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Philippe Laurencot, Christian Stinner

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**Abstract**

The nonnegative viscosity solutions to the infinite heat equation with homogeneous Dirichlet boundary conditions are shown to converge as $t \to \infty$ to a uniquely determined limit after a suitable time rescaling. The proof relies on the half-relaxed limits technique as well as interior positivity estimates and boundary estimates. The expansion of the support is also studied.

**Key words:** infinite heat equation, infinity-Laplacian, friendly giant, viscosity solution, half-relaxed limits

**MSC 2010:** 35B40, 35K65, 35K55, 35D40

## 1 Introduction

Since the pioneering work by Aronsson [4], the infinity-Laplacian $\Delta_\infty$ defined by

$$\Delta_\infty u := \langle D^2 u \nabla u, \nabla u \rangle = \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

has been the subject of several studies, in particular due to its relationship to the theory of absolutely minimizing Lipschitz extensions [4, 5, 10]. More recently, a parabolic equation involving the infinity-Laplacian (the infinite heat equation)

$$\partial_t u = \Delta_\infty u, \qquad (t,x) \in (0,\infty) \times \Omega, \tag{1.1}$$

has been considered in [1, 2, 13]. When $\Omega \subset \mathbb{R}^N$ is a bounded domain and (1.1) is supplemented with nonhomogeneous Dirichlet boundary conditions, the large time behaviour of solutions to (1.1) is investigated in [1] and convergence as $t \to \infty$ to the unique steady state is shown. Furthermore, for homogeneous Dirichlet boundary conditions

$$u = 0, \qquad (t,x) \in (0,\infty) \times \partial\Omega, \tag{1.2}$$

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and nonnegative initial condition

$$u(0,x) = u_0(x), \qquad x \in \bar{\Omega}, \tag{1.3}$$

satisfying

$$u_0 \in C_0(\bar{\Omega}) := \{f \in C(\bar{\Omega}) \,:\, f = 0 \text{ on } \partial\Omega\},\ u_0 \geq 0,\ u_0 \not\equiv 0, \tag{1.4}$$

a precise temporal decay rate is given for the $L^\infty$-norm of $u$, namely

$$C_1^{-1}(t+1)^{-1/2} \leq \|u(t,\cdot)\|_{L^\infty(\Omega)} \leq C_1(t+1)^{-1/2} \quad \text{for all } t > 0 \tag{1.5}$$

with some $C_1 \geq 1$ depending on $u_0$ and $\Omega$, the unique steady state of (1.1)-(1.2) being zero in that case.

The purpose of this note is to improve (1.5) by identifying the limit of $t^{1/2}u(t,\cdot)$ as $t \to \infty$ (see Theorem 1.2 below). We also provide additional information on the propagation of the positivity set of $u$ as time goes by.

Before stating our main result we first recall that the infinity-Laplacian is a quasilinear and degenerate elliptic operator which is not in divergence form and a suitable framework to study the well-posedness of the infinite heat equation is the theory of viscosity solutions (see e.g. [11]). Within this framework the well-posedness of (1.1)-(1.3) has been established in [2] when $\Omega$ fulfills the *uniform exterior sphere condition*:

$$\begin{aligned}&\text{For all } x_0 \in \partial\Omega \text{ there exists } y_0 \in \mathbb{R}^N \text{ such that } |x_0 - y_0| = R \text{ and}\\ &\{x \in \mathbb{R}^N \,:\, |x - y_0| < R\} \cap \Omega = \emptyset \text{ for some positive constant } R \text{ independent of } x_0.\end{aligned} \tag{1.6}$$

Introducing

$$F(s,p,X) := s - \langle Xp, p\rangle \quad \text{for } s \in \mathbb{R}, p \in \mathbb{R}^N, X \in \mathcal{S}(N), \tag{1.7}$$

where $\mathcal{S}(N)$ denotes the set of all symmetric $N \times N$ matrices, the definition of viscosity solutions to (1.1)-(1.3) reads [1, 2]:

**Definition 1** *Let $Q := (0,\infty) \times \Omega \subset \mathbb{R}^{N+1}$ and let $USC(\bar{Q})$ and $LSC(\bar{Q})$ denote the set of upper semicontinuous and lower semicontinuous functions from $\bar{Q}$ into $\mathbb{R}$, respectively. A function $u \in USC(\bar{Q})$ is a viscosity subsolution to* (1.1)-(1.3) *in $Q$ if*

*(a) $F(s,p,X) \leq 0$ is satisfied for all $(s,p,X) \in \mathcal{P}^{2,+}u(t_0,x_0)$ and all $(t_0,x_0) \in Q$, where*

$$\begin{aligned}\mathcal{P}^{2,+}u(t_0,x_0) \;:=\; \Big\{&(s,p,X) \in \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N) \,:\, u(t,x) \leq u(t_0,x_0) + s(t-t_0)\\ &+\langle p, x-x_0\rangle + \frac{1}{2}\langle X(x-x_0), x-x_0\rangle + o(|t-t_0| + |x-x_0|^2)\\ &\text{as } (t,x) \to (t_0,x_0)\Big\},\end{aligned}$$

*(b) $u \leq 0$ on $(0,\infty) \times \partial\Omega$,*

*(c) $u(0,x) \leq u_0(x)$ for $x \in \bar{\Omega}$.*

*Similarly, $u \in LSC(\bar{Q})$ is a viscosity supersolution to* (1.1)-(1.3) *in $Q$ if $F(s,p,X) \geq 0$ for all $(s,p,X) \in \mathcal{P}^{2,-}u(t_0,x_0) := -\mathcal{P}^{2,+}(-u)(t_0,x_0)$ and $(t_0,x_0) \in Q$, $u \geq 0$ on $(0,\infty) \times \partial\Omega$ and $u(0,x) \geq u_0(x)$ for $x \in \bar{\Omega}$.*

*Finally, $u \in C(\bar{Q})$ is a viscosity solution to* (1.1)-(1.3) *if it is a viscosity subsolution and a viscosity supersolution to* (1.1)-(1.3).

With this definition, the well-posedness of (1.1)-(1.3) is shown in [2, Theorems 2.3 and 2.5] and the asymptotic behaviour of nonnegative solutions is obtained in [1, Theorem 5]. We gather these results in the next theorem.

**Theorem 1.1** *([1, 2]) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that* (1.6) *is satisfied and assume* (1.4)*. Then there is a unique nonnegative viscosity solution $u$ to* (1.1)-(1.3)*. Moreover, $u(t,\cdot)$ converges to zero as $t \to \infty$ in the sense that there exists a constant $C_1 \geq 1$ such that*

$$C_1^{-1}(t+1)^{-1/2} \leq \|u(t,\cdot)\|_{L^\infty(\Omega)} \leq C_1(t+1)^{-1/2} \quad \text{for all } t > 0. \tag{1.8}$$

Our improvement of (1.8) then reads:

**Theorem 1.2** *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain fulfilling* (1.6) *and assume* (1.4)*. If $u$ denotes the viscosity solution to* (1.1)-(1.3)*, then*

$$\lim_{t\to\infty} \|t^{1/2}u(t,\cdot) - f_\infty\|_{L^\infty(\Omega)} = 0, \tag{1.9}$$

*where $f_\infty$ is the unique positive viscosity solution to*

$$-\Delta_\infty f_\infty - \frac{f_\infty}{2} = 0 \text{ in } \Omega, \quad f_\infty > 0 \text{ in } \Omega, \quad f_\infty = 0 \text{ on } \partial\Omega. \tag{1.10}$$

Theorem 1.2 not only gives the convergence of $t^{1/2}u(t,\cdot)$ as $t \to \infty$, but also provides the existence and uniqueness of the positive solution $f_\infty$ to (1.10) in $C_0(\bar{\Omega})$. An interesting consequence of (1.10) is that the function $(t,x) \mapsto t^{-1/2}f_\infty(x)$ is a separate variables solution to (1.1)-(1.2) with an initial data being identically infinite in $\Omega$. Similar solutions are already known to exist for other parabolic equations such as the porous medium equation $\partial_t u = \Delta u^m$, $m > 1$, or the $p$-Laplacian equation $\partial_t u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 2$, (see [3, 12, 14, 20, 21] for instance). They play an important role in the description of the large time dynamics [3, 16, 21] and also provide universal bounds (and are thus called *friendly giants*). The function $(t,x) \mapsto t^{-1/2}f_\infty(x)$ is a friendly giant for the infinite heat equation (1.1)-(1.3) and we have the following universal bound.

**Corollary 1.3** *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain fulfilling* (1.6) *and assume* (1.4)*. If $u$ denotes the viscosity solution to* (1.1)-(1.3)*, then*

$$u(t,x) \leq t^{-1/2}f_\infty(x) \qquad \text{for } (t,x) \in (0,\infty) \times \bar{\Omega}, \tag{1.11}$$

*the function $f_\infty$ being defined in Theorem 1.2.*

The proof of Theorem 1.2 and Corollary 1.3 involves several steps: According to (1.8) the evolution of $u(t,\cdot)$ takes place on a time scale of order $t^{-1/2}$ and we first introduce a rescaled version $v$ of $u$ defined by $u(t,x) = t^{-1/2}v(\ln t, x)$. The outcome of Theorem 1.2 is then the convergence of $v(s,\cdot)$ to the time-independent function $f_\infty$ as $s \to \infty$. To establish such a convergence, we use the half-relaxed limits technique introduced in [8] which is well-suited here as we have rather scarce information on $v(s,\cdot)$ as $s \to \infty$. This requires however a strong comparison principle for the limit problem (1.10) which will be established in Section 2, under an additional positivity assumption,

and furthermore implies the uniqueness of $f_\infty$. That the half-relaxed limits indeed enjoy this positivity property has to be proved as a preliminary step and follows from the observation that $v(s,\cdot)$ is non-decreasing with time and eventually becomes positive in $\Omega$ (see Section 3.1). At this point, boundary estimates are also needed to ensure that the half-relaxed limits vanish on $\partial\Omega$ and are shown by constructing suitable barrier functions. Thanks to these results, we deduce that the half-relaxed limits coincide, which implies that $v(s,\cdot)$ converges as $s \to \infty$ and the existence of a positive solution $f_\infty$ to (1.10) as well (see Section 3.2). We emphasize here that the existence of a positive solution to (1.10) is a consequence of the dynamical properties of $v$ and was seemingly not known previously. Finally, Corollary 1.3 is a consequence of Theorem 1.2 and the time monotonicity of $v$ (see Section 3.2).
Additionally, in Section 4 we investigate further positivity properties of the solution $u$ to (1.1)-(1.3). We show that $u(t,\cdot)$ becomes positive in $\Omega$ after a finite time if $\Omega$ satisfies an additional uniform interior sphere condition. Aside from this, $u$ may have a positive waiting time if the initial data are flat on the boundary of their support, namely the support of $u(t,\cdot)$ will be equal to that of $u_0$ for small times.

For further use, we introduce the following notation: Given $x \in \bar{\Omega}$, let $d(x,\partial\Omega) := dist(x,\partial\Omega)$ denote the distance to the boundary. Moreover, for $x \in \mathbb{R}^N$ and $r > 0$ we define $B(x,r) := \{y \in \mathbb{R}^N \,:\, |y-x| < r\}$ to be the ball of radius $r$ centered at $x$.

# 2 Uniqueness of the friendly giant

In this section we show that the friendly giant is unique. This will be a consequence of the following more general comparison lemma.

**Lemma 2.1** *Let $w \in USC(\bar{\Omega})$ and $W \in LSC(\bar{\Omega})$ be respectively a bounded viscosity subsolution and a bounded viscosity supersolution to*

$$-\Delta_\infty \zeta - \frac{\zeta}{2} = 0 \quad \textit{in } \Omega \tag{2.1}$$

*such that*

$$w(x) = W(x) = 0 \quad \textit{for } x \in \partial\Omega, \tag{2.2}$$

$$W(x) > 0 \qquad \textit{for } x \in \Omega. \tag{2.3}$$

*Then*

$$w \le W \quad \textit{in } \Omega. \tag{2.4}$$

Proof. We fix $N_0 \in \mathbb{N}$ large enough such that $\Omega_n := \{x \in \Omega \,:\, d(x,\partial\Omega) > 1/n\}$ is a nonempty open subset of $\Omega$ for all integer $n \ge N_0$. Let $n \ge N_0$. Since $\bar{\Omega}_n$ is compact and $W \in LSC(\bar{\Omega})$, $W$ has a minimum in $\bar{\Omega}_n$ and the positivity of $W$ in $\bar{\Omega}_n$ implies that

$$\mu_n := \min_{\bar{\Omega}_n} W > 0. \tag{2.5}$$

Similarly, the compactness of $\bar{\Omega} \setminus \Omega_n$ and the upper semicontinuity and boundedness of $w$ ensure that $w$ has a point of maximum $x_n$ in $\bar{\Omega} \setminus \Omega_n$ and we set

$$\eta_n := \max_{\bar{\Omega}\setminus\Omega_n} w = w(x_n) \ge 0, \tag{2.6}$$

the nonnegativity of $\eta_n$ being a consequence of the fact that $w$ vanishes of $w$ on $\partial\Omega$. We next claim that

$$\lim_{n\to\infty} \eta_n = 0. \tag{2.7}$$

Indeed, owing to the compactness of $\bar{\Omega}$ and the definition of $\Omega_n$ there are $y \in \partial\Omega$ and a subsequence of $(x_n)_{n\in\mathbb{N}}$ (not relabeled) such that $x_n \to y$ as $n \to \infty$. Since $w(y) = 0$, we deduce from the upper semicontinuity of $w$ that

$$\limsup_{x\to y} w(x) = \lim_{\varepsilon\searrow 0} \sup\{w(x) \,:\, x \in B(y,\varepsilon) \cap \bar{\Omega}\} \le 0.$$

Given $\varepsilon > 0$, there is $n_\varepsilon$ such that $x_n \in B(y,\varepsilon) \cap \bar{\Omega}$ for all $n \ge n_\varepsilon$. Hence,

$$\limsup_{n\to\infty} \eta_n \le \sup\{w(x) \,:\, x \in B(y,\varepsilon) \cap \bar{\Omega}\}$$

and letting $\varepsilon \searrow 0$ and using (2.6) allow us to conclude that

$$0 \le \limsup_{n\to\infty} \eta_n \le 0.$$

This shows that a subsequence of $(\eta_n)_{n\ge N_0}$ converges to zero and the claim (2.7) follows by noticing that $(\eta_n)_{n\ge N_0}$ is a nonincreasing sequence.
Next, fix $s \in (0,\infty)$. For $\delta > 0$ and $n \ge N_0$, we define

$$\begin{aligned} z_n(t,x) &:= (t+s)^{-1/2} w(x) - s^{-1/2}\eta_n, \qquad && (t,x) \in [0,\infty) \times \bar{\Omega}, \\ Z_\delta(t,x) &:= (t+\delta)^{-1/2} W(x), && (t,x) \in [0,\infty) \times \bar{\Omega}. \end{aligned}$$

Then $z_n$ and $Z_\delta$ are respectively a bounded usc viscosity subsolution and a bounded lsc viscosity supersolution to (1.1) with

$$Z_\delta(t,x) = 0 \ge -s^{-1/2}\eta_n = z_n(t,x), \quad (t,x) \in (0,\infty) \times \partial\Omega.$$

In addition, if

$$0 < \delta < \left( \frac{\mu_n}{1 + \|w\|_{L^\infty(\Omega)}} \right)^2 s \tag{2.8}$$

we have

$$Z_\delta(0,x) = \delta^{-1/2} W(x) \ge \delta^{-1/2}\mu_n \ge s^{-1/2}\|w\|_{L^\infty(\Omega)} \ge z_n(0,x) \quad \text{for } x \in \Omega_n$$

and

$$Z_\delta(0,x) \ge 0 \ge s^{-1/2}(w(x) - \eta_n) = z_n(0,x) \quad \text{for } x \in \bar{\Omega} \setminus \Omega_n.$$

We are then in a position to apply the comparison principle [11, Theorem 8.2] to deduce that

$$z_n(t,x) \le Z_\delta(t,x), \quad (t,x) \in [0,\infty) \times \bar{\Omega}, \tag{2.9}$$

for any $\delta > 0$ and $n \ge N_0$ satisfying (2.8). According to (2.8), the parameter $\delta$ can be taken arbitrarily small and we deduce from (2.9) that

$$(t+s)^{-1/2} w(x) - s^{-1/2}\eta_n \le t^{-1/2} W(x), \quad (t,x) \in (0,\infty) \times \bar{\Omega},$$

for $n \geq N_0$. We next pass to the limit as $n \to \infty$ with the help of (2.7) to conclude that

$$(t+s)^{-1/2} w(x) \leq t^{-1/2} W(x), \quad (t,x) \in (0,\infty) \times \bar{\Omega}.$$

Finally, as $s > 0$ is arbitrary, we may let $s \searrow 0$ and take $t = 1$ in the above inequality to complete the proof. ////

Now the uniqueness of the friendly giant is a straightforward consequence of Lemma 2.1.

**Corollary 2.2** *There is at most one positive viscosity solution to* (1.10) *in* $C_0(\bar{\Omega})$.

# 3 Large time behaviour

In this section, we assume that $\Omega$ is a bounded domain fulfilling (1.6) and that $u_0$ satisfies (1.4). Let $u$ be the corresponding viscosity solution to (1.1)-(1.3). In order to investigate the asymptotic behaviour of $u$ as stated in Theorem 1.2 we introduce the scaling variable $s = \ln t$, $t > 0$, and the rescaled unknown function $v$ defined by

$$u(t,x) = t^{-1/2} v(\ln t, x), \quad (t,x) \in (0,\infty) \times \bar{\Omega}. \tag{3.1}$$

It is easy to check that $v$ is the viscosity solution to

$$\begin{aligned} \partial_s v &= \Delta_\infty v - \frac{v}{2}, && (s,x) \in (0,\infty) \times \Omega, && (3.2)\\ v &= 0, && (s,x) \in (0,\infty) \times \partial\Omega, && (3.3)\\ v(0,x) &= v_0(x) := u(1,x), && x \in \bar{\Omega}, && (3.4) \end{aligned}$$

while it readily follows from (1.8) and (3.1) that

$$0 \leq v(s,x) \leq C_1, \qquad (s,x) \in [0,\infty) \times \bar{\Omega}. \tag{3.5}$$

## 3.1 Positivity and time monotonicity

A further property of $v$ is its time monotonicity which follows from the homogeneity of the operator $\Delta_\infty$ by a result from Bénilan & Crandall [9].

**Lemma 3.1** *For $x \in \bar{\Omega}$, $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$ such that $s_1 \leq s_2$, we have*

$$v(s_1,x) \leq v(s_2,x).$$

Proof. Theorem 1.1 provides the well-posedness of (1.1) in $C_0(\bar{\Omega})$ which is an ordered vector space. As the comparison principle is valid for (1.1)-(1.3) by [2, Theorem 2.3] and the infinity-Laplacian is homogeneous of degree 3, [9, Theorem 2] implies

$$u(t+h,x) - u(t,x) \geq \left( \left( \frac{t+h}{t} \right)^{-1/2} - 1 \right) u(t,x) \qquad \text{for } (t,x) \in (0,\infty) \times \bar{\Omega},\ h > 0. \tag{3.6}$$

Hence, for any $(s,x) \in \mathbb{R} \times \bar{\Omega}$ and $h > 0$, we obtain

$$\begin{aligned} v(s+h,x) - v(s,x) &= e^{(s+h)/2} u(e^{s+h},x) - e^{s/2} u(e^s,x) \\ &\geq e^{(s+h)/2} \Big( \frac{e^{s+h}}{e^s} \Big)^{-1/2} u(e^s,x) - e^{s/2} u(e^s,x) = 0, \end{aligned}$$

which is the expected result. ////

The monotonicity of $v$ now enables us to prove that $v$ eventually becomes positive inside $\Omega$.

**Lemma 3.2** *For any compact subset $K \subset \Omega$ there are $s_K > 0$ and $\mu_K > 0$ such that*

$$v(s,x) \geq \mu_K > 0 \qquad in\ [s_K, \infty) \times K. \tag{3.7}$$

Proof. Three steps are needed to achieve the claimed result: we first prove that if $v(s,\cdot)$ is positive at one point of $\Omega$, then it becomes positive on a "large" ball centered around this point after a finite time. The second step is to prove that $v(s,\cdot)$ becomes eventually positive in $\Omega$ as $s \to \infty$, from which we deduce (3.7) in a third step.

*Step 1:* Consider first $(t_0, x_0) \in (0,\infty) \times \Omega$ such that there are $\varepsilon > 0$ and $\delta > 0$ with $B(x_0,\varepsilon) \subset \Omega$ and

$$u(t_0,x) \geq \delta > 0 \qquad \text{for } x \in B(x_0,\varepsilon). \tag{3.8}$$

Then, choosing $\alpha := \min\{(4\delta)^{1/3}, \varepsilon^{2/3}\}$, $T := (d(x_0,\partial\Omega)^6/\alpha^9) - 1 \geq 0$, and defining

$$\mathcal{B}(t,x) := \frac{\alpha^3}{4} (t - t_0 + 1)^{-1/6} \left( 1 - \alpha^{-2} |x - x_0|^{4/3} (t - t_0 + 1)^{-2/9} \right)_+^{3/2}, \quad (t,x) \in [t_0,\infty) \times \mathbb{R}^N,$$

we deduce from [1, Proposition 1 and Corollary 1] that $\mathcal{B}$ is a viscosity solution to (1.1) in $(t_0, t_0 + T) \times \Omega$. In addition, on the one hand, we have by (3.8)

$$\mathcal{B}(t_0,x) \leq \frac{\alpha^3}{4} \leq \delta \leq u(t_0,x) \qquad \text{for } x \in B(x_0,\varepsilon)$$

and

$$\mathcal{B}(t_0,x) = 0 \leq u(t_0,x) \qquad \text{for } x \in \bar{\Omega} \setminus B(x_0,\varepsilon).$$

On the other hand, we have $u(t,x) = \mathcal{B}(t,x) = 0$ for $(t,x) \in [t_0, t_0+T] \times \partial\Omega$ thanks to the choice of $T$, $\alpha$ and the properties of $\mathcal{B}$. The comparison principle [11, Theorem 8.2] then implies $u \geq \mathcal{B}$ in $[t_0, t_0+T] \times \bar{\Omega}$. In particular, we have

$$u(t_0 + T, x) > 0 \quad \text{for } x \in B(x_0, d(x_0,\partial\Omega)), \tag{3.9}$$

where $T$ only depends on $\varepsilon$ and $\delta$, but is independent of $x_0$ and $t_0$.

*Step 2:* We next define the positivity set $\mathcal{P}(s)$ of $v(s,\cdot)$ for $s \geq 0$ by

$$\mathcal{P}(s) := \{x \in \Omega \,:\, v(s,x) > 0\}.$$

Owing to the time monotonicity of $v$ (Lemma 3.1), $(\mathcal{P}(s))_{s \geq 0}$ is a non-decreasing family of open subsets of $\Omega$ and

$$\mathcal{P}_\infty := \bigcup_{s \geq 0} \mathcal{P}(s) \text{ is an open subset of } \Omega.$$

Assume for contradiction that $\partial\mathcal{P}_\infty \cap \Omega \neq \emptyset$. Then there is $x_0 \in \partial\mathcal{P}_\infty \cap \Omega$. Since $d(x_0, \partial\Omega) > 0$ there is $y_0 \in \mathcal{P}_\infty$ such that $|y_0 - x_0| \leq d(x_0, \partial\Omega)/2 < d(y_0, \partial\Omega)$. Next, since $y_0 \in \mathcal{P}_\infty$, there is $s_0 > 0$ such that $v(s_0, y_0) > 0$, that is $u(e^{s_0}, y_0) > 0$. The previous step then guarantees the existence of $T \geq 0$, such that $u(e^{s_0} + T, x) > 0$ for $x \in B(y_0, d(y_0, \partial\Omega))$. As $x_0 \in B(y_0, d(y_0, \partial\Omega))$, we deduce from this that

$$v(\ln(e^{s_0} + T), x_0) = (e^{s_0} + T)^{1/2} u(e^{s_0} + T, x_0) > 0,$$

which contradicts the fact that $x_0 \in \partial\mathcal{P}_\infty$. Therefore, $\partial\mathcal{P}_\infty \cap \Omega = \emptyset$ and $\Omega$ is the union of the two disjoint open sets $\mathcal{P}_\infty$ and $\Omega \setminus \overline{\mathcal{P}_\infty}$. Since $\mathcal{P}_\infty \neq \emptyset$ by (1.8), the connectedness of $\Omega$ implies

$$\Omega = \mathcal{P}_\infty. \tag{3.10}$$

*Step 3:* Let $K$ be a compact subset of $\Omega$ and assume for contradiction that $K \not\subset \mathcal{P}(n)$ for each $n \geq 1$. Then there is a sequence $(x_n)_{n\geq 1}$ in $K$ such that $v(n, x_n) = 0$ for $n \geq 1$ and we may assume without loss of generality that $x_n$ converges towards $x_\infty \in K$ as $n \to \infty$, thanks to the compactness of $K$. Since $x_\infty \in \Omega$, it follows from (3.10) that there is $s_\infty > 0$ such that $v(s_\infty, x_\infty) > 0$. Owing to the continuity of $v(s_\infty, \cdot)$ there are $\varepsilon > 0$ and $\delta > 0$ such that $v(s_\infty, x) \geq \delta$ for $x \in B(x_\infty, \varepsilon) \subset \Omega$. But then for $n$ large enough we have $n \geq s_\infty$ and $x_n \in B(x_\infty, \varepsilon)$ and it follows from Lemma 3.1 and the previous bound that

$$0 = v(n, x_n) \geq v(s_\infty, x_n) \geq \delta$$

and a contradiction. Consequently, there is $n_K$ such that $K \subset \mathcal{P}(n_K)$ and

$$\mu_K := \min_{x \in K} v(n_K, x) > 0.$$

Due to the time monotonicity of $v$, this implies (3.7). ////

## 3.2 Convergence

Having studied the positivity properties of $v$, we next turn to its behaviour near the boundary of $\Omega$ and first show the following lemma which is a modification of [19, Lemma 10.1].

**Lemma 3.3** *Consider $x_0 \in \partial\Omega$, $\alpha \in (0, 1/2)$, $\delta > 0$, $B > 0$, and define*

$$\psi_{\delta,B}(r) := \delta + B \left( r - \frac{r^2}{2} \right), \quad r \in \mathbb{R}.$$

*Let $y_0 \in \mathbb{R}^N$ be such that $|x_0 - y_0| = R$ and $\Omega \cap B(y_0, R) = \emptyset$ (such a point $y_0$ exists according to the uniform exterior sphere condition* (1.6)*). Introducing*

$$U_{\alpha,x_0} := \{x \in \Omega \ : \ R < |x - y_0| < R + \alpha\}$$

*and*

$$w(s, x) := \psi_{\delta,B}(|x - y_0| - R), \quad (s, x) \in [0, \infty) \times \overline{U_{\alpha,x_0}},$$

*then $w$ is a supersolution to* (3.2) *in $(0, \infty) \times U_{\alpha,x_0}$ if $B \geq 2(1 + \delta)$.*

PROOF. To simplify notations, we set $\psi := \psi_{\delta,B}$ and $U := U_{\alpha,x_0}$. Since $\psi \in C^\infty(\mathbb{R})$ and $y_0 \notin U$, the function $w$ is $C^\infty$-smooth in $(0,\infty)\times U$ and, if $(s,x) \in (0,\infty)\times U$, we have

$$\partial_s w(s,x) - \Delta_\infty w(s,x) - \frac{w(s,x)}{2} = -\left(\psi'^2\ \psi'' + \frac{\psi}{2}\right)(|x-y_0|-R)\,. \tag{3.11}$$

Since $\alpha \in (0,1/2)$ and $B \geq 2$, we have for $r \in [0,\alpha]$

$$-\left(\psi'^2\ \psi'' + \frac{\psi}{2}\right)(r) = B^3\ (1-r)^2 - \frac{B}{2}\ \left(r - \frac{r^2}{2}\right) - \frac{\delta}{2} \geq \frac{B^3}{8} - \frac{B}{4} - \frac{\delta}{2} \geq \frac{B-2\delta}{4}\,.$$

Consequently, as $|x-y_0|-R \in [0,\alpha]$ for $(s,x) \in (0,\infty)\times U$, we deduce from (3.11) and the above inequality that

$$\partial_s w(s,x) - \Delta_\infty w(s,x) - \frac{w(s,x)}{2} \geq \frac{B-2\delta}{4} \geq 0\,,$$

the last inequality following from the choice of $B$. ////

As a consequence of Lemma 3.3, we have the following useful bound for $v$ on $\partial\Omega$.

**Lemma 3.4** *Consider $\alpha \in (0,1/2)$ and define*

$$\omega(\alpha) := \sup\left\{v(0,x)\ :\ x \in \Omega \quad \textit{and} \quad d(x,\partial\Omega) < \alpha\right\}. \tag{3.12}$$

*Then there is $\alpha_0 \in (0,1/2)$ such that, for any $\alpha \in (0,\alpha_0)$ and $x_0 \in \partial\Omega$, we have*

$$0 \leq v(s,x) \leq \omega(\alpha) + \frac{2C_1}{\alpha}\ |x-x_0|\,, \qquad (s,x) \in [0,\infty)\times(\bar\Omega \cap B(x_0,\alpha))\,, \tag{3.13}$$

*the constant $C_1$ being defined in* (3.5).

PROOF. Consider $x_0 \in \partial\Omega$ and let $y_0 \in \mathbb{R}^N$ be such that $|x_0-y_0| = R$ and $\Omega \cap B(y_0,R) = \emptyset$, the existence of such a point $y_0$ being guaranteed by the uniform exterior sphere condition (1.6). With the notations of Lemma 3.3, we define

$$w(s,x) := \psi_{\omega(\alpha),2C_1/\alpha}(|x-y_0|-R)\,, \quad (s,x) \in [0,\infty)\times\overline{U_{\alpha,x_0}}\,,$$

the constant $C_1$ being defined in (3.5) and observe that

$$B(x_0,\alpha)\cap\Omega \subset U_{\alpha,x_0} \subset \{x \in \Omega\ :\ d(x,\partial\Omega) < \alpha\}\,. \tag{3.14}$$

On the one hand, it follows from (3.12) and (3.14) that

$$w(0,x) \geq \omega(\alpha) \geq v(0,x)\,, \qquad x \in U_{\alpha,x_0}\,.$$

On the other hand, if $(s,x) \in [0,\infty)\times\partial U_{\alpha,x_0}$, we have either $x \in \partial\Omega$ and $w(s,x) \geq 0 = v(s,x)$ or $|x-y_0| = R+\alpha$ and

$$w(s,x) = \psi_{\omega(\alpha),2C_1/\alpha}(\alpha) \geq \frac{2C_1}{\alpha}\ \left(\alpha - \frac{\alpha^2}{2}\right) \geq C_1 \geq v(s,x)$$

by (3.5). Furthermore, since $v(0,x)=0$ on $\partial\Omega$, $\omega(\alpha)$ converges to 0 as $\alpha\searrow 0$ and there is thus $\alpha_0\in(0,1/2)$ such that $2C_1/\alpha\ge 2(1+\omega(\alpha))$ for $\alpha\in(0,\alpha_0)$. This condition implies that $w$ is a supersolution to (3.2) in $(0,\infty)\times U_{\alpha,x_0}$ by Lemma 3.3. According to the above analysis, we are in a position to apply the comparison principle [11, Theorem 8.2] to conclude that

$$v(s,x)\le w(s,x)\,,\qquad (s,x)\in[0,\infty)\times\overline{U_{\alpha,x_0}}\,.$$

In particular, if $(s,x)\in[0,\infty)\times(\bar\Omega\cap B(x_0,\alpha))$, the above inequality, (3.14), and the properties of $y_0$ entail that

$$\begin{aligned} v(s,x) &\le \omega(\alpha)+\frac{2C_1}{\alpha}\,(|x-y_0|-R) \\ &\le \omega(\alpha)+\frac{2C_1}{\alpha}\,(|x-x_0|+|x_0-y_0|-R) \\ &\le \omega(\alpha)+\frac{2C_1}{\alpha}\,|x-x_0|\,, \end{aligned}$$

whence (3.13). ////

Proof of Theorem 1.2. For $\varepsilon\in(0,1)$, we define

$$V_\varepsilon(s,x):=v\left(\frac{s}{\varepsilon},x\right)\,,\qquad (s,x)\in[0,\infty)\times\bar\Omega\,,$$

and the half-relaxed limits

$$V_*(x):=\liminf_{(\sigma,y,\varepsilon)\to(s,x,0)} V_\varepsilon(\sigma,y)\,,\qquad V^*(x):=\limsup_{(\sigma,y,\varepsilon)\to(s,x,0)} V_\varepsilon(\sigma,y)$$

for $(s,x)\in(0,\infty)\times\bar\Omega$. These functions are well-defined by (3.5), indeed do not depend on $s>0$, and the stability result for (discontinuous) viscosity solutions ensures that

$$V_*\quad\text{is a supersolution to}\quad -\Delta_\infty z-\frac{z}{2}=0\quad\text{in}\quad\Omega\,, \tag{3.15}$$

$$V^*\quad\text{is a subsolution to}\quad -\Delta_\infty z-\frac{z}{2}=0\quad\text{in}\quad\Omega\,. \tag{3.16}$$

In addition, it follows from (3.5) and (3.13) that

$$0\le V_*(x)\le V^*(x)\le C_1\,,\quad x\in\bar\Omega\,, \tag{3.17}$$

and, for all $(x_0,\alpha)\in\partial\Omega\times(0,\alpha_0)$,

$$0\le V_*(x)\le V^*(x)\le\omega(\alpha)+\frac{2C_1}{\alpha}\,|x-x_0|\,,\quad x\in\bar\Omega\cap B(x_0,\alpha)\,. \tag{3.18}$$

In particular, (3.18) guarantees that $0\le V_*(x_0)\le V^*(x_0)\le\omega(\alpha)$ for all $x_0\in\partial\Omega$ and $\alpha\in(0,\alpha_0)$. Since $\omega(\alpha)\to 0$ as $\alpha\searrow 0$, we end up with

$$V_*(x)=V^*(x)=0\,,\qquad x\in\partial\Omega\,. \tag{3.19}$$

We finally infer from Lemma 3.2 that

$$V_*(x) > 0 \qquad \text{for } x \in \Omega. \tag{3.20}$$

We are then in the position to apply Lemma 2.1 to obtain that $V^* \le V_*$. Recalling (3.15), (3.16), (3.17), and (3.19) we conclude that $V_* = V^* \in C_0(\bar{\Omega})$ is a viscosity solution to $-\Delta_\infty z - z/2 = 0$ in $\Omega$. We have thus proved that $f_\infty := V^*$ is a positive viscosity solution to (1.10) and it is the only one by Corollary 2.2. In addition, it follows from the identity $V^* = V_* = f_\infty$ and [7, Lemme 4.1] (see also [6, Lemma 5.1.9]) that

$$\lim_{\varepsilon \searrow 0} \|V_\varepsilon(2) - f_\infty\|_{L^\infty(\Omega)} = 0.$$

In other words,

$$\lim_{s \to \infty} \|v(s) - f_\infty\|_{L^\infty(\Omega)} = 0, \tag{3.21}$$

which is equivalent to (1.9) by (3.1). ////

Proof of Corollary 1.3. The claim now follows from Theorem 1.2 and Lemma 3.1. Indeed, we have $v(s,\cdot) \le v(\sigma,\cdot)$ for $-\infty < s \le \sigma < \infty$. Letting $\sigma \to \infty$ and using (3.21) lead us to $v(s,\cdot) \le f_\infty$ for any $s \in \mathbb{R}$, which is nothing but (1.11) once written in terms of $u$. ////

# 4 Additional positivity properties

First we state an extension of Lemma 3.2 which shows that $u$ is indeed positive in $\Omega$ after a finite time provided that $\Omega$ additionally satisfies a uniform interior sphere condition:

$$\begin{array}{l} \text{There is } R_0 > 0 \text{ such that for any } x_0 \in \partial\Omega \text{ there is } y_0 \in \Omega \\ \text{such that } |y_0 - x_0| = R_0 \text{ and } B(y_0, R_0) \subset \Omega. \end{array} \tag{4.1}$$

**Lemma 4.1** *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying* (1.6) *and* (4.1) *and that $u_0$ fulfills* (1.4)*. If $u$ denotes the viscosity solution to* (1.1)-(1.3)*, then there is $t_1 \in (0,\infty)$ such that*

$$u(t,x) > 0 \qquad \textit{in } [t_1,\infty) \times \Omega. \tag{4.2}$$

Proof. Let $v$ be defined by (3.1) and set

$$K := \left\{ x \in \Omega \,:\, d(x,\partial\Omega) \ge \frac{R_0}{2} \right\} \quad \text{and} \quad M := \{ x \in \Omega \,:\, d(x,\partial\Omega) = R_0 \}.$$

Since $K$ is a compact subset of $\Omega$, we have

$$v(s,x) \ge \mu_K > 0 \qquad \text{in } [s_K,\infty) \times K \tag{4.3}$$

for some $s_K > 0$ and $\mu_K > 0$ by Lemma 3.2. Thus, setting $t_0 := e^{s_K}$, $\varepsilon := R_0/2$ and $\delta := t_0^{-1/2}\mu_K$, (3.8) is valid for any $x_0 \in M$. Then the first step of the proof of Lemma 3.2 implies the existence of $T > 0$ which is independent of $x_0 \in M$ such that (3.9) is fulfilled for any $x_0 \in M$. Thus, we conclude that

$$v(s_0, x) > 0 \quad \text{for } x \in \tilde{M} := \bigcup_{x_0 \in M} B(x_0, R_0),$$

where $s_0 := \ln(t_0 + T) > s_K$. As (4.1) implies $\tilde{M} \cup K = \Omega$ (see e.g. [17, Section 14.6]), we deduce from Lemma 3.1 and (4.3) that

$$v(s, x) > 0 \qquad \text{in } [s_0, \infty) \times \Omega.$$

By (3.1), this shows (4.2) with $t_1 := e^{s_0}$. ////

Having shown that $u$ is positive in $\Omega$ after a finite or infinite time, we next show that the expansion of the positivity set of $u(t, \cdot)$ may take some time to be initiated.

**Proposition 4.2** *Consider $u_0 \in C_0(\bar{\Omega})$ and define its positivity set $\mathcal{P}_0$ by*

$$\mathcal{P}_0 := \{x \in \Omega \ : \ u_0(x) > 0\}.$$

*If $x_0 \in \Omega \cap \partial\mathcal{P}_0$ is such that*

$$u_0(x) \le a \, |x - x_0|^2, \qquad x \in B(x_0, \delta) \subset \Omega, \tag{4.4}$$

*for some $\delta > 0$ and $a > 0$, then there is $\tau(x_0) > 0$ such that $u(t, x_0) = 0$ for $t \in [0, \tau(x_0))$.*

In other words, the so-called waiting time

$$\tau_w(x_0) := \inf\{t > 0 \, : \, u(t, x_0) > 0\}$$

of $u$ at $x_0 \in \Omega$ is positive if $u_0$ satisfies (4.4). In addition, it is finite by Lemma 3.2. This waiting time phenomenon is typical for degenerate parabolic equations, see [15, 21] and the references therein.
The proof of Proposition 4.2 relies on the construction of supersolutions as in [18, Theorem 8.2] which we describe now.

**Lemma 4.3** *Consider $x_0 \in \Omega$ and $T > 0$ and define*

$$S_T(t, x) := \frac{|x - x_0|^2}{4(T - t)^{1/2}}, \qquad (t, x) \in [0, T) \times \bar{\Omega}.$$

*Then $S_T$ is a supersolution to (1.1) in $(0, T) \times \Omega$.*

Proof. We first note that $S_T \in C^2([0, T) \times \bar{\Omega})$. For $(t, x) \in (0, T) \times \Omega$, we compute

$$\partial_t S(t, x) - \Delta_\infty S(t, x) \quad = \quad \frac{|x - x_0|^2}{8(T - t)^{3/2}} - \frac{\langle x - x_0, x - x_0 \rangle}{8(T - t)^{3/2}} = 0$$

and readily obtain the expected result. ////

Proof of Proposition 4.2. Define

$$T := \min\left\{ \frac{1}{16a^2}, \frac{\delta^4}{16C_1^2} \right\}.$$

According to Lemma 4.3, the function $S_T$ is a supersolution to (1.1) in $(0,T) \times B(x_0,\delta)$. In addition, the choice of $T$ and (4.4) guarantee that

$$S_T(0,x) = \frac{|x-x_0|^2}{4T^{1/2}} \ge a\ |x-x_0|^2 \ge u_0(x)\,, \quad x \in B(x_0,\delta)\,,$$

while we infer from the choice of $T$ and (1.8) that, for $(t,x) \in (0,T) \times \partial B(x_0,\delta)$

$$S_T(t,x) = \frac{\delta^2}{4(T-t)^{1/2}} \ge \frac{\delta^2}{4T^{1/2}} \ge C_1 \ge u(t,x)\,.$$

The comparison principle [11, Theorem 8.2] then entails that $S_T(t,x) \ge u(t,x)$ for $(t,x) \in [0,T) \times B(x_0,\delta)$. In particular, $0 \le u(t,x_0) \le S_T(t,x_0) = 0$ for $t \in [0,T)$, and the proof of Proposition 4.2 is complete. ////

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# References

[1] G. Akagi, P. Juutinen, and R. Kajikiya, *Asymptotic behavior of viscosity solutions for a degenerate parabolic equation associated with the infinity-Laplacian*, Math. Ann. **343** (2009), 921–953.

[2] G. Akagi and K. Suzuki, *Existence and uniqueness of viscosity solutions for a degenerate parabolic equation associated with the inifinity-Laplacian*, Calc. Var. Partial Differential Equations **31** (2008), 457–471.

[3] D.G. Aronson and L.A. Peletier, *Large time behaviour of solutions of the porous medium equation in bounded domains*, J. Differential Equations **39** (1981), 378–412.

[4] G. Aronsson, *Extensions of functions satisfying Lipschitz conditions*, Ark. Mat. **6** (1967), 551–561.

[5] G. Aronsson, M.G. Crandall, and P. Juutinen, *A tour of the theory of absolutely minimizing functions*, Bull. Amer. Math. Soc. (N.S.) **41** (2004), 439–505.

[6] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Systems Control Found. Appl., Birkhäuser, Boston, 1997.

[7] G. Barles, *Solutions de Viscosité des Equations d'Hamilton-Jacobi*, Mathématiques & Applications **17**, Springer, Berlin, 1994.

[8] G. Barles and B. Perthame, *Exit time problems in optimal control and vanishing viscosity method*, SIAM J. Control Optim. **26** (1988), 1133–1148.

[9] Ph. Bénilan and M.G. Crandall, *Regularizing effects of homogeneous evolution equations*, Contributions to analysis and geometry, 23–39, John Hopkins Univ. Press, Baltimore, Md., 1981.

[10] M.G. Crandall, *A visit with the* $\infty$*-Laplacian equation,* Calculus of variations and nonlinear partial differential equations, 75–122, Lecture Notes in Math., 1927, Springer, Berlin, 2008.

[11] M.G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 1–67.

[12] M.G. Crandall, P.-L. Lions, and P.E. Souganidis, *Maximal solutions and universal bounds for some partial differential equations of evolution*, Arch. Rational Mech. Anal. **105** (1989), 163–190.

[13] M.G. Crandall and P.-Y. Wang, *Another way to say caloric*, J. Evol. Equ. **3** (2003), 653–672.

[14] B.E.J. Dahlberg and C.E. Kenig, *Nonnegative solutions of the initial-Dirichlet problem for generalized porous medium equations in cylinders*, J. Amer. Math. Soc. **1** (1988), 401–412.

[15] R. Dal Passo, L. Giacomelli, and G. Grün, *Waiting time phenomena for degenerate parabolic equations – a unifying approach*, Geometric analysis and nonlinear partial differential equations, 637–648, Springer, Berlin, 2003.

[16] V.A. Galaktionov, *Evolution completeness of separable solutions of non-linear diffusion equations in bounded domains*, Math. Methods Appl. Sci. **27** (2004), 1755–1770.

[17] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order. Reprint of the 1998 ed.*, Classics in Mathematics, Springer, Berlin, 2001.

[18] B.F. Knerr, *The porous medium equation in one dimension*, Trans. Amer. Math. Soc. **234** (1977), 381–415.

[19] G.M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, 1996.

[20] J.J. Manfredi and V. Vespri, *Large time behavior of solutions to a class of doubly nonlinear parabolic equations*, Electron. J. Differential Equations **1994**, No. 02, (1994), 1–17.

[21] J.L. Vázquez, *The porous medium equation. Mathematical theory.* Oxford Mathematical Monographs; Oxford Science Publications, Oxford University Press, 2007.