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Carleman estimates and controllability results for the one-dimensional heat equation with BV coefficients

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Abstract

We derive global Carleman estimates for one-dimensional linear parabolic operators $\partial_t \pm \partial_x(c\partial_x)$ with a coefficient c with bounded variations. These estimates are obtained by approximating c by piecewise regular coefficients, c_ε, and passing to the limit in the Carleman estimates associated to the operators defined with c_ε. Such estimates yield results of controllability to the trajectories for a classe of *semilinear* parabolic equations. *To cite this article: J. Le Rousseau, C. R. Acad. Sci. Paris, Ser. I — (—).*

Résumé

Inégalités de Carleman et résultats de contrôle pour une équation de la chaleur à coefficient BV en dimension un. On obtient des inégalités de Carleman pour des opérateurs paraboliques linéaires $\partial_t \pm \partial_x(c\partial_x)$ avec un coefficient c à variations bornées. Ces inégalités sont obtenues par approximations de c par des coefficients réguliers par morceaux, c_ε, et en passant à la limite dans les inégalités de Carleman associées aux opérateurs définis par c_ε. De telles inégalités donnent des résultats de contrôlabilité aux trajectoires pour une classe d'équations semilinéaires paraboliques. *Pour citer cet article : J. Le Rousseau, C. R. Acad. Sci. Paris, Ser. I — (—).*

Version française abrégée

On considère les opérateurs $\partial_t \pm A$ dans $Q = (0,T) \times \Omega$, avec $T > 0$ et $\Omega = (0,1)$. L'opérateur A est défini par $-\partial_x(c\partial_x)$ sur $L^2(\Omega)$, où c est à variations bornées (BV) et vérifie $0 < c_{min} \leq c \leq c_{max}$. Le domaine de A est $D(A) = \{u \in H_0^1(\Omega);\ c\partial_x u \in H^1(\Omega)\}$. On prouve l'inégalité de Carleman *globale* suivante.

Théorème 0.1 *Soit $c \in BV(\Omega)$, avec $0 < c_{min} \leq c \leq c_{max}$. Soit $\aleph$ l'ensemble des solutions (faibles) de $\partial_t q \pm \partial_x(c\partial_x q) = f$ dans Q, $q = 0$ sur $(0,T) \times \{0,1\}$, $q(T,x) = q_0(x)$ (resp. $q(0,x) = q_0(x)$) dans Ω, avec $q_0 \in L^2(\Omega)$ et $f \in L^2(Q)$.*

(i) $\exists \lambda_1 > 0$, $s_1 > 0$ et $C > 0$ tels que l'inégalité de Carleman (4) soit valable pour $s \geq s_1$, $\lambda \geq \lambda_1$ et pour tout $q \in \aleph$.

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(ii) Si c de est de classe $\mathscr{C}^1$ sur $\mathcal{O} \Subset \Omega$, avec $\mathcal{O}$ un ouvert non vide de Ω, alors on a une inégalité de Carleman avec une 'observation' sur $(0,T)\times\mathcal{O}$.

On donne une idée de la preuve du Théorème 0.1 dans le cas (i), i.e. une 'observation' frontière. Si c est $\mathscr{C}^1$ par morceaux, une inégalité de Carleman (4) est obtenue [1,2] en construisant des fonctions poids données dans (3), à partir d'une fonction $\widetilde{\beta}$, régulière par morceaux, qui à chaque point de discontinuité de c satisfait la condition (2).

Dans le cas où $c \in BV(\Omega)$, pour $\varepsilon > 0$ on choisit c_ε, constant par morceaux, de manière à avoir $\|c - c_\varepsilon\|_{L^\infty(\Omega)} \leq \varepsilon$, $V_0^1(c_\varepsilon) \leq V_0^1(c)$, où $V_a^b(\rho)$ est la variation totale de ρ sur (a,b) [3]. On construit des fonctions $\widetilde{\beta}_\varepsilon$ adaptées à c_ε : posant $[\rho]_x := \rho(x^+) - \rho(x^-)$, on exhibe une fonction f telle que i) si $[\widetilde{\beta}'_\varepsilon]_a = f([c_\varepsilon]_a)$ en chaque point de discontinuité a de c_ε, alors $\widetilde{\beta}_\varepsilon$ satisfait la condition (2) ; ii) de plus $V_0^1(\widetilde{\beta}'_\varepsilon)$ est majorée uniformément par rapport à ε. À une sous-suite près, ces fonctions $\widetilde{\beta}'_\varepsilon$ convergent partout quand ε tend vers 0, par le théorème de Helly. On déduit, par passage à la limite et intégration, une fonction $\widetilde{\beta}$.

À l'aide de $\widetilde{\beta}_\varepsilon$ et $\widetilde{\beta}$, on construit des fonctions poids selon (3). Celles associées à c_ε permettent d'obtenir des inégalités de Carleman pour $\partial_t \pm \partial_x(c_\varepsilon\partial_x)$. Une inspection détaillée de la preuve des inégalités de Carleman dans le cas $\mathscr{C}^1$ par morceaux permet de prouver l'existence d'une borne uniforme, par rapport à ε, pour la constante C à droite de (4) pour les opérateurs $\partial_t \pm \partial_x(c_\varepsilon\partial_x)$. De même, les constantes s_1 et λ_1 dans le Théorème 2.2 peuvent être choisies uniformes par rapport à ε. Ces résultats permettent un passage à la limite dans les inégalités de Carleman associées à $\partial_t \pm \partial_x(c_\varepsilon\partial_x)$: les solutions de $\partial_t q_\varepsilon \pm \partial_x(c_\varepsilon\partial_x)q_\varepsilon = f$ sont telles que $\partial_t q_\varepsilon$, q_ε, $\partial_x q_\varepsilon$, $\partial_x(c\partial_x q_\varepsilon)$ convergent vers les fonctions associées pour la solution q de $\partial_t q \pm \partial_x(c\partial_x)q = f$, pour f régulière et les conditions finales (resp. initiales) bien choisies ; les fonctions poids convergent et les constantes restent bornées. On relaxe alors la régularité sur f et les hypothèses sur les conditions finales (resp. initiales).

Le Théorème 0.1 permet d'obtenir des résultats de contrôlabilité aux trajectoires pour une classe d'équations semilinéaires paraboliques. Si $\mathscr{G}(x,y) = x\,g(x,y) + y\,G(x,y)$, avec g et G dans $L^\infty_{\text{loc}}(\mathbb{R})$.

Théorème 0.2 *Soit $T > 0$. Sous les hypothèses du Théorème 0.1, on a :*

(i) Contrôlabilité à zéro locale : Il existe $\varepsilon > 0$ tel que pour tout y_0 dans $L^2(\Omega)$ avec $\|y_0\|_{L^2(\Omega)} \leq \varepsilon$, il existe un contrôle $v \in L^\infty(0,T)$ tel que la solution de (1) satisfasse $y(T) = 0$.

(ii) Contrôlabilité à zéro globale : Si $\mathscr{G}$ satisfait de plus (7), alors pour tout $y_0 \in L^2(\Omega)$, il existe $v \in L^\infty(0,T)$ tel que la solution de (1) satisfasse $y(T) = 0$.

On obtient des résultats similaires pour le système (6) avec un ouvert non vide $\omega \Subset \Omega$, et $v \in L^\infty((0,T)\times\omega)$, en supposant une régularité $\mathscr{C}^1$ du coefficient c sur une partie ouverte non vide de ω.

Le preuve du Théorème 0.2 repose sur un argument de point fixe [4,5,6].

1. Introduction

We consider the elliptic operator A formally defined by $-\partial_x(c\partial_x)$ on $L^2(\Omega)$ in the one-dimensional bounded domain $\Omega = (0,1)$, with $0 < c_{min} \leq c \leq c_{max}$, and c is assumed to be of bounded variations (BV). The domain of A is $D(A) = \{u \in H_0^1(\Omega);\ c\partial_x u \in H^1(\Omega)\}$. Let $T > 0$. We shall use the following notations $Q = (0,T)\times\Omega$, $\Gamma = \{0,1\}$, and $\Sigma = (0,T)\times\Gamma$.

Here, we show that we can achieve *global* Carleman estimates for the operators $\partial_t \pm A$ in Q, with a boundary 'observation' or an interior 'observation'. With such Carleman estimates at hand, we treat the problem of the controllability to the trajectories for semilinear parabolic systems of the form

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) + \mathscr{G}(y,\partial_x y) = 0 & \text{in } Q, \\ y(t,0) = v(t),\ y(t,1) = 0, & \\ y(0,x) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where v is the control, $\mathscr{G} : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz and $\mathscr{G}(0,0) = 0$ (further assumptions on the nonlinear function $\mathscr{G}$ or on the initial condition will be introduced below).

A null controllability result for a *linear* parabolic equation with BV coefficients was proven in [7]. The proof relies on Russell's method [8]. However, the question of the existence of a Carleman-type observability estimate was open. In a recent work [9], a result of controllability for the one-dimensional heat equation with general bounded coefficients was proved. The method of [9] is based on that of [10]. However, with such a method, the knowledge of the observability constant is not sufficient to address the controllability of non-linear heat equations. Carleman estimates, of the form proved here, provide a more precise knowledge of this constant with respect to lower-order terms in the partial differential operator, which allows to treat the case of a class of semilinear equations following the (fixed-point) method of [4,5] (generalized in [6]). Carleman estimates also have other applications in analysis, e.g., uniqueness results, inverse problems.

The Carleman estimates derived here for the operators $\partial_t \pm \partial_x(c\partial_x)$ are obtained through a limiting process from the Carleman estimates associated for $\partial_t \pm \partial_x(c_\varepsilon\partial_x)$, for c_ε piecewise regular converging to c in $L^\infty(\Omega)$. The main issue in this limiting process is to keep both the weight functions and constants in the Carleman estimate under control. Controllability results of this type can be motivated by problems from biology for example. Note that the approximation of the BV coefficient c by some piecewise regular coefficient c_ε is closely related to numerical methods.

2. Carleman estimates in the case of a BV coefficient

We first consider a piecewise $\mathscr{C}^1$ diffusion coefficient c with a finite number of singularities. Here, we shall assume that $0 = a_0 < a_1 < \cdots < a_n = 1$ and $c_{|[a_i,a_{i+1}]} \in \mathscr{C}^1([a_i, a_{i+1}])$, $i = 0, \ldots, n-1$. We shall use the following notations $S = \{a_1, \ldots, a_{n-1}\}$, $\Omega' = \Omega \setminus S$, and $Q' = (0,T) \times \Omega'$.

Lemma 2.1 ([2]) *There exists a function $\widetilde{\beta} \in \mathscr{C}(\Omega)$ such that $\widetilde{\beta}_{|[a_i,a_{i+1}]} \in \mathscr{C}^2([a_i, a_{i+1}])$, $i = 0, \ldots, n-1$, satisfying $\widetilde{\beta} > 0$ in Ω, $\widetilde{\beta}(1) = 0$, $(\widetilde{\beta}_{|[a_i,a_{i+1}]})' < 0$, $i \in \{0, \ldots, n-1\}$, and the function $\widetilde{\beta}$ satisfies the following trace properties, for some $\alpha > 0$,*

$$(A_i u, u) \geq \alpha |u|^2, \quad \text{with } A_i = \begin{pmatrix} [\widetilde{\beta}']_{a_i} & \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i} \\ \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i} & \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']^2_{a_i} + [c^2(\widetilde{\beta}')^3]_{a_i} \end{pmatrix}, \quad u \in \mathbb{R}^2, \tag{2}$$

for $i = 1, \ldots, n-1$, and where $[\rho]_x = \rho(x^+) - \rho(x^-)$ for $x \in (0,1)$.

Choosing a function $\widetilde{\beta}$, as in the previous lemma, we introduce $\beta = \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (0,T)$, we define the following weight functions,

$$\varphi(t,x) = \frac{e^{\lambda\beta(x)}}{t(T-t)}, \qquad \eta(t,x) = \frac{e^{\lambda\overline{\beta}} - e^{\lambda\beta(x)}}{t(T-t)}, \tag{3}$$

with $\overline{\beta} = 2m\|\widetilde{\beta}\|_\infty$ (see [11,12]). We introduce

$$\aleph = \Big\{ q \in \mathscr{C}(Q,\mathbb{R}); q_{|[0,T]\times[a_i,a_{i+1}]} \in \mathscr{C}^2([0,T] \times [a_i, a_{i+1}]),\ i = 0, \ldots, n-1,\ q_{|\Sigma} = 0, \\ \text{and } q \text{ satisfies, } q(t,a_i^-) = q(t,a_i^+),\ c(a_i^-)\partial_x q(t,a_i^-) = c(a_i^+)\partial_x q(t,a_i^+),\ i = 1, \ldots, n-1, \text{ for all } t \in (0,T) \Big\}.$$

With the function $\widetilde{\beta}$ satisfying the properties of Lemma 2.1, we obtain the following Carleman estimate.

Theorem 2.2 ([2]) *There exist $\lambda_1 > 0$, $s_1 = (T + T^2)\widetilde{s_1} > 0$ and $C > 0$ so that the following estimate holds, for $s \geq s_1$, $\lambda \geq \lambda_1$ and for all $q \in \aleph$,*

$$s^{-1}\iint_Q e^{-2s\eta}\varphi^{-1}\left(|\partial_t q|^2+|\partial_x(c\partial_x q)|^2\right)dxdt+s\lambda^2\iint_Q e^{-2s\eta}\varphi\,|\partial_x q|^2\,dxdt+s^3\lambda^4\iint_Q e^{-2s\eta}\varphi^3\,|q|^2\,dxdt$$
$$\leq C\left[s\lambda\int_0^T\varphi(t,0)e^{-2s\eta(t,0)}|\partial_x q|^2(t,0)\,dt+\iint_Q e^{-2s\eta}\,|\partial_t q\pm\partial_x(c\partial_x q)|^2\,dxdt\right]. \quad (4)$$

We now consider a diffusion coefficient $c\in BV(\Omega)$, with $0<c_{min}\leq c\leq c_{max}$. We denote the total variation of c on $(0,1)$ by $\vartheta:=V_0^1(c)$. Let $\varepsilon>0$. There exists a *piecewise-constant* function c_ε such that $\|c-c_\varepsilon\|_{L^\infty(\Omega)}\leq\varepsilon$, $V_0^1(c_\varepsilon)\leq\vartheta$ (see e.g. [13]).

We denote by $a_1,\dots,a_k$ the points of discontinuity of c_ε. We then have $\sum_{i=1}^k|c_\varepsilon(a_i^+)-c_\varepsilon(a_i^-)|\leq\vartheta$. Let $Y_i=c_\varepsilon(a_i^+)/c_\varepsilon(a_i^-)$ and X_i, $i=1,\dots,k$, be defined by $X_i=f(Y_i)$, with $f(s)=s$ if $s\geq 1$ and $f(s)=2-s$ if $s<1$. We define the *piecewise-constant* function γ_ε as $\gamma_\varepsilon(x):=\gamma(1)\prod_{x<a_j}X_j$, for $x\notin\{a_1,\dots,a_k\}$, for some fixed $\gamma(1)<0$. We set the function $\widetilde{\beta}_\varepsilon(x):=\int_1^x\gamma_\varepsilon(y)\,dy$, which satisfies the properties listed in Lemma 2.1 by the jump conditions imposed on $\gamma_\varepsilon=\widetilde{\beta}_\varepsilon'$ at $a_1,\dots,a_k$. Concerning the total variation of the functions γ_ε we have the following lemma.

Lemma 2.3 *There exist $K>0$ and $\varepsilon_0>0$ that solely depend on the diffusion coefficient $c\in BV(\Omega)$ such that, for all $0<\varepsilon\leq\varepsilon_0$, $V_0^1(\gamma_\varepsilon)\leq K\,|\gamma(1)|$.*

By Helly's theorem [3,13], up to a subsequence, the functions γ_ε converge *everywhere* to a function γ as ε goes to 0. Moreover, this function satisfies $V_0^1(\gamma)\leq K|\gamma(1)|$. By dominated convergence, the associated functions $\widetilde{\beta}_\varepsilon$ converge everywhere to the continuous function $\widetilde{\beta}(x):=\int_1^x\gamma(y)dy$.

Let a be any point of discontinuity of c_ε. As above, we set $Y=\frac{c_\varepsilon(a^+)}{c_\varepsilon(a^-)}$ and define the matrix A as in Lemma 2.1. Then we have the following crucial lemma.

Lemma 2.4 *The eigenvalues ν_1, ν_2 of the matrix A satisfy $\nu_i\geq C|Y-1|$, $i=1,2$, with C uniform w.r.t. ε and the considered singularity of c_ε.*

With $\widetilde{\beta}_\varepsilon$ and $\widetilde{\beta}$, we define weight functions according to (3). With a detailed inspection of the proof of the Carleman estimate in the case of a piecewise $\mathscr{C}^1$ coefficient [2], the previous lemma yields

Proposition 2.5 *The constant C on the r.h.s. of the Carleman estimate (4) for $\partial_t\pm\partial_x(c_\varepsilon\partial_x)$ and the constants s_1 and λ_1 can be chosen uniformly w.r.t. ε for $0<\varepsilon\leq\varepsilon_0$, with ε_0 sufficiently small.*

The previous results show that both the weight functions and the constants in the Carleman estimate remain 'well behaved', as ε goes to zero. We consider q and q_ε (weak) solutions to

$$\begin{cases}\partial_t q\pm\partial_x(c\partial_x q)=f & \text{in } Q,\\ q=0 & \text{on } \Sigma,\\ q(T,x)=q_0(x)\ (\text{resp. } q(0,x)=q_0(x)) & \text{in } \Omega.\end{cases}\qquad\begin{cases}\partial_t q_\varepsilon\pm\partial_x(c_\varepsilon\partial_x q_\varepsilon)=f & \text{in } Q,\\ q_\varepsilon=0 & \text{on } \Sigma,\\ q_\varepsilon(T,x)=q_{0,\varepsilon}(x)\ (\text{resp. } q_\varepsilon(0,x)=q_{0,\varepsilon}(x)) & \text{in } \Omega,\end{cases}$$

with the initial data q_0 and $q_{0,\varepsilon}$ given by $\partial_x(c\partial_x q_0)=\mu$ and $\partial_x(c_\varepsilon\partial_x q_{0,\varepsilon})=\mu$, with the same μ, where μ and f are sufficiently regular. This choice of initial data, yields the convergence of $q_{0,\varepsilon}$ to q_0 in $H_0^1(\Omega)$ as ε goes to zero. Since $\partial_t q_\varepsilon$, q_ε, $\partial_x q_\varepsilon$ and $\partial_x(c\partial_x q_\varepsilon)$ converge to $\partial_t q$, q, $\partial_x q$ and $\partial_x(c\partial_x q)$ in L^2 norm, the previous results show that the Carleman estimate for q_ε and the operators $\partial_t\pm\partial_x(c_\varepsilon\partial_x)$ yield a similar estimate for q and the operators $\partial_t\pm\partial_x(c\partial_x)$ with the same constants, as ε goes to zero. We can now relax the assumptions on μ and f. We have thus obtained the following theorem.

Theorem 2.6 *Let $c\in BV(\Omega)$ with $0<c_{min}\leq c\leq c_{max}$. There exists $\lambda_1>0$, $s_1=(T+T^2)\widetilde{s_1}>0$ and $C>0$ so that the Carleman estimate (4) holds for $s\geq s_1$, $\lambda\geq\lambda_1$ and for all q (weak) solution to $\partial_t q\pm\partial_x(c\partial_x q)=f$ in Q, $q=0$ on Σ, $q(T,x)=q_0(x)$ (resp. $q(0,x)=q_0(x)$) in Ω, with $q_0\in L^2(\Omega)$ and $f\in L^2(Q)$.*

In the case where $O\Subset\Omega$, with O a non-empty open set and c is of class $\mathscr{C}^1$ in O, we obtain Carleman estimates, for the operators $\partial_t\pm\partial_x(c\partial_x)$, with an interior 'observation' on $(0,T)\times O$ i.e. with the term

$$s^3\lambda^4 \iint_{(0,T)\times O} e^{-2s\eta}\varphi^3\, |q|^2\, dxdt$$

replacing the time integral on the r.h.s. of (4).

In the case of an interior 'observation', the proof is similar to the previous one. The function $\widetilde{\beta}_\varepsilon$ is constructed on both sides of O, through its derivative, and the regularity of c in O allows to smoothly connect the two parts of the function, with some convergence in $\mathscr{C}^2(\overline{O})$ as ε goes to zero. As above, we then pass to the limit in a Carleman estimate for $\partial_t \pm \partial_x(c_\varepsilon\partial_x)$, with c_ε piecewise $\mathscr{C}^1$, with an interior 'observation', proven in [2].

3. A Carleman estimate for the heat equation with a r.h.s. in $L^2(0,T,H^{-1}(\Omega))$

Following [14], we obtain a Carleman estimate for $\partial_t q \pm \partial_x(c\partial_x q) = f$ if $f \in H^{-1}$. We set

$$\widetilde{\aleph}_\pm \quad = \quad \left\{q \in \mathscr{C}([0,T], H_0^1(\Omega)); q(t) \in D(A)\ \forall\, t \in [0,T],\ \partial_t q \pm \partial_x(c\partial_x q) = F_0 + \partial_x F_1,\ F_0, F_1 \in L^2(Q)\right\}.$$

Theorem 3.1 *Let $O \Subset \Omega$ be a non-empty open set and $c \in BV(\Omega)$ with $0 < c_{min} \le c \le c_{max}$ and c of class $\mathscr{C}^1$ in O. There exists $\lambda_2 = \lambda_2(O,c) > 0$, $s_2 = (T+T^2)s_2(O,c) > 0$ and a positive constant $C = C(O,c)$ so that the following estimate holds, for $s \ge s_2$, $\lambda \ge \lambda_2$ and for all $q \in \widetilde{\aleph}_\pm$,*

$$s\lambda^2 \iint_Q e^{-2s\eta}\varphi\, |\partial_x q|^2\, dxdt + s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3\, |q|^2\, dxdt \le C\Bigg[s^3\lambda^4 \iint_{(0,T)\times O} e^{-2s\eta}\varphi^3\, |q|^2\, dxdt + \iint_Q e^{-2s\eta}\, |F_0|^2\, dxdt + s^2\lambda^2 \iint_Q e^{-2s\eta}\, \varphi^2 |F_1|^2\, dxdt\Bigg]. \quad (5)$$

4. Controllability results

The Carleman estimate (5) proven in the previous section allows to give observability estimates that yield results of controllability to the trajectories for classes of semilinear heat equations. We let $\omega \Subset \Omega$ be a non-empty open set and $c \in BV(\Omega)$ with $0 < c_{min} \le c \le c_{max}$ and c of class $\mathscr{C}^1$ in some non-empty open subset of ω. We let a and b be in $L^\infty(Q)$ and $q_T \in L^2(\Omega)$. From Carleman estimate (5) we obtain the following lemma.

Lemma 4.1 *The solution q to $-\partial_t q - \partial_x(c\partial_x q) + aq - \partial_x(bq) = 0$ in Q, $q = 0$ on Σ, and $q(T) = q_T$ satisfies $\|q(0)\|^2_{L^2(\Omega)} \le e^{CH}\left(\iint_{(0,T)\times\omega} |q|\, dxdt\right)^2$, where $H = H(T, \|a\|_\infty, \|b\|_\infty) = 1 + \frac{1}{T} + T + (T+T^{1/2})\|a\|_\infty + \|a\|_\infty^{2/3} + (1+T)\|b\|_\infty^2$.*

Since the coefficient c is $\mathscr{C}^1$ in some open subset of ω, the proof of [6, Theorem 2.5, Lemma 2.5] can be adapted. See also [12, Proposition 4.2, Lemma 4.3]. Such an observability estimate yields the null controllability of the semilinear parabolic system (1), as well as the following system

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) + \mathscr{G}(y, \partial_x y) = 1_\omega v & \text{in } Q, \\ y(t,.) = 0 & \text{on} \Sigma, \\ y(0,x) = y_0(x) & \text{in } \Omega, \end{cases} \quad (6)$$

where $\mathscr{G} : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz and $\mathscr{G}(0,0) = 0$, which implies that $\mathscr{G}(x,y) = x\, g(x,y) + y\, G(x,y)$, with g and G in $L^\infty_{\text{loc}}(\mathbb{R})$.

Theorem 4.2 *Let $T > 0$. Let $c \in BV(\Omega)$ with $0 < c_{min} \le c \le c_{max}$.*

(i) Local null controllability: There exists $\varepsilon > 0$ such that for all y_0 in $L^2(\Omega)$ with $\|y_0\|_{L^2(\Omega)} \le \varepsilon$, there exists a control $v \in L^\infty(0,T)$ such that the solution to (1) satisfies $y(T) = 0$.

(ii) Global null controllability: Let $\mathscr{G}$ satisfy in addition

$$\lim_{|(x,y)|\to\infty} \frac{|g(x,y)|}{\ln^{3/2}(1+|x|+|y|)} = 0 \quad \textit{and} \lim_{|(x,y)|\to\infty} \frac{|G(x,y)|}{\ln^{1/2}(1+|x|+|y|)} = 0. \tag{7}$$

For all $y_0 \in L^2(\Omega)$, there exists $v \in L^\infty(0,T)$ such that the solution to (1) satisfies $y(T) = 0$.
We have similar results for system (6) with a non-empty open subset $\omega \Subset \Omega$, and $v \in L^\infty((0,T)\times\omega)$, assuming that c of class $\mathscr{C}^1$ in some non-empty open subset of ω.

The proof is based on a fixed-point argument and is along the same lines as that in [6] and originates from [4,5]. Note that as usual, $y(T) = y^*(T)$ can replace $y(T) = 0$ in the previous statements, where y^* is any trajectory defined in $[0,T]$ of system (1), corresponding to some initial data $y_0^* \in L^2(\Omega)$ and any v^* in $L^\infty(0,T)$ ($L^\infty((0,T)\times\omega)$ in the case of an interior control). For the local controllability result, one has to assume $\|y_0 - y_0^*\|_{L^2(\Omega)} \le \varepsilon$, with ε sufficiently small.

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