}{p}} \leq \left(\|f\|_{H^2}^2 + 4 \|f'\|_{H^2}^2 \right)^{\frac{1}{p'}} \|f\|_{H_\alpha}^{\frac{2}{p}} \leq \\
&\leq \left(\|f\|_{H^2}^2 + 4C_{n,r}^2 \|f\|_{H^2}^2 \right)^{\frac{1}{p'}} \|f\|_{H_\alpha}^{\frac{2}{p}} = (1 + 4C_{n,r}^2)^{\frac{1}{p'}} \|f\|_{H^2}^{\frac{2}{p'}} \|f\|_{H_\alpha}^{\frac{2}{p}},
\end{aligned}$$

using the definition of $C_{n,r}$. This gives

$$\|f\|_{H^2}^{2-\frac{2}{p'}} \leq (1 + 4C_{n,r}^2)^{\frac{1}{p'}} \|f\|_{H_\alpha}^{\frac{2}{p}},$$

which means

$$\|f\|_{H^2}^{\frac{2}{p}} \leq (1 + 4C_{n,r}^2)^{\frac{1}{p'}} \|f\|_{H_\alpha}^{\frac{2}{p}}.$$

Finally,

$$\begin{aligned}
\|f'\|_{H_\alpha}^2 &\leq 2^{\frac{\alpha}{p'}} C_{n,r}^{\frac{2}{p}} \|f\|_{H^2}^{\frac{2}{p}} \|f\|_{H_\alpha}^{\frac{2}{p'}} \leq \\
&\leq 2^{\frac{\alpha}{p'}} C_{n,r}^{\frac{2}{p}} (1 + 4C_{n,r}^2)^{\frac{1}{p'}} \|f\|_{H_\alpha}^{\frac{2}{p}} \|f\|_{H_\alpha}^{\frac{2}{p'}} = \\
&= 2^{\frac{\alpha}{p'}} C_{n,r}^{\frac{2}{p}} (1 + 4C_{n,r}^2)^{\frac{1}{p'}} \|f\|_{H_\alpha}^2 = \\
&= 2^{\frac{\alpha}{p'}} C_{n,r}^{\frac{2}{p}} C_{n,r}^{\frac{2}{p'}} \left(4 + \frac{1}{C_{n,r}^2} \right)^{\frac{1}{p'}} \|f\|_{H_\alpha}^2.
\end{aligned}$$

But we already know (see the proof of Theorem 4.1), that

$$\frac{1}{C_{n,r}} \leq 4.$$

As a result,

$$\|f'\|_{H_\alpha}^2 \leq 2^{\frac{\alpha}{p'}} C_{n,r}^2 (4 + 16)^{\frac{1}{p'}} \|f\|_{H_\alpha}^2,$$

or equivalently

$$\|f'\|_{H_\alpha} \leq 2^{\frac{\alpha}{2(\alpha+1)}} 20^{\frac{\alpha}{2(\alpha+1)}} C_{n,r} \|f\|_{H_\alpha}.$$

In order to prove that this inequality is asymptotically sharp as $r \rightarrow 1$, it is sufficient to consider again $f_n = \frac{1}{(1-rz)^n}$, with $n > \alpha + \frac{1}{2}$. Then

$$\frac{\|f'_n\|_{H_\alpha}}{\|f_n\|_{H_\alpha}} \sim \frac{1}{2} \frac{n}{1-r},$$

as $r \rightarrow 1$. Indeed, recalling that

$$\|g\|_{H_\alpha}^2 = \int_{\mathbb{D}} |g(z)|^2 (1 - |z|^2)^{2\alpha-1} dA,$$

for all $g \in H_\alpha$, and setting

$$I_{c,t}(\lambda) = \int_{\mathbb{D}} \frac{(1 - |z|^2)^t}{|1 - \bar{\lambda}z|^{2+t+c}} dA,$$

we get

$$\frac{\|f'_n\|_{H_\alpha}^2}{\|f_n\|_{H_\alpha}^2} = r^2 n^2 \frac{I_{2(n-\alpha)+1, 2\alpha-1}(r)}{I_{2(n-\alpha)-1, 2\alpha-1}(r)} \sim n^2 \frac{\frac{1}{(1-r^2)^{2(n-\alpha)+1}}}{\frac{1}{(1-r^2)^{2(n-\alpha)-1}}} \sim \left(\frac{n}{1-r^2} \right)^2,$$

as r tends to 1, (see [Zhu] Chapter 4, or equivalently [HeKoZh]). This gives

$$\frac{\|f'_n\|_{H_\alpha}}{\|f_n\|_{H_\alpha}} \sim \frac{1}{1+r} \frac{n}{1-r} \sim \frac{1}{2} \frac{n}{1-r},$$

as r tends to 1.

□

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