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COMPUTATIONAL PROOF OF THE MACKEY FORMULA FOR $q > 2$

CÉDRIC BONNAFÉ & JEAN MICHEL

ABSTRACT. Let $\mathbf{G}$ be a connected reductive group defined over a finite field with q elements. We prove that the Mackey formula for the Lusztig induction and restriction holds in $\mathbf{G}$ whenever $q > 2$ or $\mathbf{G}$ does not have a component of type E.

Let $\mathbf{G}$ be a connected reductive group defined over an algebraic closure $\mathbb{F}$ of a finite field of characteristic $p > 0$ and let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism endowing $\mathbf{G}$ with an $\mathbb{F}_q$-structure, where q is a power of p and $\mathbb{F}_q$ is the finite subfield of $\mathbb{F}$ of cardinal q. By *the Mackey formula for Lusztig induction and restriction*, we mean the following formula

$$(\mathscr{M}_{\mathbf{G},\mathbf{L},\mathbf{P},\mathbf{M},\mathbf{Q}}) \quad {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}} = \sum_{g\in\mathbf{L}^F\backslash\mathscr{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} R^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{L}\cap{}^g\mathbf{Q}} \circ {}^*R^{{}^g\mathbf{M}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{P}\cap{}^g\mathbf{M}} \circ (\operatorname{ad} g)_{\mathbf{M}}.$$

Here, $\mathbf{P}$ and $\mathbf{Q}$ are two parabolic subgroups of $\mathbf{G}$, $\mathbf{L}$ and $\mathbf{M}$ are F-stable Levi complements of $\mathbf{P}$ and $\mathbf{Q}$ respectively, $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ denote respectively the Lusztig induction and restriction maps, $(\operatorname{ad} g)_{\mathbf{M}}$ is the map between class functions on $\mathbf{M}^F$ and ${}^g\mathbf{M}^F$ induced by conjugacy by g, and $\mathscr{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})$ is the set of elements $g \in \mathbf{G}$ such that $\mathbf{L}$ and ${}^g\mathbf{M}$ have a common maximal torus.

It is conjectured that the Mackey formula always holds. This paper is a contribution towards a solution to this conjecture. Our aim is to prove the last two lines of the following theorem:

Theorem. *Assume that one of the following holds:*

(1) $\mathbf{P}$ *and* $\mathbf{Q}$ *are F-stable (Deligne* [13, Theorem 2.5]*).*
(2) $\mathbf{L}$ *or* $\mathbf{M}$ *is a maximal torus of* $\mathbf{G}$ *(Deligne and Lusztig* [9, Theorem 7]*).*
(3) $q > 2$.
(4) $\mathbf{G}$ *does not contain an F-stable quasi-simple component of type* ${}^2\mathrm{E}_6$, E_7 *or* E_8.

Then the Mackey formula $(\mathscr{M}_{\mathbf{G},\mathbf{L},\mathbf{P},\mathbf{M},\mathbf{Q}})$ *holds.*

While the proofs of (1) and (2) work in full generality and are pretty elegant, our proof of (3) and (4) is as ugly as possible. It follows an induction argument suggested by Deligne and Lusztig [8, Proof of Theorems 6.8 and 6.9] (and improved in [1] and [2]) that shows that if the semisimple elements of $\mathbf{G}^F$ satisfy some strange properties (see

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Proposition 2.1 for the list of properties) then the Mackey formula holds: then, checking Proposition 2.1 in cases (3) and (4) of the above theorem is done by a case-by-case analysis together with computer calculations using the `CHEVIE` package (in `GAP3`).

Even when a proof of some important result requires a case-by-case analysis, one might expect to get some interesting intermediate mathematical results: this is not even the case in this paper. The interest of this paper is of two kinds: the result (not its proof) and the development of the `CHEVIE` package for computing with (semisimple elements of) algebraic groups. This extension of the `CHEVIE` package, together with some application to our problems, is presented in an appendix at the end of this paper.

Remark - In fact, our proof shows that, if the Mackey formula $(\mathscr{M}_{\mathbf{G},\mathbf{L},\mathbf{P},\mathbf{M},\mathbf{Q}})$ holds whenever $(\mathbf{G}, F)$ is the semisimple and simply-connected group of type ${}^2\mathrm{E}_6(2)$ and $\mathbf{M}$ is of type $\mathrm{A}_2 \times \mathrm{A}_2$, then the Mackey formula holds in general (see Remark 3.9). □

1. Notation, recollection

Algebraic groups. We fix a prime number p, an algebraic closure $\mathbb{F}$ of the finite field with p elements $\mathbb{F}_p$, a power q of p and a connected reductive group $\mathbf{G}$ (over $\mathbb{F}$) endowed with an $\mathbb{F}_q$-structure determined by a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ (here, $\mathbb{F}_q$ denotes the subfield of $\mathbb{F}$ with q elements). We also fix a pair $(\mathbf{G}^*, F^*)$ dual to $(\mathbf{G}, F)$ and we denote by $\pi : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ the simply-connected covering of the derived subgroup of $\mathbf{G}^*$. Then there exists a unique $\mathbb{F}_q$-Frobenius endomorphism $F^* : \tilde{\mathbf{G}}^* \to \tilde{\mathbf{G}}^*$ such that π is defined over $\mathbb{F}_q$.

In this paper, if $\mathbf{H}$ is an algebraic group, we denote by $\mathbf{H}^\circ$ its neutral component. If $\mathbf{U}$ denotes the unipotent radical of $\mathbf{H}$, a *Levi complement* of $\mathbf{H}$ is a subgroup $\mathbf{L}$ of $\mathbf{H}$ such that $\mathbf{H} = \mathbf{L} \ltimes \mathbf{U}$. We shall define a *Levi subgroup* of $\mathbf{G}$ to be a Levi complement of some parabolic subgroup of $\mathbf{G}$. The centre of $\mathbf{H}$ will be denoted by $\mathbf{Z}(\mathbf{H})$ and we set $\mathscr{Z}(\mathbf{H}) = \mathbf{Z}(\mathbf{H})/\mathbf{Z}(\mathbf{H})^\circ$. If $g \in \mathbf{H}$, the order of g will be denoted by $o(g)$.

If $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$, then the morphism $h_{\mathbf{L}}^{\mathbf{G}} : \mathscr{Z}(\mathbf{G}) \to \mathscr{Z}(\mathbf{L})$ is surjective (see [10, Lemma 1.4]) and its kernel has been completely computed in [3, Proposition 2.8 and Table 2.17]). If $\mathbf{M}$ is another Levi subgroup of $\mathbf{G}$, we denote by $\mathscr{S}_{\mathbf{G}}(\mathbf{L}, \mathbf{M})$ the set of elements $g \in \mathbf{G}$ such that $\mathbf{L}$ and ${}^g\mathbf{M}$ have a common maximal torus. Recall that this implies that $\mathbf{L} \cap {}^g\mathbf{M}$ is a Levi complement of $\mathbf{L} \cap {}^g\mathbf{Q}$, as well as a Levi complement of $\mathbf{P} \cap {}^g\mathbf{M}$.

If $\mathbf{Z}$ is an F-stable subgroup of the centre $\mathbf{Z}(\mathbf{G})$ of $\mathbf{G}$, we also fix a pair $((\mathbf{G}/\mathbf{Z})^*, F^*)$ dual to $(\mathbf{G}/\mathbf{Z}, F)$ and we denote by $\pi_{\mathbf{Z}} : \tilde{\mathbf{G}}^* \to (\mathbf{G}/\mathbf{Z})^*$ the induced morphism: note that it is defined over $\mathbb{F}_q$. There exists a unique morphism $\tau_{\mathbf{Z}} : (\mathbf{G}/\mathbf{Z})^* \to \mathbf{G}^*$ such that $\pi = \tau_{\mathbf{Z}} \circ \pi_{\mathbf{Z}}$: it is also defined over $\mathbb{F}_q$. Finally, we set $\mathbf{Z}^* = \operatorname{Ker} \pi_{\mathbf{Z}}$: it is an F^*-stable subgroup of $\mathbf{Z}(\tilde{\mathbf{G}}^*)$, which should not be confused with a dual (in any sense) of $\mathbf{Z}$.

We also recall the following definition:

Definition 1.1. *A semisimple element $s \in \mathbf{G}^*$ is said to be **isolated** (respectively **quasi-isolated**) if its connected centralizer $C^\circ_{\mathbf{G}^*}(s)$ (respectively its centralizer $C_{\mathbf{G}^*}(s)$) is not contained in a proper Levi subgroup of $\mathbf{G}$.*

Class functions. We fix a prime number $\ell \neq p$ and we denote by $\overline{\mathbb{Q}}_\ell$ an algebraic closure of the ℓ-adic field $\mathbb{Q}_\ell$. If Γ is a finite group, the $\overline{\mathbb{Q}}_\ell$-vector space of class functions $\Gamma \to \overline{\mathbb{Q}}_\ell$ is denoted by $\mathrm{Class}(\Gamma)$. This vector space is endowed with the canonical scalar product $\langle , \rangle_\Gamma$, for which the set of irreducible characters $\mathrm{Irr}\,\Gamma$ of Γ is an orthonormal basis.

If $\mathbf{L}$ is an F-stable Levi complement of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, let $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathrm{Class}(\mathbf{L}^F) \to \mathrm{Class}(\mathbf{G}^F)$ and ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathrm{Class}(\mathbf{G}^F) \to \mathrm{Class}(\mathbf{L}^F)$ denote respectively the Lusztig induction and restriction maps. They are adjoint with respect to the scalar products $\langle , \rangle_{\mathbf{L}^F}$ and $\langle , \rangle_{\mathbf{G}^F}$. If $g \in \mathbf{G}^F$, we denote by $(\mathrm{ad}\, g)_{\mathbf{L}} : \mathrm{Class}(\mathbf{L}^F) \to \mathrm{Class}({}^g\mathbf{L}^F)$, $\lambda \mapsto ({}^g\lambda : l \mapsto \lambda(g^{-1}lg))$.

If $s \in \mathbf{G}^F$ is a semisimple element and if $f \in \mathrm{Class}(\mathbf{G}^F)$, we define

$$\begin{array}{rccl} d_s^{\mathbf{G}} f : & C^\circ_{\mathbf{G}}(s)^F & \longrightarrow & \overline{\mathbb{Q}}_\ell \\ & u & \longmapsto & \begin{cases} f(su) & \text{if } u \text{ is unipotent,} \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Note that $d_s^{\mathbf{G}} f \in \mathrm{Class}(C^\circ_{\mathbf{G}}(s)^F)$, so that we have defined a $\overline{\mathbb{Q}}_\ell$-linear map

$$d_s^{\mathbf{G}} : \mathrm{Class}(\mathbf{G}^F) \longrightarrow \mathrm{Class}(C^\circ_{\mathbf{G}}(s)^F).$$

Tori over finite fields. If $\mathbf{S}$ is a torus defined over $\mathbb{F}_q$, we denote by $X(\mathbf{S})$ (respectively $Y(\mathbf{S})$) the lattice of rational characters $\mathbf{S} \to \mathbb{F}^\times$ (respectively of one-parameter subgroups $\mathbb{F}^\times \to \mathbf{S}$). Let $\langle , \rangle_{\mathbf{S}} : X(\mathbf{S}) \times Y(\mathbf{S}) \to \mathbb{Z}$ denote the canonical perfect pairing.

If moreover $\mathbf{S}$ is defined over $\mathbb{F}_q$, with corresponding Frobenius endomorphism $F : \mathbf{S} \to \mathbf{S}$, then there exists a unique automorphism $\phi : Y(\mathbf{S}) \to Y(\mathbf{S})$ of *finite order* such that $F(\lambda) = q\phi(\lambda)$ for all $\lambda \in Y(\mathbf{S})$. The characteristic polynomial of ϕ will be denoted by $\chi_{\mathbf{S},F}$: since ϕ has finite order, $\chi_{\mathbf{S},F}$ is a product of cyclotomic polynomials. Note that

$$\deg \chi_{\mathbf{S},F} = \dim \mathbf{S} \tag{1.2}$$

and that [11, Proposition 13.7 (ii)]

$$\mathbf{S}^F \simeq Y(\mathbf{S})/(F - \mathrm{Id}_{Y(\mathbf{S})})(Y(\mathbf{S})) \qquad \text{and} \qquad |\mathbf{S}^F| = \chi_{\mathbf{S},F}(q). \tag{1.3}$$

If m is a non-zero natural number, we denote by Φ_m the m-th cyclotomic polynomial. We shall recall here the notions of Φ_m-torus, as defined in [7, Definition 3.2]:

Definition 1.4 (Broué-Malle-Michel)**.** *We say that $(\mathbf{S}, F)$ is a **Φ_m-torus** if the characteristic polynomial of ϕ is a power of Φ_m.*

*If $\mathbf{S}'$ is an F-stable subtorus of $\mathbf{S}$, we say that $(\mathbf{S}', F)$ is a **Sylow Φ_m-subtorus** of $(\mathbf{S}, F)$ if $\chi_{\mathbf{S}',F}$ is exactly the highest power of Φ_m dividing $\chi_{\mathbf{S},F}$.*

Recall [7, Theorem 3.4 (1)] that there always exists a Sylow Φ_m-subtorus, that it is unique and that, if $(\mathbf{S}, F)$ is itself a Φ_m-torus (with $\chi_{\mathbf{S},F} = \Phi_m^r$), then [7, Proposition 3.3 (3)]

$$\mathbf{S}^F \simeq \big(\mathbb{Z}/\Phi_m(q)\mathbb{Z}\big)^r. \tag{1.5}$$

It follows from the definition that $(\mathbf{S}, F)$ is a Φ_1-torus if and only if $(\mathbf{S}, F)$ is a split torus.

Lemma 1.6. *If* $\dim \mathbf{S} = 2$, *then* $\mathbf{S}^F$ *is isomorphic to one of the following groups*

$$\big(\mathbb{Z}/(q-1)\mathbb{Z}\big)^2, \quad \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/(q+1)\mathbb{Z}, \quad \big(\mathbb{Z}/(q+1)\mathbb{Z}\big)^2,$$

$$\mathbb{Z}/(q^2-1)\mathbb{Z}, \quad \mathbb{Z}/(q^2+q+1)\mathbb{Z} \quad \textit{or} \quad \mathbb{Z}/(q^2-q+1)\mathbb{Z}.$$

Proof. Since the only cyclotomic polynomials of degree $\leqslant 2$ are Φ_1, Φ_2, Φ_3 and Φ_6, it follows from 1.2 that $(\mathbf{S}, F)$ is a Φ_m-torus for $m \in \{1, 2, 3, 6\}$ or that $\chi_{\mathbf{S},F} = \Phi_1\Phi_2$. In the first case, the result follows from 1.5.

So let us examine now the case where $\chi_{\mathbf{S},F} = \Phi_1\Phi_2$. So $\phi^2 = \mathrm{Id}_{Y(\mathbf{S})}$. For $m \in \{1, 2\}$, let $\mathbf{S}_m$ denote the Sylow Φ_m-subtorus of $\mathbf{S}$ and let λ_m be a generator of $Y(\mathbf{S}_m) = \{\lambda \in Y(\mathbf{S}) \mid \Phi_m(\phi)(\lambda) = 0\}$. So (λ_1, λ_2) is a basis of $\mathbb{Q} \otimes_{\mathbb{Z}} Y(\mathbf{S})$. Note that $Y(\mathbf{S})/Y(\mathbf{S}_m)$ is torsion-free. Two cases may occur:

• If (λ_1, λ_2) is a $\mathbb{Z}$-basis of $Y(\mathbf{S})$. Then $\mathbf{S} \simeq \mathbf{S}_1 \times \mathbf{S}_2$ and $\mathbf{S}^F \simeq \mathbf{S}_1^F \times \mathbf{S}_2^F \simeq \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/(q+1)\mathbb{Z}$.

• If (λ_1, λ_2) is not a basis of $Y(\mathbf{S})$, let $\lambda = (\lambda_1 + \lambda_2)/2 \in \mathbb{Q} \otimes_{\mathbb{Z}} Y(\mathbf{S})$ and let $\mu = \phi(\lambda) = (\lambda_1 - \lambda_2)/2 \in \mathbb{Q} \otimes_{\mathbb{Z}} Y(\mathbf{S})$. First of all, let us show that $\lambda \in Y(\mathbf{S})$. Indeed, there exists a and b in $\mathbb{Q}$ such that $\eta_0 = a_0\lambda_1 + b_0\lambda_2 \in Y(\mathbf{S})$ and $(a_0, b_0) \notin \mathbb{Z} \times \mathbb{Z}$. Since $\eta_0 + \phi(\eta_0) \in Y(\mathbf{S}_1)$ and $\eta_0 - \phi(\eta_0) \in Y(\mathbf{S})$, we get that $2a_0 \in \mathbb{Z}$ and $2b_0 \in \mathbb{Z}$. By replacing a_0 and b_0 by $a_0 - a_0'$ and $b_0 - b_0'$ with $b_0, b_0' \in \mathbb{Z}$, we may (and we will) assume that $a_0, b_0 \in \{0, 1/2\}$. But $\lambda_1/2 \notin Y(\mathbf{S})$ and $\lambda_2/2 \notin Y(\mathbf{S})$. So $(a_0, b_0) = (1/2, 1/2)$. So $\lambda \in Y(\mathbf{S})$.

Similarly, $\mu = \phi(\lambda) \in Y(\mathbf{S})$. Now, if $\eta \in Y(\mathbf{S})$, then there exists a, $b \in \mathbb{Z}$ such that $\eta + \phi(\eta) = a\lambda_1$ and $\eta - \phi(\eta) = b\lambda_2$. Therefore, $\eta = (a\lambda_1 + b\lambda_2)/2 = (a+b)\lambda + (a-b)\mu$. So (λ, μ) is a $\mathbb{Z}$-basis of $Y(\mathbf{S})$. In this basis, the matrix representing F is

$$\begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}.$$

It then follows from 1.3 that $\mathbf{S}^F$ is cyclic of order $q^2 - 1$. □

2. A property of quasi-isolated semisimple elements

The aim of this section is to prove the following proposition, from which the Mackey formula in the cases (3) and (4) of the Theorem will be deduced.

Proposition 2.1. *Let $\mathbf{M}$ be an F-stable Levi complement of a parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$. Assume that the quadruple $(\mathbf{G}, F, \mathbf{M}, \mathbf{Q})$ satisfies all of the following conditions:*

(P1) *$\mathbf{G}$ is semisimple and simply-connected and F permutes transitively the quasi-simple components of $\mathbf{G}$;*
(P2) *The quasi-simple components of $\mathbf{G}$ are not of type A;*
(P3) *$\mathbf{M}$ is not a maximal torus of $\mathbf{G}$ and $\mathbf{M} \neq \mathbf{G}$;*
(P4) *There exists an F-stable unipotent class of $\mathbf{M}$ which supports an F-stable cuspidal local system;*
(P5) *$\mathbf{Q}$ is not contained in an F-stable proper parabolic subgroup of $\mathbf{G}$;*
(P6) *For every F-stable subgroup $\mathbf{Z}$ of $\mathbf{Z}(\mathbf{G}) \cap \mathbf{Z}(\mathbf{M})^\circ$, there exists a semisimple element $s \in (\mathbf{M}/\mathbf{Z})^{*F^*}$ which is quasi-isolated in $(\mathbf{M}/\mathbf{Z})^*$ and $(\mathbf{G}/\mathbf{Z})^*$ and such that, for every $z \in \pi_{\mathbf{Z}}(\tilde{\mathbf{G}}^{*F^*}) \cap \mathbf{Z}((\mathbf{M}/\mathbf{Z})^*)^{F^*}$, s and sz are conjugate in $(\mathbf{G}/\mathbf{Z})^{*F^*}$.*

Then $\mathbf{G}$ is quasi-simple, the pair $(\mathbf{G}, F)$ is of type ${}^2\mathrm{E}_6$, $q = 2$ and $\mathbf{M}$ is of type $\mathrm{A}_2 \times \mathrm{A}_2$.

The rest of this section is devoted to the proof of this proposition. This will be done through a case-by-base analysis, relying on some computer calculation using the `CHEVIE` package (in `GAP3`). Before starting the case-by-case analysis, we gather some consequences of properties (Pk), $1 \leqslant k \leqslant 6$, that hold in all groups.

So, from now on, and until the end of this section 2, we fix an F-stable Levi complement $\mathbf{M}$ of a parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$ such that the quadruple $(\mathbf{G}, F, \mathbf{M}, \mathbf{Q})$ satisfies the statements (P1), (P2), (P3), (P4), (P5) and (P6) of the Proposition 2.1. Using (P1), let us write

$$\mathbf{G} = \underbrace{\mathbf{G}_0 \times \cdots \times \mathbf{G}_0}_{d \text{ times}},$$

where $\mathbf{G}_0$ is a semisimple, simply-connected and quasi-simple group defined over $\mathbb{F}_{q^d}$ and F permutes transitively the quasi-simple components of $\mathbf{G}$. In particular, F^d stabilizes $\mathbf{G}_0$ and $\mathbf{G}^F \simeq \mathbf{G}_0^{F^d}$. Let $\mathbf{M}_0$ denote the F^d-stable Levi subgroup of $\mathbf{G}_0$ such that $\mathbf{M} = \mathbf{M}_0 \times {}^F\mathbf{M}_0 \times \cdots \times {}^{F^{d-1}}\mathbf{M}_0$. We denote by $\pi_0 : \tilde{\mathbf{G}}_0^* \to \mathbf{G}_0^*$ the restriction of π to the first component.

We write

$$\mathbf{G}^* = \underbrace{\mathbf{G}_0^* \times \cdots \times \mathbf{G}_0^*}_{d \text{ times}}, \quad \tilde{\mathbf{G}}^* = \underbrace{\tilde{\mathbf{G}}_0^* \times \cdots \times \tilde{\mathbf{G}}_0^*}_{d \text{ times}}$$

$$\mathbf{M}^* = \mathbf{M}_0^* \times {}^{F^*}\mathbf{M}_0^* \times \cdots \times {}^{F^{*d-1}}\mathbf{M}_0^*,$$

$$\tilde{\mathbf{M}}^* = \pi^{-1}(\mathbf{M}^*) \qquad \text{and} \qquad \tilde{\mathbf{M}}_0^* = \pi_0^{-1}(\mathbf{M}_0^*).$$

If $Z \subset \mathbf{Z}(\mathbf{G}_0) \cap \mathbf{Z}(\mathbf{M}_0)^\circ$ is F^d-stable, we set $\mathbf{Z} = Z \times {}^FZ \times \cdots \times {}^{F^{d-1}}Z$, $S_Z = \mathbf{Z}((\mathbf{M}/\mathbf{Z})^*)^{\circ F^*}$, $S'_Z = S_Z \cap \pi_{\mathbf{Z}}(\mathbf{Z}(\tilde{\mathbf{M}}^*)^{\circ F^*})$, $e_Z = |S_Z|/|S'_Z|$ and we denote by s_Z a semisimple element

of $(\mathbf{M}/\mathbf{Z})^{*F^*}$ which is quasi-isolated in $(\mathbf{M}/\mathbf{Z})^*$ and $(\mathbf{G}/\mathbf{Z})^*$ and such that, for every $z \in S'_Z$, $s_Z z$ and s_Z are conjugate in $(\mathbf{G}/\mathbf{Z})^{*F^*}$. Note that such an element exists by (P6). If $Z = 1$, we set $s_Z = s$, $S_Z = S$, $S'_Z = S'$ and $e_Z = e$ for simplification. We denote by s_0 the projection of s on the first component $\mathbf{G}_0$ of $\mathbf{G}$. Recall that $\mathbf{Z}^*$ is the kernel of $\pi_{\mathbf{Z}}$: we denote by $\mathbf{Z}_0^*$ the projection of $\mathbf{Z}^*$ on the first component $\mathbf{G}_0^*$.

Note that, if $Z = \mathbf{Z}(\mathbf{G}_0)$ (which might happen only if $\mathbf{Z}(\mathbf{G}_0) \subseteq \mathbf{Z}(\mathbf{M}_0)^\circ$), then $\mathbf{Z} = \mathbf{Z}(\mathbf{G})$, $(\mathbf{G}/\mathbf{Z})^* = \tilde{\mathbf{G}}^*$, $\mathbf{Z}^* = 1$ and $\pi_{\mathbf{Z}(\mathbf{G}_0)}$ is the identity. Then:

Lemma 2.2. *The properties* (Pk), $1 \leqslant k \leqslant 6$, *have the following consequences:*

(a) *There exists an F^d-stable unipotent class of $\mathbf{M}_0$ which supports an F^d-stable cuspidal local system.*
(b) *$\mathbf{Z}(\mathbf{M})^\circ$ is not an F-split torus (for the action of F).*
(c) *$e_Z = |H^1(F^*, \mathbf{Z}^* \cap \operatorname{Ker} h^{\tilde{\mathbf{G}}^*}_{\tilde{\mathbf{M}}^*})| = |H^1(F^{*d}, \mathbf{Z}_0^* \cap \operatorname{Ker} h^{\tilde{\mathbf{G}}_0^*}_{\tilde{\mathbf{M}}_0^*})|$. In particular, $e_{\mathbf{Z}(\mathbf{G}_0)} = 1$ and $e_1 = |H^1(F^{*d}, \operatorname{Ker} h^{\tilde{\mathbf{G}}_0^*}_{\tilde{\mathbf{M}}_0^*})|$.*
(d) *S_Z contains an element of order $\geqslant \max(q^d - 1, q + 1)$.*
(e) *All the elements of S'_Z have order dividing the order of s_Z.*
(f) *If $\mathbf{M}_0$ is of type* B, C *or* D, *then $p \neq 2$.*
(g) *If $\dim \mathbf{Z}(\mathbf{M}_0^*) = 1$, then S_Z is isomorphic to $\mathbb{Z}/(q^d - 1)\mathbb{Z}$ or $\mathbb{Z}/(q^d + 1)\mathbb{Z}$.*
(h) *If $\dim \mathbf{Z}(\mathbf{M}_0^*) = 2$, then S_Z is isomorphic to one of the following groups*

$$\big(\mathbb{Z}/(q^d - 1)\mathbb{Z}\big)^2, \quad \mathbb{Z}/(q^d - 1)\mathbb{Z} \times \mathbb{Z}/(q^d + 1)\mathbb{Z}, \quad \big(\mathbb{Z}/(q^d + 1)\mathbb{Z}\big)^2,$$
$$\mathbb{Z}/(q^{2d} - 1)\mathbb{Z}, \quad \mathbb{Z}/(q^{2d} + q^d + 1)\mathbb{Z} \quad \textit{or} \quad \mathbb{Z}/(q^{2d} - q^d + 1)\mathbb{Z}.$$

Proof of lemma 2.2. (a) follows immediately from (P4). (b) follows from (P5) and the following well-known result:

Lemma 2.3. *Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and let $\mathbf{L}$ be a Levi complement of $\mathbf{P}$. Assume that $\mathbf{L}$ is F-stable and that $\mathbf{Z}(\mathbf{L})^\circ$ is F-split. Then $\mathbf{P}$ is F-stable.*

Proof of Lemma 2.3. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{L}$. Let $\Phi \subset X(\mathbf{T})$ denote the root system of $\mathbf{G}$ with respect to $\mathbf{T}$. If $\alpha \in \Phi$, we denote by $\mathbf{U}_\alpha$ the associated one-parameter unipotent subgroup. If $\lambda \in Y(\mathbf{T})$, we denote by $\mathbf{P}(\lambda)$ the subgroup of $\mathbf{G}$ generated by $\mathbf{T}$ and the $\mathbf{U}_\alpha$'s such that $\langle \alpha, \lambda \rangle \geqslant 0$. Then $\mathbf{P}(\lambda)$ is a parabolic subgroup and $F(\mathbf{P}(\lambda)) = \mathbf{P}(F(\lambda))$.

Since $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ admitting $\mathbf{L}$ as a Levi complement, there exists $\lambda \in Y(\mathbf{Z}(\mathbf{L})^\circ) \subset Y(\mathbf{T})$ such that $\mathbf{P} = \mathbf{P}(\lambda)$. Now, $F(\mathbf{P}) = \mathbf{P}(F(\lambda))$ and $F(\lambda) = q\lambda$ because F is split on $\mathbf{Z}(\mathbf{L})^\circ$. So $F(\mathbf{P}) = \mathbf{P}$, as expected. □

(c) Let $K = \operatorname{Ker} \pi_{\mathbf{Z}} \cap \mathbf{Z}(\tilde{\mathbf{M}}^*)^\circ = \mathbf{Z}^* \cap \operatorname{Ker} h^{\tilde{\mathbf{G}}^*}_{\tilde{\mathbf{M}}^*}$. From the natural exact sequence $1 \to K \to \mathbf{Z}(\tilde{\mathbf{M}}^*)^\circ \to \mathbf{Z}((\mathbf{M}/\mathbf{Z})^*)^\circ \to 1$, we deduce an exact sequence of cohomology groups

$$1 \longrightarrow K^{F^*} \longrightarrow \mathbf{Z}(\tilde{\mathbf{M}}^*)^{\circ F^*} \to \mathbf{Z}((\mathbf{M}/\mathbf{Z})^*)^{\circ F^*} \longrightarrow H^1(F^*, K) \longrightarrow H^1(F^*, \mathbf{Z}(\tilde{\mathbf{M}}^*)^\circ) = 1.$$

The first equality in (c) then follows immediately. The second is straightforward.

(d) Note that $\Phi_m(q) \geqslant q-1 \geqslant 1$ if $m \geqslant 1$ and $\Phi_m(q) \geqslant q+1 \geqslant 3$ if $m \geqslant 2$ (note, however, that $\Phi_6(2) = \Phi_2(2) = 3$). Let $\mathbf{S}_Z = \mathbf{Z}((\mathbf{M}_0/Z)^*)^\circ$. We have $S_Z \simeq \mathbf{S}_Z^{F^{*d}}$. Since $\mathbf{M} \neq \mathbf{G}$, we have that $\dim \mathbf{S}_Z \geqslant 1$. So there exists $m \geqslant 1$ such that Φ_m divides $\chi_{\mathbf{S}_Z,F^{*d}}$. By 1.5, there exists an element of S_Z of order $\Phi_m(q^d) \geqslant q^d - 1$.

If $d \geqslant 2$, then $q^d - 1 \geqslant q+1$. If $d = 1$, then $(\mathbf{S}_Z, F^*)$ is not split by (b), so there exists $m \geqslant 2$ such that Φ_m divides $\chi_{\mathbf{S}_Z,F^*}$. So, again by 1.5, there exists an element of S_Z of order $\Phi_m(q) \geqslant q+1$.

(e) Let $z \in S'_Z$. Let d denote the order of s_Z. Since $s_Z z$ and s_Z are conjugate in $(\mathbf{G}/Z)^*$, we have $(s_Z z)^d = 1$. But $s_Z z = z s_Z$ and $s_Z^d = 1$, so $z^d = 1$.

(f) Assume that $\mathbf{M}_0$ is of type B, C or D and that $p = 2$. Then $\mathbf{Z}(\mathbf{M}) = \mathbf{Z}(\mathbf{M})^\circ$ and $\mathbf{Z}(\mathbf{M}^*) = \mathbf{Z}(\mathbf{M}^*)^\circ$. So $\mathbf{Z}(\mathbf{G}) \subset \mathbf{Z}(\mathbf{M})^\circ$. Therefore, by [4, Example 4.8], $s_{\mathbf{Z}(\mathbf{G}_0)}$ is central in $\tilde{\mathbf{M}}^*$. Moreover, $S_{\mathbf{Z}(\mathbf{G}_0)} = S'_{\mathbf{Z}(\mathbf{G}_0)}$ by (c), and $s_{\mathbf{Z}(\mathbf{G}_0)} \in S_{\mathbf{Z}(\mathbf{G}_0)}$. So, by Lemma 2.2 (d), there exists an element z of $S'_{\mathbf{Z}(\mathbf{G}_0)}$ different from 1. If $s_{\mathbf{Z}(\mathbf{G}_0)} = 1$, then $s_{\mathbf{Z}(\mathbf{G}_0)}$ and $s_{\mathbf{Z}(\mathbf{G}_0)} z$ are not conjugate in $\tilde{\mathbf{G}}^{*F^*}$, contradicting (P6). If $s_{\mathbf{Z}(\mathbf{G}_0)} \neq 1$, then $s_{\mathbf{Z}(\mathbf{G}_0)}$ and $s_{\mathbf{Z}(\mathbf{G}_0)} s_{\mathbf{Z}(\mathbf{G}_0)}^{-1} = 1$ are not conjugate in $\tilde{\mathbf{G}}^{*F^*}$, contradicting again (P6).

(g) follows from 1.5 and (h) follows from Lemma 1.6. □

We can now start our case-by-case analysis, that will be done as a long sequence of lemmas.

Lemma FG. *The group $\mathbf{G}_0$ is not of type F_4 or G_2.*

Proof of Lemma FG. Assume first that $\mathbf{G}_0$ is of type G_2. Then by Lemma 2.2 (a) and [12, §15.5], we get that $\mathbf{M}_0 = \mathbf{G}_0$ or that $\mathbf{M}_0$ is a torus. This contradicts (P3).

Assume now that $\mathbf{G}_0$ is of type F_4. Then by Lemma 2.2 (a) and [12, §15.4], we get that $\mathbf{M}_0 = \mathbf{G}_0$ or that $\mathbf{M}_0$ is a torus or that $p = 2$ and $\mathbf{M}_0$ is of type B_2. This contradicts (P3) and Lemma 2.2 (f). □

Lemma BCD2. *If $\mathbf{G}_0$ is of type* B, C *or* D, *then $p > 2$.*

Proof of Lemma BCD2. Assume that $\mathbf{G}$ is of type B, C or D and that $p = 2$. Then $s = 1$ (see [4, Example 4.8]), $e = 1$ (since $\mathbf{Z}(\tilde{\mathbf{G}}^*) = 1$), and, if z denotes an element of $S = S'$ different from 1 (such an element exists by Lemma 2.2 (d)), then s and sz are not conjugate in $\mathbf{G}^{*F^*}$. □ □

Lemma C. *The group $\mathbf{G}_0$ is not of type* C.

Proof of Lemma C. Assume that $\mathbf{G}_0$ is of type C_n with $n \geqslant 2$. Note that $p \neq 2$ by Lemma BCD2 (so $q \geqslant 3$). Since $\mathbf{G}_0^*$ is a special orthogonal group, we have $o(s) = 1$ or 2 (see [4, Proposition 4.11]). Moreover, since $|\mathbf{Z}(\tilde{\mathbf{G}}^*)| = 2$, we have $e \leqslant 2$ by Lemma 2.2 (c). Therefore, if S contains a non-trivial element z of odd order, then $z \in S'$ and s and sz are not conjugate in $\mathbf{G}^*$. This shows that every element of S is a 2-element.

On the other hand, if S contains a non-trivial element z of order greater than or equal to 8, then $z^2 \in S'$ and $o(z^2) \geqslant 4 > 2$. So s and sz^2 are not conjugate in $\mathbf{G}^*$, contradicting (P6). So, every element of S has order 1, 2 or 4. In particular, by Lemma 2.2 (d), we get that $d = 1$ and $q = 3$, so $\mathbf{G} = \mathbf{Sp}_{2n}(\mathbb{F})$. In particular, $\mathbf{G}$ is split.

By (P4) and by [12, §10.4], we have $\mathbf{G}^* \simeq \mathbf{SO}_{2n+1}(\mathbb{F})$ and $\mathbf{M}^* \simeq \mathbf{SO}_{2m+1}(\mathbb{F}) \times (\mathbb{F}^\times)^r$ with $n = m + r$ and $r \geqslant 1$. Note that $\mathbf{Z}(\mathbf{M}^*) = (\mathbb{F}^\times)^r$. Since every element of S has order 1, 2 or 4, this means that F^* acts on $\mathbf{M}^*$ (through the previous isomorphism) via the following formula :

$$F^*(\sigma, t_1, \ldots, t_r) = (F^*(\sigma), t_1^{3\varepsilon_1}, \ldots, t_r^{3\varepsilon_r})$$

for every $\sigma \in \mathbf{SO}_{2m+1}(\mathbb{F})$ and $t_j \in \mathbb{F}^\times$. Here, $\varepsilon_j \in \{1, -1\}$. Moreover, by Lemma 2.2 (d), S contains at least one element of order 4, so there exists j such that $\varepsilon_j = -1$. Let i denote a fourth root of unity in $\mathbb{F}^\times$. Then

$$z = (\mathrm{Id}, 1, \ldots, 1, \underbrace{i}_{\substack{j-\text{th}\\ \text{position}}}, 1, \ldots, 1) \in \mathbf{SO}_{2m+1}(\mathbb{F}) \times (\mathbb{F}^\times)^r \simeq \mathbf{M}^*$$

is an element of S. In particular, $z^2 \in S'$ (since $e \leqslant 2$). Let us write $s = (s', \xi_1, \ldots, \xi_r) \in \mathbf{M}^{*F^*}$ with $s' \in \mathbf{SO}_{2m+1}(\mathbb{F})$ and $\xi_i \in \mathbb{F}^\times$. Since $s^2 = 1$, we have $s'^2 = 1$ and $\xi_i^2 = 1$. It is now easy to check that s and sz^2 are not conjugate in $\mathbf{G}^*$. □

Lemma SO. *There does not exist a subgroup Z of $\mathbf{Z}(\mathbf{G}_0) \cap \mathbf{Z}(\mathbf{M}_0)^\circ$ such that $\mathbf{G}_0/Z \simeq \mathbf{SO}_N(\mathbb{F})$ for some $N \geqslant 7$.*

Proof of Lemma SO. Assume that $\mathbf{G}_0 \simeq \mathbf{Spin}_N(\mathbb{F})$, with $N \geqslant 7$, and that there exists a subgroup Z of $\mathbf{Z}(\mathbf{G}_0) \cap \mathbf{Z}(\mathbf{M}_0)^\circ$ such that $\mathbf{G}_0/Z \simeq \mathbf{SO}_N(\mathbb{F})$. Note that $p \geqslant 3$ by Lemma BCD2. Then, by (P3), by Lemma 2.2 (a) and by [12, §§10.6 and 14], we have $Z = \mathbf{Z}(\mathbf{M}_0)^\circ \cap \mathbf{Z}(\mathbf{G}_0)$. In particular, Z is F^d-stable. Then $(\mathbf{G}_0/Z)^*$ is a special orthogonal or a symplectic group. Thus, $s_Z^2 = 1$ by [4, Example 4.10 and Proposition 4.11]. Moreover, by [12, §10.6] and [3, Table 2.17], $e_Z = 1$. Therefore, S'_Z contains an element z of order greater than or equal to $q + 1 \geqslant 4$. So s_Z and $s_Z z$ are not conjugate in $(\mathbf{G}/Z)^*$. This contradicts (P6). □

Corollary D4. *The group $\mathbf{G}_0$ is not of type D_4.*

Proof of Corollary D4. Assume that $\mathbf{G}_0$ is of type D_4. Then, by (P3), by Lemma 2.2 (a) and by [12, §§10.6 and 14], we have that $\mathbf{M}_0$ is of type $\mathrm{A}_1 \times \mathrm{A}_1$. Now, let $Z = \mathbf{Z}(\mathbf{G}_0) \cap \mathbf{Z}(\mathbf{M}_0)^\circ$. Then $|Z| = 2$ and $(\mathbf{G}_0/Z)^*$ is a special orthogonal group. This is impossible by Lemma SO. □

Lemma BD. *The group $\mathbf{G}_0$ is not of type B or D.*

Proof of Lemma BD. Assume that $\mathbf{G}_0 \simeq \mathbf{Spin}_N(\mathbb{F})$. By Lemma BCD2, we have $p \geqslant 3$. Moreover, by (P2) and by Lemma C (and since $\mathbf{Spin}_N(\mathbb{F}) \simeq \mathbf{Sp}_4(\mathbb{F})$, we have that $N \geqslant 7$. Let n denote the rank of $\mathbf{G}_0$. We have $n = [N/2]$. Then $\mathbf{G}_0$ is of type $\diamondsuit_n$, with $\diamondsuit \in \{B, D\}$. By Lemma 2.2 (a), by Lemma SO and by [12, §§10.6 and 14], $\mathbf{M}_0$ is of type $\diamondsuit_m \times (\mathrm{A}_1)^r$ with $m \geqslant 0$ and $n = m + 2r$.

Note that $N \neq 8$ by Lemma D4, that $e \leqslant 2$, and that $s^4 = 1$ (see [4, Table 2]). So, by (P6), every element of S has order dividing 8. By Lemma 2.2 (d), this implies that $d \in \{1, 2\}$ and $q \in \{3, 7\}$. Assume first that $d = 2$. Then necessarily $q = 3$ (by Lemma 2.2 (d)) and $S \simeq (\mathbb{Z}/8\mathbb{Z})^r$. If $r \geqslant 2$, then S' contains an element z of order 8 (because S' has index at most 2 in S) and s and sz are not conjugate in $\mathbf{G}^*$: this contradicts (P6). So $r = 1$, but then $e = 1$ so $S = S'$ contains an element of order 8: this contradicts again (P6). Therefore, $d = 1$, $\mathbf{G} = \mathbf{G}_0$ and $\mathbf{M} = \mathbf{M}_0$ and $q \in \{3, 7\}$.

Let Z be the subgroup of $\mathbf{Z}(\mathbf{G})$ of order 2 such that $\mathbf{G}/Z \simeq \mathbf{SO}_N(\mathbb{F})$. Then Z is F-stable. Write $\overline{\mathbf{G}} = \mathbf{G}/Z$ and $\overline{\mathbf{M}} = \mathbf{M}/Z$ and let $\bar{s}$ be an element of $\overline{\mathbf{G}}^*$ such that $\tau_Z(\bar{s}) = s$. Then, by [4, Table 2 and Proposition 5.5 (a)], $\bar{s}^4 = 1$. Moreover, $\overline{\mathbf{M}}^* \simeq \mathbf{H} \times (\mathbf{GL}_2(\mathbb{F}))^r$, where $\mathbf{H} \simeq \mathbf{Sp}_{2m}(\mathbb{F})$ if $\diamondsuit = B$ and $\mathbf{H} \simeq \mathbf{SO}_{2m}(\mathbb{F})$ if $\diamondsuit = D$.

Write $\bar{s} = (\bar{s}', t_1, \ldots, t_r)$ where $\bar{s}' \in \mathbf{H}$ and $t_i \in \mathbf{GL}_2(\mathbb{F})$. Since s is quasi-isolated in $\mathbf{G}$, the eigenvalues (ξ_i, ξ_i') of t_i, which are fourth roots of unity, satisfy $\xi_i \xi_i' \in \{1, -1\}$.

Now, F^* permutes the r factors $\mathbf{GL}_2(\mathbb{F})$. Assume that F^* has an orbit of length greater than or equal to 3. Then S contains an element of order greater than or equal to $3^3 - 1$, which is impossible. Now let $\overline{\mathbf{S}}$ denote the center of a product of $\mathbf{GL}_2(\mathbb{F})$ factors which are in the same orbit (we denote by l the length of this orbit: we have $l \in \{1, 2\}$). Let $\tilde{\mathbf{S}}$ denote the torus of $\tilde{\mathbf{G}}^*$ such that $\pi_Z(\tilde{\mathbf{S}}) = \overline{\mathbf{S}}$ and let $\mathbf{S} = \tau_Z(\overline{\mathbf{S}})$. Since $l \leqslant 2$ and $n \neq 8$, the map $\pi : \tilde{\mathbf{S}} \to \mathbf{S}$ is an isomorphism of groups. Note that $\mathbf{S} \subset \mathbf{Z}(\mathbf{M})^\circ$, so $\tilde{\mathbf{S}}^{F^*} \simeq \pi(\tilde{\mathbf{S}}^{F^*}) \subset S'$. But, if $l = 2$, then $\tilde{\mathbf{S}}^{F^*}$ contains an element of order greater than or equal to 8 so S' contains an element of order greater than or equal to 8. This contradicts (P6). If $l = 1$, let z denote an element of $\tilde{\mathbf{S}}^{F^*}$ of order 4. Let $i \in \{1, 2, \ldots, r\}$ denote the place of the $\mathbf{GL}_2(\mathbb{F})$ factor we are considering. Write $\bar{s}\pi_Z(z) = (\bar{s}', t_1, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_r)$ and let ζ_i and ζ_i' denote the eigenvalues of t_i'. Since $\xi_i \xi_i' \in \{1, -1\}$, we have $\{\xi_i, \xi_i'\} \neq \{\zeta_i, \zeta_i'\}$ so s and $s\pi(z)$ are not conjugate in $\mathbf{G}^*$. This contradicts again (P6). □

Before going on our investigation of the remaining cases (types E_6, E_7 and E_8), we introduce the following property of the quadruple $(\mathbf{G}_0, \mathbf{M}_0, Z, n)$, where Z is a subgroup of $\mathbf{Z}(\mathbf{G}_0) \cap \mathbf{Z}(\mathbf{M}_0)^\circ$ and n is a non-zero natural number.

($\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,Z,\mathrm{n}}$) *If s is a semisimple element of $(\mathbf{M}_0/Z)^*$ which is quasi-isolated in $(\mathbf{M}_0/Z)^*$ and in $(\mathbf{G}_0/Z)^*$, then there exists an element $z \in \mathbf{Z}((\mathbf{M}_0/Z)^*)^\circ$ of order dividing n such that s and sz are not conjugate in $(\mathbf{G}_0/Z)^*$.*

The property ($\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,Z,n}$) does not always hold (for instance, $\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,Z,1}$ never holds) but it can be tested with an algorithm using the `CHEVIE` package: this will be explained in the appendix. In the appendix, we will also present some examples for which property ($\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,Z,n}$) holds (see Lemma A.1) and that will be used in the proof of the next lemmas (we will also give non-trivial examples in which ($\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,Z,n}$) does not hold).

Lemma E6. *If $\mathbf{G}_0$ is of type E_6, then $q = 2$, $d = 1$, $\mathbf{M} = \mathbf{M}_0$ is of type $\mathrm{A}_2 \times \mathrm{A}_2$ and $(\mathbf{G}, F)$ is not split.*

Proof of Lemma E6. We assume in this subsection, and only in this subsection, that $\mathbf{G}_0$ is of type E_6. Then, by Lemma 2.2 (a) and by [12, §15.1], this implies that we are in one of the following cases:

- $\mathbf{M}_0$ is a maximal torus.
- $\mathbf{M}_0$ is of type D_4 and $p = 2$.
- $\mathbf{M}_0$ is of type $\mathrm{A}_2 \times \mathrm{A}_2$, $p \neq 3$
- $\mathbf{M}_0 = \mathbf{G}_0$.

By (P3), the first and the last cases are excluded. By Lemma 2.2 (f), the second case is excluded. So $\mathbf{M}_0$ is of type $\mathrm{A}_2 \times \mathrm{A}_2$ and $p \neq 3$. Note that $\mathscr{Z}(\mathbf{M}_0) \simeq \mathbb{Z}/3\mathbb{Z} \simeq \mathscr{Z}(\mathbf{G}_0)$ (see [3, Table 2.17]) and that the graph automorphism of $\mathbf{G}_0$ acts non trivially on $\mathscr{Z}(\mathbf{G}_0)$. So $e = 1$ (by Lemma 2.2 (c)) and, if we denote by $\varepsilon \in \{1, -1\}$ the element defined by the condition $\varepsilon = 1$ if and only if the graph automorphism of $\mathbf{G}_0$ induced by F^d is trivial, then F^d acts on $\mathscr{Z}(\mathbf{M}_0) \simeq \mathbb{Z}/3\mathbb{Z}$ by multiplication by εq^d. But, if we denote by χ the linear character of $\mathscr{Z}(\mathbf{M}_0)$ associated with the cuspidal local system on $\mathbf{M}_0$, then χ is faithful [12, §15.1] and F^d-stable, so $\varepsilon q^d \equiv 1 \bmod 3$. We can summarize these facts in the following statement:

($\mathrm{E}_6^{(1)}$) *$\mathbf{M}_0$ is of type $\mathrm{A}_2 \times \mathrm{A}_2$, $p \neq 3$, $e = 1$ and $\varepsilon q^d \equiv 1 \bmod 3$.*

In particular, it follows from (P6) that

($\mathrm{E}_6^{(2)}$) *s_0 is quasi-isolated in $\mathbf{G}_0^*$ and $\mathbf{M}_0^*$ and s_0 is $\mathbf{G}_0^*$-conjugate to $s_0 z$ for all $z \in \mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$.*

This implies that

($\mathrm{E}_6^{(3)}$) *The order of s_0 divides 6.*

($\mathrm{E}_6^{(4)}$) *All elements of $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$ have order dividing 6.*

($\mathrm{E}_6^{(5)}$) *$\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$ does not contain all elements of order 3 of $\mathbf{Z}(\mathbf{M}_0^*)^\circ$.*

Indeed, ($\mathrm{E}_6^{(3)}$) follows from [4, Table 3], ($\mathrm{E}_6^{(4)}$) follows from ($\mathrm{E}_6^{(3)}$) and Lemma 2.2 (e), while ($\mathrm{E}_6^{(5)}$) follows from the fact that ($\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,1,3}$) holds (see Lemma A.1 (1)), which is proved by computer calculation in the Appendix A.

Now, by Lemma 2.2 (d) and $(\mathrm{E}_6^{(4)})$, we get that $q^d \leqslant 7$ and $q \leqslant 5$. Recall also that $p \neq 3$ by $(\mathrm{E}_6^{(1)})$. So

$(\mathrm{E}_6^{(6)})$ *The pair* (q, d) *belongs to* $\{(2,1), (4,1), (5,1), (2,2)\}$.

Moreover, $\dim \mathbf{Z}(\mathbf{M}_0^*)^\circ = 2$. If $q^d = 4$, then Lemma 2.2 (h) and $(\mathrm{E}_6^{(4)})$ force that $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, so $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$ contains all elements of order 3 of $\mathbf{Z}(\mathbf{M}_0^*)^\circ$. This contradicts $(\mathrm{E}_6^{(5)})$.

Similarly, if $q^d = 5$, then Lemma 2.2 (h) and $(\mathrm{E}_6^{(4)})$ forces that $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}} \simeq \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, so $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$ contains all elements of order 3 of $\mathbf{Z}(\mathbf{M}_0^*)^\circ$. Again, this contradicts $(\mathrm{E}_6^{(5)})$. Therefore $q = 2$ and $d = 1$: in particular, $\varepsilon = -1$ by $(\mathrm{E}_6^{(1)})$.

$(\mathrm{E}_6^{(7)})$ $q = 2$, $d = 1$ *and the pair* $(\mathbf{G}, F)$ *is not split.*

So Lemma E6 follows from $(\mathrm{E}_6^{(1)})$ and $(\mathrm{E}_6^{(7)})$. □

Lemma E7. *The group* $\mathbf{G}_0$ *is not of type* E_7.

Proof of Lemma E7. We assume in this subsection, and only in this subsection, that $\mathbf{G}_0$ is of type E_7. Then, by (4), the group $\mathbf{M}_0$ admits an F^d-stable cuspidal local system supported by a unipotent class. By [12, §15.1], this implies that we are in one of the following cases:

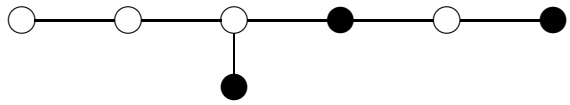
- $\mathbf{M}_0$ is a maximal torus.
- $\mathbf{M}_0$ is of type $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$, $\ker h_{\mathbf{M}_0}^{\mathbf{G}_0} = 1$, and $p \neq 2$.
- $\mathbf{M}_0$ is of type D_4 and $p = 2$.
- $\mathbf{M}_0$ is of type E_6 and $p = 3$.
- $\mathbf{M}_0 = \mathbf{G}_0$.

By (P3), the first and the last cases are excluded. By Lemma 2.2 (f), the third case is excluded. We will now investigate the two remaining cases.

• Assume first that $\mathbf{M}_0$ is of type E_6 and that $p = 3$. By [3, Table 2.17], we get that $\mathscr{Z}(\mathbf{M}_0) = 1$ so $\mathbf{Z}(\mathbf{G}_0) \subseteq \mathbf{Z}(\mathbf{M}_0)^\circ$. Now, $(\mathbf{G}_0/\mathbf{Z}(\mathbf{G}_0))^* = \tilde{\mathbf{G}}_0^*$, so $s_{\mathbf{Z}(\mathbf{G}_0)}$ is isolated in $\tilde{\mathbf{G}}_0^*$: in particular, its order belongs to $\{1, 2, 3, 4\}$. So it follows from Lemma 2.2 (c) and (d) that $q^d \leqslant 3$. So $q = 3$ and $d = 1$. Since $\dim \mathbf{Z}(\tilde{\mathbf{M}}^*)^\circ = 1$ and since $\mathbf{Z}(\tilde{\mathbf{M}}^*)^\circ$ is not split by Lemma 2.2 (b) and (g), this forces $S_{\mathbf{Z}(\mathbf{G}_0)} \simeq \mathbb{Z}/4\mathbb{Z}$. So $S'_{\mathbf{Z}(\mathbf{G}_0)} = S_{\mathbf{Z}(\mathbf{G}_0)}$ contains all the elements of $\mathbf{Z}(\tilde{\mathbf{M}}^*)^\circ$ of order dividing 4. But then Lemma A.1 (2) contradicts (P6). So this case cannot occur.

• So assume now that $\mathbf{M}_0$ is of type $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$, that $\mathrm{Ker}\, h_{\mathbf{M}_0}^{\mathbf{G}_0} = 1$ and that $p \neq 2$. Note that these conditions describe completely the type of the pair $(\mathbf{G}_0, \mathbf{M}_0)$. Indeed, it is given by the following diagram, where the three black nodes correspond to the simple roots of $\mathbf{G}_0$ which are simple roots of $\mathbf{M}_0$:

$(\mathrm{E}_7[\mathrm{A}_1^3]^\#)$



By [12, §15.2] and [3, Table 2.17], we get that $e = 1$. By [4, Table 3], $o(s) \in \{1,2,3,4,6\}$. So it follows from Lemma 2.2 (c) and (d) that $q^d \leqslant 7$ and $q \leqslant 5$. Moreover, since $p > 2$, we get:

$(\mathrm{E}_7^{(1)})$ *$d = e = 1$ and $q \in \{3,5\}$.*

Moreover, by Lemma A.1 (3), we get that $(\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,1,4})$ and $(\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,1,6})$ hold, so

$(\mathrm{E}_7^{(2)})$ *$\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^*}$ does not contain all elements of order 4 of $\mathbf{Z}(\mathbf{M}_0^*)^\circ$.*

$(\mathrm{E}_7^{(2')})$ *$\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^*}$ does not contain all elements of order 6 of $\mathbf{Z}(\mathbf{M}_0^*)^\circ$.*

Since $o(s) \in \{1,2,3,4,6\}$, we get that

$(\mathrm{E}_7^{(3)})$ *Every element of $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^*}$ has order in $\{1,2,3,4,6\}$.*

For simplification, let $\chi = \chi_{\mathbf{Z}(\mathbf{M}_0^*)^\circ,F^*}$. If χ contains some factor Φ_m with $m \geqslant 3$, then it follows from Lemma 1.5 that S_Z contains an element of order $\Phi_m(q) \geqslant \Phi_m(3) \geqslant \Phi_6(3) = 7$, so this contradicts $(\mathrm{E}_7^{(3)})$. Since $\dim \mathbf{Z}(\mathbf{M}_0^*) = 4$, $\chi = \Phi_1^a\Phi_2^b$, with $a + b = 4$. If $a, b \geqslant 1$, then it follows from Lemma A.2 of the Appendix that $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^*}$ contains an element of order $\geqslant 8$, which is impossible by $(\mathrm{E}_7^{(3)})$. Moreover $b \neq 0$ by Lemma 2.2 (b). So $\chi = \Phi_2^4$. But then $(\mathbf{Z}(\mathbf{M}_0^*)^\circ, F^*)$ is a Φ_2-torus. It then follows from 1.5 that

$$S_Z \simeq \big(\mathbb{Z}/(q+1)\mathbb{Z}\big)^4.$$

Since $q \in \{3,5\}$, this contradicts $(\mathrm{E}_7^{(2)})$ and $(\mathrm{E}_7^{(2')})$. □

Lemma E8. *The group $\mathbf{G}_0$ is not of type E_8.*

Proof of Lemma E8. First, note that $\mathbf{G}_0$ is simply-connected and adjoint, so $e = 1$. It then follows from (P6) that

$(\mathrm{E}_8^{(1)})$ $1 \leqslant o(s_0) \leqslant 6$.

$(\mathrm{E}_8^{(2)})$ *All the elements of $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$ have order in $\{1,2,3,4,5,6\}$.*

Indeed, $(\mathrm{E}_8^{(1)})$ follows from [4, Proposition 4.9], $(\mathrm{E}_8^{(2)})$ follows from $(\mathrm{E}_8^{(1)})$ and (P6). Moreover, by Lemma 2.2 (a) and [12, §15.3], we are in one of the following cases:

- $\mathbf{M}_0$ is a maximal torus.
- $\mathbf{M}_0$ is of type D_4 and $p = 2$.
- $\mathbf{M}_0$ is of type E_6 and $p = 3$.
- $\mathbf{M}_0$ is of type E_7 and $p = 2$.
- $\mathbf{M}_0 = \mathbf{G}_0$.

By (P3), the first and the last cases are excluded. By Lemma 2.2 (f), the second case is excluded. We will investigate the two remaining cases.

• Assume first that $\mathbf{M}_0$ is of type E_6 and that $p = 3$. Then, since $(\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,1,2})$ holds by Lemma A.1 (5), it follows that

$(\mathrm{E}_8^{(3)})$ $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$ *does not contain all elements of order* 2 *of* $\mathbf{Z}(\mathbf{M}_0^*)^\circ$.

By Lemma 2.2 (c), $(\mathrm{E}_8^{(2)})$ forces $q^d \leqslant 7$. Since $p = 3$, we get that $q = 3$ and $d = 1$. Moreover, since $\dim \mathbf{Z}(\mathbf{M}^*)^\circ = 2$, we get that $\mathbf{Z}(\mathbf{M}^*)^{\circ F^*}$ is isomorphic to one of the following groups

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},$$

$$\mathbb{Z}/(3^2-1)\mathbb{Z} = \mathbb{Z}/8\mathbb{Z}, \quad \mathbb{Z}/\Phi_3(3)\mathbb{Z} = \mathbb{Z}/13\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/\Phi_6(3)\mathbb{Z} = \mathbb{Z}/7\mathbb{Z}.$$

But this contradicts the fact that $(\mathrm{E}_8^{(2)})$ and $(\mathrm{E}_8^{(3)})$ both hold.

• Assume now that $\mathbf{M}_0$ is of type E_7 and that $p = 2$. It then follows from Lemma A.1 (4) that

$(\mathrm{E}_8^{(4)})$ $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$ *does not contain all elements of order* 3 *of* $\mathbf{Z}(\mathbf{M}_0^*)^\circ$.
$(\mathrm{E}_8^{(4')})$ $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$ *does not contain all elements of order* 5 *of* $\mathbf{Z}(\mathbf{M}_0^*)^\circ$.

By Lemma 2.2 (c), $(\mathrm{E}_8^{(2)})$ forces $q^d \leqslant 7$. Since $p = 2$, this implies that $q^d \in \{2, 4\}$. But $\dim \mathbf{Z}(\mathbf{M}_0^*)^\circ = 1$, so $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$ is isomorphic to one of the following groups

$$\mathbb{Z}/(q^d-1)\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/(q^d+1)\mathbb{Z}.$$

In other words, $\mathbf{Z}(\mathbf{M}_0^*)^{\circ F^{*d}}$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/5\mathbb{Z}$. This contradicts $(\mathrm{E}_8^{(4)})$ or $(\mathrm{E}_8^{(4')})$. □

Now, the proof of the Proposition 2.1 is complete by (P2) and the Lemmas C, BD, FG, E6, E7 and E8.

3. Application to the Mackey formula for Lusztig induction and restriction

This section is devoted to the proof of the main result of this paper, namely the Theorem stated in the introduction. We shall fix some notation: if $\mathbf{L}$ and $\mathbf{M}$ are respective F-stable Levi complements of parabolic subgroups $\mathbf{P}$ and $\mathbf{Q}$ of $\mathbf{G}$, we set

$$\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}} - \sum_{g\in\mathbf{L}^F\backslash\mathscr{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} R^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{L}\cap{}^g\mathbf{Q}} \circ {}^*R^{{}^g\mathbf{M}}_{\mathbf{L}\cap{}^g\mathbf{M}\subset\mathbf{P}\cap{}^g\mathbf{M}} \circ (\operatorname{ad} g)_{\mathbf{M}}.$$

The Mackey formula $(\mathscr{M}_{\mathbf{G},\mathbf{L},\mathbf{P},\mathbf{M},\mathbf{Q}})$ is then equivalent to the equality

$$(\mathscr{M}_{\mathbf{G},\mathbf{L},\mathbf{P},\mathbf{M},\mathbf{Q}}) \qquad \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = 0.$$

3.A. **Preliminaries.** First, note that

(3.1) $$f = 0 \textit{ if and only if } d_s^{\mathbf{G}} f = 0 \textit{ for all semisimple elements } s \in \mathbf{G}^F.$$

We now recall some results from [2]: these are some properties of the maps $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}$ which can be proved a priori (see [2]).

First of all

(3.2) $$\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} \textit{ and } \Delta^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q},\mathbf{L}\subset\mathbf{P}} \textit{ are adjoint for the scalar products } \langle,\rangle_{\mathbf{L}^F} \textit{ and } \langle,\rangle_{\mathbf{M}^F}.$$

Let $\mathbf{P}'$ and $\mathbf{Q}'$ be parabolic subgroups of $\mathbf{G}$ and let $\mathbf{L}'$ and $\mathbf{M}'$ be the unique Levi complement of $\mathbf{P}'$ and $\mathbf{Q}'$: we assume that $\mathbf{L}'$ and $\mathbf{M}'$ are F-stable and that $\mathbf{L} \subset \mathbf{L}'$, $\mathbf{P} \subset \mathbf{P}'$, $\mathbf{M} \subset \mathbf{M}'$ and $\mathbf{Q} \subset \mathbf{Q}'$. Then [2, Proposition 1 (c) of the Corrigenda]

(3.3) $$\begin{aligned}\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} \;=\; & {}^*R^{\mathbf{L}'}_{\mathbf{L}\subset\mathbf{P}\cap\mathbf{L}'} \circ \Delta^{\mathbf{G}}_{\mathbf{L}'\subset\mathbf{P}',\mathbf{M}'\subset\mathbf{Q}'} \circ R^{\mathbf{M}'}_{\mathbf{M}\subset\mathbf{Q}\cap\mathbf{M}'} \\ & + \sum_{g\in\mathbf{L}'^F\backslash\mathscr{S}_{\mathbf{G}}(\mathbf{L}',\mathbf{M}')/\mathbf{M}'^F} {}^*R^{\mathbf{L}'}_{\mathbf{L}\subset\mathbf{P}\cap\mathbf{L}'} \circ R^{\mathbf{L}'}_{\mathbf{L}'\cap{}^g\mathbf{M}'\subset\mathbf{L}'\cap{}^g\mathbf{Q}'} \\ & \circ \Delta^{{}^g\mathbf{M}'}_{\mathbf{L}'\cap{}^g\mathbf{M}'\subset\mathbf{P}'\cap{}^g\mathbf{M}',{}^g\mathbf{M}\subset{}^g(\mathbf{Q}\cap\mathbf{M}')} \circ (\operatorname{ad} g)_{\mathbf{M}} \\ & + \sum_{g\in\mathbf{L}'^F\backslash\mathscr{S}_{\mathbf{G}}(\mathbf{L}',\mathbf{M})^F/\mathbf{M}^F} \Delta^{\mathbf{L}'}_{\mathbf{L}\subset\mathbf{P}\cap\mathbf{L}',\mathbf{L}'\cap{}^g\mathbf{M}\subset\mathbf{L}'\cap{}^g\mathbf{Q}} \\ & \circ R^{{}^g\mathbf{M}}_{\mathbf{L}'\cap{}^g\mathbf{M}\subset\mathbf{P}'\cap{}^g\mathbf{M}} \circ (\operatorname{ad} g)_{\mathbf{M}}.\end{aligned}$$

Also, if $s \in \mathbf{L}^F$ is semisimple, then [2, 5.1.5]

(3.4) $$d_s^{\mathbf{L}} \circ \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = \sum_{\substack{g\in\mathbf{G}^F \\ \text{such that } s\in{}^g\mathbf{M}}} \frac{|C^\circ_{{}^g\mathbf{M}}(s)^F|}{|\mathbf{M}^F|\cdot|C^\circ_{\mathbf{G}}(s)^F|} \Delta^{C^\circ_{\mathbf{G}}(s)}_{C^\circ_{\mathbf{L}}(s)\subset C^\circ_{\mathbf{P}}(s),C^\circ_{{}^g\mathbf{M}}\subset C^\circ_{{}^g\mathbf{Q}}(s)} \circ d_s^{{}^g\mathbf{M}} \circ (\operatorname{ad} g)_{\mathbf{M}}.$$

In particular, if $z \in \mathbf{Z}(\mathbf{G})^F$, then [2, 5.1.6]

(3.5) $$d_z^{\mathbf{L}} \circ \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} \circ d_z^{\mathbf{M}}.$$

Finally, if $\hat{\mathbf{G}}$ denotes a connected reductive group endowed with an $\mathbb{F}_q$-Frobenius endomorphism (still denoted by F) and if $i : \hat{\mathbf{G}} \to \mathbf{G}$ is a morphism of algebraic groups defined over $\mathbb{F}_q$ and such that $\operatorname{Ker} i$ is central in $\hat{\mathbf{G}}$ and $\operatorname{Im} i$ contains the derived subgroup of $\mathbf{G}$, then [5, Proposition 1.1]

(3.6) $$\operatorname{Res}^{\mathbf{L}^F}_{\hat{\mathbf{L}}^F} \circ \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = \Delta^{\mathbf{G}}_{\hat{\mathbf{L}}\subset\hat{\mathbf{P}},\hat{\mathbf{M}}\subset\hat{\mathbf{Q}}} \circ \operatorname{Res}^{\mathbf{M}^F}_{\hat{\mathbf{M}}^F}.$$

Here, $\hat{?} = i^{-1}(?)$ for $? \in \{\mathbf{L},\mathbf{P},\mathbf{M},\mathbf{Q}\}$ and $\operatorname{Res}^{\mathbf{L}^F}_{\hat{\mathbf{L}}^F} : \operatorname{Class}(\mathbf{L}^F) \to \operatorname{Class}(\hat{\mathbf{L}}^F)$, $f \mapsto f \circ i$. In this last situation, we shall need the following lemma:

Lemma 3.7. *If* $\operatorname{Ker} i \subseteq \mathbf{Z}(\hat{\mathbf{G}})^\circ$ *and if u and v are two* ***unipotent*** *elements of* $\hat{\mathbf{G}}^F$*, then u and v are conjugate in* $\hat{\mathbf{G}}^F$ *if and only if $i(u)$ and $i(v)$ are conjugate in* $\mathbf{G}^F$.

Proof. Of course, if u and v are conjugate in $\hat{\mathbf{G}}^F$, then $i(u)$ and $i(v)$ are conjugate in $\mathbf{G}^F$. Conversely, assume that $i(u)$ and $i(v)$ are conjugate in $\mathbf{G}^F$. Then there exists $g \in \hat{\mathbf{G}}$ such that $i(g) \in \mathbf{G}^F$ and $i(gug^{-1}) = i(v)$. So there exists $z \in \operatorname{Ker} i$ such that $gug^{-1} = zv$. Since u and v are unipotent, this forces $z = 1$. So $gug^{-1} = v$.

On the other hand, as $i(g) \in \mathbf{G}^F$, we get that $g^{-1}F(g) \in \operatorname{Ker} i \subseteq \mathbf{Z}(\hat{\mathbf{G}})^\circ$. By Lang's Theorem, there exists $z_\circ \in \mathbf{Z}(\hat{\mathbf{G}})^\circ$ such that $z_\circ F(z_\circ^{-1}) = g^{-1}F(g)$. Then $z_\circ g \in \hat{\mathbf{G}}^F$ and $(z_\circ g)u(z_\circ g)^{-1} = v$. □

3.B. **Main Theorem.** We are now ready to prove the Theorem announced in the introduction:

Theorem 3.8. *Assume that one of the following holds:*

(1) $\mathbf{P}$ *and* $\mathbf{Q}$ *are F-stable (Deligne* [13, Theorem 2.5]*).*
(2) $\mathbf{L}$ *or* $\mathbf{M}$ *is a maximal torus of* $\mathbf{G}$ *(Deligne and Lusztig* [9, Theorem 7]*).*
(3) $q > 2$.
(4) $\mathbf{G}$ *does not contain an F-stable quasi-simple component of type* ${}^2\mathrm{E}_6$, E_7 *or* E_8.

Then the Mackey formula $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = 0$ *holds.*

Proof. For simplification, we will denote by (P0) the following assertion on $\mathbf{G}$:

(P0) $q > 2$ *or* $\mathbf{G}$ *does not contain an F-stable quasi-simple component of type* ${}^2\mathrm{E}_6$, E_7 *or* E_8.

In other words, (P0) is equivalent to say that $\mathbf{G}$ satisfies at least one of the assertions (3) or (4) of the Theorem 3.8.

Our proof of Theorem 3.8 follows an induction argument. We denote by $\chi(\mathbf{G})$ the order of the torsion group of $Y(\mathbf{T})/\langle\Phi^\vee\rangle$, where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$ and $\Phi^\vee \subset Y(\mathbf{T})$ is its coroot system. We set

$$\mathbf{n}_{\mathbf{G},\mathbf{L},\mathbf{M}} = (\dim \mathbf{G}, \dim \mathbf{L} + \dim \mathbf{M}, \chi(\mathbf{G})) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}.$$

We shall denote by $\preccurlyeq$ the lexicographic order on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and we assume that we have found a sextuple $(\mathbf{G}, F, \mathbf{L}, \mathbf{P}, \mathbf{M}, \mathbf{Q})$ which satisfies (P0) and such that $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} \neq 0$ with $\mathbf{n}_{\mathbf{G},\mathbf{L},\mathbf{M}}$ is minimal (for the lexicographic order $\preccurlyeq$). Our aim is to show that $(\mathbf{G}, F, \mathbf{L}, \mathbf{P})$ or $(\mathbf{G}, F, \mathbf{M}, \mathbf{Q})$ satisfies all the properties (Pk), $1 \leqslant k \leqslant 6$. Then we get a contradiction, since Proposition 2.1 shows that there is no quadruple $(\mathbf{G}, F, \mathbf{M}, \mathbf{Q})$ satisfying (P0), (P1), (P2), (P3), (P4), (P5) and (P6) together.

For this purpose, we shall need the following trivial remark, that will allow to use an induction argument:

(IND) *If* $\mathbf{G}$ *satisfies* (P0) *and if* $\mathbf{H}$ *is a connected reductive subgroup of* $\mathbf{G}$ *of the same rank, then* $\mathbf{H}$ *satisfies also* (P0).

• *First step: proof of* (P5). Assume that $\mathbf{P}$ and $\mathbf{Q}$ are contained in proper F-stable parabolic subgroups $\mathbf{P}'$ and $\mathbf{Q}'$ respectively. Let $\mathbf{L}'$ (respectively $\mathbf{M}'$) be the unique Levi complement of $\mathbf{P}'$ (respectively $\mathbf{Q}'$) containing $\mathbf{L}$ (respectively $\mathbf{M}$). Then $\mathbf{L}'$ and $\mathbf{M}'$ are F-stable by uniqueness and $\Delta^{\mathbf{G}}_{\mathbf{L}'\subset\mathbf{P}',\mathbf{M}'\subset\mathbf{Q}'} = 0$ by Theorem 3.8 (1). So it follows from the minimality of $\mathbf{n}_{\mathbf{G},\mathbf{L},\mathbf{M}}$, from 3.3 and from (IND) that $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = 0$, contrarily to our hypothesis.

Therefore, $\mathbf{P}$ or $\mathbf{Q}$ is not contained in a proper F-stable parabolic subgroup of $\mathbf{G}$. By 3.2, we have also that $\Delta^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q},\mathbf{L}\subset\mathbf{P}} \neq 0$, so we may assume that $\mathbf{Q}$ is not contained in a proper F-stable parabolic subgroup of $\mathbf{G}$.

Therefore, from now on, we will prove that $(\mathbf{G}, F, \mathbf{M}, \mathbf{Q})$ satisfies all of the properties (Pk), $1 \leqslant k \leqslant 6$. We have just proved (P5).

• *Second step: proof of* (P3). This follows immediately from Theorem 3.8 (2).

• *Third step: proof of* (P1). Let $\mu \in \mathrm{Class}(\mathbf{G}^F)$ be such that $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(\mu) \neq 0$ and let $s \in \mathbf{L}^F$ be semisimple. By the minimality of $\dim \mathbf{G}$ and by (IND), it follows from 3.4 that $d^{\mathbf{G}}_s \circ \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}} = 0$ if $s \notin \mathbf{Z}(\mathbf{G})^F$. So, by 3.1, there exists $z \in \mathbf{Z}(\mathbf{G})^F$ such that $d^{\mathbf{L}}_z(\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(\mu)) \neq 0$. Therefore, $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(d^{\mathbf{M}}_z\mu) = 0$ by 3.5. In other words, if we replace μ by $d^{\mathbf{M}}_z\mu$, this means that we may (and we will) assume that μ has support on unipotent elements of $\mathbf{M}^F$.

Now, let $i : \hat{\mathbf{G}} \to \mathbf{G}$ be the simply-connected covering of the derived subgroup of $\mathbf{G}$ and let $F : \hat{\mathbf{G}} \to \hat{\mathbf{G}}$ denote the unique $\mathbb{F}_q$-Frobenius endomorphism such that i is defined over $\mathbb{F}_q$. Let $\hat{?} = i^{-1}(?)$, for $? \in \{\mathbf{L}, \mathbf{P}, \mathbf{M}, \mathbf{Q}\}$. Since μ has unipotent support, $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(\mu)$ has also a unipotent support. Moreover, since i induces a bijective morphism between the unipotent varieties, it induces a bijection between unipotent elements of $\hat{\mathbf{G}}^F$ and unipotent elements of $\mathbf{G}^F$. Therefore, $\mathrm{Res}^{\mathbf{L}^F}_{\hat{\mathbf{L}}^F} \Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(\mu) \neq 0$. By 3.6, this means that $\Delta^{\hat{\mathbf{G}}}_{\hat{\mathbf{L}}\subset\hat{\mathbf{P}},\hat{\mathbf{M}}\subset\hat{\mathbf{Q}}} \neq 0$. But $\mathbf{n}_{\hat{\mathbf{G}},\hat{\mathbf{L}},\hat{\mathbf{M}}} \preccurlyeq \mathbf{n}_{\mathbf{G},\mathbf{L},\mathbf{M}}$, with equality if and only if i is an isomorphism. By the minimality of $\mathbf{n}_{\mathbf{G},\mathbf{L},\mathbf{M}}$, we get that i is an isomorphism. So $\mathbf{G}$ is semisimple and simply-connected.

By writing $\mathbf{G}$ as the product of its quasi-simple components, one can write $\mathbf{G}$ as a direct product of semisimple simply-connected F-stable groups $\mathbf{G}_j$, $j \in J$ (J being some indexing set), the Frobenius endomorphism acting transitively on the quasi-simple components of $\mathbf{G}_j$. Since Lusztig functors are compatible with direct products, the minimality of $\mathbf{n}_{\mathbf{G},\mathbf{L},\mathbf{M}}$ implies that $\mathbf{G}$ is one of these $\mathbf{G}_j$'s. Therefore, $\mathbf{G}$ is semisimple and simply-connected, and F permutes transitively the quasi-simple components of $\mathbf{G}$. This completes the proof of (P1).

• *Fourth step: proof of* (P2). This follows immediately from [2, Theorem 5.2.1].

• *Fifth step: proof of* (P4). Recall that we have found a class function μ on $\mathbf{M}^F$, with unipotent support, and such that $\Delta^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}(\mu) \neq 0$. Let $\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)$ denotes the subspace of $\mathrm{Class}(\mathbf{G}^F)$ consisting of functions with unipotent support. In other words, $\mathrm{Class}_{\mathrm{uni}}(\mathbf{G}^F)$ is the image of $d^{\mathbf{G}}_1$.

Let $\mathscr{E}_{\mathbf{M}}$ denote the subspace of $\mathrm{Class}_{\mathrm{uni}}(\mathbf{M}^F)$ generated by all the $R_{\mathbf{M}'\subset\mathbf{Q}'}^{\mathbf{M}}(\mu')$, where $\mathbf{M}'$ is an F-stable Levi complement of a *proper* parabolic subgroup $\mathbf{Q}'$ of $\mathbf{M}$ and $\mu' \in \mathrm{Class}_{\mathrm{uni}}(\mathbf{M}'^F)$. Then it follows from the minimality of $\mathbf{n}_{\mathbf{G},\mathbf{L},\mathbf{M}}$, from (IND) and from 3.3 (see also [2, 5.1.8 and Proposition 1 (a) of the Corrigenda] for a particular form of this formula) that $\Delta_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}^{\mathbf{G}}(\mathscr{E}_{\mathbf{M}}) = 0$. So, if we write $\mu = \mu_c + \mu'$, with $\mu' \in \mathscr{E}_{\mathbf{M}}$ and $\mu_c \in \mathscr{E}_{\mathbf{M}}^{\perp}$, then $\Delta_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}^{\mathbf{G}}(\mu) = \Delta_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}^{\mathbf{G}}(\mu_c) \neq 0$. This means that the vector space $\mathscr{E}_{\mathbf{M}}^{\perp}$ is non-zero and that we may assume that $\mu = \mu_c$. But $\mathscr{E}_{\mathbf{M}}^{\perp}$ is the space of *absolutely cuspidal functions on* $\mathbf{M}^F$ *with unipotent support* (as defined in [2, §3.1]: it was denoted by $\mathrm{Cus}_{\mathrm{uni}}(\mathbf{M}^F)$ in this paper).

Now, by the minimality of $\mathbf{n}_{\mathbf{G},\mathbf{L},\mathbf{M}}$, it follows that *the Mackey formula holds in* $\mathbf{M}$ (in the sense of [2, Definition 1.4.2]). So it follows from [2, Corollary 8 of the Corrigenda] that there exists an F-stable unipotent class of $\mathbf{M}$ which supports a cuspidal local system. This shows (P4).

• *Sixth step: proof of* (P6). Now, let $\mathbf{Z}$ be an F-stable subgroup of $\mathbf{Z}(\mathbf{M})^\circ \cap \mathbf{Z}(\mathbf{G})$. Note that $\mathbf{Z}$ is finite (since $\mathbf{G}$ is semisimple). Let $\overline{\mathbf{G}} = \mathbf{G}/\mathbf{Z}$. If $? \in \{\mathbf{L},\mathbf{P},\mathbf{M},\mathbf{Q}\}$, we set $\overline{?} = ? \cap \overline{\mathbf{G}}$. Note that $\dim\overline{\mathbf{G}} = \dim\mathbf{G}$ and $\dim\overline{\mathbf{L}} + \dim\overline{\mathbf{M}} = \dim\mathbf{L} + \dim\mathbf{M}$. Let $\tau : \mathbf{G} \to \overline{\mathbf{G}}$ denote the canonical morphism. Let u and v be two unipotent elements of $\mathbf{M}^F$. Since $\mathbf{Z} \subset \mathbf{Z}(\mathbf{M})^\circ$, u and v are conjugate in $\mathbf{M}^F$ if and only if $\tau(u)$ and $\tau(v)$ are conjugate in $\overline{\mathbf{M}}^F$ (see Lemma 3.7). So there exists a unique $\bar{f} \in \mathrm{Class}(\overline{\mathbf{G}}^F)$ such that $f = d_1^{\mathbf{G}}\,\mathrm{Res}_{\mathbf{M}^F}^{\overline{\mathbf{M}}^F}\bar{f}$. Moreover, since $\mathbf{Z} \subset \mathbf{Z}(\mathbf{M})^\circ$ and by 3.5 and 3.6,

$$d_1^{\mathbf{L}}\,\mathrm{Res}_{\mathbf{L}^F}^{\overline{\mathbf{L}}^F} \circ \Delta_{\overline{\mathbf{L}}\subset\overline{\mathbf{P}},\overline{\mathbf{M}}\subset\overline{\mathbf{Q}}}^{\overline{\mathbf{G}}}(\bar{f}) = \Delta_{\mathbf{L}\subset\mathbf{P},\mathbf{M}\subset\mathbf{Q}}^{\mathbf{G}}(f).$$

So $\Delta_{\overline{\mathbf{L}}\subset\overline{\mathbf{P}},\overline{\mathbf{M}}\subset\overline{\mathbf{Q}}}^{\overline{\mathbf{G}}}(\bar{f}) \neq 0$. In particular, there exists an irreducible character μ of $\overline{\mathbf{M}}^F$ such that $\Delta_{\overline{\mathbf{L}}\subset\overline{\mathbf{P}},\overline{\mathbf{M}}\subset\overline{\mathbf{Q}}}^{\overline{\mathbf{G}}}(\mu) \neq 0$. Let $s \in \overline{\mathbf{M}}^{*F^*}$ be semisimple and such that $\mu \in \mathscr{E}(\overline{\mathbf{M}}^F, [s]_{\overline{\mathbf{M}}^{*F^*}})$. By the argument in [1, Lemmas 5.1.3 and 5.1.4], and by the minimality of $(\dim\overline{\mathbf{G}}, \dim\overline{\mathbf{L}} + \dim\overline{\mathbf{M}}) = (\dim\mathbf{G}, \dim\mathbf{L} + \dim\mathbf{M}) \in \mathbb{N}\times\mathbb{N}$ (where $\mathbb{N}\times\mathbb{N}$ is ordered lexicographically), s is quasi-isolated in $\overline{\mathbf{M}}$ and in $\overline{\mathbf{G}}$. Moreover, by the argument at the end of the proof of [1, Theorem 5.1.1], s is conjugate to sz (in $\mathbf{G}^*$) for every $z \in \mathbf{Z}(\overline{\mathbf{M}}^*)^{F^*} \cap \pi_Z^*(\tilde{\mathbf{G}}^{*F^*})$, where $\pi_Z^* : \tilde{\mathbf{G}}^* \to \overline{\mathbf{G}}^*$ is the simply connected covering of $\overline{\mathbf{G}}^*$. So we have proved (P6). □

Remark 3.9 - In fact, our proof shows that, if we are able to prove the Mackey formula $(\mathscr{M}_{\mathbf{G},\mathbf{L},\mathbf{P},\mathbf{M},\mathbf{Q}})$ whenever $(\mathbf{G}, F)$ is semisimple and simply-connected of type ${}^2\mathrm{E}_6$, $q = 2$ and $\mathbf{M}$ is of type $\mathrm{A}_2 \times \mathrm{A}_2$, then the Mackey formula would hold for any pair $(\mathbf{G}, F)$, where $\mathbf{G}$ is a connected reductive algebraic group and $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism.

Actually it can be shown that the problem reduces to prove that the scalar product $\langle R_{\mathbf{M}}^{\mathbf{G}}\Gamma_\zeta^{\mathbf{M}}, R_{\mathbf{M}}^{\mathbf{G}}\Gamma_\zeta^{\mathbf{M}}\rangle$ has the value predicted by the Mackey formula, where ζ is a faithful character of $H^1(F, \mathbf{Z}(\mathbf{M}))$ and $\Gamma_\zeta^{\mathbf{M}}$ is the corresponding Mellin transform of a Gelfand-Graev character (see [2, Theorem 6.2.1]). We were unfortunately unable to do this. □

A. Appendix: computations with semisimple elements using CHEVIE

In this Appendix, we present briefly some algorithms and some programs using the CHEVIE package for computing with semisimple elements in reductive groups. We also present some applications that were used in the proof of Proposition 2.1 (see Lemmas E6, E7 and E8).

Let $\mathbf{S}$ be a torus defined over $\mathbb{F}$. The map $\mathbb{F}^\times \otimes_{\mathbb{Z}} Y(\mathbf{S}) \to \mathbf{S}$ given by $x \otimes \lambda \mapsto \lambda(x)$ is an isomorphism, where we identify $\mathbf{S}$ to the group of its points over $\mathbb{F}$. Thus, if we choose an isomorphism $\mathbb{F}^\times \simeq (\mathbb{Q}/\mathbb{Z})_{p'}$, we get an isomorphism $(\mathbb{Q}/\mathbb{Z})_{p'} \otimes_{\mathbb{Z}} Y(\mathbf{S}) \to \mathbf{S}$. Thus, if $\dim \mathbf{S} = r$, an element of $\mathbf{S}$ can be represented by an element of $(\mathbb{Q}/\mathbb{Z})^r$ as soon as we choose a basis of $Y(\mathbf{S})$.

If $\mathbf{S}$ is a subtorus of $\mathbf{T}$, then the inclusion $\mathbf{S} \subset \mathbf{T}$ is determined by giving a basis of the sublattice $Y(\mathbf{S})$ inside $Y(\mathbf{T})$. These are the basic ideas used to represent semi-simple elements in CHEVIE.

A.A. **Representing reductive groups.** A reductive group $\mathbf{G}$ over $\mathbb{F}$ is determined up to isomorphism by the *root datum* $(X(\mathbf{T}), \Phi, Y(\mathbf{T}), \Phi^\vee)$ where $\Phi \subset X(\mathbf{T})$ are the roots with respect to the maximal torus $\mathbf{T}$ and $\Phi^\vee \subset Y(\mathbf{T})$ are the corresponding coroots. This determines the Weyl group, a finite reflection group $W \subset \mathrm{GL}(Y(\mathbf{T}))$.

In CHEVIE, to specify $\mathbf{G}$, we give an integral matrix R whose lines represent the simple roots in terms of a basis of $X(\mathbf{T})$, and an integral matrix $R^\vee$ whose lines represent the simple coroots in terms of a basis of $Y(\mathbf{T})$. It is assumed that the bases of $X(\mathbf{T})$ and $Y(\mathbf{T})$ are chosen such that the canonical pairing is given by $\langle x, y\rangle_{\mathbf{T}} = \sum_i x_i y_i$.

For convenience, two particular cases are implemented in CHEVIE where the user just has to specify the Coxeter type of the Weyl group W. If $\mathbf{G}$ is adjoint then R is the identity matrix and $R^\vee$ is the Cartan matrix of the root system given by $\{\alpha^\vee(\beta)\}_{\alpha,\beta}$ where $\alpha^\vee$ (resp. β) runs over the simple coroots (resp. simple roots). If $\mathbf{G}$ is semi-simple simply connected, then $\mathbf{G}^*$ is adjoint thus the situation is reversed: $R^\vee$ is the identity matrix and R the Cartan matrix. In all cases, the function we use constructs a particular integral representation of a Coxeter group, so it is called `CoxeterGroup`.

By default, the adjoint group is returned. To illustrate this, the group PGL_3 is obtained by

```
gap> PGL:=CoxeterGroup("A",2);
CoxeterGroup("A",2)
gap> PGL.simpleRoots;
[ [ 1, 0 ], [ 0, 1 ] ]
gap> PGL.simpleCoroots;
[ [ 2, -1 ], [ -1, 2 ] ]
```

To get the semi-simple simply connected group, the additional parameter `"sc"` has to be given. For instance, SL_3 is obtained by

```
gap> SL:=CoxeterGroup("A",2,"sc");
```

```
CoxeterGroup("A",2,"sc")
gap> SL.simpleRoots;
[ [ 2, -1 ], [ -1, 2 ] ]
gap> SL.simpleCoroots;
[ [ 1, 0 ], [ 0, 1 ] ]
```

To get GL_3 we must use the general form by giving R and $R^\vee$:

```
gap> GL := CoxeterGroup( [ [ -1, 1, 0], [ 0, -1, 1 ] ],
> [ [ -1, 1, 0], [ 0, -1, 1 ] ] );
CoxeterGroup([ [ -1, 1, 0 ], [ 0, -1, 1 ] ],[ [ -1, 1, 0 ], [ 0, -1, 1 ] ])
gap> GL.simpleRoots;
[ [ -1, 1, 0 ], [ 0, -1, 1 ] ]
gap> GL.simpleCoroots;
[ [ -1, 1, 0 ], [ 0, -1, 1 ] ]
```

More features of CHEVIE will be illustrated when describing the computations with semi-simple elements below.

A.B. **Some application.** Recall that, in order to prove Lemmas E6, E7 and E8, we had introduced the following property:

($\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,\mathrm{Z},\mathrm{n}}$) *If s is a semisimple element of $(\mathbf{M}_0/Z)^*$ which is quasi-isolated in $(\mathbf{M}_0/Z)^*$ and in $(\mathbf{G}_0/Z)^*$, there exists an element $z \in \mathbf{Z}((\mathbf{M}_0/Z)^*)^\circ$ of order dividing n such that s and sz are not conjugate in $(\mathbf{G}_0/Z)^*$.*

The aim of this subsection is to show how to use the CHEVIE package to check the following lemma:

Lemma A.1. *Assume that one of the following holds:*

(1) $\mathbf{G}_0$ *is of type* E_6, $\mathbf{M}_0$ *is of type* $\mathrm{A}_2 \times \mathrm{A}_2$, $Z = 1$ *and* $n = 3$.
(2) $\mathbf{G}_0$ *is of type* E_7, $\mathbf{M}_0$ *is of type* E_6, $Z = \mathbf{Z}(\mathbf{G}_0)$ *and* $n = 4$.
(3) $\mathbf{G}_0$ *is of type* E_7, $\mathbf{M}_0$ *is of type* $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$ *as in diagram* $(\mathrm{E}_7[\mathrm{A}_1^3]^\#)$, $Z = 1$ *and* $n \in \{4, 6\}$.
(4) $\mathbf{G}_0$ *is of type* E_8, $\mathbf{M}_0$ *is of type* E_7, $Z = 1$ *and* $n \in \{3, 5\}$.
(5) $\mathbf{G}_0$ *is of type* E_8, $\mathbf{M}_0$ *is of type* E_6, $Z = 1$ *and* $n = 2$.

Then $\mathscr{S}_{\mathbf{G}_0,\mathbf{M}_0,Z,n}$ *holds.*

Proof. Note that in cases (1), (3), (4), (5), $(\mathbf{G}_0/Z)^*$ is the adjoint group of type E_n, while in case (2) it is the semi-simple simply connected group of type E_7.

In what follows, to simplify notations, we will set $\mathbf{G} = (\mathbf{G}_0/Z)^*$ and $\mathbf{M} = (\mathbf{M}_0/Z)^*$. We show the complete computation corresponding to case (1). The other cases are treated by completely similar code, excepted that in case (2), the group should be defined via `G:=CoxeterGroup("E",7,"sc");`.

We want to show that for any s which is quasi-isolated in $\mathbf{M}$ of type $A_2 \times A_2$ and $\mathbf{G}$ of type E_6, there is an element of order 3 of $\mathbf{Z}(\mathbf{M})$ which is not conjugate to s.

We first compute the list of elements of order 3 of $\mathbf{Z}(\mathbf{M})$. The first thing is to specify $\mathbf{M}$.

```
gap> G:=CoxeterGroup("E",6);;PrintDiagram(G);
E6      2
        |
1 - 3 - 4 - 5 - 6
gap> M:=ReflectionSubgroup(G,[1,3,5,6]);
ReflectionSubgroup(CoxeterGroup("E",6), [ 1, 3, 5, 6 ])
```

In GAP, the result of a command which ends with a double semicolon is not printed. The command `PrintDiagram` shows the numbering of the simple roots.

We now compute the torus $Z(\mathbf{M})^\circ = Z(\mathbf{M})$.

```
gap> ZM:=AlgebraicCentre(M).Z0;
[ [ 0, 1, 0, -1, 0, 0 ], [ 0, 0, 0, 1, 0, 0 ] ]
```

The torus $Z(\mathbf{M})^\circ$ is represented by giving a basis of $Y(Z(\mathbf{M})^\circ)$ inside $Y(\mathbf{T})$.

We now ask for the subgroup of elements of order 3 of $Z(\mathbf{M})^\circ$.

```
gap> Z3:=SemisimpleSubgroup(G,ZM,3);
Group( <0,1/3,0,2/3,0,0>, <0,0,0,1/3,0,0> )
```

This group is represented as a subgroup of $\mathbf{T}$; elements of $\mathbf{T}$, which is of dimension 6, are represented as lists of 6 elements of $\mathbb{Q}/\mathbb{Z}$ in angle brackets; elements of $\mathbb{Q}/\mathbb{Z}$ are themselves represented as fractions r such that $0 \leqslant r < 1$. The subgroup of elements of order 3 of $Z(\mathbf{M})^\circ$ is generated by 2 elements which are given above. We may ask for the list of all elements of this group.

```
gap> Z3:=Elements(Z3);
[ <0,0,0,0,0,0>, <0,0,0,1/3,0,0>, <0,0,0,2/3,0,0>, <0,1/3,0,2/3,0,0>,
  <0,1/3,0,0,0,0>, <0,1/3,0,1/3,0,0>, <0,2/3,0,1/3,0,0>, <0,2/3,0,2/3,0,0>,
  <0,2/3,0,0,0,0> ]
```

We now compute the list of elements quasi-isolated in both $\mathbf{G}$ and $\mathbf{M}$.

```
gap> reps:=QuasiIsolatedRepresentatives(G);
[ <0,0,0,0,0,0>, <0,0,0,0,1/2,0>, <0,0,0,1/3,0,0>, <0,1/6,1/6,0,1/6,0>,
  <1/3,0,0,0,0,1/3> ]
```

The list `reps` now contains representatives of $\mathbf{G}$-orbits of quasi-isolated elements. The algorithm to get these was described in [4]. To get all the quasi-isolated elements in $\mathbf{T}$, we need to take the orbits under the Weyl group:

```
gap> qi:=List(reps,s->Orbit(G,s));;
gap> List(qi,Length);
[ 1, 36, 80, 1080, 90 ]
```

We have not displayed the orbits since they are quite large: the first orbit is that of the identity element, which is trivial, but the fourth contains 1080 elements. We now filter each orbit by the condition to be quasi-isolated also in $\mathbf{M}$.

```
gap> qi:=List(qi,x->Filtered(x,y->IsQuasiIsolated(M,y)));;
gap> List(qi,Length);
[ 1, 3, 26, 36, 12 ]
gap> qi[2];
[ <0,0,0,1/2,0,0>, <0,1/2,0,1/2,0,0>, <0,1/2,0,0,0,0> ]
```

There is a way to do the same computation which does not need to compute the large intermediate orbits under the Weyl group of $\mathbf{G}$. The idea is to compute first the orbit of a semi-simple quasi-isolated representative s under representatives of the double cosets $C_{\mathbf{G}}(s)\backslash\mathbf{G}/\mathbf{M}$, which are not too many, then test for being quasi-isolated in $\mathbf{M}$, and finally take the orbits under the Weyl group of $\mathbf{M}$. So starting with `reps` as above, we first compute:

```
ce:=List(reps,s->SemisimpleCentralizer(G,s));;ce[5];
Extended(ReflectionSubgroup(CoxeterGroup("E",6), [ 2, 3, 4, 5 ]),<(2,5,3)>)
gap> ce[5].group;
ReflectionSubgroup(CoxeterGroup("E",6), [ 2, 3, 4, 5 ])
gap> ce[5].permauts;
Group( ( 1,72, 6)( 2, 5, 3)( 7,71,11)( 8,10, 9)(12,70,16)(13,14,15)(17,68,21)
(18,69,20)(22,66,25)(23,67,65)(26,63,28)(27,64,62)(29,59,31)(30,61,58)
(32,57,53)(33,56,54)(34,52,48)(35,47,43)(36,42,37)(38,41,39)(44,46,45)
(49,50,51) )
```

The first command above computes the groups $C_{\mathbf{G}}(s)$, which are possibly disconnected groups. We show for the 5th element of `reps` how such a group is represented: it contains a reflection subgroup of the Weyl group of $\mathbf{G}$, the Weyl group of $C^{\circ}_{\mathbf{G}}(s)$, obtained above as `ce[5].group`, extended by the group of diagram automorphisms induced on it by $C_{\mathbf{G}}(s)$, obtained above as `ce[5].permauts`; these automorphisms are denoted by the permutation of the simple roots of $C^{\circ}_{\mathbf{G}}(s)$ they induce.

To get the whole Weyl group of $C_{\mathbf{G}}(s)$ we need to combine these two pieces. For this we define a `GAP` function:

```
TotalGroup:=g->Subgroup(G,Concatenation(g.group.generators,g.permauts.generators));
```

We then compute representatives of the double cosets $C_{\mathbf{G}}(s)\backslash\mathbf{G}/\mathbf{M}$, we apply them to `reps`, keep the ones still quasi-simple in $\mathbf{M}$:

```
dreps:=List(ce,g->List(DoubleCosets(G,TotalGroup(g),M),Representative));;
```

```
qi:=List([1..Length(reps)],i->List(dreps[i],w->reps[i]^w));;
qi:=List(qi,x->Filtered(x,y->IsQuasiIsolated(M,y)));
[ [ <0,0,0,0,0,0> ], [ <0,1/2,0,0,0,0>, <0,0,0,1/2,0,0>, <0,1/2,0,1/2,0,0> ],
  [ <0,0,0,1/3,0,0>, <0,0,0,2/3,0,0>, <1/3,2/3,1/3,0,2/3,2/3>,
      <1/3,1/3,1/3,0,2/3,2/3> ],
  [ <1/3,1/2,1/3,1/2,2/3,2/3>, <1/3,1/2,1/3,0,2/3,2/3>,
      <2/3,0,2/3,5/6,2/3,2/3> ], [ <1/3,0,1/3,0,1/3,1/3> ] ]
```

We get a list such that the **M**-orbits of the sublists give the same list as before. We will need this previous list of all **G**-conjugates which are **M**-quasi-isolated, so if we did not keep it we recompute this list containing the **M**-orbits of the sublists by

```
qim:=List(qi,l->Union(List(l,s->Orbit(M,s))));;
```

We now ask, for each element s of each our orbits, how many elements z of `Z3` are such that s and sz are not **G**-conjugate. The test for being conjugate is that sz is in the same **G**-orbit. We need to make the test only for our representatives of the **M**-orbits, since if s is **G**-conjugate to sz with $z \in \mathbf{Z}(\mathbf{M})$, then msm^{-1} is **G**-conjugate to $msm^{-1}z = mszm^{-1}$.

```
gap> List([1..Length(qi)],i->List(qi[i],s->Number(Z3,
  z->PositionProperty(qim,o->s*z in o)<>i)));
[ [ 8 ], [ 8, 8, 8 ], [ 7, 7, 3, 3 ], [ 6, 6, 6 ], [ 6 ] ]
```

and we find indeed that there is always more than 0 elements z which work. Note that the function `PositionProperty` returns `false` when no element is found satisfying the given property, thus the number counted is the z such that s and sz are in a different orbit, as well as the cases when sz is not quasi-isolated in **G**. □

A.C. **Rational structures.** We now show the `CHEVIE` code for the following lemma. Here again to simplify notations we note **G** for $(\mathbf{G}_0/Z)^*$ and **M** for $(\mathbf{M}_0)^*$. We are going to show the `CHEVIE` code to prove the following lemma:

Lemma A.2. *If* **G** *is adjoint of type* E_7*, if* **M** *is of type* $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$ *as in diagram* $(\mathrm{E}_7[\mathrm{A}_1^3]^\#)$*, if* $q \in \{3, 5\}$ *and if* $\chi_{\mathbf{Z}(\mathbf{M}),F} = \Phi_1^a \Phi_2^b$ *with* $a, b \geqslant 1$*, then* $\mathbf{Z}(\mathbf{M})^F$ *contains an element of order* 8.

In `CHEVIE`, to specify an $\mathbb{F}_q$-structure on a reductive group, we must in addition give an element $\phi \in \mathrm{GL}(Y(\mathbf{T}))$ such that $F = q\phi$. We may chose ϕ such that it stabilizes the set of simple roots. Such an element ϕ is determined by the coset $W\phi \subset \mathrm{GL}(Y(\mathbf{T}))$, so the structure which represents it in `CHEVIE` is called a *Coxeter coset*.

Further, if $\mathbf{M}'$ is an F-stable **G**-conjugate of the Levi subgroup **M**, the pair $(\mathbf{M}', F)$ is isomorphic to $(\mathbf{M}, wF)$ for some $w \in W$ (determined by $\mathbf{M}'$ up to F-conjugacy). So, given a Coxeter coset $W\phi$, an F-stable conjugate of a Levi subgroup whose Weyl group is a standard parabolic subgroup W_I is represented by a subcoset of the form $W_I w\phi$, where $w\phi$ normalizes W_I.

To check the lemma, we first compute the list of elements of order 8 of $\mathbf{Z}(\mathbf{M})$, using the same commands as shown before.

```
gap> G:=CoxeterGroup("E",7);;PrintDiagram(G);
E7      2
        |
1 - 3 - 4 - 5 - 6 - 7
gap> M:=ReflectionSubgroup(G,[2,5,7]);;
gap> ZM:=AlgebraicCentre(M);;
gap> Z8:=SemisimpleSubgroup(G,ZM.Z0,8);
Group( <1/8,0,0,0,0,0,0,0>, <0,0,1/8,7/8,0,1/8,0>, <0,0,0,1/8,0,7/8,0>,
<0,0,0,0,0,1/8,0> )
gap> Z8:=Elements(Z8);;Length(Z8);
4096
```

We now ask for representatives of the $\mathbf{G}^F$-classes of F-stable $\mathbf{G}$-conjugates of $\mathbf{M}$. The group $\mathbf{G}$ is split, so ϕ is trivial. Thus an F-stable-conjugate of $\mathbf{M}$ is represented by a coset of the form $W_I w$. We first ask for the list of all possible such twistings of $\mathbf{M}$:

```
gap> Mtwists:=Twistings(G,M);
[ A1<2>xA1<5>xA1<7>.(q-1)^4, (A1xA1xA1)<2,5,7>.(q-1)^2*(q^2+q+1),
  A1<2>xA1<5>xA1<7>.(q-1)^2*(q^2+q+1), (A1xA1xA1)<2,5,7>.(q^2+q+1)^2,
  (A1xA1xA1)<2,7,5>.(q-1)*(q+1)*(q^2+q+1),
  (A1xA1xA1)<2,7,5>.(q-1)*(q+1)*(q^2-q+1), (A1xA1xA1)<2,7,5>.(q+1)^2*(q^2-q+1)
    , A1<2>xA1<5>xA1<7>.(q+1)^2*(q^2-q+1),
  (A1xA1)<2,7>xA1<5>.(q-1)*(q+1)*(q^2+q+1),
  (A1xA1)<2,7>xA1<5>.(q-1)*(q+1)*(q^2-q+1), (A1xA1xA1)<2,7,5>.(q^2-q+1)^2,
  (A1xA1xA1)<2,5,7>.(q^4-q^2+1), A1<2>xA1<5>xA1<7>.(q+1)^4,
  A1<2>xA1<5>xA1<7>.(q^2+1)^2, (A1xA1)<2,7>xA1<5>.(q^4+1),
  A1<2>xA1<5>xA1<7>.(q-1)^2*(q+1)^2, (A1xA1)<2,7>xA1<5>.(q+1)^2*(q^2+1),
  (A1xA1)<2,7>xA1<5>.(q-1)^2*(q^2+1), A1<2>xA1<5>xA1<7>.(q-1)*(q+1)*(q^2+1),
  (A1xA1)<2,7>xA1<5>.(q-1)*(q+1)*(q^2+1), (A1xA1)<2,7>xA1<5>.(q-1)^3*(q+1),
  A1<2>xA1<5>xA1<7>.(q-1)*(q+1)^3, A1<2>xA1<5>xA1<7>.(q-1)^3*(q+1),
  (A1xA1)<2,7>xA1<5>.(q-1)*(q+1)^3, (A1xA1)<2,7>xA1<5>.(q-1)^2*(q+1)^2 ]
```

In the above list, brackets around pairs or triples of A_1 denote an orbit of the Frobenius on the components. The element w is not displayed, but the order of $|\mathbf{Z}(\mathbf{M})^{wF}|$ is displayed. We want to keep the sublist where that order is a product of $\Phi_1(q)$ and $\Phi_2(q)$.

```
gap> Mtwists:=Filtered(Mtwists,MF->Set(PhiFactors(MF))=[-1,1]);
[ A1<2>xA1<5>xA1<7>.(q+1)^4, A1<2>xA1<5>xA1<7>.(q-1)^2*(q+1)^2,
  (A1xA1)<2,7>xA1<5>.(q-1)^3*(q+1), A1<2>xA1<5>xA1<7>.(q-1)*(q+1)^3,
  A1<2>xA1<5>xA1<7>.(q-1)^3*(q+1), (A1xA1)<2,7>xA1<5>.(q-1)*(q+1)^3,
  (A1xA1)<2,7>xA1<5>.(q-1)^2*(q+1)^2 ]
```

Here `PhiFactors` gives the eigenvalues of w on the invariants of the Weyl group of $\mathbf{M}$ acting on the symmetric algebra of $X(\mathbf{T}) \otimes \mathbb{C}$. The cases we want is when these eigenvalues are all equal to 1 or -1 (actually this gives use one extra case, where $|\mathbf{Z}(\mathbf{M})^{wF}| = (q+1)^4$ since the eigenvalues on the complement of $\mathbf{Z}(\mathbf{M})$ are always 1; we will just have to disregard the first entry of `Mtwists`).

Now for each of the remaining `Mtwists` we compute the fixed points of wF on `Z8`, and look at the maximal order of an element in there. We first illustrate the necessary commands one by one on an example before showing a line of code which combines them.

```
gap> Z8F:=Filtered(Z8,s->Frobenius(Mtwists[3])(s)^3=s);
[ <0,0,0,0,0,0,0>, <0,0,0,1/4,0,1/4,0>, <0,0,0,1/2,0,1/2,0>,
  <0,0,0,3/4,0,3/4,0>, <0,0,1/2,0,0,1/2,0>, <0,0,1/2,1/4,0,3/4,0>,
  <0,0,1/2,1/2,0,0,0>, <0,0,1/2,3/4,0,1/4,0>, <1/4,0,1/4,1/8,0,7/8,0>,
  <1/4,0,1/4,3/8,0,1/8,0>, <1/4,0,1/4,5/8,0,3/8,0>, <1/4,0,1/4,7/8,0,5/8,0>,
  <1/4,0,3/4,1/8,0,3/8,0>, <1/4,0,3/4,3/8,0,5/8,0>, <1/4,0,3/4,5/8,0,7/8,0>,
  <1/4,0,3/4,7/8,0,1/8,0>, <1/2,0,0,0,0,0,0>, <1/2,0,0,1/4,0,1/4,0>,
  <1/2,0,0,1/2,0,1/2,0>, <1/2,0,0,3/4,0,3/4,0>, <1/2,0,1/2,0,0,1/2,0>,
  <1/2,0,1/2,1/4,0,3/4,0>, <1/2,0,1/2,1/2,0,0,0>, <1/2,0,1/2,3/4,0,1/4,0>,
  <3/4,0,1/4,1/8,0,7/8,0>, <3/4,0,1/4,3/8,0,1/8,0>, <3/4,0,1/4,5/8,0,3/8,0>,
  <3/4,0,1/4,7/8,0,5/8,0>, <3/4,0,3/4,1/8,0,3/8,0>, <3/4,0,3/4,3/8,0,5/8,0>,
  <3/4,0,3/4,5/8,0,7/8,0>, <3/4,0,3/4,7/8,0,1/8,0> ]
```

The expression `Frobenius(Mtwists[3])` returns a function which applies the w for the 3rd twisting of $\mathbf{M}$, described as `(A1xA1)<2,7>xA1<5>.(q-1)^3*(q+1)`, to its argument. To compute wF we still have to raise to the third power since $q = 3$. We can see from the denominators that some elements in the resulting list of wF-stable elements of `Z8` are of order 8. We can make this easier to see by writing a small function:

```
gap> OrderSemisimple:=s->Lcm(List(s.v,Denominator));
gap> List(Z8F,OrderSemisimple);
[ 1, 4, 2, 4, 2, 4, 2, 4, 8, 8, 8, 8, 8, 8, 8, 8, 2, 4, 2, 4, 2, 4, 2, 4, 8,
  8, 8, 8, 8, 8, 8, 8 ]
gap> Set(last);
[ 1, 2, 4, 8 ]
```

We now do the computation simultaneously for all cosets:

```
gap> List(Mtwists,MF->Set(List(Filtered(Z8,s->Frobenius(MF)(s)^3=s),
> OrderSemisimple)));
[ [ 1, 2, 4 ], [ 1, 2, 4, 8 ], [ 1, 2, 4, 8 ], [ 1, 2, 4, 8 ],
  [ 1, 2, 4, 8 ], [ 1, 2, 4, 8 ], [ 1, 2, 4, 8 ] ]
```

and we see that indeed, apart from the first twist which should be disregarded, for all twists the fixed points of `Z8` still contain elements of order 8.

References

[1] C. Bonnafé, Formule de Mackey pour q grand, *J. Algebra* **201** (1998), 207-232.

[2] C. Bonnafé, Mackey formula in type A, *Proc. London Math. Soc.* **80** (2000), 545-574; Corrigenda: "Mackey formula in type A", *Proc. of the London Math. Soc.* **86** (2003), 435-442.

[3] C. Bonnafé, Éléments unipotents réguliers des sous-groupes de Levi, *Canad. J. Math.* **56** (2004), 246-276.

[4] C. Bonnafé, Quasi-isolated elements in reductive groups, *Comm. in Algebra* **33** (2005), 2315-2337.

[5] C. Bonnafé & R. Rouquier, Coxeter orbits and modular representations, *Nagoya J. Math.* **183** (2006), 1-34.

[6] A. Borel & J. Tits, Groupes réductifs, *Publ. Math. de l'IHES* **27** (1965), 55-160.

[7] M. Broué & G. Malle, Théorèmes de Sylow génériques pour les groupes réductifs sur les corps finis, *Math. Ann.* **292** (1992), 241-262.

[8] P. Deligne & G. Lusztig, Representations of finite reductive groups, *Ann. of Math.* **103** (1976), 103-161.

[9] P. Deligne & G. Lusztig, Duality for representations of a reductive group over a finite field, *J. of Alg.* **81** (1983), 540-545.

[10] F. Digne, G. Lehrer & J. Michel, The characters of the group of rational points of a reductive group with non-connected centre, *J. für die Reine und Angew. Math.* **425** (1992), 155-192.

[11] F. Digne & J. Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts **21**, Cambridge University Press, Cambridge (1991), 159 pages.

[12] G. Lusztig, Intersection cohomology complexes on a reductive group, *Invent. Math.* **75** (1984), 205-272.

[13] G. Lusztig & N. Spaltenstein, Induced unipotent classes, *J. of the London. Math. Soc.* **19** (1979), 41-52.

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