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FLAG VARIETIES, TORIC VARIETIES, AND SUSPENSIONS: THREE INSTANCES OF INFINITE TRANSITIVITY

I.V. ARZHANTSEV, K. KUYUMZHIYAN, M. ZAIDENBERG

ABSTRACT. We say that a group G acts infinitely transitively on a set X if for every $m \in \mathbb{N}$ the induced diagonal action of G is transitive on the cartesian m th power $X^m \setminus \Delta$ with the diagonals removed. We describe three classes of affine algebraic varieties such that their automorphism groups act infinitely transitively on their smooth loci. The first class consists of affine cones over flag varieties, the second of non-degenerate affine toric varieties, and the third of iterated suspensions over affine varieties with infinitely transitive automorphism groups of a reinforced type.

INTRODUCTION

All varieties in this paper are assumed being reduced and irreducible. Unless we explicitly precise a setting, the base field \mathbb{k} is supposed to be algebraically closed of characteristic zero.

An effective action of the additive group $G_a(\mathbb{k})$ on an algebraic variety X defines a one-parameter unipotent subgroup of the automorphism group $\text{Aut}(X)$. We let $\text{SAut}(X)$ denote the subgroup of $\text{Aut}(X)$ generated by all its one-parameter unipotent subgroups. In the sequel, we adopt the following definitions.

Definition 0.1. Let X be an algebraic variety over \mathbb{k} . We say that a point $x \in X$ is *flexible* if the tangent space $T_x X$ is spanned by the tangent vectors to the orbits $H.x$ of one-parameter unipotent subgroups $H \subseteq \text{Aut}(X)$. The variety X is called *flexible* if every smooth point $x \in X_{\text{reg}}$ is.

Definition 0.2. An action of a group G on a set A is said to be *m -transitive* if for every two tuples of pairwise distinct points (a_1, a_2, \dots, a_m) and (b_1, b_2, \dots, b_m) in A there exists $g \in G$ such that $g(a_i) = b_i$, $i = 1, 2, \dots, m$. The action which is m -transitive for all $m \in \mathbb{N}$ will be called *infinitely transitive*.

Clearly, for $m > \dim G$ the Lie group G cannot act m -transitively on a connected variety X . In fact, it cannot act even 4-transitively on a smooth simply connected variety, due to the following theorem.

Theorem 0.1. (A. Borel [5, Théorèmes 5-6]) *There is no 3-transitive action of a real Lie group G on a simply connected non-compact variety, and such 2-transitive actions exist only*

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on the Euclidean spaces \mathbb{R}^n ($n \geq 2$). There is no 4-transitive action of G on a compact simply connected variety, and such 3-transitive actions exist only on the spheres S^n ($n \geq 2$).

See e.g., [23] and [37] for a classification of 2- and 3-transitive group actions. See also [32] for algebraic group actions that are transitive on Zariski dense open subsets of cartesian products.

The infinite transitivity of the full automorphism group is well known for the affine space \mathbb{A}^n over \mathbb{k} , where $n \geq 2$; see also [2] and [36] for an analytic counterpart. This phenomenon takes place as well for any hypersurface X in \mathbb{A}^{n+1} given by equation $uv + f(x_1, \dots, x_{n-1}) = 0$, where $n \geq 3$ and $f \in \mathbb{k}[x_1, \dots, x_{n-1}]$ is a non-constant polynomial [21, §5]. In the sequel we call such a variety X a *suspension* over $Y = \mathbb{A}^{n-1}$. In analytic category, similar phenomenons were studied in the spirit of the Andersen-Lempert-Varolin theory [2], [39], see e.g., [13], [19], [35], and [38]. This concerns, in particular, the infinite transitivity on smooth affine algebraic varieties of certain subgroups of biholomorphic transformations generated by complete regular vector fields, see a survey [20], especially §2(B) and Remark 2.2.

Following [21, §5] we are interested here in algebraic varieties X such that the special automorphism group $\text{SAut}(X)$ acts infinitely transitively on the smooth locus X_{reg} of X . We show, in particular, that this property is preserved when passing to a suspension.

Definition 0.3. We call a *suspension* over an affine variety Y the hypersurface $X \subseteq Y \times \mathbb{A}^2$ given by equation $uv - f(y) = 0$, where $\mathbb{A}^2 = \text{Spec } \mathbb{k}[u, v]$ and $f \in \mathbb{k}[Y]$ is non-constant. In particular, $\dim X = 1 + \dim Y$.

In this paper we deal with three classes of flexible affine algebraic varieties whose special automorphism group $\text{SAut}(X)$ is infinitely transitive. Our main results can be formulated as follows.

- Theorem 0.2.**
1. Consider a flag variety G/P over \mathbb{k} . Then every normal affine cone X over G/P is flexible and its special automorphism group $\text{SAut}(X)$ acts infinitely transitively on the smooth locus X_{reg} .
 2. The same conclusion holds if X is any non-degenerate¹ affine toric variety over \mathbb{k} of dimension at least 2.
 3. Suppose that an affine variety X over \mathbb{k} is flexible and either $X = \mathbb{A}^1$, or $\dim X \geq 2$ and the special automorphism group $\text{SAut}(X)$ acts infinitely transitively on the smooth locus X_{reg} . Then all iterated suspensions over X have the same properties².

Theorem 0.2(n) is proven in Section n , where $n = 1, 2, 3$, respectively. Smooth compact real algebraic surfaces with infinitely transitive automorphism groups were classified in [3], [4], and [17]; cf. also [18]. In Theorem 3.3 we extend part 3 of Theorem 0.2 to real algebraic varieties, under certain additional restrictions. In the particular case of a suspension over the affine line, it remains valid over an arbitrary field of characteristic zero, see Theorem 3.1.

The concluding remarks in Section 4 concern the Makar-Limanov invariant. Besides, we formulate some open questions.

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¹This means that does not have a 1-torus as a factor.

²of flexibility and infinite transitivity of the special automorphism group.

1. AFFINE CONES OVER FLAG VARIETIES

Given a connected, simply connected, semisimple linear algebraic group G over \mathbb{k} , we consider an irreducible representation $V(\lambda)$ of G with highest weight λ and a highest weight vector $v_0 \in V(\lambda)$. Let

$$Y = G[v_0] \subseteq \mathbb{P}(V(\lambda))$$

be the closed G -orbit of $[v_0]$ in the associate projective representation, and

$$X = \text{AffCone}(Y) = \overline{Gv_0} = Gv_0 \cup \{0\}$$

be the affine cone over Y (such a cone X is called an *HV-variety* in terminology of [33]).

Remarks 1.1. 1. Actually every projective embedding $\varphi : G/P \xrightarrow{\cong} Y \subseteq \mathbb{P}^n$ with a projectively normal image Y , where $P \subseteq G$ is a parabolic subgroup, arises in this way. Indeed, being projectively normal Y is as well linearly normal i.e., $\varphi = \varphi_{|D|}$, where $D \in \text{Pic}(G/P)$ is very ample. Hence $D \sim \sum_{i=1}^s a_i D_i$, where D_1, \dots, D_s are the Schubert divisorial cycles on G/P , and $a_i > 0 \forall i = 1, \dots, s$. Then $V(\lambda) = H^0(G/P, \mathcal{O}_{G/P}(D))^\vee$ is a simple G -module with a highest weight $\lambda = \sum_{i=1}^s a_i \omega_i$, where $\omega_1, \dots, \omega_s$ are fundamental weights, and $Y = G[v_0]$ for a highest weight vector v_0 ; see e.g., [30, Theorem 5].

2. Notice that for $\mathbb{k} = \mathbb{C}$ the homogeneous variety $X \setminus \{0\}$ has two ends in the Freudenthal sense. Let G be a connected Lie group and $H \subseteq G$ a closed connected subgroup. According to A. Borel [5, Théorème 2] the homogeneous space G/H has at most two ends. D. Akhiezer [1]³ showed that if G is a linear algebraic group over \mathbb{C} and $H \subseteq G$ is an algebraic subgroup (not necessarily connected) then G/H has exactly two ends if and only if H is the kernel of a nontrivial character $\varphi : P \rightarrow \mathbb{C}^*$, where P is a parabolic subgroup of G . The homogeneous fibration $G/H \rightarrow G/P$ realizes G/H as a principal \mathbb{C}^* -bundle over the homogeneous projective variety G/P .⁴ Furthermore, G/H admits a projective completion by two disjoint divisors E_0 and E_∞ , where $E_0 \cong E_\infty \cong G/P$, and $\tilde{X} := G/H \cup E_0 \rightarrow G/P$ represents a line bundle, say, L over G/P . Its zero section E_0 is contractible if and only if the dual line bundle L^{-1} is ample, and then also very ample. In the latter case the contraction of E_0 yields the affine cone X over the image $Y = \varphi_{|L^{-1}|}(G/P) \subseteq \mathbb{P}^n$. For $\mathbb{k} = \mathbb{C}$, any affine cone X as in Theorem 1.1 arises in this way, with $H \subseteq P$ being the stabilizer of a smooth point of the cone X , \tilde{X} the blowup of X at the vertex, and E_0 the exceptional divisor.

The next result provides part 1 of Theorem 0.2.

Theorem 1.1. *Let X be the affine cone over a flag variety G/P under an embedding $G/P \hookrightarrow \mathbb{P}^N$ with projectively normal image. Then X is flexible and the group $\text{SAut}(X)$ acts m -transitively on $X \setminus \{0\}$ for any $m \in \mathbb{N}$.*

The flexibility follows from the next general observation.

Proposition 1.1. *If a semisimple linear algebraic group G acts on an affine variety X and this action is transitive in X_{reg} then X is flexible.*

³See also [16] and [25] for the complex analytic counterpart, in the context of Lie groups.

⁴In this general setting, neither the pair (G, P) nor (G, H) is uniquely defined by the flag variety G/P .

Proof. The group G acts on X with an open orbit $X_{\text{reg}} = G.x_0$. The dominant morphism onto this orbit $\varphi : G \rightarrow X$, $g \mapsto g.x_0$, yields a surjection $d\varphi : \mathfrak{g} \rightarrow T_{x_0}X$, where $\mathfrak{g} = \text{Lie}(G)$. We claim that \mathfrak{g} is spanned over \mathbb{k} by nilpotent elements, which implies the assertion. Indeed, consider the decomposition $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$ of \mathfrak{g} into simple subalgebras. Let \mathfrak{h} be the span of the set of all nilpotent elements in \mathfrak{g} . This is an ad-submodule of \mathfrak{g} and so an ideal of \mathfrak{g} , hence a direct sum of some of the simple ideals \mathfrak{g}_i . However every simple ideal \mathfrak{g}_i , $i = 1, \dots, k$ contains at least one nonzero nilpotent element. Therefore $\mathfrak{h} = \mathfrak{g}$, as claimed. \square

In the setting of Theorem 1.1, $X_{\text{reg}} = X$ if $X \cong \mathbb{A}^n$ and $X_{\text{reg}} = X \setminus \{0\}$ otherwise. Anyhow, the group G acts transitively on $X \setminus \{0\}$, see e.g., [33, Theorem 1]. Hence by Proposition 1.1 X is flexible.

Before passing to the proof of infinite transitivity we need some preparation.

Let $P \subseteq G$ be the stabilizer of the line $\langle v_0 \rangle \subseteq V(\lambda)$, $B = TB_u \subseteq P$ be a Borel subgroup of G with the maximal torus T and the unipotent radical B_u , and $\mathbb{X}(T)$ be the character lattice of T . Consider the T -invariant (weight) decompositions

$$V(\lambda) = \bigoplus_{\nu \in \mathbb{X}(T)} V(\lambda)_\nu = \langle v_0 \rangle \oplus H(\lambda),$$

where $\langle v_0 \rangle = V(\lambda)_\lambda$ and $H(\lambda) \subseteq V(\lambda)$ is the hyperplane

$$H(\lambda) = \bigoplus_{\nu \in \mathbb{X}(T) \setminus \{\lambda\}} V(\lambda)_\nu.$$

The coordinate function $l_\lambda \in V(\lambda)^*$ of the first projection $p_1 : v \mapsto l_\lambda(v)v_0$ defines a non-trivial character of P .

Let $B^- = TB_u^-$ be the Borel subgroup of G opposite to $B = B^+$. The flag variety G/P contains an open B^- -orbit (the big Schubert cell) isomorphic to the affine space \mathbb{A}^n , where $n = \dim G/P$. Its complement is a union of the divisorial Schubert cycles D_1, D_2, \dots, D_s , see e.g. [24, pp. 22–24].

The orbit map $G \rightarrow \mathbb{P}(V(\lambda))$, $g \mapsto g.[v_0]$, embeds G/P onto a subvariety $Y \subseteq \mathbb{P}(V(\lambda))$. We let $\omega_\lambda \subseteq Y$ be the image of the big Schubert cell under this embedding. By [30, Theorem 2] the hyperplane

$$\mathcal{H}(\lambda) = \mathbb{P}(H(\lambda)) = l_\lambda^{-1}(0) \subseteq \mathbb{P}(V(\lambda))$$

is supported by the union of the Schubert divisors $\bigcup_{i=1}^s D_i$. In particular $\omega_\lambda = Y \setminus \mathcal{H}(\lambda)$.

Let $\sigma : \widehat{X} \rightarrow X$ be the blow-up of the cone X at the vertex 0 . The exceptional divisor $E \subseteq \widehat{X}$ is isomorphic to Y . Moreover, the natural map $\pi : X \setminus \{0\} \rightarrow Y$ yields the projection $p : \widehat{X} \rightarrow Y$ of the line bundle $\mathcal{O}_Y(-1)$ on Y with E being the zero section. Since $\omega_\lambda \cong \mathbb{A}^n$, the restriction $p : p^{-1}(\omega_\lambda) \rightarrow \omega_\lambda$ is a trivial line bundle. Hence the open set

$$\Omega_\lambda := \pi^{-1}(\omega_\lambda) = X \setminus H(\lambda) \subseteq X \setminus \{0\} \cong \widehat{X} \setminus E$$

is isomorphic to $\mathbb{A}^n \times \mathbb{A}_*^1$.

For every $c \in \mathbb{A}_*^1$ the invertible function $l_\lambda(\cdot, c)$ is constant on the affine space \mathbb{A}^n . Thus on $\Omega_\lambda \cong \mathbb{A}^n \times \mathbb{A}_*^1$ we have $l_\lambda = az^k$ for some $a \in \mathbb{k}^\times$, where z is a coordinate in \mathbb{A}_*^1 . Here actually $k = 1$ since l_λ gives a coordinate on $\langle v_0 \rangle$. We may assume that also $a = 1$ and so $l_\lambda|_{\Omega_\lambda} : \Omega_\lambda \rightarrow \mathbb{A}_*^1$ is the second projection.

To prove the infinite transitivity of the group $\mathrm{SAut}(X)$ in $X \setminus \{0\}$ as stated in Theorem 1.1, let us show first the infinite transitivity of $\mathrm{SAut}(X)$ in each hyperplane section $\Omega_\lambda(c_0) := l_\lambda^{-1}(c_0) \subseteq X$, where $c_0 \neq 0$; cf. [21, Lemma 5.6]. More precisely, given a k -tuple of distinct points $c_1, \dots, c_k \in \mathbb{k}$ different from c_0 we consider the subgroup $\mathrm{Stab}_{c_1, \dots, c_k}^\lambda \subseteq \mathrm{SAut}(X)$ of all automorphisms fixing pointwise the subvarieties $\Omega_\lambda(c_i)$ for all $i = 1, \dots, k$ and leaving invariant the function l_λ .

Proposition 1.2. *In the notation as before, for any $n \geq 2$ and any $m \in \mathbb{N}$ the group $\mathrm{Stab}_{c_1, \dots, c_k}^\lambda$ acts m -transitively on $\Omega_\lambda(c_0) \cong \mathbb{A}^n$.*

Proof. Let Q_1, Q_2, \dots, Q_m and Q'_1, Q'_2, \dots, Q'_m be two tuples of pairwise distinct points in $\Omega_\lambda(c_0)$. For any $n \geq 2$ the group $\mathrm{SAut}(\mathbb{A}^n)$ acts m -transitively on \mathbb{A}^n ; see e.g. [21, Lemma 5.5]. Since $\Omega_\lambda(c_0) \cong \mathbb{A}^n$, we can find $g \in \mathrm{SAut}(\Omega_\lambda(c_0))$ mapping (Q_1, Q_2, \dots, Q_m) to $(Q'_1, Q'_2, \dots, Q'_m)$. By definition, $g = \delta_1(1)\delta_2(1)\dots\delta_s(1)$ for some one-parameter unipotent subgroups $\delta_1, \delta_2, \dots, \delta_s \subseteq \mathrm{SAut}(\Omega_\lambda(c_0))$. Let $\partial_1, \partial_2, \dots, \partial_s$ be the corresponding locally nilpotent derivations⁵ (LNDs for short). First we extend them to LNDs $\bar{\partial}_1, \bar{\partial}_2, \dots, \bar{\partial}_s$ of $\mathbb{k}[\Omega_\lambda] \cong \mathbb{k}[\mathbb{A}^n \times \mathbb{A}_*^1]$ by putting $\bar{\partial}_i(l_\lambda) = 0$.

Recall that Ω_λ is a principal Zariski open subset in X defined by the function l_λ . In particular, for every $i = 1, \dots, s$ we have $\bar{\partial}_i: \mathbb{k}[X] \rightarrow \mathbb{k}[X][1/l_\lambda]$. Since $\mathbb{k}[X]$ is finitely generated, there exists $N \in \mathbb{N}$ such that $(l_\lambda)^N \bar{\partial}_i$ is an LND of $\mathbb{k}[X]$ for all $i = 1, \dots, s$; cf. [22, Proposition 3.5].

Let $q[z] \in \mathbb{k}[z]$ be a polynomial with $q(c_0) = 1$ which has simple roots at c_1, \dots, c_k and a root $z = 0$ of multiplicity N (we recall that $c_0 \neq 0$). Then for every $i = 1, \dots, s$, $q(l_\lambda)\bar{\partial}_i$ is an LND of $\mathbb{k}[X]$ such that the corresponding one-parameter subgroup in $\mathrm{Stab}_{c_1, \dots, c_k}^\lambda$ extends the subgroup δ_i . Thus g extends to an element of the group $\mathrm{Stab}_{c_1, \dots, c_k}^\lambda$. Now the assertion follows. \square

Let μ be an extremal weight of the simple G -module $V(\lambda)$ different from λ . Then μ defines a parabolic subgroup P' conjugated to P , the corresponding linear form $l_\mu \in V(\lambda)^*$, and the principal Zariski open subset $\Omega_\mu = \{l_\mu \neq 0\}$ of X , where $X \setminus \Omega_\mu = H(\mu) := l_\mu^{-1}(0)$.

Lemma 1.1. *For every set of m distinct points $Q_1, Q_2, \dots, Q_m \in X \setminus \{0\}$ there exists $g \in \mathrm{SAut}(X)$ such that $g(Q_i) \in \Omega_\mu$ for all $i = 1, \dots, m$.*

Proof. Since the group G is semisimple, it is contained in $\mathrm{SAut}(X)$ (see [31, Lemma 1.1]). Clearly, $G_i := \{g \in G \mid g(Q_i) \in H(\mu)\}$ ($i = 1, \dots, m$) are proper closed subsets of G . Hence the conclusion of the lemma holds for any $g \in G \setminus (G_1 \cup \dots \cup G_m)$. \square

Lemma 1.2. *For every $c \neq 0$ the restriction $l_\lambda|_{\Omega_\mu(c)}$ is non-constant.*

Proof. If the restriction $l_\lambda|_{\Omega_\mu(c)}$ were a constant equal, say, a , then the cone X would be contained in the hyperplane $al_\mu - cl_\lambda = 0$ in $V(\lambda)$, which is not the case. \square

Proof of Theorem 1.1. If $n = \dim G/P = 1$ and so $G/P \cong \mathbb{P}^1$, then X is a normal affine toric surface (a Veronese cone). The infinite transitivity in this case follows from Theorem 2.1 in Section 2 below.

⁵A derivation ∂ of a ring A is called locally nilpotent if $\forall a \in A, \partial^n a = 0$ for $n \gg 1$.

From now on we suppose that $n \geq 2$. Given $m \in \mathbb{N}$, we fix an m -tuple of distinct points $Q_1^0, Q_2^0, \dots, Q_m^0 \in \Omega_\lambda(1)$. Let us show that for any m -tuple of distinct points $Q_1, Q_2, \dots, Q_m \in X \setminus \{0\}$ there exists $\psi \in \text{SAut}(X)$ such that $\psi(Q_1) = Q_1^0, \dots, \psi(Q_m) = Q_m^0$.

According to Lemma 1.1 we may suppose that $Q_i \in \Omega_\mu$ for all $i = 1, \dots, m$. Divide the set $\{Q_1, Q_2, \dots, Q_m\}$ into several pieces according to the values of $l_\mu(Q_i)$:

$$\{Q_1, Q_2, \dots, Q_m\} = \bigcup_{j=1}^k M_j, \quad M_j = \{Q_i \mid Q_i \in \Omega_\mu(c_j)\},$$

where $c_1, \dots, c_k \in \mathbb{k}^\times = \mathbb{k} \setminus \{0\}$ are distinct. By Lemma 1.2, every intersection $\Omega_\lambda(1) \cap \Omega_\mu(c_i)$ contains infinitely many points. Acting with the subgroups $\text{Stab}_{c_1, \dots, \hat{c}_i, \dots, c_k}^\mu \subseteq \text{SAut}(X)$ (see Proposition 1.2), we can successively send the pieces M_i ($i = 1, \dots, k$) to the affine hyperplane section $\Omega_\lambda(1)$. The resulting m -tuple can be sent further to the standard one $(Q_1^0, Q_2^0, \dots, Q_m^0)$ using an automorphism from Proposition 1.2 with $c_0 = 1$ and arbitrary c_1, \dots, c_k . Now the proof is completed. \square

2. AUTOMORPHISMS OF AFFINE TORIC VARIETIES

As before, \mathbb{k} stands for an algebraically closed field of characteristic zero. In this section we consider an affine toric variety X over \mathbb{k} with a torus \mathbb{T} acting effectively on X . We assume that X is non-degenerate i.e., the only invertible regular functions on X are constants or, which is equivalent, that $X \not\cong Y \times \mathbb{A}_*^1$, where $\mathbb{A}_*^1 = \text{Spec } \mathbb{k}[t, t^{-1}] \cong \mathbb{k}^\times$. The following result yields part 2 of Theorem 0.2.

Theorem 2.1. *Every non-degenerate affine toric variety X of dimension $n \geq 1$ is flexible. Furthermore, for $n \geq 2$ and for any $m \in \mathbb{N}$ the group $\text{SAut}(X)$ acts m -transitively on the smooth locus X_{reg} of X .*

We note that X is flexible if $\text{Aut}(X)$ acts transitively on X_{reg} and at least one smooth point is flexible on X . Both properties will be established below. Let us recall first some necessary generalities on toric varieties.

1. Ray generators.

Let N be the lattice of one-parameter subgroups of the torus \mathbb{T} , $M = \mathbb{X}(\mathbb{T})$ the dual lattice of characters, and $\langle \cdot, \cdot \rangle : N \times M \rightarrow \mathbb{Z}$ the natural pairing. Letting χ^m denote the character of \mathbb{T} which corresponds to a lattice point $m \in M$ (so that $\chi^m \chi^{m'} = \chi^{m+m'}$), the group algebra

$$\mathbb{k}[M] := \bigoplus_{m \in M} \mathbb{k}\chi^m$$

can be identified with the algebra $\mathbb{k}[\mathbb{T}]$ of regular functions on the torus \mathbb{T} . Let $\mathbb{T}.x_0$ be the open \mathbb{T} -orbit in X . Since the orbit map $\mathbb{T} \rightarrow X$, $t \mapsto t.x_0$, is dominant, we may identify $\mathbb{k}[X]$ with a subalgebra of $\mathbb{k}[M]$. More precisely, there exists a convex polyhedral cone $\sigma^\vee \subseteq M_\mathbb{Q} := M \otimes_\mathbb{Z} \mathbb{Q}$ such that $\mathbb{k}[X]$ coincides with the semigroup algebra of σ^\vee i.e.,

$$(1) \quad \mathbb{k}[X] = \bigoplus_{m \in \sigma^\vee \cap M} \mathbb{k}\chi^m,$$

see [14] for details. We let $\sigma \subseteq N_\mathbb{Q}$ denote the cone dual to σ^\vee . The cone σ is pointed and of full dimension in $N_\mathbb{Q}$. Let $\Xi = \{\rho_1, \dots, \rho_r\}$ be the set of ray generators i.e., the primitive

vectors on extremal rays of the cone σ . Given a ray generator $\rho \in \Xi$, we let R_ρ denote the associate one-parameter subgroup of \mathbb{T} .

2. *The Orbit-Cone correspondence* [6, §3.2].

There exist two natural one to one correspondences $\delta \xleftrightarrow{1 \div 1} \tau \xleftrightarrow{1 \div 1} O_\tau$ between the faces δ of σ , the dual faces $\tau = \delta^\perp$ ⁶ of σ^\vee , and the \mathbb{T} -orbits O_τ on X such that $\dim O_\tau = \dim \tau = \dim \sigma - \dim \delta$. In particular, the unique fixed point of \mathbb{T} in X corresponds to the vertex of σ^\vee , and the open \mathbb{T} -orbit to the cone σ^\vee itself. These correspondences respect the inclusions: the \mathbb{T} -orbit O_μ meets the orbit closure $\overline{O_\tau}$ if and only if $O_\mu \subseteq \overline{O_\tau}$, if and only if $\mu \subseteq \tau$, if and only if $\mu^\perp \supseteq \tau^\perp$; cf. [6, §3.2].

Given a face $\tau \subseteq \sigma^\vee$ there is a direct sum decomposition

$$\mathbb{k}[X] = \mathbb{k}[\overline{O_\tau}] \oplus I(\overline{O_\tau}),$$

where

$$(2) \quad \mathbb{k}[\overline{O_\tau}] = \bigoplus_{m \in \tau \cap M} \mathbb{k}\chi^m, \quad \text{and} \quad I(\overline{O_\tau}) = \bigoplus_{m \in (\sigma^\vee \setminus \tau) \cap M} \mathbb{k}\chi^m$$

is the graded ideal of $\overline{O_\tau}$ in $\mathbb{k}[X]$.

A stabilizer $\mathbb{T}_p = \text{Stab}_{\mathbb{T}}(p)$ of any point $p \in X$ is connected, hence $\mathbb{T}_p \subseteq \mathbb{T}$ is a subtorus. Furthermore, $\mathbb{T}_p \subseteq \mathbb{T}_q$ if and only if $\mathbb{T}.q \subseteq \mathbb{T}.p$. Moreover $\overline{\mathbb{T}.p} = X^{\mathbb{T}_p}$ (here X^G stands, as usual, for the set of fixed points of the group G acting on X).

3. *Roots and associate one parameter unipotent subgroups.*

Definition 2.1. (M. Demazure [9]) A *root* of the cone σ is a vector $e \in M$ such that for some index i with $1 \leq i \leq r$, where $r = \text{card } \Xi$, we have

$$(3) \quad \langle \rho_i, e \rangle = -1 \quad \text{and} \quad \langle \rho_j, e \rangle \geq 0 \quad \text{for every } j \neq i.$$

We let $\mathcal{R}(\sigma)$ denote the set of all roots of the cone σ . There is a one to one correspondence $e \xleftrightarrow{1 \div 1} H_e$ between the roots of σ and the one-parameter unipotent subgroups of $\text{Aut}(X)$ normalized by the torus, see [9] or [26]. Letting $\rho_e := \rho_i$, the root $e \in \mathcal{R}(\sigma)$ defines an LND ∂_e of the M -graded algebra $\mathbb{k}[X]$ given by

$$(4) \quad \partial_e(\chi^m) = \langle \rho_e, m \rangle \chi^{m+e}.$$

Its kernel is a (finitely generated) graded subalgebra of $\mathbb{k}[X]$ (see [26]):

$$(5) \quad \ker \partial_e = \bigoplus_{m \in \rho_e^\perp \cap M} \mathbb{k}\chi^m,$$

where $\rho_e^\perp = \{m \in \sigma^\vee \cap M, \langle \rho_e, m \rangle = 0\}$ is the facet⁷ of σ^\vee orthogonal to ρ_e .

Definition 2.2. (see [12], [26]) Two roots e and e' with $\rho_e = \rho_{e'}$ are called *equivalent*; we write $e \sim e'$. Two roots e and e' are equivalent if and only if $\ker \partial_e = \ker \partial_{e'}$.

⁶By abuse of notation, here $\delta^\perp \cap \sigma^\vee$ is denoted simply by δ^\perp .

⁷Facets of the cone σ^\vee are its codimension one faces.

Remark 2.1. Enumerating the ray generators $\Xi = \{\rho_1, \dots, \rho_r\}$ yields a disjoint partition

$$\mathcal{R}(\sigma) = \bigcup_{i=1}^r \mathcal{R}_i, \quad \text{where } \mathcal{R}_i = \{e \in \mathcal{R}(\sigma) \mid \rho_e = \rho_i\}$$

are nonempty. Indeed, consider the facet τ_i of σ^\vee orthogonal to the ray generator ρ_i . For every v in the relative interior $\text{Int}_{\text{rel}}(\tau_i)$ the inequalities $\langle \rho_j, m \rangle > 0$ hold for all $j \neq i$. Let $e_0 \in M$ be such that $\langle \rho_i, e_0 \rangle = -1$, and let $v_0 \in \text{Int}_{\text{rel}}(\tau_i) \cap M$. Letting $e = e_0 + kv_0$ with $k \gg 1$ we obtain $\langle \rho_j, e \rangle > 0$ for all $j \neq i$ and $\langle \rho_i, e \rangle = -1$. Hence $e \in \mathcal{R}_i$ i.e., $\rho_e = \rho_i$.

For instance, let $X = \mathbb{A}^2$ be the affine plane with the standard 2-torus action so that σ and σ^\vee coincide with the first quadrants. Then $\mathcal{R}(\sigma)$ consists of two equivalence classes

$$\mathcal{R}_1 = \{(x, -1) \mid x \in \mathbb{Z}_{\geq 0}\} \quad \text{and} \quad \mathcal{R}_2 = \{(-1, y) \mid y \in \mathbb{Z}_{\geq 0}\}.$$

4. One-parameter groups of automorphisms.

The derivation ∂_e generates a one-parameter unipotent subgroup $H_e = \lambda_e(G_a(\mathbb{k})) \subseteq \text{Aut}(X)$, where $\lambda_e : t \mapsto \exp(t\partial_e)$. The algebra of invariants $\mathbb{k}[X]^{H_e}$ coincides with $\ker \partial_e$. The inclusion $\mathbb{k}[X]^{H_e} \subseteq \mathbb{k}[X]$ induces a morphism $\pi : X \rightarrow Z = \text{Spec } \mathbb{k}[X]^{H_e}$ whose general fibers are one-dimensional H_e -orbits isomorphic to \mathbb{A}^1 , cf. [34, Theorems 2.3 and 3.3]. The torus \mathbb{T} normalizes the subgroup H_e i.e., $\mathbb{T} \subseteq N(H_e)$. Hence \mathbb{T} stabilizes the fixed point set X^{H_e} .

Letting $R_e = R_{\rho_e} \subseteq \mathbb{T}$ be the one-parameter subgroup which corresponds to the vector $\rho_e \in N$, the action of R_e on the graded algebra $\mathbb{k}[X]$ can be given, under a suitable parametrization $\rho_e : G_m(\mathbb{k}) \ni t \mapsto \rho_e(t) \in R_e$, by

$$(6) \quad t \cdot \chi^m = t^{\langle \rho_e, m \rangle} \chi^m, \quad t \in G_m(\mathbb{k}).$$

In particular $\mathbb{k}[X]^{R_e} = \mathbb{k}[X]^{H_e}$. Hence the morphism $\pi : X \rightarrow Z$ coincides with the quotient map $X \rightarrow X//R_e$. So the general H_e -orbits are closures of general R_e -orbits. By Proposition 2.1 below the latter holds actually for every one-dimensional H_e -orbit.⁸

There is a direct sum decomposition

$$(7) \quad \mathbb{k}[X] = \mathbb{k}[X]^{R_e} \oplus \bigoplus_{m \in \sigma^\vee \cap M \setminus \rho_e^\perp} \mathbb{k}\chi^m = \mathbb{k}[X]^{R_e} \oplus I(D_e),$$

where $D_e := X^{R_e} \cong Z$. The divisor D_e coincides with the attractive set of the action of R_e on X . So every one-dimensional R_e -orbit has a limit point on D_e . The following simple lemma completes the picture.

Lemma 2.1. *Let τ be a face of σ^\vee , \mathcal{O}_τ the corresponding orbit, \mathbb{T}_τ the stabilizer of a point in \mathcal{O}_τ , and Ξ_τ the set of ray generators orthogonal to τ i.e., the ray generators of the dual face $\tau^\perp \subseteq \sigma$. Then the following hold.*

(a) *The orbit closure $\overline{\mathcal{O}_\tau}$ is stable under H_e if and only if*

$$(8) \quad m + e \in \sigma^\vee \setminus \tau \quad \forall m \in (\sigma^\vee \setminus \tau) \cap M \quad \text{such that } \langle \rho_e, m \rangle > 0.$$

(b) *The closure $\overline{\mathcal{O}_\tau}$ is $H_{e'}$ -stable for any root $e' \sim e$ of σ if one of the following equivalent conditions is fulfilled:*

(i) $\rho_e \notin \Xi_\tau$,

⁸If X is a surface then all fibers of π are R_e -orbit closures isomorphic to \mathbb{A}^1 (see [11], the parabolic case). While for a toric affine 3-fold X some degenerate fibers of π can be two-dimensional.

- (ii) $\overline{\mathcal{O}_\tau} \not\subseteq D_e$,
- (iii) $R_e \not\subseteq \mathbb{T}_\tau$.

Proof. (a) By virtue of (4) the ideal $I(\overline{\mathcal{O}_\tau})$ is ∂_e -invariant if and only if

$$(9) \quad \chi^{m+e} \in I(\overline{\mathcal{O}_\tau}) \quad \forall \chi^m \in I(\overline{\mathcal{O}_\tau}) \quad \text{such that} \quad \langle \rho_e, m \rangle > 0,$$

which is equivalent to (8) (see (2)). This proves (a).

(b) For $m \in M$,

$$m \in \sigma^\vee \setminus \tau \Leftrightarrow \langle \rho, m \rangle \geq 0 \quad \forall \rho \in \Xi \quad \text{and} \quad \exists \rho \in \Xi_\tau : \langle \rho, m \rangle > 0.$$

For any $\rho \neq \rho_e$ we have $\langle \rho, m+e \rangle \geq \langle \rho, m \rangle$. Hence (i) \Rightarrow (8).

We have $\overline{\mathcal{O}_\tau} = X^{\mathbb{T}_\tau}$ and $D_e = X^{R_e} = \overline{\mathcal{O}_{\rho_e^\perp}}$, where $\rho_e^\perp = (\mathbb{R}_+\rho_e)^\perp$. So the equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii) are due to the Orbit-Cone correspondence. \square

Remark 2.2. Consequently, $\rho_e \in \Xi_\tau$ if $\overline{\mathcal{O}_\tau}$ is not H_e -stable. The converse is not true, in general. For instance, let $X = \mathbb{A}^2$ be the plane with the standard torus action so that $\Xi = \{(1,0), (0,1)\}$. Let $\tau = \{\bar{0}\}$ and $e = (0,-1)$, $e' = (a,-1) \sim e$, where $a > 0$ and $\rho_e = (0,1)$. Then (8) holds for $H_{e'}$ and not for H_e . Hence $\rho_e \in \Xi_\tau$ and the \mathbb{T} -fixed point $\overline{\mathcal{O}_\tau} = \{(0,0)\}$ is $H_{e'}$ -stable but not H_e -stable.

One can also construct an example with $\dim X = 4$ such that the closure $\overline{\mathcal{O}_\tau}$ is $H_{e'}$ -stable for every root $e' \sim e$, whereas the equivalent conditions (i)-(iii) are not fulfilled.

For the proof of infinite transitivity we need somewhat more precise information concerning one-parameter group actions on toric varieties, see Proposition 2.1 and Lemmas 2.2-2.3 below.

Proposition 2.1. *Given a root $e \in \mathcal{R}(\sigma)$ we let as before $H_e \subseteq \text{SAut}(X)$ be the associate one-parameter unipotent subgroup. Then the following hold.*

- (a) *For every point $x \in X \setminus X^{H_e}$ the orbit $H_e \cdot x$ meets exactly two \mathbb{T} -orbits \mathcal{O}_1 and \mathcal{O}_2 on X , where $\dim \mathcal{O}_1 = 1 + \dim \mathcal{O}_2$.*
- (b) *The intersection $\mathcal{O}_2 \cap H_e \cdot x$ consists of a single point, while*

$$\mathcal{O}_1 \cap H_e \cdot x = R_e \cdot y \quad \forall y \in \mathcal{O}_1 \cap H_e \cdot x.$$

Proof. (a) The number of \mathbb{T} -orbits in X being finite, there exists a \mathbb{T} -orbit \mathcal{O}_1 such that $\mathcal{O}_1 \cap H_e \cdot x$ is a Zariski open subset of $H_e \cdot x$. So $H_e \cdot x \subseteq \overline{\mathcal{O}_1}$. There is also another \mathbb{T} -orbit \mathcal{O}_2 that meets $H_e \cdot x$. Indeed, otherwise $H_e \cdot x \cong \mathbb{A}^1$ would be contained in a single \mathbb{T} -orbit \mathcal{O}_1 . However, this is impossible because the algebra of regular functions on \mathcal{O}_1 is generated by invertible elements. Since \mathcal{O}_2 meets $\overline{\mathcal{O}_1}$ we have $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$ and so $\dim \mathcal{O}_2 < \dim \mathcal{O}_1$.

The torus \mathbb{T} normalizes the unipotent subgroup H_e . Hence the elements of \mathbb{T} send the H_e -orbits into H_e -orbits. In particular, for every point $q \in H_e \cdot x$ the stabilizer \mathbb{T}_q preserves the orbit $H_e \cdot x$. For all $q \in \mathcal{O}_1 \cap H_e \cdot x$ the stabilizer is the same. Since $H_e \cdot x \subseteq \overline{\mathcal{O}_1} = X^{\mathbb{T}_q}$, this stabilizer acts trivially on $H_e \cdot x$. Thus $\mathbb{T}_r \supseteq \mathbb{T}_q$ for any point $r \in H_e \cdot x$, and $\mathbb{T}_r = \mathbb{T}_q$ if and only if $r \in \mathcal{O}_1 \cap H_e \cdot x$.

Fix a point $p \in \mathcal{O}_2 \cap H_e \cdot x$. If $\mathbb{T}_p \subseteq \mathbb{T}_q$ then $\mathbb{T}_p = \mathbb{T}_q$ and so $\dim \mathcal{O}_2 = \dim \mathcal{O}_1$, a contradiction. Consequently, the stabilizer \mathbb{T}_p acts on $H_e \cdot x$ with two orbits i.e., $H_e \cdot x = \mathbb{T}_p \cdot q \cup \{p\}$, where $q \in H_e \cdot x \setminus \{p\}$. From the exact sequence

$$1 \rightarrow \mathbb{T}_q \rightarrow \mathbb{T}_p \rightarrow G_m(\mathbb{k}) \rightarrow 1$$

we deduce that $\dim \mathbb{T}_p = 1 + \dim \mathbb{T}_q$. Finally $H_e.x \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ and $\dim \mathcal{O}_1 = 1 + \dim \mathcal{O}_2$, as stated in (a).

(b) We may assume that $\mathcal{O}_1 = \mathbb{T}.x$. Since $H_e.x \subseteq \overline{\mathcal{O}_1}$ and the torus \mathbb{T} normalizes the subgroup H_e we have $H_e(\mathcal{O}_1) \subseteq \overline{\mathcal{O}_1}$. Thus $\overline{\mathcal{O}_1}$ is H_e -stable. On the other hand, since $H_e.p = H_e.x \not\subseteq \overline{\mathcal{O}_2}$ the closure $\overline{\mathcal{O}_2}$ is not H_e -stable. In particular, by Lemma 2.1 $\rho_e \in \Xi_{\tau_2}$, where τ_i is the face of σ^\vee which corresponds to $\mathcal{O}_i = \mathcal{O}_{\tau_i}$, $i = 1, 2$. Hence by the same lemma $R_e \subseteq \mathbb{T}_{\tau_2} = \mathbb{T}_p$. Let us show that $R_e \not\subseteq \mathbb{T}_{\tau_1} = \mathbb{T}_q$, where $q \in H_e.x \setminus \{p\}$.

Applying again Lemma 2.1 we obtain that (8) holds for $\tau = \tau_1$ but not for $\tau = \tau_2$. Since $\tau_2 \subseteq \tau_1$ this implies that (8) does not hold for some $m \in \tau_1 \setminus \tau_2$. The latter is only possible if $\rho_e \in \tau_2^\perp \setminus \tau_1^\perp$. Thus by Lemma 2.1 $R_e \not\subseteq \mathbb{T}_{\tau_1} = \mathbb{T}_q$.

Finally, the one-dimensional orbit $\mathbb{T}_p.q$ coincides with $R_e.q$. This ends the proof. \square

Definition 2.3. We say that a pair of \mathbb{T} -orbits $(\mathcal{O}_1, \mathcal{O}_2)$ in X is H_e -connected if $H_e.x \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ for some $x \in X \setminus X^{H_e}$. By Proposition 2.1 for such a pair (up to a permutation) $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$ and $\dim \mathcal{O}_2 = 1 + \dim \mathcal{O}_1$. Clearly, we can choose a point x as above on the orbit \mathcal{O}_2 . Since the torus normalizes the subgroup H_e any point of \mathcal{O}_2 can actually serve as such a point x .

Example 2.1. Given a root $e \in \mathcal{R}(\sigma)$, the derivation ∂_e as in (4) extends to an LND of a bigger graded algebra

$$A(\rho_e) = \bigoplus_{m \in M, \langle \rho_e, m \rangle \geq 0} \mathbb{k}\chi^m.$$

Indeed, letting $k = \langle \rho_e, m \rangle \geq 0$ yields $\langle \rho_e, m + ke \rangle = 0$ and so $\partial_e^k(\chi^m) \in \ker \partial_e$. This provides a \mathbb{T} - and H_e -stable open subset

$$U = \text{Spec } A(\rho_e) \cong (\mathbb{A}_*^1)^{n-1} \times \mathbb{A}^1 \subseteq X,$$

where $n = \dim X$, $\mathbb{A}_*^1 = \text{Spec } \mathbb{k}[t, t^{-1}]$, $\mathbb{A}^1 = \text{Spec } \mathbb{k}[u]$, where $u = \chi^{-e}$, and H_e acts by the shifts along the second factor. The only \mathbb{T} -orbits in U are the open orbit $\mathcal{O}_1 = \{u \neq 0\}$ (which corresponds to the vertex of σ) and the codimension one orbit $\mathcal{O}_2 = \{u = 0\}$ (which corresponds to the ray $\mathbb{k}\rho_e$ of σ). It is easily seen that the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e -connected.

From Proposition 2.1 and its proof we deduce the following criterion of H_e -connectedness.

Lemma 2.2. *Let $(\mathcal{O}_1, \mathcal{O}_2)$ be a pair of \mathbb{T} -orbits on X with $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$, where $\mathcal{O}_i = \mathcal{O}_{\sigma_i^\perp}$ for a face σ_i of σ , $i = 1, 2$. Given a root $e \in \mathcal{R}(\sigma)$, the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e -connected if and only if $e|_{\sigma_2} \leq 0$ and σ_1 is a facet of σ_2 given by equation $\langle v, e \rangle = 0$.*

Proof. In course of the proof of Proposition 2.1(b) we established that the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e -connected if and only if $\overline{\mathcal{O}_1}$ is H_e -invariant, $\overline{\mathcal{O}_2}$ is not, and $\dim \mathcal{O}_1 = 1 + \dim \mathcal{O}_2$. Moreover, if $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e -connected then σ_2^\perp is a facet of σ_1^\perp (and so σ_1 is a facet of σ_2), and there exists $m_0 \in \sigma_1^\perp \setminus \sigma_2^\perp$ such that

$$\langle \rho_e, m_0 \rangle > 0 \quad \text{and} \quad m_0 + e \in \sigma_2^\perp.$$

Since $\langle \rho_i, e \rangle \geq 0 \forall \rho_i \neq \rho_e$, we obtain that $\sigma_2 = \text{Cone}(\sigma_1, \rho_e)$. We have also $e|_{\sigma_1} = 0$ because $e = m_0 + e - m_0 \in \text{span } \sigma_1^\perp$. Thus $e|_{\sigma_2} \leq 0$ and σ_1 is given in σ_2 by equation $\langle v, e \rangle = 0$.

Conversely, assume that $e|_{\sigma_2} \leq 0$ and σ_1 is given in σ_2 by equation $\langle v, e \rangle = 0$. Then for any $m \in \sigma^\vee \setminus \sigma_1^\perp$ with $\langle \rho_e, m \rangle > 0$ we have $m + e \notin \sigma_1^\perp$ (indeed, $e|_{\sigma_1} = 0$ and so $e \in \sigma_1^\perp$).

Thus (8) holds for σ_1^\perp . Furthermore, $\langle \rho_e, m' \rangle > 0$ for any $m' \in \sigma_1^\perp \setminus \sigma_2^\perp$. It follows that

$$m_0 := m' + (\langle \rho_e, m' \rangle - 1) \cdot e \in \sigma_1^\perp \setminus \sigma_2^\perp \quad \text{and} \quad m_0 + e \in \sigma_2^\perp.$$

Indeed, $\langle \rho_e, m_0 \rangle = 1$ and $\langle \rho_e, m_0 + e \rangle = 0$ while $\langle \rho_i, m_0 \rangle \geq 0$ and $\langle \rho_i, m_0 + e \rangle \geq 0$ for every $\rho_i \neq \rho_e$. Therefore (8) is fulfilled for σ_1^\perp and not for σ_2^\perp . Consequently, the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e -connected. \square

Remark 2.3. Given a one-parameter subgroup $R \subseteq \mathbb{T}$ and a point $x \in X \setminus X^R$, the orbit closure $\overline{R.x}$ coincides with an H_e -orbit if and only if $\overline{R.x}$ is covered by a pair of H_e -connected \mathbb{T} -orbits. For instance, for $X = \mathbb{A}^2$ with the standard torus action and $R \subseteq \mathbb{T}$ being the subgroup of scalar matrices, the latter condition holds only for the points $x \neq 0$ on one of the coordinate axis, which are \mathbb{T} -orbits as well.

Lemma 2.3. *For any $x \in X_{\text{reg}} \setminus \mathcal{O}_{\sigma^\vee}$ ⁹ there is a root $e \in \mathcal{R}(\sigma)$ such that*

$$\dim \mathbb{T}.y > \dim \mathbb{T}.x$$

for a general point $y \in H_e.x$. In particular, the pair $(\mathbb{T}.y, \mathbb{T}.x)$ is H_e -connected.

Proof. Since $x \notin \mathcal{O}_{\sigma^\vee}$, by the Orbit-Cone Correspondence there exists a proper face, say, $\sigma_2 \subseteq \sigma$ such that $\mathbb{T}.x = \mathcal{O}_{\sigma_2^\perp}$. The point $x \in X$ being regular the ray generators, say, ρ_1, \dots, ρ_s of σ_2 form a base of a primitive sublattice $N' \subseteq N$, see [14, §2.1]. We let σ_1 be the facet of σ_2 spanned by ρ_2, \dots, ρ_s . Again by the Orbit-Cone Correspondence, $\mathcal{O}_{\sigma_2^\perp} \subseteq \overline{\mathcal{O}_{\sigma_1^\perp}}$ and $\dim \mathcal{O}_{\sigma_1^\perp} = 1 + \dim \mathcal{O}_{\sigma_2^\perp}$. Let us show that the pair $(\mathcal{O}_{\sigma_1^\perp}, \mathcal{O}_{\sigma_2^\perp})$ is H_e -connected for some root $e \in \mathcal{R}(\sigma)$ satisfying the assumptions of Lemma 2.2.

Choosing a σ -supporting hyperplane $L \subseteq N_{\mathbb{Q}}$ such that $\sigma_2 = \sigma \cap L$ we obtain a splitting $N = N' \oplus N'' \oplus N'''$, where $N \cap L = N' \oplus N''$ and $N''' \cong \mathbb{Z}$. Consider a linear form e_1 on N' defined by

$$\langle \rho_1, e_1 \rangle = -1 \quad \text{and} \quad \langle \rho_2, e_1 \rangle = \dots = \langle \rho_s, e_1 \rangle = 0.$$

Let e_2 be a non-zero linear form on N''' . Extending e_1 and e_2 to the whole lattice N by zero on the complementary sublattices we obtain a linear form $e = e_1 + e_2$ on N . Multiplying e_2 by a suitable integer we can achieve that $\langle \rho_j, e \rangle > 0$ for every $\rho_j \notin \sigma_2$. Then e is a root of the cone σ such that $\rho_e = \rho_1$ and the condition of Lemma 2.2 holds for e . By Lemma 2.2 the pair $(\mathcal{O}_{\sigma_1^\perp}, \mathcal{O}_{\sigma_2^\perp})$ is H_e -connected, as claimed. Since $\mathbb{T}.x = \mathcal{O}_{\sigma_2^\perp}$ and the torus \mathbb{T} normalizes the group H_e , the desired conclusion follows from Proposition 2.1 and the observation in Definition 2.3. \square

The proof of infinite transitivity in Theorem 2.1 is based on Lemmas 2.4-2.8 below.

Lemma 2.4. *For any collection of m distinct points $Q_1, \dots, Q_m \in X_{\text{reg}}$ there exists an automorphism $\phi \in \text{SAut}(X)$ such that the images $\phi(Q_1), \dots, \phi(Q_m)$ are contained in the open \mathbb{T} -orbit.*

Proof. Letting

$$d(Q_1, \dots, Q_m) = \dim \mathbb{T}.Q_1 + \dots + \dim \mathbb{T}.Q_m,$$

we assume that $\dim \mathbb{T}.Q_i < \dim X$ for some i . By Lemma 2.3 there exists a root $e \in \mathcal{R}(\sigma)$ such that $\dim \mathbb{T}.P_i > \dim \mathbb{T}.Q_i$ for a general point $P_i \in H_e.Q_i$. Fix an isomorphism $\lambda_e : G_a(\mathbb{k}) \xrightarrow{\cong} \mathbb{T}$

⁹Recall that $\mathcal{O}_{\sigma^\vee}$ is the open \mathbb{T} -orbit in X .

H_e . There is a finite set of values $t \in G_a(\mathbb{k})$ such that $\dim \mathbb{T} \cdot (\lambda_e(t) \cdot Q_j) < \dim \mathbb{T} \cdot Q_j$ for some $j \neq i$. Thus for a general $t \in G_a(\mathbb{k})$,

$$d(\lambda_e(t) \cdot Q_1, \dots, \lambda_e(t) \cdot Q_m) > d(Q_1, \dots, Q_m).$$

Applying recursion we get the result. \square

From now on we assume that Q_1, \dots, Q_m are contained in the open \mathbb{T} -orbit, say, $\mathbb{T} \cdot x_0$. We fix a maximal subset of linearly independent ray generators $\{\rho_1, \dots, \rho_n\} =: \Xi^{(0)} \subseteq \Xi$, where $n = \dim X$. For every $i = 1, \dots, n$ we choose an isomorphism $\rho_i : G_m(\mathbb{k}) \xrightarrow{\cong} R_{\rho_i}$ (denoted by the same letter as the ray generator). Recall that for a root $e \in \mathcal{R}(\sigma)$ the inclusion $\mathbb{k}[X]^{H_e} \subseteq \mathbb{k}[X]$ induces a morphism $\tau : X \rightarrow Z$, where $Z = \text{Spec } \mathbb{k}[X]^{H_e}$.

Lemma 2.5. *Given $\rho_i \in \Xi^{(0)}$ and a root $e \in \mathcal{R}(\sigma)$ with $\rho_e = \rho_i$, for every finite set $\mathcal{T}_0, \dots, \mathcal{T}_k$ of pairwise distinct R_e -orbits in $\mathbb{T} \cdot x_0$ there exists a regular invariant $q \in \mathbb{k}[X]^{H_e}$ which equals identically 1 on \mathcal{T}_0 and vanishes on $\mathcal{T}_1, \dots, \mathcal{T}_k$.*

Proof. The quotient morphism $\tau : X \rightarrow Z$ separates typical H_e -orbits, see [34, Theorems 2.3 and 3.3]. Since the torus \mathbb{T} normalizes the group H_e , there is a \mathbb{T} -action on Z such that the morphism τ is \mathbb{T} -equivariant. In particular, for every $x \in \mathbb{T} \cdot x_0$ the fiber of τ through x is an H_e -orbit. According to Proposition 2.1 the R_e -orbits $\mathcal{T}_0, \dots, \mathcal{T}_k$ are intersections of the corresponding H_e -orbits with the open orbit $\mathbb{T} \cdot x_0$. Thus for every $j = 1, \dots, k$ there exists an invariant $q_j \in \mathbb{k}[X]^{H_e}$ that vanishes on \mathcal{T}_j and restricts to \mathcal{T}_0 as the constant function 1. It is easily seen that the product $q = q_1 \cdot \dots \cdot q_k \in \mathbb{k}[X]^{H_e}$ has the desired properties. \square

In the notation of Lemma 2.5 we let $\text{Stab}_{\mathcal{T}_1, \dots, \mathcal{T}_k}(\mathcal{T}_0) \subseteq \text{SAut}(X)$ denote the subgroup of all transformations that fix pointwise the orbits $\mathcal{T}_1, \dots, \mathcal{T}_k$ and stabilize the closure $\overline{\mathcal{T}_0}$ in X .

Lemma 2.6. *There exists a one-parameter unipotent subgroup $H \subseteq \text{Stab}_{\mathcal{T}_1, \dots, \mathcal{T}_k}(\mathcal{T}_0)$ which acts transitively on $\overline{\mathcal{T}_0}$.*

Proof. We let as before $e \in \mathcal{R}(\sigma)$ be a root with $\rho_e = \rho_i$, and q be a regular H_e -invariant as in Lemma 2.5. The LND $q\partial_e \in \text{Der } \mathbb{k}[X]$ defines a one-parameter unipotent subgroup $H \subseteq \text{Stab}_{\mathcal{T}_1, \dots, \mathcal{T}_k}(\mathcal{T}_0)$. Clearly the restriction $H|_{\mathcal{T}_0} = H_e|_{\mathcal{T}_0}$ acts transitively on $\overline{\mathcal{T}_0} \cong \mathbb{A}^1$ by shifts. \square

In the remaining part of the proof of Theorem 2.1 we use the following notation. For a basis $\Xi^{(0)} = \{\rho_1, \dots, \rho_n\}$ in $N_{\mathbb{Q}}$ formed by ray generators, we consider the homomorphism $\theta : (\mathbb{k}^\times)^n \rightarrow \mathbb{T}$ of the standard n -torus to \mathbb{T} given by

$$(10) \quad \theta : (t_1, \dots, t_n) \longmapsto (\rho_1(t_1) \cdot \dots \cdot \rho_n(t_n)).$$

It is easily seen that θ is surjective and its kernel $\Theta = \ker(\theta)$ is a finite subgroup in $(\mathbb{k}^\times)^n$. We consider as well the induced surjective morphism of $(\mathbb{k}^\times)^n$ to the open orbit $\mathbb{T} \cdot x_0$. In particular, given m distinct points $Q_1, \dots, Q_m \in \mathbb{T} \cdot x_0$ we can write

$$(11) \quad Q_j = \theta(t_{1,j}, \dots, t_{n,j}) \cdot x_0, \quad j = 1, \dots, m,$$

where the point $(t_{1,j}, \dots, t_{n,j}) \in (\mathbb{k}^\times)^n$ is determined by Q_j up to the diagonal action of Θ on $(\mathbb{k}^\times)^n$:

$$(12) \quad \vartheta \cdot (t_1, \dots, t_n) = (\vartheta_1 t_1, \dots, \vartheta_n t_n), \quad \text{where } \vartheta = (\vartheta_1, \dots, \vartheta_n) \in \Theta.$$

Letting $\kappa = \text{ord } \Theta$, by the Lagrange Theorem we have $\vartheta_i^\kappa = 1 \ \forall i = 1, \dots, n$.

We fix a standard collection of m points in $\mathbb{T}.x_0$:

$$(13) \quad Q_j^0 = \theta(j, \dots, j).x_0, \quad j = 1, \dots, m.$$

Since by our assumptions, $\text{Char}(\mathbb{k}) = 0$ and $\overline{\mathbb{k}} = \mathbb{k}$, these points are distinct. It remains to find a special automorphism $\varphi \in \text{SAut}(X)$ such that $\varphi(Q_j) = Q_j^0$ for every $j = 1, \dots, m$. To this end we use the following Lemmas 2.7 and 2.8.

We will say that $t, t' \in \mathbb{k}^\times$ are κ -equivalent if $t' = \varepsilon t$ for some κ th root of unity $\varepsilon \in \mathbb{k}^\times$.

Lemma 2.7. (a) *For any distinct elements $t_1, \dots, t_n \in \mathbb{k}^\times$ the set of values $a \in \mathbb{k}$ such that $t_i + a$ and $t_j + a$ are κ -equivalent for some $i \neq j$ is finite.*

(b) *Fix $s \in \{1, \dots, n\}$. If the points Q_i and Q_j lie on the same R_s -orbit then their r th components $t_{i,r}$ and $t_{j,r}$ are κ -equivalent for every $r \neq s$.*

(c) *Suppose that the points Q_{j_1}, \dots, Q_{j_l} lie on the same R_s -orbit $\mathcal{T}^{(s)}$. Then their images under a general shift on the line $\overline{\mathcal{T}^{(s)}} \cong \mathbb{A}^1$ belong to distinct R_r -orbits for every $r \neq s$.*

Proof. (a) Given a κ th root of unity ε the linear equation

$$t_i + a = \varepsilon(t_j + a)$$

is satisfied for at most one value of a . Now (a) follows.

The assertion of (b) holds since the R_s -action on X lifted via (10) affects only the component $t_{i,s}$ of Q_i in (11), while the Θ -action on $(\mathbb{k}^\times)^n$ replaces the component $t_{i,r}$ ($r \neq s$) by a κ -equivalent one.

Now (c) is immediate from (a) and (b). Indeed, for $i \neq j$ the intersection of any R_i - and R_j -orbits is at most finite. \square

Lemma 2.8. *In the notation as above there exists $\psi \in \text{SAut}(X)$ such that the points $\psi(Q_1), \dots, \psi(Q_m)$ lie in different R_1 -orbits in $\mathbb{T}.x_0$.*

Proof. By our assumption $n \geq 2$, so there is an R_2 -action on X . Let $\mathcal{T}_0^{(2)}, \dots, \mathcal{T}_k^{(2)}$ be the distinct R_2 -orbits passing through the points Q_1, \dots, Q_m so that this collection splits into $k+1$ disjoint pieces. We may assume that the piece on $\mathcal{T}_0^{(2)}$ is Q_1, \dots, Q_l . Applying Lemma 2.6 with $\rho_e = \rho_2$ we can find a one-parameter unipotent subgroup $H \subseteq \text{Stab}_{\mathcal{T}_1^{(2)}, \dots, \mathcal{T}_k^{(2)}}(\mathcal{T}_0^{(2)})$

acting by shifts on $\overline{\mathcal{T}_0^{(2)}} \cong \mathbb{A}^1$. By Lemma 2.7 the images of Q_1, \dots, Q_l under a general such shift lie in different R_r -orbits for every $r \neq 2$, while all the other points Q_j ($j > l$) remain fixed. Applying the same procedure subsequently to the other pieces we obtain finally a special automorphism $\psi \in \text{SAut}(X)$ such that the points $\psi(Q_1), \dots, \psi(Q_m)$ belong to different R_r -orbits for every $r \neq 2$. \square

Proof of infinite transitivity in Theorem 2.1. By virtue of Lemma 2.8 we may assume that the orbits $\mathcal{T}_j^{(1)} = R_1.Q_j$, $j = 1, \dots, m$, are all distinct. By Lemma 2.6 we can change the component $t_{1,j}$ of a point Q_j arbitrarily while fixing the other components and as well the other points of our collection. Thus we can achieve that $t_{1,j} = j$ for all $j = 1, \dots, m$. This guarantees that for any $l \geq 2$ the orbits $R_l.Q_1, \dots, R_l.Q_m$ are pairwise distinct. Applying Lemma 2.6 again to every R_l -orbit for $l = 2, \dots, n$ we can reach the standard collection $Q_1^{(0)}, \dots, Q_m^{(0)}$ as in (13) with $t_{l,j} = j$ for all $j = 1, \dots, m$, $l = 1, \dots, n$. This proves the infinite transitivity statement in Theorem 2.1. For the proof of flexibility, see the next lemma. \square

Lemma 2.9. *X is flexible.*

Proof. If $\dim X = 1$ then $X \cong \mathbb{A}^1$ and the assertion is evidently true. Suppose further that $\dim X \geq 2$. We know already that the group $\text{SAut}(X)$ acts (infinitely) transitively on X_{reg} . Hence it is enough to find just one flexible point in X_{reg} . Let us show that the point x_0 in the open \mathbb{T} -orbit is flexible. Consider the action of the standard torus $(\mathbb{k}^\times)^n$ on X induced by the \mathbb{T} -action on X via (10). The stabilizer $\text{Stab}(x_0) \subseteq (\mathbb{k}^\times)^n$ being finite, the tangent map $T_g(\mathbb{k}^\times)^n \rightarrow T_{x_0}X$ at each point $g \in \text{Stab}(x_0)$ is surjective. Hence the tangent vectors at x_0 to the orbits $R_i.x_0$, $i = 1, \dots, n$, span the tangent space $T_{x_0}X$. By Remark 2.1 for every $i = 1, \dots, n$ there exists a root $e_i \in \mathcal{R}(\sigma)$ such that $\rho_i = \rho_{e_i}$. Since x_0 cannot be fixed by the one-parameter unipotent subgroup H_{e_i} , by virtue of Proposition 2.1 $H_{e_i}.x_0 = \overline{R_i.x_0}$. Parameterizing properly these two orbits, their velocity vectors at x_0 coincide. Therefore, $T_{x_0}X$ is spanned as well by the tangent vectors of the orbits $H_{e_i}.x_0$, $i = 1, \dots, n$, which means that the point x_0 is flexible on X (see Definition 0.1). \square

Now the proof of Theorem 2.1 is completed.

Example 2.2. Consider an affine toric surface $X_{d,e} = \mathbb{A}^2/G_d$, where d and e are coprime integers with $0 < e < d$, and G_d is the cyclic group generated by a primitive d th root of unity ζ acting on the plane \mathbb{A}^2 via $\zeta.(x, y) = (\zeta x, \zeta^e y)$. It is well known [11], [15], [29] that for $e \geq 2$ the smooth locus $(X_{d,e})_{\text{reg}} = X_{d,e} \setminus \{0\}$ is not isomorphic to a homogeneous space of any affine algebraic group. However $X_{d,e} \setminus \{0\}$ is homogeneous under the action of the infinite dimensional group $\text{SAut}(X)$.

3. AFFINE SUSPENSIONS

In this section we prove part 3 of Theorem 0.2 in the Introduction. Let us first recall necessary notions.

Definition 3.1. Let $X^{(0)}$ be an affine variety. By a *cylinder* over $X^{(0)}$ we mean the product $X^{(0)} \times \mathbb{A}^1$. Given a nonconstant regular function $f_1 \in \mathbb{k}[X^{(0)}]$ we define a new affine variety

$$X^{(1)} = \text{Susp}(X^{(0)}, f_1) := \{f_1(x) - uv = 0\} \subseteq X^{(0)} \times \mathbb{A}^2$$

called a *suspension* over $X^{(0)}$. By recursion, for any $l \in \mathbb{N}$ we obtain the iterated suspension $X^{(l)} = \text{Susp}(X^{(l-1)}, f_l)$.

For instance, starting with $X^{(0)} = \mathbb{A}^k$ we arrive at the l th suspension $X^{(l)}$ given in the affine space $\mathbb{A}^{k+2l} = \text{Spec } \mathbb{k}[x_1, \dots, x_k, u_1, v_1, \dots, u_l, v_l]$ by equations

$$(14) \quad \begin{cases} u_1 v_1 - f_1(x_1, x_2, \dots, x_k) = 0 \\ u_2 v_2 - f_2(x_1, x_2, \dots, x_k, u_1, v_1) = 0 \\ \vdots \\ u_l v_l - f_l(x_1, x_2, \dots, x_k, u_1, v_1, u_2, v_2, \dots, u_{l-1}, v_{l-1}) = 0, \end{cases}$$

where for every $i = 1, 2, \dots, l$ the polynomial $f_i \in \mathbb{k}[x_1, \dots, x_k, u_1, v_1, \dots, u_{i-1}, v_{i-1}]$ is non-constant modulo the ideal $(u_1 v_1 - f_1, \dots, u_{i-1} v_{i-1} - f_{i-1})$.

We separate the two cases in part 3 of Theorem 0.2 according to the dimension of the base. In dimension 1, the assertion holds true over an arbitrary field of characteristic zero, under an additional restriction on the function $f = f_1$.

Theorem 3.1. *Let \mathbb{k} be a field of characteristic zero. Given a polynomial $f \in \mathbb{k}[x]$ with $f(\mathbb{k}) = \mathbb{k}$, consider a surface $X \subseteq \mathbb{A}_{\mathbb{k}}^3$ with equation $f(x) - uv = 0$. Then X is flexible and the special automorphism group $\text{SAut}(X)$ acts m -transitively on X_{reg} for every $m \in \mathbb{N}$.*

In higher dimensions, part 3 of Theorem 0.2 can be restated as follows.

Theorem 3.2. *Let \mathbb{k} be an algebraically closed field of characteristic zero. Let $X^{(0)}$ be a flexible affine variety. Assume that either $X^{(0)} \cong \mathbb{A}^1$, or $\dim X^{(0)} \geq 2$ and the special automorphism group $\text{SAut}(X^{(0)})$ acts m -transitively on $X_{\text{reg}}^{(0)}$ for every $m \in \mathbb{N}$. Then any iterated suspension $X^{(l)}$ over $X^{(0)}$, $l \geq 1$, is flexible and the special automorphism group $\text{SAut}(X^{(l)})$ acts m -transitively on $X_{\text{reg}}^{(l)}$ for every $m \in \mathbb{N}$.*

Since the assumptions of the theorem are fulfilled for the affine space $X^{(0)} = \mathbb{A}^k$, $k \geq 1$, we can conclude that for every $k, l \geq 1$ the affine variety $X^{(l)} \subseteq \mathbb{A}^{k+2l}$ defined by (14) is flexible and the group $\text{SAut}(X^{(l)})$ acts infinitely transitively in $X_{\text{reg}}^{(l)}$.

The proof of Theorem 3.2, with minor changes, works also for real algebraic varieties and leads to the following result.

Theorem 3.3. *Let $X^{(0)}$ be a flexible real algebraic variety. Suppose that the smooth locus $X_{\text{reg}}^{(0)}$ is connected and the special automorphism group $\text{SAut}(X^{(0)})$ acts m -transitively on $X_{\text{reg}}^{(0)}$ for every $m \in \mathbb{N}$. Consider the iterated suspensions $X^{(i)} = \text{Susp}(X^{(i-1)}, f_i)$, where the functions $f_i \in \mathbb{R}[X^{(i-1)}]$ satisfy the conditions $f_i(X_{\text{reg}}^{(i-1)}) = \mathbb{R}$, $i = 1, \dots, l$. Then for every $i = 1, \dots, l$ the variety $X^{(i)}$ is flexible and the special automorphism group $\text{SAut}(X^{(i)})$ acts m -transitively on $X_{\text{reg}}^{(i)}$ for every $m \in \mathbb{N}$.*

The infinite transitivity in Theorems 3.1, 3.2, and 3.3 is proven in Subsections 3.1, 3.2, and 3.3, respectively. The flexibility in all three cases is established in Subsection 3.4.

3.1. Suspension over a line. Here we prove the infinite transitivity in Theorem 3.1. The proof is elementary and based on some explicit formulae from [27].

Proof. We may assume that $d = \deg f \geq 2$. According to [27]¹⁰, in our case the special automorphism group $\text{SAut}(X)$ contains the abelian subgroups G_u and G_v generated, respectively, by the one parameter unipotent subgroups

$$(15) \quad H_u(q) : (x, u, v) \mapsto \left(x + tq(u), u, v + \frac{f(x + tq(u)) - f(x)}{u} \right)$$

and

$$(16) \quad H_v(q) : (x, u, v) \mapsto \left(x + tq(v), u + \frac{f(x + tq(v)) - f(x)}{v}, v \right),$$

where $q(z) \in \mathbb{k}[z]$, $q(0) = 0$, and $t \in \mathbb{k}$. So $u \in \mathbb{k}[X]^{G_u}$ and $v \in \mathbb{k}[X]^{G_v}$. We claim that the subgroup $G = \langle G_u, G_v \rangle \subseteq \text{SAut}(X)$ acts m -transitively on X_{reg} for every $m \in \mathbb{N}$. So given an m -tuple of distinct points

$$Q_1 = (x_1, u_1, v_1), \dots, Q_m = (x_m, u_m, v_m) \in X_{\text{reg}},$$

¹⁰Cf. also [7], [10], and [28].

our aim is to find an automorphism $\phi \in G$ which sends this m -tuple to a standard m -tuple

$$Q_i^{(0)} = (x_i^{(0)}, u_i^{(0)}, v_i^{(0)}), \quad i = 1, \dots, m,$$

chosen in such a way that all $v_i^{(0)}$ are nonzero and distinct.

Step 1. Acting with G_u we can replace the original m -tuple by another one such that $v_i \neq 0 \forall i = 1, \dots, m$. Indeed, the polynomial in t

$$\frac{f(x+tu) - f(x)}{u} = \frac{f'(x)}{1!}t + \dots + \frac{f^{(d)}(x)}{d!}u^{d-1}t^d \in \mathbb{k}[x, u][t],$$

where $f^{(d)}(x)$ is a nonzero constant, is non-constant provided that either $f'(x) \neq 0$ or $u \neq 0$. Since the point $Q_s \in X$ is smooth the equalities $u_s = 0, v_s = 0, f'(x_s) = 0$ cannot hold simultaneously. Hence acting by (15) with $q = z$ and a general t does change the coordinate $v_s = 0$, while keeping nonzero those v_i that were already nonzero. Now the claim follows.

Step 2. Suppose further that $v_i \neq 0 \forall i = 1, \dots, m$. Then acting with G_v we can send our m -tuple to another one where all the u_i ($i = 1, \dots, m$) are nonzero and distinct. Indeed, let

$$F(Q_i, q, t) = \frac{f'(x_i)q(v_i)}{1!v_i}t + \dots + \frac{f^{(d)}(x_i)q(v_i)^d}{d!v_i}t^d \in \mathbb{k}[t].$$

We have $(x_i, v_i) \neq (x_j, v_j) \forall i \neq j$ because $(x_i, u_i, v_i) \neq (x_j, u_j, v_j)$ while $u_i = f(x_i)/v_i$ and $u_j = f(x_j)/v_j$. If $v_i = v_j$ then $f^{(d-1)}(x_i) \neq f^{(d-1)}(x_j)$ since the linear form $f^{(d-1)}(x)$ is nonzero. Thus for a suitable $q \in \mathbb{k}[z]$ such that $q(v_i) \neq 0 \forall i$, the polynomials $F(Q_i, q, t)$ and $F(Q_j, q, t)$ are different for every $i \neq j$. Applying an automorphism $H_v(q)$ in (16) with a general t , we obtain the result.

Step 3. We assume now that all the coordinates u_j are nonzero and distinct. Let us show that it is possible to achieve the standard values $v_s^{(0)}, s = 1, \dots, m$, acting by G_u . To this end, we construct an automorphism that preserves all the points but Q_i and sends Q_i to a new point Q'_i with $v'_i = v_i^{(0)}$. Namely, fix a polynomial $q(z)$ with $q(0) = 0, q(u_i) \neq 0$ and $q(u_j) = 0$ for all $j \neq i$. Our assumption on $f(x)$ guarantees that the equation $f(x) = u_i(v_i^{(0)} - v_i) + f(x_i)$ has a root $x = a_i$, where $a_i \in \mathbb{k}$. Applying $H_u(q)$ in (15) with $t = (a_i - x_i)/q(u_i)$, we obtain the required.

Step 4. Suppose finally that $v_i = v_i^{(0)}$ for all i . It suffices to reach the values $x_i = x_i^{(0)}$ for all i acting by an automorphism from G_v (indeed, then also $u_i = f(x_i)/v_i = f(x_i^{(0)})/v_i^{(0)} = u_i^{(0)}$). This can be done by applying $H_u(q)$ as in (16) with $t = 1$ and a polynomial q satisfying $q(0) = 0$ and $q(v_i^{(0)}) = x_i^{(0)} - x_i$ for all i . Now the proof is completed. \square

3.2. Infinite transitivity in higher dimensions. Clearly, it is enough to prove Theorems 3.2-3.3 for $l = 1$. Before passing to the proofs we establish in Lemmas 3.1-3.3 below some necessary elementary facts concerning suspensions. In Lemmas 3.1-3.4 we consider an arbitrary field \mathbb{k} of characteristic zero.

Lemma 3.1. *If $X^{(0)}$ is irreducible then the suspension $X^{(1)} = \text{Susp}(X^{(0)}, f)$ is.*

Proof. Suppose to the contrary that there exist nonzero elements

$$F_1, F_2 \in \mathbb{k}[X^{(1)}] = \mathbb{k}[X^{(0)}][u, v]/(uv - f) \quad \text{such that} \quad F_1 F_2 = 0.$$

We may assume that $\deg_{u,v}(F_1) + \deg_{u,v}(F_2)$ is minimal and no monomial in F_i contains the product uv , since otherwise we can replace this product by f according to Definition 3.1. If

u occurs in both F_1 and F_2 then $\deg_u(F_1F_2) > 0$ since the leading term in u cannot cancel. Hence, up to twisting u and v , we may assume that F_1 does not contain v and F_2 does not contain u . Let us write

$$F_1 = \sum_{i=0}^k a_i u^i \quad \text{and} \quad F_2 = \sum_{j=0}^l b_j v^j,$$

where $a_i, b_j \in \mathbb{k}[X^{(0)}]$, and $k + l$ is minimal.

If neither u nor v occurs i.e., $k = l = 0$, then $F_1, F_2 \in \mathbb{k}[X^{(0)}]$ are zero divisors, which contradicts the irreducibility of $X^{(0)}$. So $k + l > 0$.

If $a_0 = b_0 = 0$ then we can decrease the degree $k + l$ by dividing out u and v . This contradicts the minimality assumption. So we may suppose that $a_0 \neq 0$. Then the product F_1F_2 contains a nonzero term $a_0b_lv^l$, which gives again a contradiction. \square

Lemma 3.2. *We have $\pi(X_{\text{reg}}^{(1)}) = X_{\text{reg}}^{(0)}$, where $\pi : X^{(1)} \rightarrow X^{(0)}$ is the restriction of the canonical projection $X^{(0)} \times \mathbb{A}^2 \rightarrow X^{(0)}$ to $X^{(1)}$.*

Proof. Let $f_1, f_2, \dots, f_m \in \mathbb{k}[x_1, x_2, \dots, x_s]$ generate the ideal of $X^{(0)} \subseteq \mathbb{A}^s$. A point $P \in X^{(0)}$ is regular if and only if the rank of the Jacobian matrix

$$D_0 = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_s} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_s} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_s} \end{pmatrix}$$

attains its maximal value $s - \dim X^{(0)}$ at P . The corresponding matrix for $X^{(1)}$ is

$$D_1 = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_s} & 0 & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_s} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_s} & 0 & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_s} & -v & -u \end{pmatrix}.$$

Obviously, $\text{rk } D_1 \leq 1 + \text{rk } D_0$ at every point. Since $\dim X^{(1)} = 1 + \dim X^{(0)}$, any regular point of $X^{(1)}$ is mapped via π to a regular point of $X^{(0)}$. On the other hand, let a square submatrix M of D_0 and a point $P \in X_{\text{reg}}^{(0)}$ be such that $M(P)$ is of rank $r = s - \dim X^{(0)}$ equal to its order. We extend M to a square submatrix M' of order $r + 1$ by adding the last line and one of the two extra columns of D_1 in such a way that $\text{rk } M'(P, u, v) = 1 + \text{rk } M(P) = r + 1$ for some $(u, v) \neq (0, 0)$, where $(P, u, v) \in X^{(1)}$. Hence $(P, u, v) \in X_{\text{reg}}^{(1)}$. Now the assertion follows. \square

Remark 3.1. Recall [21] that an *affine modification* of an affine algebra A with center (I, v) , where $I \subseteq A$ is an ideal and $v \in I$ is not a zero divisor, is the quotient algebra $A[It]/(1 - vt)$, where

$$\text{Bl}_I(A) = A[It] = A \oplus \bigoplus_{n=1}^{\infty} (It)^n \cong A \oplus I \oplus I^2 \oplus \dots$$

is the blow-up (or the Rees) algebra of the pair (A, I) and t is a formal symbol.

Geometrically, the variety $\text{Spec}(A[It]/(1 - vt))$ is obtained from $X = \text{Spec} A$ as follows. Performing a blowup of X with center I we remove the proper transform in $\text{Bl}_I(X)$ of the zero divisor $V(v)$ in X , which results again in an affine variety. (We note that this proper transform meets the exceptional divisor E , since $v \in I$.) See [21, §1] for more details.

According to [21, Example 1.4 and §5], the suspension $X^{(1)} = \text{Susp}(X^{(0)}, f)$ can be viewed as an affine modification of $X^{(0)} \times \mathbb{A}^1$ (where $\mathbb{A}^1 = \text{Spec} \mathbb{k}[v]$) with center $(I_1 = (v, f), v)$ along the divisor $v = 0$. Interchanging v and u , the variety $X^{(1)}$ can be regarded also as an affine modification of the product $X^{(0)} \times \mathbb{A}^1$ (where this time $\mathbb{A}^1 = \text{Spec} \mathbb{k}[u]$) with center $(I_2 = (u, f), u)$ along the divisor $u = 0$. The exceptional divisors $v = 0$ and $u = 0$, respectively, of these two modifications are both isomorphic to $X^{(0)} \times \mathbb{A}^1$ but different as subvarieties of $X^{(1)}$. In the sequel for every $c \in \mathbb{k}$ we consider the level hypersurfaces $U_c = \{u = c\}$ and $V_c = \{v = c\}$ in $X^{(1)}$.

In [21, §2] a method was developed which allows to extend an LND ∂ to the affine modification provided that ∂ stabilizes the center of the modification. In Lemma 3.3 below we concretize this in our particular case of affine suspensions.

Given an LND δ_0 of an affine domain A_0 and a polynomial $q \in \mathbb{k}[z]$ with $q(0) = 0$, we can define a new LND $\delta' = \delta'(\delta_0, q)$ on $A' = A_0 \otimes \mathbb{k}[v]$, where v is a new variable, as follows. First we extend δ_0 to A' by letting $\delta_0(v) = 0$, and then we multiply δ_0 by the element $q(v) \in \ker \delta_0$. Suppose that A_0 is generated by x_1, x_2, \dots, x_s . Then δ' is given in coordinates by

$$(17) \quad \delta'(x_i) = q(v)\delta_0(x_i), \quad i = 1, 2, \dots, s, \quad \delta'(v) = 0.$$

Let now u be yet another variable and $f \in A_0$ be nonzero. Consider the structure algebra A_1 of the suspension over A_0 :

$$A_1 = (A_0 \otimes \mathbb{k}[u, v]) / (uv - f).$$

Lemma 3.3. *In the notation as above, the LND $\delta' \in \text{Der} A'$ can be transformed into an LND $\delta_1 = \delta_1(\delta_0, q) \in \text{Der} A_1$ by letting*

$$(18) \quad \begin{aligned} \delta_1(x_i) &= \delta'(x_i), \quad i = 1, 2, \dots, s, \\ \delta_1(u) &= \frac{q(v)}{v} \delta_0(f), \\ \delta_1(v) &= \delta'(v) = 0. \end{aligned}$$

Proof. We check first that these formulae extend δ' to $\delta_1 = \delta_1(\delta_0, q) \in \text{Der}(A_0 \otimes \mathbb{k}[u, v])$, where δ_1 preserves the ideal $(uv - f)$. Indeed, since $\frac{q(v)}{v} \in \mathbb{k}[v]$ by our choice of q , the derivation δ_1 is well defined on the generators of $A_0 \otimes \mathbb{k}[u, v]$. It is easily seen that δ_1 is still locally nilpotent. The straightforward calculation shows that $\delta_1(uv - f) = 0$. Hence δ_1 descends to an LND of the quotient algebra A_1 denoted by the same symbol δ_1 . \square

Definition 3.2. We let G_v denote the subgroup of the special automorphism group $\text{SAut}(X^{(1)})$ generated by all one-parameter unipotent subgroups

$$H_v(\delta_0, q) = \exp(t\delta_1), \quad \text{where } t \in \mathbb{k}_+ \quad \text{and} \quad \delta_1 = \delta_1(\delta_0, q),$$

with δ_0 and $q(t)$ as above¹¹. Interchanging the roles of v and u we obtain the second subgroup $G_u \subseteq \text{SAut}(X^{(1)})$. Thus $u \in \mathbb{k}[X^{(1)}]^{G_u}$ and $v \in \mathbb{k}[X^{(1)}]^{G_v}$. We will show that the subgroup $G \subseteq \text{SAut}(X^{(1)})$ generated by G_u and G_v acts infinitely transitively in $X_{\text{reg}}^{(1)}$.

Given k distinct constants $c_1, \dots, c_k \in \mathbb{k}$ we let $\text{Stab}_{c_1 \dots c_k}^v$ denote the subgroup of the group G_v fixing pointwise the hypersurfaces $V_{c_s} \subseteq X^{(1)}$, $s = 1, \dots, k$.

Lemma 3.4. *Suppose that the group $\text{SAut}(X^{(0)})$ acts m -transitively on $X_{\text{reg}}^{(0)}$. Then for any distinct constants $c_0, c_1, \dots, c_k \in \mathbb{k}^\times$ the group $\text{Stab}_{c_1 \dots c_k}^v$ acts m -transitively on $V_{c_0} \cap X_{\text{reg}}^{(1)}$.*

Proof. Given two collections of m distinct points P'_1, \dots, P'_m and Q'_1, \dots, Q'_m in $V_{c_0} \cap X_{\text{reg}}^{(1)}$ we let P_1, \dots, P_m and Q_1, \dots, Q_m denote their π -projections to $X^{(0)}$. Notice that the hypersurface $V_{c_0} \subseteq X^{(1)}$ is mapped via π isomorphically onto $X^{(0)}$, while by Lemma 3.2 we have $\pi(V_{c_0} \cap X_{\text{reg}}^{(1)}) \subseteq X_{\text{reg}}^{(0)}$. Indeed, a point $P' \in V_{c_0}$ can be written as $P' = (P, u, c_0)$, where $P = \pi(P') \in X^{(0)}$ and $u = u(P') = f(P)/c_0$. Conversely, these formulae give an isomorphism $X^{(0)} \xrightarrow{\cong} V_{c_0}$ which sends P to P' .

Since by our assumption the group $\text{SAut}(X^{(0)})$ acts m -transitively on $X_{\text{reg}}^{(0)}$, there exists an automorphism $\psi_0 \in \text{SAut}(X^{(0)})$ which sends the ordered collection (P_1, \dots, P_m) to (Q_1, \dots, Q_m) . It can be written as a product

$$\psi_0 = \prod_{i=1}^k \exp(\delta_0^{(i)})$$

for some LNDs $\delta_0^{(1)}, \dots, \delta_0^{(k)} \in \text{Der } \mathbb{k}[X^{(0)}]$.

Letting $q = \alpha z(z - c_1) \dots (z - c_k)$, where $\alpha \in \mathbb{k}^\times$ is such that $q(c_0) = 1$, by Lemma 3.3 we can lift the $\delta_0^{(i)}$ to the LNDs

$$\delta_1^{(i)} = \delta_1^{(i)}(\delta_0^{(i)}, q) \in \text{Der } \mathbb{k}[X^{(1)}], \quad i = 1, \dots, k.$$

Respectively, ψ_0 can be lifted to a special automorphism

$$\psi_1 = \prod_{i=1}^k \exp(\delta_1^{(i)}) \in G_v \subseteq \text{SAut}(X^{(1)}).$$

By virtue of (17) it is easily seen that the actions on $X^{(1)}$ of the corresponding one-parameter unipotent subgroups $H_v(\delta_0^{(i)}, q)$ restrict to the original actions on $V_{c_0} \cong X^{(0)}$. So the automorphism $\psi_1|_{V_{c_0}} = \psi_0$ sends (P'_1, \dots, P'_m) to (Q'_1, \dots, Q'_m) . Due to our choice of $q(z)$, this automorphism fixes all the other hypersurfaces V_{c_s} pointwise. \square

Lemma 3.5. *Let \mathbb{k} be an algebraically closed field of characteristic zero. Suppose as before that the group $\text{SAut}(X^{(0)})$ acts m -transitively on $X_{\text{reg}}^{(0)}$. Then for any set of distinct points $Q'_1, \dots, Q'_m \in X_{\text{reg}}^{(1)}$ there exists an automorphism $\varphi \in \text{SAut}(X^{(1)})$ such that $\varphi(Q'_i) \notin U_0 \cup V_0$ for all $i = 1, 2, \dots, m$.*

Proof. We say that the point $Q'_i = (Q_i, u_i, v_i) \in X^{(1)}$ is *hyperbolic* if $u_i v_i \neq 0$ i.e., $Q'_i \notin U_0 \cup V_0$. We have to show that the original collection can be moved by means of a special automorphism so that all the points become hyperbolic. Suppose that Q'_1, \dots, Q'_l are already hyperbolic while

¹¹Notice that for $X^{(0)} = \mathbb{A}^1 = \text{Spec } \mathbb{k}[z]$ and $\delta_0 = d/dz$ we have $H_v(\delta_0, q) = H_v(q)$ from (16).

Q'_{l+1} is not, where $l \geq 0$. By recursion, it is sufficient to move Q'_{l+1} off $U_0 \cup V_0$ while leaving the points Q'_1, \dots, Q'_l hyperbolic. It is enough to consider the following two cases:

Case 1: $u_{l+1} = 0$, $v_{l+1} \neq 0$, and

Case 2: $u_{l+1} = v_{l+1} = 0$.

We claim that there exists an automorphism $\varphi \in \text{SAut}(X^{(1)})$ leaving Q'_1, \dots, Q'_l hyperbolic such that in Case 1 the point $\varphi(Q'_{l+1})$ is hyperbolic as well, and in Case 2 this point satisfies the assumptions of Case 1.

In Case 1 we divide Q'_1, \dots, Q'_{l+1} into several disjoint pieces M_0, \dots, M_k according to different values of v so that $Q'_i \in M_j \Leftrightarrow v_i = c_j$, where $c_j \neq 0$. Assuming that $M_0 = \{Q'_{i_1}, \dots, Q'_{i_r}, Q'_{l+1}\}$, where $i_k \leq l$ for all $k = 1, \dots, r$, we can choose an extra point $Q''_{l+1} \in (V_{c_0} \cap X_{\text{reg}}^{(1)}) \setminus U_0$. Indeed, since $c_0 = v_{l+1} \neq 0$ we have $V_{c_0} \cong X^{(0)}$. Under the assumptions of Theorem 3.2 $\dim X^{(0)} \geq 2$, hence also $\dim(V_{c_0} \cap X_{\text{reg}}^{(1)}) \setminus U_0 = \dim X^{(0)} \geq 2$.

By Lemma 3.4 the subgroup $\text{Stab}_{c_1, \dots, c_k}^v \subseteq G_v$ acts $(r+1)$ -transitively on $V_{c_0} \cap X_{\text{reg}}^{(1)}$. Therefore we can send the $(r+1)$ -tuple $(Q'_{i_1}, \dots, Q'_{i_r}, Q'_{l+1})$ to $(Q'_{i_1}, \dots, Q'_{i_r}, Q''_{l+1})$ fixing the remaining points of $M_1 \cup \dots \cup M_k$. This confirms our claim in Case 1.

In Case 2 we have $Q'_{l+1} = (Q_{l+1}, 0, 0) \in X_{\text{reg}}^{(1)}$. From Lemma 3.2 and its proof it follows that $Q_{l+1} = \pi(Q'_{l+1}) \in X_{\text{reg}}^{(0)}$ and $df(Q_{l+1}) \neq 0$ in the cotangent space $T_{Q_{l+1}}^* X^{(0)}$. The variety $X^{(0)}$ being flexible, there exists an LND $\partial_0 \in \text{Der } \mathbb{k}[X^{(0)}]$ such that $\partial_0(f)(Q_{l+1}) \neq 0$. Letting $q(v) = v(v - v_1)(v - v_2) \dots (v - v_l) \in \mathbb{k}[v]$ and choosing a set of generators x_1, \dots, x_s of the algebra $\mathbb{k}[X^{(0)}]$, similarly as in (18) we extend ∂_0 to $\partial_1 \in \text{Der } \mathbb{k}[X^{(1)}]$ via

$$(19) \quad \begin{aligned} \partial_1(x_i) &= q(v)\partial_0(x_i), \quad i = 1, 2, \dots, s, \\ \partial_1(u) &= \frac{q(v)}{v} \partial_0(f), \\ \partial_1(v) &= 0. \end{aligned}$$

Due to our choice, $\partial_1(u)(Q'_{l+1}) \neq 0$. Hence the action of the associate one-parameter unipotent subgroup $H_v(\partial_0, q) = \exp(t\partial_1)$ pushes the point Q'_{l+1} out of U_0 . So the orbit $H_v(\partial_0, q).Q'_{l+1}$ meets the hypersurface $U_0 \subseteq X^{(1)}$ in finitely many points. Similarly, for every $j = 1, 2, \dots, l$ the orbit $H_v(\partial_0, q).Q_j \not\subseteq U_0$ meets U_0 in finitely many points. Letting now $\varphi = \exp(t_0\partial_1) \in H_v(\partial_0, q) \subseteq G_v$, we conclude that for a general value of $t_0 \in \mathbb{k}$ the image $\varphi(Q'_j)$ lies outside U_0 for all $j = 1, 2, \dots, l+1$. Since the group $H_v(\partial_0, q)$ preserves the coordinate v , the points $\varphi(Q'_1), \dots, \varphi(Q'_l)$ are still hyperbolic. Interchanging the role of u and v ¹² we can achieve that the assumptions of Case 1 are fulfilled for the new collection $\varphi(Q'_1), \dots, \varphi(Q'_l), \varphi(Q'_{l+1})$, as required. \square

Proof of infinite transitivity in Theorem 3.2. If $X^{(0)} = \mathbb{A}^1$ then the assertion follows from Theorem 3.1. Let now $\dim X^{(0)} \geq 2$. To show that the action of the group $\text{SAut}(X^{(1)})$ on $X_{\text{reg}}^{(1)}$ is m -transitive for every $m \in \mathbb{N}$, we fix a standard collection of m distinct points $P'_1, \dots, P'_m \in U_1 \cap X_{\text{reg}}^{(1)}$. It suffices to move any other m -tuple of distinct points $Q'_1, \dots, Q'_m \in X_{\text{reg}}^{(1)}$ to the position of P'_1, \dots, P'_m by means of a special automorphism $\psi \in \text{SAut}(X^{(1)})$. In view of Lemma 3.5 we may suppose that $Q'_i \notin U_0 \cup V_0$ for all $i = 1, \dots, m$. Similarly as in

¹²We have not done this earlier in order to keep our previous notation.

the proof of Lemma 3.5 we divide the collection Q'_1, \dots, Q'_m into disjoint pieces M_1, \dots, M_k according to the values of v so that $M_j \subseteq V_{c_j}$, where $c_j \in \mathbb{k}^\times$ for all $j = 1, \dots, k$.

By our assumption the variety $X^{(0)}$ is flexible. It follows that the only units in $\mathbb{k}[X^{(0)}]$ are constants. Consequently, since f is non-constant we have $f(X^{(0)}) = \mathbb{k}$. In particular, $U_c \cap V_d \neq \emptyset$ for any $c, d \in \mathbb{k}$. Since $\dim X^{(1)} = 1 + \dim X^{(0)} \geq 3$ the intersection $U_c \cap V_d$ has positive dimension, hence is infinite.

Therefore acting with the subgroups $\text{Stab}_{c_1 \dots c_i \dots c_l}^v \subseteq G_v$, by Lemma 3.4 we can send M_i to $U_1 \cap V_{c_i} \cap X_{\text{reg}}^{(1)}$ fixing the other pieces M_j ($j \neq i$) pointwise. So we may assume that $Q'_1, \dots, Q'_m \in U_1 \cap X_{\text{reg}}$. Applying Lemma 3.4 again with u and v interchanged, $k = 0$, and $c_0 = 1$, i.e., acting with the subgroup G_u , we can send the resulting collection to the standard one P'_1, \dots, P'_m . Now the proof is completed. \square

3.3. Suspensions over real varieties. Here we prove Theorem 3.3. We need the following elementary lemma.

Lemma 3.6. *Let Y be a smooth, connected real manifold of dimension at least two. Then for any continuous function $f : Y \rightarrow \mathbb{R}$ the level set $f^{-1}(c)$ is infinite for each $c \in \text{Int } f(Y)$.*

Proof. Under our assumptions $\text{Int } f(Y) \subseteq \mathbb{R}$ is an open interval. Choosing two points $y_1, y_2 \in Y$ such that $f(y_1) = c_1 < c$ and $f(y_2) = c_2 > c$, we can join them in Y by a smooth path l . There exists a tubular neighborhood U of l diffeomorphic to a cylinder $\Delta \times I$, where $I = [0, 1]$ and Δ is a ball of dimension $\dim \Delta = \dim Y - 1 \geq 1$. So there is a continuous family of paths joining y_1 and y_2 within U such that any two of them meet only at their ends y_1 and y_2 . Since the hypersurface $f^{-1}(c)$ separates Y , each of these paths crosses it. In particular, $f^{-1}(c)$ is infinite. \square

The proof of Theorem 3.3 differs just slightly from that of Theorem 3.2. Hence it is enough to indicate the necessary changes.

Sketch of the proof of Theorem 3.3. The assumption that the field \mathbb{k} is algebraically closed was actually used in the proof of Theorem 3.2 only on two occasions. Namely, in the proofs of Lemma 3.5 and of the infinite transitivity in Theorem 3.2 we exploited the fact that under our assumptions the level sets $(V_{c_k} \cap X_{\text{reg}}^{(1)}) \setminus U_0$ and $U_1 \cap V_{c_i} \cap X_{\text{reg}}^{(1)}$ are of positive dimension, hence are infinite. For $\mathbb{k} = \overline{\mathbb{k}}$ the latter follows from the Krull theorem and the dimension count. In the case where $\mathbb{k} = \mathbb{R}$, we can deduce the same conclusion using Lemma 3.6. Indeed, in the notation as before, for every $c_i \neq 0$ the restrictions

$$\pi : V_{c_i} \cap X_{\text{reg}}^{(1)} = (V_{c_i})_{\text{reg}} \rightarrow X_{\text{reg}}^{(0)} \quad \text{and} \quad \pi : U_1 \cap V_{c_i} \cap X_{\text{reg}}^{(1)} \rightarrow f^{-1}(c_i) \cap X_{\text{reg}}^{(0)}$$

are isomorphisms. Under the assumptions of Theorem 3.3 the smooth real manifold $X_{\text{reg}}^{(0)}$ of dimension ≥ 2 is connected. Since $f(X_{\text{reg}}^{(0)}) = \mathbb{R}$, by Lemma 3.6 the level set $f^{-1}(c_i) \cap X_{\text{reg}}^{(0)}$ is infinite. Since $c_k \neq 0$, the set $(V_{c_k} \cap X_{\text{reg}}^{(1)}) \setminus U_0 \supseteq U_1 \cap V_{c_k} \cap X_{\text{reg}}^{(1)}$ is infinite too.

À posteriori, the manifold $X_{\text{reg}}^{(1)}$ is connected too. Hence by recursion we can apply the argument to the iterated suspensions $X^{(i)}$ over $X^{(0)}$, $i = 1, \dots, l$. \square

3.4. Flexibility. To complete the proofs of Theorems 3.1-3.3, it remains to establish the flexibility of $X^{(1)}$.

Lemma 3.7. *Under the assumptions of any one of Theorems 3.1-3.3 the variety $X^{(1)}$ is flexible.*

Proof. We know already that the group $\text{SAut}(X^{(1)})$ is transitive in $X_{\text{reg}}^{(1)}$. Hence, similarly as in the proof of Lemma 2.9, it suffices to find just one flexible point $P' = (P, u, v) \in X_{\text{reg}}^{(1)}$.

The function $f \in \mathbb{k}[X^{(0)}]$ being non-constant, $df(P) \neq 0$ at some point $P \in X_{\text{reg}}^{(0)}$ with $f(P) \neq 0$. Due to our assumption $X^{(0)}$ is flexible. Hence there exist n locally nilpotent derivations $\partial_0^{(1)}, \dots, \partial_0^{(n)} \in \text{Der } \mathbb{k}[X^{(0)}]$, where $n = \dim X^{(0)}$, such that the corresponding vector fields ξ_1, \dots, ξ_n span the tangent space $T_P X^{(0)}$ i.e.,

$$\text{rk} \begin{pmatrix} \xi_1(P) \\ \vdots \\ \xi_n(P) \end{pmatrix} = n.$$

It follows that $\partial_0^{(i)}(f)(P) \neq 0$ for some index $i \in \{1, \dots, n\}$.

Let now $P' = (P, u_0, v_0) \in X_{\text{reg}}^{(1)}$ be a point such that $\pi(P') = P$. Since $u_0 v_0 = f(P) \neq 0$, the point P' is hyperbolic. Letting $q(v) = v$ in Lemma 3.3 we obtain n LNDs

$$\partial_1^{(1)}, \dots, \partial_1^{(n)} \in \text{Der } \mathbb{k}[X^{(1)}], \quad \text{where } \partial_1^{(j)} = \partial_1^{(j)}(\partial_0^{(j)}, v).$$

Interchanging u and v and letting $j = i$ gives yet another LND

$$\partial_2^{(i)} = \partial_2^{(i)}(\partial_0^{(i)}, u) \in \text{Der } \mathbb{k}[X^{(1)}].$$

Let us show that the corresponding $n + 1$ vector fields span the tangent space $T_{P'} X^{(1)}$ at P' , as required. We can view $\partial_1^{(1)}, \dots, \partial_1^{(n)}, \partial_2^{(i)}$ as LNDs in $\text{Der } \mathbb{k}[X^{(0)}][u, v]$ preserving the ideal $(uv - f)$, so that the corresponding vector fields are tangent to the hypersurface

$$X^{(1)} = \{uv - f(P) = 0\} \subseteq X^{(0)} \times \mathbb{A}^2.$$

The values of these vector fields at the point $P' \in X_{\text{reg}}^{(1)}$ yield an $(n + 1) \times (n + 2)$ -matrix

$$E = \begin{pmatrix} v_0 \xi_1(P) & \partial_0^{(1)}(f)(P) & 0 \\ \vdots & \vdots & \vdots \\ v_0 \xi_n(P) & \partial_0^{(n)}(f)(P) & 0 \\ u_0 \xi_i(P) & 0 & \partial_0^{(i)}(f)(P) \end{pmatrix}.$$

The first n rows of E are linearly independent, and the last one is independent from the preceding since $\partial_0^{(i)}(f)(P) \neq 0$. Therefore $\text{rk}(E) = n + 1 = \dim X^{(1)}$. So our locally nilpotent vector fields indeed span the tangent space $T_{P'} X^{(1)}$ at P' , as claimed. \square

Now the proofs of Theorems 3.1-3.3 are completed.

4. CONCLUDING REMARKS

4.1. The Makar-Limanov invariant. Recall [12, §9] that the Makar-Limanov invariant $\text{ML}(X)$ of an affine variety X is the intersection of the kernels of all locally nilpotent derivations of $\mathbb{k}[X]$, or, in other words, the subalgebra in $\mathbb{k}[X]$ of common invariants for all one parameter unipotent subgroups of $\text{Aut}(X)$. From this definition it is straightforward that $\text{ML}(X) = \mathbb{k}[X]^{\text{SAut}(X)}$. Hence the Makar-Limanov invariant of X is trivial (that

is $\text{ML}(X) = \mathbb{k}$) provided that the special automorphism group $\text{SAut}(X)$ acts on X with a dense open orbit (cf. [31]). This holds in particular for the varieties in all three classes from Theorem 0.2 (for the first two of them, see also [22, 3.16], [26], and [31]). On the other hand, $\text{ML}(X)$ is trivial if X is flexible. Indeed, if $f \in \mathbb{k}[X]^{\text{SAut}(X)}$ then the differential df vanishes along the orbits of any unipotent subgroup, hence it vanishes on the tangent space at any flexible point of X_{reg} . Since X is flexible, f is constant.

4.2. Some open questions. It would be interesting to clarify relations between flexibility and infinite transitivity. Of course, the affine line $X = \mathbb{A}^1$ is flexible but the affine group $\text{Aut}(\mathbb{A}^1)$ is just 2-transitive. However, we do not dispose a similar example in higher dimensions. It is plausible that in higher dimensions, for varieties over an algebraically closed field of characteristic zero, *transitivity of the group $\text{SAut}(X)$ implies flexibility of X , and flexibility of X implies in turn infinite transitivity of $\text{SAut}(X)$* ¹³. In other direction, *how can one characterize the class of affine varieties such that the group $\text{SAut}(X)$ acts infinitely transitively on a Zariski open subset?*

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