



Refined asymptotics for the infinite heat equation with homogeneous Dirichlet boundary conditions

Philippe Laurencot, Christian Stinner

► To cite this version:

Philippe Laurencot, Christian Stinner. Refined asymptotics for the infinite heat equation with homogeneous Dirichlet boundary conditions. Communications in Partial Differential Equations, 2011, 36, pp.532-546. 10.1080/03605302.2010.498493 . hal-00463067

HAL Id: hal-00463067

<https://hal.science/hal-00463067>

Submitted on 11 Mar 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Refined asymptotics for the infinite heat equation with homogeneous Dirichlet boundary conditions

Philippe Laurençot* & Christian Stinner†

March 11, 2010

Abstract

The nonnegative viscosity solutions to the infinite heat equation with homogeneous Dirichlet boundary conditions are shown to converge as $t \rightarrow \infty$ to a uniquely determined limit after a suitable time rescaling. The proof relies on the half-relaxed limits technique as well as interior positivity estimates and boundary estimates. The expansion of the support is also studied.

Key words: infinite heat equation, infinity-Laplacian, friendly giant, viscosity solution, half-relaxed limits

MSC 2010: 35B40, 35K65, 35K55, 35D40

1 Introduction

Since the pioneering work by Aronsson [4], the infinity-Laplacian Δ_∞ defined by

$$\Delta_\infty u := \langle D^2 u \nabla u, \nabla u \rangle = \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

has been the subject of several studies, in particular due to its relationship to the theory of absolutely minimizing Lipschitz extensions [4, 5, 10]. More recently, a parabolic equation involving the infinity-Laplacian (the infinite heat equation)

$$\partial_t u = \Delta_\infty u, \quad (t, x) \in (0, \infty) \times \Omega, \quad (1.1)$$

has been considered in [1, 2, 13]. When $\Omega \subset \mathbb{R}^N$ is a bounded domain and (1.1) is supplemented with nonhomogeneous Dirichlet boundary conditions, the large time behaviour of solutions to (1.1) is investigated in [1] and convergence as $t \rightarrow \infty$ to the unique steady state is shown. Furthermore, for homogeneous Dirichlet boundary conditions

$$u = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega, \quad (1.2)$$

*Institut de Mathématiques de Toulouse, CNRS UMR 5219, Université de Toulouse, F-31062 Toulouse cedex 9, France. E-mail: laurenco@math.univ-toulouse.fr

†Fakultät für Mathematik, Universität Duisburg-Essen, D-45117 Essen, Germany. E-mail: christian.stinner@uni-due.de

and nonnegative initial condition

$$u(0, x) = u_0(x), \quad x \in \bar{\Omega}, \quad (1.3)$$

satisfying

$$u_0 \in C_0(\bar{\Omega}) := \{f \in C(\bar{\Omega}) : f = 0 \text{ on } \partial\Omega\}, \quad u_0 \geq 0, \quad u_0 \not\equiv 0, \quad (1.4)$$

a precise temporal decay rate is given for the L^∞ -norm of u , namely

$$C_1^{-1}(t+1)^{-1/2} \leq \|u(t, \cdot)\|_{L^\infty(\Omega)} \leq C_1(t+1)^{-1/2} \quad \text{for all } t > 0 \quad (1.5)$$

with some $C_1 \geq 1$ depending on u_0 and Ω , the unique steady state of (1.1)-(1.2) being zero in that case.

The purpose of this note is to improve (1.5) by identifying the limit of $t^{1/2}u(t, \cdot)$ as $t \rightarrow \infty$ (see Theorem 1.2 below). We also provide additional information on the propagation of the positivity set of u as time goes by.

Before stating our main result we first recall that the infinity-Laplacian is a quasilinear and degenerate elliptic operator which is not in divergence form and a suitable framework to study the well-posedness of the infinite heat equation is the theory of viscosity solutions (see e.g. [11]). Within this framework the well-posedness of (1.1)-(1.3) has been established in [2] when Ω fulfills the *uniform exterior sphere condition*:

$$\begin{aligned} &\text{For all } x_0 \in \partial\Omega \text{ there exists } y_0 \in \mathbb{R}^N \text{ such that } |x_0 - y_0| = R \text{ and} \\ &\{x \in \mathbb{R}^N : |x - y_0| < R\} \cap \Omega = \emptyset \text{ for some positive constant } R \text{ independent of } x_0. \end{aligned} \quad (1.6)$$

Introducing

$$F(s, p, X) := s - \langle Xp, p \rangle \quad \text{for } s \in \mathbb{R}, p \in \mathbb{R}^N, X \in \mathcal{S}(N), \quad (1.7)$$

where $\mathcal{S}(N)$ denotes the set of all symmetric $N \times N$ matrices, the definition of viscosity solutions to (1.1)-(1.3) reads [1, 2]:

Definition 1 Let $Q := (0, \infty) \times \Omega \subset \mathbb{R}^{N+1}$ and let $USC(\bar{Q})$ and $LSC(\bar{Q})$ denote the set of upper semicontinuous and lower semicontinuous functions from \bar{Q} into \mathbb{R} , respectively. A function $u \in USC(\bar{Q})$ is a viscosity subsolution to (1.1)-(1.3) in Q if

(a) $F(s, p, X) \leq 0$ is satisfied for all $(s, p, X) \in \mathcal{P}^{2,+}u(t_0, x_0)$ and all $(t_0, x_0) \in Q$, where

$$\begin{aligned} \mathcal{P}^{2,+}u(t_0, x_0) &:= \left\{ (s, p, X) \in \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N) : u(t, x) \leq u(t_0, x_0) + s(t - t_0) \right. \\ &\quad \left. + \langle p, x - x_0 \rangle + \frac{1}{2} \langle X(x - x_0), x - x_0 \rangle + o(|t - t_0| + |x - x_0|^2) \right. \\ &\quad \left. \text{as } (t, x) \rightarrow (t_0, x_0) \right\}, \end{aligned}$$

(b) $u \leq 0$ on $(0, \infty) \times \partial\Omega$,

(c) $u(0, x) \leq u_0(x)$ for $x \in \bar{\Omega}$.

Similarly, $u \in LSC(\bar{Q})$ is a viscosity supersolution to (1.1)-(1.3) in Q if $F(s, p, X) \geq 0$ for all $(s, p, X) \in \mathcal{P}^{2,-}u(t_0, x_0) := -\mathcal{P}^{2,+}(-u)(t_0, x_0)$ and $(t_0, x_0) \in Q$, $u \geq 0$ on $(0, \infty) \times \partial\Omega$ and $u(0, x) \geq u_0(x)$ for $x \in \bar{\Omega}$.

Finally, $u \in C(\bar{Q})$ is a viscosity solution to (1.1)-(1.3) if it is a viscosity subsolution and a viscosity supersolution to (1.1)-(1.3).

With this definition, the well-posedness of (1.1)-(1.3) is shown in [2, Theorems 2.3 and 2.5] and the asymptotic behaviour of nonnegative solutions is obtained in [1, Theorem 5]. We gather these results in the next theorem.

Theorem 1.1 (*[1, 2]*) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that (1.6) is satisfied and assume (1.4). Then there is a unique nonnegative viscosity solution u to (1.1)-(1.3). Moreover, $u(t, \cdot)$ converges to zero as $t \rightarrow \infty$ in the sense that there exists a constant $C_1 \geq 1$ such that*

$$C_1^{-1}(t+1)^{-1/2} \leq \|u(t, \cdot)\|_{L^\infty(\Omega)} \leq C_1(t+1)^{-1/2} \quad \text{for all } t > 0. \quad (1.8)$$

Our improvement of (1.8) then reads:

Theorem 1.2 *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain fulfilling (1.6) and assume (1.4). If u denotes the viscosity solution to (1.1)-(1.3), then*

$$\lim_{t \rightarrow \infty} \|t^{1/2}u(t, \cdot) - f_\infty\|_{L^\infty(\Omega)} = 0, \quad (1.9)$$

where f_∞ is the unique positive viscosity solution to

$$-\Delta_\infty f_\infty - \frac{f_\infty}{2} = 0 \text{ in } \Omega, \quad f_\infty > 0 \text{ in } \Omega, \quad f_\infty = 0 \text{ on } \partial\Omega. \quad (1.10)$$

Theorem 1.2 not only gives the convergence of $t^{1/2}u(t, \cdot)$ as $t \rightarrow \infty$, but also provides the existence and uniqueness of the positive solution f_∞ to (1.10) in $C_0(\bar{\Omega})$. An interesting consequence of (1.10) is that the function $(t, x) \mapsto t^{-1/2}f_\infty(x)$ is a separate variables solution to (1.1)-(1.2) with an initial data being identically infinite in Ω . Similar solutions are already known to exist for other parabolic equations such as the porous medium equation $\partial_t u = \Delta u^m$, $m > 1$, or the p -Laplacian equation $\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 2$, (see [3, 12, 14, 20, 21] for instance). They play an important role in the description of the large time dynamics [3, 16, 21] and also provide universal bounds (and are thus called *friendly giants*). The function $(t, x) \mapsto t^{-1/2}f_\infty(x)$ is a friendly giant for the infinite heat equation (1.1)-(1.3) and we have the following universal bound.

Corollary 1.3 *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain fulfilling (1.6) and assume (1.4). If u denotes the viscosity solution to (1.1)-(1.3), then*

$$u(t, x) \leq t^{-1/2}f_\infty(x) \quad \text{for } (t, x) \in (0, \infty) \times \bar{\Omega}, \quad (1.11)$$

the function f_∞ being defined in Theorem 1.2.

The proof of Theorem 1.2 and Corollary 1.3 involves several steps: According to (1.8) the evolution of $u(t, \cdot)$ takes place on a time scale of order $t^{-1/2}$ and we first introduce a rescaled version v of u defined by $u(t, x) = t^{-1/2}v(\ln t, x)$. The outcome of Theorem 1.2 is then the convergence of $v(s, \cdot)$ to the time-independent function f_∞ as $s \rightarrow \infty$. To establish such a convergence, we use the half-relaxed limits technique introduced in [8] which is well-suited here as we have rather scarce information on $v(s, \cdot)$ as $s \rightarrow \infty$. This requires however a strong comparison principle for the limit problem (1.10) which will be established in Section 2, under an additional positivity assumption,

and furthermore implies the uniqueness of f_∞ . That the half-relaxed limits indeed enjoy this positivity property has to be proved as a preliminary step and follows from the observation that $v(s, \cdot)$ is non-decreasing with time and eventually becomes positive in Ω (see Section 3.1). At this point, boundary estimates are also needed to ensure that the half-relaxed limits vanish on $\partial\Omega$ and are shown by constructing suitable barrier functions. Thanks to these results, we deduce that the half-relaxed limits coincide, which implies that $v(s, \cdot)$ converges as $s \rightarrow \infty$ and the existence of a positive solution f_∞ to (1.10) as well (see Section 3.2). We emphasize here that the existence of a positive solution to (1.10) is a consequence of the dynamical properties of v and was seemingly not known previously. Finally, Corollary 1.3 is a consequence of Theorem 1.2 and the time monotonicity of v (see Section 3.2).

Additionally, in Section 4 we investigate further positivity properties of the solution u to (1.1)-(1.3). We show that $u(t, \cdot)$ becomes positive in Ω after a finite time if Ω satisfies an additional uniform interior sphere condition. Aside from this, u may have a positive waiting time if the initial data are flat on the boundary of their support, namely the support of $u(t, \cdot)$ will be equal to that of u_0 for small times.

For further use, we introduce the following notation: Given $x \in \bar{\Omega}$, let $d(x, \partial\Omega) := \text{dist}(x, \partial\Omega)$ denote the distance to the boundary. Moreover, for $x \in \mathbb{R}^N$ and $r > 0$ we define $B(x, r) := \{y \in \mathbb{R}^N : |y - x| < r\}$ to be the ball of radius r centered at x .

2 Uniqueness of the friendly giant

In this section we show that the friendly giant is unique. This will be a consequence of the following more general comparison lemma.

Lemma 2.1 *Let $w \in USC(\bar{\Omega})$ and $W \in LSC(\bar{\Omega})$ be respectively a bounded viscosity subsolution and a bounded viscosity supersolution to*

$$-\Delta_\infty \zeta - \frac{\zeta}{2} = 0 \quad \text{in } \Omega \quad (2.1)$$

such that

$$w(x) = W(x) = 0 \quad \text{for } x \in \partial\Omega, \quad (2.2)$$

$$W(x) > 0 \quad \text{for } x \in \Omega. \quad (2.3)$$

Then

$$w \leq W \quad \text{in } \Omega. \quad (2.4)$$

PROOF. We fix $N_0 \in \mathbb{N}$ large enough such that $\Omega_n := \{x \in \Omega : d(x, \partial\Omega) > 1/n\}$ is a nonempty open subset of Ω for all integer $n \geq N_0$. Let $n \geq N_0$. Since $\bar{\Omega}_n$ is compact and $W \in LSC(\bar{\Omega})$, W has a minimum in $\bar{\Omega}_n$ and the positivity of W in $\bar{\Omega}_n$ implies that

$$\mu_n := \min_{\bar{\Omega}_n} W > 0. \quad (2.5)$$

Similarly, the compactness of $\bar{\Omega} \setminus \Omega_n$ and the upper semicontinuity and boundedness of w ensure that w has a point of maximum x_n in $\bar{\Omega} \setminus \Omega_n$ and we set

$$\eta_n := \max_{\bar{\Omega} \setminus \Omega_n} w = w(x_n) \geq 0, \quad (2.6)$$

the nonnegativity of η_n being a consequence of the fact that w vanishes on $\partial\Omega$. We next claim that

$$\lim_{n \rightarrow \infty} \eta_n = 0. \quad (2.7)$$

Indeed, owing to the compactness of $\bar{\Omega}$ and the definition of Ω_n there are $y \in \partial\Omega$ and a subsequence of $(x_n)_{n \in \mathbb{N}}$ (not relabeled) such that $x_n \rightarrow y$ as $n \rightarrow \infty$. Since $w(y) = 0$, we deduce from the upper semicontinuity of w that

$$\limsup_{x \rightarrow y} w(x) = \lim_{\varepsilon \searrow 0} \sup\{w(x) : x \in B(y, \varepsilon) \cap \bar{\Omega}\} \leq 0.$$

Given $\varepsilon > 0$, there is n_ε such that $x_n \in B(y, \varepsilon) \cap \bar{\Omega}$ for all $n \geq n_\varepsilon$. Hence,

$$\limsup_{n \rightarrow \infty} \eta_n \leq \sup\{w(x) : x \in B(y, \varepsilon) \cap \bar{\Omega}\}$$

and letting $\varepsilon \searrow 0$ and using (2.6) allow us to conclude that

$$0 \leq \limsup_{n \rightarrow \infty} \eta_n \leq 0.$$

This shows that a subsequence of $(\eta_n)_{n \geq N_0}$ converges to zero and the claim (2.7) follows by noticing that $(\eta_n)_{n \geq N_0}$ is a nonincreasing sequence.

Next, fix $s \in (0, \infty)$. For $\delta > 0$ and $n \geq N_0$, we define

$$\begin{aligned} z_n(t, x) &:= (t + s)^{-1/2} w(x) - s^{-1/2} \eta_n, & (t, x) &\in [0, \infty) \times \bar{\Omega}, \\ Z_\delta(t, x) &:= (t + \delta)^{-1/2} W(x), & (t, x) &\in [0, \infty) \times \bar{\Omega}. \end{aligned}$$

Then z_n and Z_δ are respectively a bounded usc viscosity subsolution and a bounded lsc viscosity supersolution to (1.1) with

$$Z_\delta(t, x) = 0 \geq -s^{-1/2} \eta_n = z_n(t, x), \quad (t, x) \in (0, \infty) \times \partial\Omega.$$

In addition, if

$$0 < \delta < \left(\frac{\mu_n}{1 + \|w\|_{L^\infty(\Omega)}} \right)^2 s \quad (2.8)$$

we have

$$Z_\delta(0, x) = \delta^{-1/2} W(x) \geq \delta^{-1/2} \mu_n \geq s^{-1/2} \|w\|_{L^\infty(\Omega)} \geq z_n(0, x) \quad \text{for } x \in \Omega_n$$

and

$$Z_\delta(0, x) \geq 0 \geq s^{-1/2} (w(x) - \eta_n) = z_n(0, x) \quad \text{for } x \in \bar{\Omega} \setminus \Omega_n.$$

We are then in a position to apply the comparison principle [11, Theorem 8.2] to deduce that

$$z_n(t, x) \leq Z_\delta(t, x), \quad (t, x) \in [0, \infty) \times \bar{\Omega}, \quad (2.9)$$

for any $\delta > 0$ and $n \geq N_0$ satisfying (2.8). According to (2.8), the parameter δ can be taken arbitrarily small and we deduce from (2.9) that

$$(t + s)^{-1/2} w(x) - s^{-1/2} \eta_n \leq t^{-1/2} W(x), \quad (t, x) \in (0, \infty) \times \bar{\Omega},$$

for $n \geq N_0$. We next pass to the limit as $n \rightarrow \infty$ with the help of (2.7) to conclude that

$$(t+s)^{-1/2}w(x) \leq t^{-1/2}W(x), \quad (t,x) \in (0,\infty) \times \bar{\Omega}.$$

Finally, as $s > 0$ is arbitrary, we may let $s \searrow 0$ and take $t = 1$ in the above inequality to complete the proof. ////

Now the uniqueness of the friendly giant is a straightforward consequence of Lemma 2.1.

Corollary 2.2 *There is at most one positive viscosity solution to (1.10) in $C_0(\bar{\Omega})$.*

3 Large time behaviour

In this section, we assume that Ω is a bounded domain fulfilling (1.6) and that u_0 satisfies (1.4). Let u be the corresponding viscosity solution to (1.1)-(1.3). In order to investigate the asymptotic behaviour of u as stated in Theorem 1.2 we introduce the scaling variable $s = \ln t$, $t > 0$, and the rescaled unknown function v defined by

$$u(t,x) = t^{-1/2}v(\ln t, x), \quad (t,x) \in (0,\infty) \times \bar{\Omega}. \quad (3.1)$$

It is easy to check that v is the viscosity solution to

$$\partial_s v = \Delta_\infty v - \frac{v}{2}, \quad (s,x) \in (0,\infty) \times \Omega, \quad (3.2)$$

$$v = 0, \quad (s,x) \in (0,\infty) \times \partial\Omega, \quad (3.3)$$

$$v(0,x) = v_0(x) := u(1,x), \quad x \in \bar{\Omega}, \quad (3.4)$$

while it readily follows from (1.8) and (3.1) that

$$0 \leq v(s,x) \leq C_1, \quad (s,x) \in [0,\infty) \times \bar{\Omega}. \quad (3.5)$$

3.1 Positivity and time monotonicity

A further property of v is its time monotonicity which follows from the homogeneity of the operator Δ_∞ by a result from B enilan & Crandall [9].

Lemma 3.1 *For $x \in \bar{\Omega}$, $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$ such that $s_1 \leq s_2$, we have*

$$v(s_1, x) \leq v(s_2, x).$$

PROOF. Theorem 1.1 provides the well-posedness of (1.1) in $C_0(\bar{\Omega})$ which is an ordered vector space. As the comparison principle is valid for (1.1)-(1.3) by [2, Theorem 2.3] and the infinity-Laplacian is homogeneous of degree 3, [9, Theorem 2] implies

$$u(t+h, x) - u(t, x) \geq \left(\left(\frac{t+h}{t} \right)^{-1/2} - 1 \right) u(t, x) \quad \text{for } (t,x) \in (0,\infty) \times \bar{\Omega}, h > 0. \quad (3.6)$$

Hence, for any $(s, x) \in \mathbb{R} \times \bar{\Omega}$ and $h > 0$, we obtain

$$\begin{aligned} v(s+h, x) - v(s, x) &= e^{(s+h)/2} u(e^{s+h}, x) - e^{s/2} u(e^s, x) \\ &\geq e^{(s+h)/2} \left(\frac{e^{s+h}}{e^s} \right)^{-1/2} u(e^s, x) - e^{s/2} u(e^s, x) = 0, \end{aligned}$$

which is the expected result. ////

The monotonicity of v now enables us to prove that v eventually becomes positive inside Ω .

Lemma 3.2 *For any compact subset $K \subset \Omega$ there are $s_K > 0$ and $\mu_K > 0$ such that*

$$v(s, x) \geq \mu_K > 0 \quad \text{in } [s_K, \infty) \times K. \quad (3.7)$$

PROOF. Three steps are needed to achieve the claimed result: we first prove that if $v(s, \cdot)$ is positive at one point of Ω , then it becomes positive on a “large” ball centered around this point after a finite time. The second step is to prove that $v(s, \cdot)$ becomes eventually positive in Ω as $s \rightarrow \infty$, from which we deduce (3.7) in a third step.

Step 1: Consider first $(t_0, x_0) \in (0, \infty) \times \Omega$ such that there are $\varepsilon > 0$ and $\delta > 0$ with $B(x_0, \varepsilon) \subset \Omega$ and

$$u(t_0, x) \geq \delta > 0 \quad \text{for } x \in B(x_0, \varepsilon). \quad (3.8)$$

Then, choosing $\alpha := \min\{(4\delta)^{1/3}, \varepsilon^{2/3}\}$, $T := (d(x_0, \partial\Omega)^6/\alpha^9) - 1 \geq 0$, and defining

$$\mathcal{B}(t, x) := \frac{\alpha^3}{4} (t - t_0 + 1)^{-1/6} \left(1 - \alpha^{-2} |x - x_0|^{4/3} (t - t_0 + 1)^{-2/9} \right)_+^{3/2}, \quad (t, x) \in [t_0, \infty) \times \mathbb{R}^N,$$

we deduce from [1, Proposition 1 and Corollary 1] that \mathcal{B} is a viscosity solution to (1.1) in $(t_0, t_0 + T) \times \Omega$. In addition, on the one hand, we have by (3.8)

$$\mathcal{B}(t_0, x) \leq \frac{\alpha^3}{4} \leq \delta \leq u(t_0, x) \quad \text{for } x \in B(x_0, \varepsilon)$$

and

$$\mathcal{B}(t_0, x) = 0 \leq u(t_0, x) \quad \text{for } x \in \bar{\Omega} \setminus B(x_0, \varepsilon).$$

On the other hand, we have $u(t, x) = \mathcal{B}(t, x) = 0$ for $(t, x) \in [t_0, t_0 + T] \times \partial\Omega$ thanks to the choice of T , α and the properties of \mathcal{B} . The comparison principle [11, Theorem 8.2] then implies $u \geq \mathcal{B}$ in $[t_0, t_0 + T] \times \bar{\Omega}$. In particular, we have

$$u(t_0 + T, x) > 0 \quad \text{for } x \in B(x_0, d(x_0, \partial\Omega)), \quad (3.9)$$

where T only depends on ε and δ , but is independent of x_0 and t_0 .

Step 2: We next define the positivity set $\mathcal{P}(s)$ of $v(s, \cdot)$ for $s \geq 0$ by

$$\mathcal{P}(s) := \{x \in \Omega : v(s, x) > 0\}.$$

Owing to the time monotonicity of v (Lemma 3.1), $(\mathcal{P}(s))_{s \geq 0}$ is a non-decreasing family of open subsets of Ω and

$$\mathcal{P}_\infty := \bigcup_{s \geq 0} \mathcal{P}(s) \text{ is an open subset of } \Omega.$$

Assume for contradiction that $\partial\mathcal{P}_\infty \cap \Omega \neq \emptyset$. Then there is $x_0 \in \partial\mathcal{P}_\infty \cap \Omega$. Since $d(x_0, \partial\Omega) > 0$ there is $y_0 \in \mathcal{P}_\infty$ such that $|y_0 - x_0| \leq d(x_0, \partial\Omega)/2 < d(y_0, \partial\Omega)$. Next, since $y_0 \in \mathcal{P}_\infty$, there is $s_0 > 0$ such that $v(s_0, y_0) > 0$, that is $u(e^{s_0}, y_0) > 0$. The previous step then guarantees the existence of $T \geq 0$, such that $u(e^{s_0} + T, x) > 0$ for $x \in B(y_0, d(y_0, \partial\Omega))$. As $x_0 \in B(y_0, d(y_0, \partial\Omega))$, we deduce from this that

$$v(\ln(e^{s_0} + T), x_0) = (e^{s_0} + T)^{1/2} u(e^{s_0} + T, x_0) > 0,$$

which contradicts the fact that $x_0 \in \partial\mathcal{P}_\infty$. Therefore, $\partial\mathcal{P}_\infty \cap \Omega = \emptyset$ and Ω is the union of the two disjoint open sets \mathcal{P}_∞ and $\Omega \setminus \overline{\mathcal{P}_\infty}$. Since $\mathcal{P}_\infty \neq \emptyset$ by (1.8), the connectedness of Ω implies

$$\Omega = \mathcal{P}_\infty. \quad (3.10)$$

Step 3: Let K be a compact subset of Ω and assume for contradiction that $K \not\subset \mathcal{P}(n)$ for each $n \geq 1$. Then there is a sequence $(x_n)_{n \geq 1}$ in K such that $v(n, x_n) = 0$ for $n \geq 1$ and we may assume without loss of generality that x_n converges towards $x_\infty \in K$ as $n \rightarrow \infty$, thanks to the compactness of K . Since $x_\infty \in \Omega$, it follows from (3.10) that there is $s_\infty > 0$ such that $v(s_\infty, x_\infty) > 0$. Owing to the continuity of $v(s_\infty, \cdot)$ there are $\varepsilon > 0$ and $\delta > 0$ such that $v(s_\infty, x) \geq \delta$ for $x \in B(x_\infty, \varepsilon) \subset \Omega$. But then for n large enough we have $n \geq s_\infty$ and $x_n \in B(x_\infty, \varepsilon)$ and it follows from Lemma 3.1 and the previous bound that

$$0 = v(n, x_n) \geq v(s_\infty, x_n) \geq \delta$$

and a contradiction. Consequently, there is n_K such that $K \subset \mathcal{P}(n_K)$ and

$$\mu_K := \min_{x \in K} v(n_K, x) > 0.$$

Due to the time monotonicity of v , this implies (3.7). ////

3.2 Convergence

Having studied the positivity properties of v , we next turn to its behaviour near the boundary of Ω and first show the following lemma which is a modification of [19, Lemma 10.1].

Lemma 3.3 *Consider $x_0 \in \partial\Omega$, $\alpha \in (0, 1/2)$, $\delta > 0$, $B > 0$, and define*

$$\psi_{\delta, B}(r) := \delta + B \left(r - \frac{r^2}{2} \right), \quad r \in \mathbb{R}.$$

Let $y_0 \in \mathbb{R}^N$ be such that $|x_0 - y_0| = R$ and $\Omega \cap B(y_0, R) = \emptyset$ (such a point y_0 exists according to the uniform exterior sphere condition (1.6)). Introducing

$$U_{\alpha, x_0} := \{x \in \Omega : R < |x - y_0| < R + \alpha\}$$

and

$$w(s, x) := \psi_{\delta, B}(|x - y_0| - R), \quad (s, x) \in [0, \infty) \times \overline{U_{\alpha, x_0}},$$

then w is a supersolution to (3.2) in $(0, \infty) \times U_{\alpha, x_0}$ if $B \geq 2(1 + \delta)$.

PROOF. To simplify notations, we set $\psi := \psi_{\delta, B}$ and $U := U_{\alpha, x_0}$. Since $\psi \in C^\infty(\mathbb{R})$ and $y_0 \notin U$, the function w is C^∞ -smooth in $(0, \infty) \times U$ and, if $(s, x) \in (0, \infty) \times U$, we have

$$\partial_s w(s, x) - \Delta_\infty w(s, x) - \frac{w(s, x)}{2} = - \left(\psi'^2 \psi'' + \frac{\psi}{2} \right) (|x - y_0| - R). \quad (3.11)$$

Since $\alpha \in (0, 1/2)$ and $B \geq 2$, we have for $r \in [0, \alpha]$

$$- \left(\psi'^2 \psi'' + \frac{\psi}{2} \right) (r) = B^3 (1 - r)^2 - \frac{B}{2} \left(r - \frac{r^2}{2} \right) - \frac{\delta}{2} \geq \frac{B^3}{8} - \frac{B}{4} - \frac{\delta}{2} \geq \frac{B - 2\delta}{4}.$$

Consequently, as $|x - y_0| - R \in [0, \alpha]$ for $(s, x) \in (0, \infty) \times U$, we deduce from (3.11) and the above inequality that

$$\partial_s w(s, x) - \Delta_\infty w(s, x) - \frac{w(s, x)}{2} \geq \frac{B - 2\delta}{4} \geq 0,$$

the last inequality following from the choice of B . ////

As a consequence of Lemma 3.3, we have the following useful bound for v on $\partial\Omega$.

Lemma 3.4 *Consider $\alpha \in (0, 1/2)$ and define*

$$\omega(\alpha) := \sup \{ v(0, x) : x \in \Omega \text{ and } d(x, \partial\Omega) < \alpha \}. \quad (3.12)$$

Then there is $\alpha_0 \in (0, 1/2)$ such that, for any $\alpha \in (0, \alpha_0)$ and $x_0 \in \partial\Omega$, we have

$$0 \leq v(s, x) \leq \omega(\alpha) + \frac{2C_1}{\alpha} |x - x_0|, \quad (s, x) \in [0, \infty) \times (\bar{\Omega} \cap B(x_0, \alpha)), \quad (3.13)$$

the constant C_1 being defined in (3.5).

PROOF. Consider $x_0 \in \partial\Omega$ and let $y_0 \in \mathbb{R}^N$ be such that $|x_0 - y_0| = R$ and $\Omega \cap B(y_0, R) = \emptyset$, the existence of such a point y_0 being guaranteed by the uniform exterior sphere condition (1.6). With the notations of Lemma 3.3, we define

$$w(s, x) := \psi_{\omega(\alpha), 2C_1/\alpha}(|x - y_0| - R), \quad (s, x) \in [0, \infty) \times \overline{U_{\alpha, x_0}},$$

the constant C_1 being defined in (3.5) and observe that

$$B(x_0, \alpha) \cap \Omega \subset U_{\alpha, x_0} \subset \{x \in \Omega : d(x, \partial\Omega) < \alpha\}. \quad (3.14)$$

On the one hand, it follows from (3.12) and (3.14) that

$$w(0, x) \geq \omega(\alpha) \geq v(0, x), \quad x \in U_{\alpha, x_0}.$$

On the other hand, if $(s, x) \in [0, \infty) \times \partial U_{\alpha, x_0}$, we have either $x \in \partial\Omega$ and $w(s, x) \geq 0 = v(s, x)$ or $|x - y_0| = R + \alpha$ and

$$w(s, x) = \psi_{\omega(\alpha), 2C_1/\alpha}(\alpha) \geq \frac{2C_1}{\alpha} \left(\alpha - \frac{\alpha^2}{2} \right) \geq C_1 \geq v(s, x)$$

by (3.5). Furthermore, since $v(0, x) = 0$ on $\partial\Omega$, $\omega(\alpha)$ converges to 0 as $\alpha \searrow 0$ and there is thus $\alpha_0 \in (0, 1/2)$ such that $2C_1/\alpha \geq 2(1 + \omega(\alpha))$ for $\alpha \in (0, \alpha_0)$. This condition implies that w is a supersolution to (3.2) in $(0, \infty) \times U_{\alpha, x_0}$ by Lemma 3.3. According to the above analysis, we are in a position to apply the comparison principle [11, Theorem 8.2] to conclude that

$$v(s, x) \leq w(s, x), \quad (s, x) \in [0, \infty) \times \overline{U_{\alpha, x_0}}.$$

In particular, if $(s, x) \in [0, \infty) \times (\bar{\Omega} \cap B(x_0, \alpha))$, the above inequality, (3.14), and the properties of y_0 entail that

$$\begin{aligned} v(s, x) &\leq \omega(\alpha) + \frac{2C_1}{\alpha} (|x - y_0| - R) \\ &\leq \omega(\alpha) + \frac{2C_1}{\alpha} (|x - x_0| + |x_0 - y_0| - R) \\ &\leq \omega(\alpha) + \frac{2C_1}{\alpha} |x - x_0|, \end{aligned}$$

whence (3.13). ////

PROOF OF THEOREM 1.2. For $\varepsilon \in (0, 1)$, we define

$$V_\varepsilon(s, x) := v\left(\frac{s}{\varepsilon}, x\right), \quad (s, x) \in [0, \infty) \times \bar{\Omega},$$

and the half-relaxed limits

$$V_*(x) := \liminf_{(\sigma, y, \varepsilon) \rightarrow (s, x, 0)} V_\varepsilon(\sigma, y), \quad V^*(x) := \limsup_{(\sigma, y, \varepsilon) \rightarrow (s, x, 0)} V_\varepsilon(\sigma, y)$$

for $(s, x) \in (0, \infty) \times \bar{\Omega}$. These functions are well-defined by (3.5), indeed do not depend on $s > 0$, and the stability result for (discontinuous) viscosity solutions ensures that

$$V_* \text{ is a supersolution to } -\Delta_\infty z - \frac{z}{2} = 0 \text{ in } \Omega, \quad (3.15)$$

$$V^* \text{ is a subsolution to } -\Delta_\infty z - \frac{z}{2} = 0 \text{ in } \Omega. \quad (3.16)$$

In addition, it follows from (3.5) and (3.13) that

$$0 \leq V_*(x) \leq V^*(x) \leq C_1, \quad x \in \bar{\Omega}, \quad (3.17)$$

and, for all $(x_0, \alpha) \in \partial\Omega \times (0, \alpha_0)$,

$$0 \leq V_*(x) \leq V^*(x) \leq \omega(\alpha) + \frac{2C_1}{\alpha} |x - x_0|, \quad x \in \bar{\Omega} \cap B(x_0, \alpha). \quad (3.18)$$

In particular, (3.18) guarantees that $0 \leq V_*(x_0) \leq V^*(x_0) \leq \omega(\alpha)$ for all $x_0 \in \partial\Omega$ and $\alpha \in (0, \alpha_0)$. Since $\omega(\alpha) \rightarrow 0$ as $\alpha \searrow 0$, we end up with

$$V_*(x) = V^*(x) = 0, \quad x \in \partial\Omega. \quad (3.19)$$

We finally infer from Lemma 3.2 that

$$V_*(x) > 0 \quad \text{for } x \in \Omega. \quad (3.20)$$

We are then in the position to apply Lemma 2.1 to obtain that $V^* \leq V_*$. Recalling (3.15), (3.16), (3.17), and (3.19) we conclude that $V_* = V^* \in C_0(\bar{\Omega})$ is a viscosity solution to $-\Delta_\infty z - z/2 = 0$ in Ω . We have thus proved that $f_\infty := V^*$ is a positive viscosity solution to (1.10) and it is the only one by Corollary 2.2. In addition, it follows from the identity $V^* = V_* = f_\infty$ and [7, Lemme 4.1] (see also [6, Lemma 5.1.9]) that

$$\lim_{\varepsilon \searrow 0} \|V_\varepsilon(2) - f_\infty\|_{L^\infty(\Omega)} = 0.$$

In other words,

$$\lim_{s \rightarrow \infty} \|v(s) - f_\infty\|_{L^\infty(\Omega)} = 0, \quad (3.21)$$

which is equivalent to (1.9) by (3.1). /////

PROOF OF COROLLARY 1.3. The claim now follows from Theorem 1.2 and Lemma 3.1. Indeed, we have $v(s, \cdot) \leq v(\sigma, \cdot)$ for $-\infty < s \leq \sigma < \infty$. Letting $\sigma \rightarrow \infty$ and using (3.21) lead us to $v(s, \cdot) \leq f_\infty$ for any $s \in \mathbb{R}$, which is nothing but (1.11) once written in terms of u . /////

4 Additional positivity properties

First we state an extension of Lemma 3.2 which shows that u is indeed positive in Ω after a finite time provided that Ω additionally satisfies a uniform interior sphere condition:

$$\begin{aligned} &\text{There is } R_0 > 0 \text{ such that for any } x_0 \in \partial\Omega \text{ there is } y_0 \in \Omega \\ &\text{such that } |y_0 - x_0| = R_0 \text{ and } B(y_0, R_0) \subset \Omega. \end{aligned} \quad (4.1)$$

Lemma 4.1 *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying (1.6) and (4.1) and that u_0 fulfills (1.4). If u denotes the viscosity solution to (1.1)-(1.3), then there is $t_1 \in (0, \infty)$ such that*

$$u(t, x) > 0 \quad \text{in } [t_1, \infty) \times \Omega. \quad (4.2)$$

PROOF. Let v be defined by (3.1) and set

$$K := \left\{ x \in \Omega : d(x, \partial\Omega) \geq \frac{R_0}{2} \right\} \quad \text{and} \quad M := \{ x \in \Omega : d(x, \partial\Omega) = R_0 \}.$$

Since K is a compact subset of Ω , we have

$$v(s, x) \geq \mu_K > 0 \quad \text{in } [s_K, \infty) \times K \quad (4.3)$$

for some $s_K > 0$ and $\mu_K > 0$ by Lemma 3.2. Thus, setting $t_0 := e^{s_K}$, $\varepsilon := R_0/2$ and $\delta := t_0^{-1/2} \mu_K$, (3.8) is valid for any $x_0 \in M$. Then the first step of the proof of Lemma 3.2 implies the existence

of $T > 0$ which is independent of $x_0 \in M$ such that (3.9) is fulfilled for any $x_0 \in M$. Thus, we conclude that

$$v(s_0, x) > 0 \quad \text{for } x \in \tilde{M} := \bigcup_{x_0 \in M} B(x_0, R_0),$$

where $s_0 := \ln(t_0 + T) > s_K$. As (4.1) implies $\tilde{M} \cup K = \Omega$ (see e.g. [17, Section 14.6]), we deduce from Lemma 3.1 and (4.3) that

$$v(s, x) > 0 \quad \text{in } [s_0, \infty) \times \Omega.$$

By (3.1), this shows (4.2) with $t_1 := e^{s_0}$. ////

Having shown that u is positive in Ω after a finite or infinite time, we next show that the expansion of the positivity set of $u(t, \cdot)$ may take some time to be initiated.

Proposition 4.2 *Consider $u_0 \in C_0(\bar{\Omega})$ and define its positivity set \mathcal{P}_0 by*

$$\mathcal{P}_0 := \{x \in \Omega : u_0(x) > 0\}.$$

If $x_0 \in \Omega \cap \partial\mathcal{P}_0$ is such that

$$u_0(x) \leq a |x - x_0|^2, \quad x \in B(x_0, \delta) \subset \Omega, \quad (4.4)$$

for some $\delta > 0$ and $a > 0$, then there is $\tau(x_0) > 0$ such that $u(t, x_0) = 0$ for $t \in [0, \tau(x_0))$.

In other words, the so-called waiting time

$$\tau_w(x_0) := \inf\{t > 0 : u(t, x_0) > 0\}$$

of u at $x_0 \in \Omega$ is positive if u_0 satisfies (4.4). In addition, it is finite by Lemma 3.2. This waiting time phenomenon is typical for degenerate parabolic equations, see [15, 21] and the references therein.

The proof of Proposition 4.2 relies on the construction of supersolutions as in [18, Theorem 8.2] which we describe now.

Lemma 4.3 *Consider $x_0 \in \Omega$ and $T > 0$ and define*

$$S_T(t, x) := \frac{|x - x_0|^2}{4(T - t)^{1/2}}, \quad (t, x) \in [0, T) \times \bar{\Omega}.$$

Then S_T is a supersolution to (1.1) in $(0, T) \times \Omega$.

PROOF. We first note that $S_T \in C^2([0, T) \times \bar{\Omega})$. For $(t, x) \in (0, T) \times \Omega$, we compute

$$\partial_t S(t, x) - \Delta_\infty S(t, x) = \frac{|x - x_0|^2}{8(T - t)^{3/2}} - \frac{\langle x - x_0, x - x_0 \rangle}{8(T - t)^{3/2}} = 0$$

and readily obtain the expected result. ////

PROOF OF PROPOSITION 4.2. Define

$$T := \min \left\{ \frac{1}{16a^2}, \frac{\delta^4}{16C_1^2} \right\}.$$

According to Lemma 4.3, the function S_T is a supersolution to (1.1) in $(0, T) \times B(x_0, \delta)$. In addition, the choice of T and (4.4) guarantee that

$$S_T(0, x) = \frac{|x - x_0|^2}{4T^{1/2}} \geq a |x - x_0|^2 \geq u_0(x), \quad x \in B(x_0, \delta),$$

while we infer from the choice of T and (1.8) that, for $(t, x) \in (0, T) \times \partial B(x_0, \delta)$

$$S_T(t, x) = \frac{\delta^2}{4(T-t)^{1/2}} \geq \frac{\delta^2}{4T^{1/2}} \geq C_1 \geq u(t, x).$$

The comparison principle [11, Theorem 8.2] then entails that $S_T(t, x) \geq u(t, x)$ for $(t, x) \in [0, T) \times B(x_0, \delta)$. In particular, $0 \leq u(t, x_0) \leq S_T(t, x_0) = 0$ for $t \in [0, T)$, and the proof of Proposition 4.2 is complete. ////

Acknowledgements

This work was done while C. Stinner held a one month invited position at the Institut de Mathématiques de Toulouse, Université Paul Sabatier - Toulouse III. He would like to express his gratitude for the invitation, support, and hospitality.

References

- [1] G. Akagi, P. Juutinen, and R. Kajikiya, *Asymptotic behavior of viscosity solutions for a degenerate parabolic equation associated with the infinity-Laplacian*, Math. Ann. **343** (2009), 921–953.
- [2] G. Akagi and K. Suzuki, *Existence and uniqueness of viscosity solutions for a degenerate parabolic equation associated with the infinity-Laplacian*, Calc. Var. Partial Differential Equations **31** (2008), 457–471.
- [3] D.G. Aronson and L.A. Peletier, *Large time behaviour of solutions of the porous medium equation in bounded domains*, J. Differential Equations **39** (1981), 378–412.
- [4] G. Aronsson, *Extensions of functions satisfying Lipschitz conditions*, Ark. Mat. **6** (1967), 551–561.
- [5] G. Aronsson, M.G. Crandall, and P. Juutinen, *A tour of the theory of absolutely minimizing functions*, Bull. Amer. Math. Soc. (N.S.) **41** (2004), 439–505.
- [6] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Systems Control Found. Appl., Birkhäuser, Boston, 1997.

- [7] G. Barles, *Solutions de Viscosité des Equations d'Hamilton-Jacobi*, Mathématiques & Applications **17**, Springer, Berlin, 1994.
- [8] G. Barles and B. Perthame, *Exit time problems in optimal control and vanishing viscosity method*, SIAM J. Control Optim. **26** (1988), 1133–1148.
- [9] Ph. Bénilan and M.G. Crandall, *Regularizing effects of homogeneous evolution equations*, Contributions to analysis and geometry, 23–39, John Hopkins Univ. Press, Baltimore, Md., 1981.
- [10] M.G. Crandall, *A visit with the ∞ -Laplacian equation*, Calculus of variations and nonlinear partial differential equations, 75–122, Lecture Notes in Math., 1927, Springer, Berlin, 2008.
- [11] M.G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 1–67.
- [12] M.G. Crandall, P.-L. Lions, and P.E. Souganidis, *Maximal solutions and universal bounds for some partial differential equations of evolution*, Arch. Rational Mech. Anal. **105** (1989), 163–190.
- [13] M.G. Crandall and P.-Y. Wang, *Another way to say caloric*, J. Evol. Equ. **3** (2003), 653–672.
- [14] B.E.J. Dahlberg and C.E. Kenig, *Nonnegative solutions of the initial-Dirichlet problem for generalized porous medium equations in cylinders*, J. Amer. Math. Soc. **1** (1988), 401–412.
- [15] R. Dal Passo, L. Giacomelli, and G. Grün, *Waiting time phenomena for degenerate parabolic equations – a unifying approach*, Geometric analysis and nonlinear partial differential equations, 637–648, Springer, Berlin, 2003.
- [16] V.A. Galaktionov, *Evolution completeness of separable solutions of non-linear diffusion equations in bounded domains*, Math. Methods Appl. Sci. **27** (2004), 1755–1770.
- [17] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order. Reprint of the 1998 ed.*, Classics in Mathematics, Springer, Berlin, 2001.
- [18] B.F. Knerr, *The porous medium equation in one dimension*, Trans. Amer. Math. Soc. **234** (1977), 381–415.
- [19] G.M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, 1996.
- [20] J.J. Manfredi and V. Vespri, *Large time behavior of solutions to a class of doubly nonlinear parabolic equations*, Electron. J. Differential Equations **1994**, No. 02, (1994), 1–17.
- [21] J.L. Vázquez, *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs; Oxford Science Publications, Oxford University Press, 2007.