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A NON-COORDINATIZABLE SECTIONALLY COMPLEMENTED MODULAR LATTICE WITH A LARGE JÓNSSON FOUR-FRAME

FRIEDRICH WEHRUNG

ABSTRACT. A sectionally complemented modular lattice L is *coordinatizable* if it is isomorphic to the lattice $\mathbb{L}(R)$ of all principal right ideals of a von Neumann regular (not necessarily unital) ring R . We say that L has a *large 4-frame* if it has a homogeneous sequence (a_0, a_1, a_2, a_3) such that the neutral ideal generated by a_0 is L . Jónsson proved in 1962 that if L has a countable cofinal sequence and a large 4-frame, then it is coordinatizable; whether the cofinal sequence assumption could be dispensed with was left open. We solve this problem by finding a non-coordinatizable sectionally complemented modular lattice L with a large 4-frame; it has cardinality \aleph_1 . Furthermore, L is an ideal in a complemented modular lattice L' with a spanning 5-frame (in particular, L' is coordinatizable).

Our proof uses *Banaschewski functions*. A Banaschewski function on a bounded lattice L is an antitone self-map of L that picks a complement for each element of L . In an earlier paper, we proved that every *countable* complemented modular lattice has a Banaschewski function. We prove that there exists a unit-regular ring R of cardinality \aleph_1 and index of nilpotence 3 such that $\mathbb{L}(R)$ has no Banaschewski function.

1. INTRODUCTION

1.1. History of the problem. The set $\mathbb{L}(R)$ of all principal right ideals of a (not necessarily unital) von Neumann regular ring R , ordered by inclusion, is a sublattice of the lattice of all ideals of L ; hence it satisfies the *modular law*,

$$X \supseteq Z \implies X \cap (Y + Z) = (X \cap Y) + Z.$$

(Here $+$ denotes the addition of ideals.) Moreover, $\mathbb{L}(R)$ is *sectionally complemented*, that is, for all principal right ideals X and Y such that $X \subseteq Y$, there exists a principal right ideal Z such that $X \oplus Z = Y$. A lattice is *coordinatizable* if it is isomorphic to $\mathbb{L}(R)$ for some von Neumann regular ring R . In particular, every coordinatizable lattice is sectionally complemented modular. (For precise definitions we refer the reader to Section 2.) In his monograph [25], John von Neumann proved the following result:

Von Neumann's Coordinatization Theorem. *Every complemented modular lattice that admits a spanning n -frame, with $n \geq 4$, is coordinatizable.*

It is not hard to find non-coordinatizable complemented modular lattices. The easiest one to describe is the lattice M_7 of length two with seven atoms. Although

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von Neumann’s original proof is very long and technical (about 150 pages), its basic idea is fairly simple: namely, assume a sufficiently rich lattice-theoretical version of a coordinate system (the spanning n -frame, richness being measured by the condition $n \geq 4$) to carry over the ideas in projective geometry underlying the construction of “von Staudt’s algebra of throws” that makes it possible to go from *synthetic geometry* (geometry described by incidence axioms on “flats”) to *analytic geometry* (prove statements of geometry by using coordinates and algebra), see [12, Section IV.5]. Instead of constructing (a matrix ring over) a field, von Neumann’s method yields a regular ring.

A powerful generalization of von Neumann’s Coordinatization Theorem was obtained by Bjarni Jónsson in 1960, see [19]:

Jónsson’s Coordinatization Theorem. *Every complemented modular lattice L that admits a large n -frame, with $n \geq 4$ (or $n \geq 3$ if L is Arguesian), is coordinatizable.*

There have been many simplifications, mainly due to I. Halperin [13, 14, 15], of the proof of von Neumann’s Coordinatization Theorem. A substantial simplification of the proof of Jónsson’s Coordinatization Theorem has been achieved by Christian Herrmann [16]—*assuming the basic Coordinatization Theorem for Projective Geometries*, and thus reducing most of the complicated lattice calculations of both von Neumann’s proof and Jónsson’s proof to linear algebra. Now the Coordinatization Theorem for Projective Geometries is traditionally credited to Hilbert and to Veblen and Young, however, it is unclear whether a complete proof was published before von Neumann’s breakthrough in [25]. A very interesting discussion of the matter can be found in Israel Halperin’s review of Jónsson’s paper [19], cf. MR 0120175 (22 #10932).

On the other hand, there is in some sense no “Ultimate Coordinatization Theorem” for complemented modular lattices, as the author proved that there is no first-order axiomatization for the class of all coordinatizable lattices with unit [27].

While Von Neumann’s sufficient condition for coordinatizability requires the lattices have a unit (a spanning n -frame joins, by definition, to the unit of the lattice), Jónsson’s sufficient condition leaves more room for improvement. While Jónsson assumes a unit in his above-cited Coordinatization Theorem, a large n -frame does not imply the existence of a unit.

And indeed, Jónsson published in 1962 a new Coordinatization Theorem [20], assuming a large n -frame where $n \geq 4$, where the lattice L is no longer assumed to have a unit (it is still sectionally complemented)... *but* where the conclusion is weakened to L being isomorphic to the lattice of all finitely generated submodules of some *locally projective module* over a regular ring. He also proved that if L is countable, or, more generally, has a countable cofinal sequence, then, still under the existence of a large n -frame, it is coordinatizable. The question whether full coordinatizability could be reached in general was left open.

In the present paper we solve the latter problem, in the negative. Our counterexample is a non-coordinatizable sectionally complemented modular lattice L , of cardinality \aleph_1 , with a large 4-frame. Furthermore, L is isomorphic to an ideal in a complemented modular lattice L' with a spanning 5-frame (in particular, L' is coordinatizable).

Although our counterexample is constructed explicitly, our road to it is quite indirect. It starts with a discovery made in 1957, by Bernhard Banaschewski [1],

that on every vector space V , over an arbitrary division ring, there exists an *order-reversing* (we say *antitone*) map that sends any subspace X of V to a complement of X in V . Such a function was then used in order to find a simple proof of Hahn's Embedding Theorem that states that every totally ordered abelian group embeds into a generalized lexicographic power of the reals.

1.2. Banaschewski functions on lattices and rings. By analogy with Banaschewski's result, we define a *Banaschewski function* on a bounded lattice L as an antitone self-map of L that picks a complement for each element of L (Definition 3.1). Hence Banaschewski's above-mentioned result from [1] states that the subspace lattice of every vector space has a Banaschewski function. This result is extended to all *geometric* (not necessarily modular) lattices in Saarimäki and Sorjonen [26].

We proved in [28, Theorem 4.1] that *Every countable complemented modular lattice has a Banaschewski function*. In the present paper, we construct in Proposition 4.4 a unital regular ring $S_{\mathbb{F}}$ such that $\mathbb{L}(S_{\mathbb{F}})$ has no Banaschewski function. The ring $S_{\mathbb{F}}$ has the optimal cardinality \aleph_1 . Furthermore, $S_{\mathbb{F}}$ has index 3 (Proposition 4.5); in particular, it is unit-regular.

The construction of the ring $S_{\mathbb{F}}$ involves a parameter \mathbb{F} , which is any *countable* field, and $S_{\mathbb{F}}$ is a “ \mathbb{F} -algebra with quasi-inversion defined by generators and relations” in any large enough variety. Related structures have been considered in Goodearl, Menal, and Moncasi [11] and in Herrmann and Semenova [17].

1.3. From non-existence of Banaschewski functions to failure of coordinatizability. As we are aiming to a counterexample to the above-mentioned problem on coordinatization, we prove in Theorem 6.4 a stronger negative result, namely the non-existence of any “Banaschewski measure” on a certain increasing ω_1 -sequence of elements in L .

A modification of this example, based on the 5×5 matrix ring over $S_{\mathbb{F}}$, yields (Lemma 7.4) an ω_1 -increasing chain $\vec{A} = (A_{\xi} \mid \xi < \omega_1)$ of countable sectionally complemented modular lattices, all with the same large 4-frame, that cannot be lifted, with respect to the \mathbb{L} functor, by any ω_1 -chain of regular rings (Lemma 7.4). Our final conclusion follows from a use of a general categorical result, called the *Condensate Lifting Lemma* (CLL), introduced in a paper by Pierre Gillibert and the author [9], designed to relate liftings of *diagrams* and liftings of *objects*. Here, CLL will turn the *diagram counterexample* of Lemma 7.4 to the *object counterexample* of Theorem 7.5. This counterexample is a so-called *condensate* of the diagram \vec{A} by a suitable “ ω_1 -scaled Boolean algebra”. It has cardinality \aleph_1 (cf. Theorem 7.5). Furthermore, it is isomorphic to an ideal of a complemented modular lattice L' with a spanning 5-frame (so L' is coordinatizable).

2. BASIC CONCEPTS

2.1. Partially ordered sets and lattices. Let P be a partially ordered set. We denote by 0_P (respectively, 1_P) the least element (respectively, largest element) of P when they exist, also called *zero* (respectively, *unit*) of P , and we simply write 0 (respectively, 1) in case P is understood. Furthermore, we set $P^- := P \setminus \{0_P\}$. We

set

$$\begin{aligned} U \downarrow X &:= \{u \in U \mid (\exists x \in X)(u \leq x)\}, \\ U \uparrow X &:= \{u \in U \mid (\exists x \in X)(u \geq x)\}, \end{aligned}$$

for any subsets U and X of P , and we set $U \downarrow x := U \downarrow \{x\}$, $U \uparrow x := U \uparrow \{x\}$, for any $x \in P$. We say that U is a *lower subset* of P if $U = P \downarrow U$. We say that P is *upward directed* if every pair of elements of P is contained in $P \downarrow x$ for some $x \in P$. We say that U is *cofinal* in P if $P \downarrow U = P$. We define p^U the least element of $U \uparrow p$ if it exists, and we define p_U dually, for each $p \in P$. An *ideal* of P is a nonempty, upward directed, lower subset of P . We set

$$P^{[2]} := \{(x, y) \in P \times P \mid x \leq y\}.$$

For subsets X and Y of P , let $X < Y$ hold if $x < y$ holds for all $(x, y) \in X \times Y$. We shall also write $X < p$ (respectively, $p < X$) instead of $X < \{p\}$ (respectively, $\{p\} < X$), for each $p \in P$. For partially ordered sets P and Q , a map $f: P \rightarrow Q$ is *isotone* (*antitone*, *strictly isotone*, respectively) if $x < y$ implies that $f(x) \leq f(y)$ ($f(y) \leq f(x)$, $f(x) < f(y)$, respectively), for all $x, y \in P$.

We refer to Birkhoff [2] or Grätzer [12] for basic notions of lattice theory. We recall here a sample of needed notation, terminology, and results. In any lattice L with zero, a family $(a_i \mid i \in I)$ is *independent* if the equality

$$\bigvee(a_i \mid i \in X) \wedge \bigvee(a_i \mid i \in Y) = \bigvee(a_i \mid i \in X \cap Y)$$

holds for all finite subsets X and Y of I . In case L is modular and $I = \{0, 1, \dots, n-1\}$ for a positive integer n , this amounts to verifying that $a_k \wedge \bigvee_{i < k} a_i = 0$ for each $k < n$. We denote by \oplus the operation of finite independent sum in L , so $a = \bigoplus(a_i \mid i \in I)$ means that I is finite, $(a_i \mid i \in I)$ is independent, and $a = \bigvee_{i < n} a_i$. If L is modular, then \oplus is both commutative and associative in the strongest possible sense for a partial operation, see [22, Section II.1].

A lattice L with zero is *sectionally complemented* if for all $a \leq b$ in L there exists $x \in L$ such that $b = a \oplus x$. For elements $a, x, b \in L$, let $a \sim_x b$ (respectively, $a \lesssim_x b$) hold if $a \oplus x = b \oplus x$ (respectively, $a \oplus x \leq b \oplus x$). We say that a is *perspective* (respectively, *subperspective*) to b , in notation $a \sim b$ (respectively, $a \lesssim b$), if there exists $x \in L$ such that $a \sim_x b$ (respectively, $a \lesssim_x b$). We say that L is *complemented* if it has a unit and every element $a \in L$ has a *complement*, that is, an element $x \in L$ such that $1 = a \oplus x$. A bounded modular lattice is complemented if and only if it is sectionally complemented.

An ideal I of a lattice L is *neutral* if $\{I, X, Y\}$ generates a distributive sublattice of $\text{Id } L$ for all ideals X and Y of L . In case L is sectionally complemented modular, this is equivalent to the statement that every element of L perspective to some element of I belongs to I . In that case, the assignment that to a congruence θ associates the θ -block of 0 is an isomorphism from the congruence lattice of L onto the lattice of all neutral ideals of L .

An independent finite sequence $(a_i \mid i < n)$ in a lattice L with zero is *homogeneous* if the elements a_i are pairwise perspective. An element $x \in L$ is *large* if the neutral ideal generated by x is L . A family $((a_i \mid 0 \leq i < n), (c_i \mid 1 \leq i < n))$, with $(a_i \mid 0 \leq i < n)$ independent, is a

- *n-frame* if $a_0 \sim_{c_i} a_i$ for each i with $1 \leq i < n$;
- *large n-frame* if it is an *n-frame* and a_0 is large.

- *spanning n -frame* if it is a frame, L has a unit, and $1 = \bigoplus_{i < n} a_i$.

In a lattice with unit, every spanning n -frame is large; the converse fails for trivial examples. A *large partial n -frame* of a complemented modular lattice, as defined in Jónsson [19], consists of a large n -frame as defined above, together with a finite collection of elements of L joining to the unit of L and satisfying part of the relations defining frames, so that, for instance, all of them are subspective to a_0 . In particular, in a complemented modular lattice, the existence of a large partial n -frame (as defined by Jónsson) is equivalent to the existence of a large n -frame (as defined here).

Definition 2.1. Let m and n be positive integers with $m \geq n$. A modular lattice L with zero is *n/m -entire* if L has an ideal I and a homogeneous sequence $(a_i \mid i < m)$ such that, setting $a := \bigoplus_{i < n} a_i$,

- (i) each element $x \in I$ is a join of $m - n$ elements subspective to a_0 ; furthermore, $x \wedge a = 0$;
- (ii) $\{a \vee x \mid x \in I\}$ is cofinal in L .

Evidently, L has a spanning n -frame if and only if it is n/n -entire. Furthermore, if L is n/m -entire, then it has a large n -frame.

2.2. Set theory. By “countable” we will always mean “at most countable”. We denote by ω the first infinite ordinal and we identify it with the set of all non-negative integers. More generally, any ordinal α is identified with the set of all ordinals smaller than α . Cardinals are initial ordinals. For any ordinal α , we denote by ω_α the α th infinite cardinal. Following the usual set-theoretical convention, we also denote it by \aleph_α whenever we wish to view it as a cardinal in the “naive” sense.

Šanin’s classical Δ -Lemma (cf. [18, Lemma 22.6]) is the following.

Δ -Lemma. *Let \mathcal{W} be an uncountable collection of finite sets. Then there are an uncountable subset \mathcal{Z} of \mathcal{W} and a set Z (the root of \mathcal{Z}) such that $X \cap Y = Z$ for all distinct $X, Y \in \mathcal{Z}$.*

We shall require the following slight strengthening of the Δ -Lemma.

Lemma 2.2. *Let C be an uncountable subset of ω_1 and let $(S_\alpha \mid \alpha \in C)$ be a family of finite subsets of ω_1 . Then there are an uncountable subset W of C and a set Z such that*

$$S_\alpha \cap S_\beta = Z \quad \text{and} \quad Z < S_\alpha \setminus Z < S_\beta \setminus Z \quad \text{for all } \alpha < \beta \text{ in } W.$$

Proof. By a first application of the Δ -Lemma, we may assume that there exists a set Z such that $S_\alpha \cap S_\beta = Z$ for all distinct $\alpha, \beta \in C$. Put $X_\xi := S_\xi \setminus Z$, for each $\xi \in C$.

Claim. *For every countable $D \subset \omega_1$, there exists $\alpha \in C$ such that $D < X_\eta$ for each $\eta \in C \uparrow \alpha$.*

Proof of Claim. Let $\theta < \omega_1$ containing $D \cup Z$. For each $\xi \in \omega_1 \setminus Z$, there exists at most one element $f(\xi) \in C$ such that $\xi \in S_{f(\xi)}$. Any $\alpha \in C$, such that $f(\xi) < \alpha$ for each $\xi < \theta$ in the domain of f , satisfies the required condition. \square Claim.

By applying the Claim to $D := Z$, we get $\bar{\alpha} \in C$ such that $Z < X_\eta$ for each $\eta \in C \uparrow \bar{\alpha}$. Now let $\xi < \omega_1$ and suppose having constructed a strictly increasing ξ -sequence $(\alpha_\eta \mid \eta < \xi)$ in $C \uparrow \bar{\alpha}$ such that $\eta < \eta' < \xi$ implies that $X_{\alpha_\eta} < X_{\alpha_{\eta'}}$.

By applying the Claim to $\bigcup(X_{\alpha_\eta} \mid \eta < \xi)$, we obtain $\alpha_\xi \in C$, which can be taken above both $\bar{\alpha}$ and $\bigcup(\alpha_\eta \mid \eta < \xi)$, such that $X_{\alpha_\eta} < X_\zeta$ for each $\eta < \xi$ and each $\zeta \geq \alpha_\xi$. Take $W := \{\alpha_\xi \mid \xi < \omega_1\}$. \square

2.3. Von Neumann regular rings. All our rings will be associative but not necessarily unital. A ring R is (von Neumann) *regular* if for all $x \in R$ there exists $y \in R$ such that $xyx = x$. We shall call such an element y a *quasi-inverse* of x .

We shall need the following classical result (see [10, Theorem 1.7], or [7, Section 3.6] for the general, non-unital case).

Proposition 2.3. *For any regular ring R and any positive integer n , the ring $R^{n \times n}$ of all $n \times n$ matrices with entries in R is regular.*

For any regular ring R , we set $\mathbb{L}(R) := \{xR \mid x \in R\}$. If y is a quasi-inverse of x , then $xR = xyR$ and xy is idempotent, thus $\mathbb{L}(R) = \{eR \mid e \in R \text{ idempotent}\}$. It is well known that $\mathbb{L}(R)$ is a sectionally complemented sublattice of the (modular) lattice of all right ideals of R (cf. [6, Section 3.2]). The proof implies that \mathbb{L} defines a *functor* from the category of all regular rings with ring homomorphisms to the category of sectionally complemented modular lattices with 0-lattice homomorphisms (cf. Micol [24] for details). This functor preserves directed colimits.

Lemma 2.4 (folklore). *A regular ring R is unital if and only if $\mathbb{L}(R)$ has a largest element.*

Proof. We prove the non-trivial direction. Let $e \in R$ idempotent such that eR is the largest element of $\mathbb{L}(R)$. For each $x \in R$ with quasi-inverse y , observe that $x = xyx \in xR$, thus, as $xR \subseteq eR$ and by the idempotence of e , we get $x = ex$. Let y be a quasi-inverse of $x - xe$. From $y = ey$ it follows that $xy - xey = 0$, thus

$$x - xe = (x - xe)y(x - xe) = (xy - xey)(x - xe) = 0,$$

so $x = xe$. Therefore, e is the unit of R . \square

Denote by $\text{Idemp } R$ the set of all idempotent elements in a ring R . Define the *orthogonal sum* in $\text{Idemp } R$ by

$$a = \bigoplus_{i < n} a_i \iff \left(a = \sum_{i < n} a_i \text{ and } a_i a_j = 0 \text{ for all distinct } i, j < n \right).$$

For idempotents a and b in a ring R , let $a \trianglelefteq b$ hold if $a = ab = ba$; equivalently, there exists an idempotent x such that $a \oplus x = b$; and equivalently, $a \in bRb$.

We shall need the following well known (and easy) result.

Lemma 2.5 (folklore). *Let A and B be right ideals in a ring R and let e be an idempotent element of R . If $eR = A \oplus B$, then there exists a unique pair $(a, b) \in A \times B$ such that $e = a + b$. Furthermore, both a and b are idempotent, $e = a \oplus b$, $A = aR$, and $B = bR$.*

2.4. Category theory. For a partially ordered set I and a category \mathcal{A} , an *I -indexed diagram from \mathcal{A}* is a system $(A_i, f_i^j \mid i \leq j \text{ in } I)$, where all A_i are objects in \mathcal{A} , $f_i^j : A_i \rightarrow A_j$ in \mathcal{A} , and $f_i^i = \text{id}_{A_i}$ together with $f_i^k = f_j^k \circ f_i^j$ for $i \leq j \leq k$ in I . Such an object can of course be identified with a functor from I , viewed as a category the usual way, to \mathcal{A} . If \mathcal{B} is a category, $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is a functor, and \vec{B} is an I -indexed diagram from \mathcal{B} , we say that an I -indexed diagram \vec{A} from \mathcal{A} *lifts \vec{B} with respect to Φ* if there is a natural equivalence from $\Phi \vec{A}$ to \vec{B} (in notation $\Phi \vec{A} \cong \vec{B}$).

3. BANASCHEWSKI FUNCTIONS ON LATTICES AND RINGS

In the present section we recall some definitions and results from [28].

Definition 3.1. Let X be a subset in a bounded lattice L . A *partial Banaschewski function on X in L* is an antitone map $f: X \rightarrow L$ such that $x \oplus f(x) = 1$ for each $x \in X$. In case $X = L$, we say that f is a *Banaschewski function on L* .

Definition 3.2. Let X be a subset in a ring R . A *partial Banaschewski function on X in R* is a mapping $\varepsilon: X \rightarrow \text{Idemp } R$ such that

- (i) $xR = \varepsilon(x)R$ for each $x \in X$.
- (ii) $xR \subseteq yR$ implies that $\varepsilon(x) \leq \varepsilon(y)$, for all $x, y \in X$.

In case $X = R$ we say that f is a *Banaschewski function on R* .

In the context of Definition 3.2, we put

$$\mathbb{L}_R(X) := \{xR \mid x \in X\}. \quad (3.1)$$

We proved the following result in [28, Lemma 3.5].

Lemma 3.3. *Let R be a unital regular ring and let $X \subseteq R$. Then the following are equivalent:*

- (i) *There exists a partial Banaschewski function on $\mathbb{L}_R(X)$ in $\mathbb{L}(R)$.*
- (ii) *There exists a partial Banaschewski function on X in R .*

4. A COORDINATIZABLE COMPLEMENTED MODULAR LATTICE WITHOUT A BANASCHEWSKI FUNCTION

For a field \mathbb{F} , we consider the similarity type $\Sigma_{\mathbb{F}} = (0, 1, -, \cdot, ', (h_{\lambda} \mid \lambda \in \mathbb{F}))$ that consists of two symbols of constant 0 and 1, two binary operation symbols $-$ (difference) and \cdot (multiplication), one unary operation symbol $'$ (quasi-inversion), and a family of unary operations h_{λ} , for $\lambda \in \mathbb{F}$ (left multiplications by the elements in \mathbb{F}). We consider the variety $\mathbf{Reg}_{\mathbb{F}}$ of all unital \mathbb{F} -algebras with a distinguished operation $x \mapsto x'$ in which the identity $xx'x = x$ holds (i.e., $x \mapsto x'$ is a quasi-inversion). We shall call $\mathbf{Reg}_{\mathbb{F}}$ the *variety of all \mathbb{F} -algebras with quasi-inversion*. Of course, all the ring reducts of the structures in $\mathbf{Reg}_{\mathbb{F}}$ are regular, and the reducts of such structures to the subtype $\Sigma := (0, -, \cdot, ',)$ are *regular rings with quasi-inversion*.

Until Proposition 4.3 we shall fix a variety (i.e., the class of all the structures satisfying a given set of identities) \mathbf{V} of $\Sigma_{\mathbb{F}}$ -structures contained in $\mathbf{Reg}_{\mathbb{F}}$. By [23, Theorem V.11.2.4], it is possible to construct “objects defined by generators and relations” in any (quasi-)variety.

Definition 4.1. For any (possibly empty) chain Λ , we shall denote by $\mathcal{R}_{\mathbf{V}}(\Lambda)$ the \mathbf{V} -object defined by generators $\tilde{\alpha}$, for $\alpha \in \Lambda$, and the relations

$$\tilde{\alpha} = \tilde{\beta} \cdot \tilde{\alpha}, \quad \text{for all } \alpha \leq \beta \text{ in } \Lambda. \quad (4.1)$$

We shall write $\tilde{\alpha}^{\Lambda}$ instead of $\tilde{\alpha}$ in case Λ needs to be specified.

Observe, in particular, that the $(0, 1, -, \cdot, ', (h_{\lambda} \mid \lambda \in \mathbb{F}))$ -reduct of $\mathcal{R}_{\mathbf{V}}(\Lambda)$ is a regular \mathbb{F} -algebra.

For a chain Λ , denote by $\Lambda \sqcup \{0^b, 1^b\}$ the chain obtained by adjoining to Λ a new smallest element 0^b and a new largest element 1^b . Likewise, define $\Lambda \sqcup \{0^b\}$ and $\Lambda \sqcup \{1^b\}$. We extend the meaning of $\tilde{\alpha}$, for $\alpha \in \Lambda \sqcup \{0^b, 1^b\}$, by setting

$$\tilde{0}^b = 0 \text{ and } \tilde{1}^b = 1. \quad (4.2)$$

The equations (4.1) are still satisfied for all $\alpha \leq \beta$ in $\Lambda \sqcup \{0^b, 1^b\}$.

Denote by **Ch** the category whose objects are all the (possibly empty) chains and where, for chains A and B , a *morphism* from A to B is an isotone map from $A \sqcup \{0^b, 1^b\}$ to $B \sqcup \{0^b, 1^b\}$ fixing both 0^b and 1^b . In particular, we identify every isotone map from A to B with its extension that fixes both 0^b and 1^b . This occurs, in particular, in case A is a subchain of B and $f := e_A^B$ is the inclusion map from A into B ; in this case, we put $e_A^B := \mathcal{R}_V(e_A^B)$, the canonical $\Sigma_{\mathbb{F}}$ -morphism from $\mathcal{R}_V(A)$ to $\mathcal{R}_V(B)$.

Every morphism $f: A \rightarrow B$ in **Ch** induces a (unique) $\Sigma_{\mathbb{F}}$ -homomorphism $\mathcal{R}_V(f): \mathcal{R}_V(A) \rightarrow \mathcal{R}_V(B)$ by the rule

$$\mathcal{R}_V(f)(\tilde{\alpha}^A) = \widetilde{f(\alpha)}^B, \quad \text{for each } \alpha \in A \quad (4.3)$$

(use (4.1) and (4.2)). The assignments $\Lambda \mapsto \mathcal{R}_V(\Lambda)$, $f \mapsto \mathcal{R}_V(f)$ define a functor from **Ch** to **V**. For a chain Λ and an element $x \in \mathcal{R}_V(\Lambda)$, there are a $\Sigma_{\mathbb{F}}$ -term t and finitely many elements $\xi_1, \dots, \xi_n \in \Lambda$ such that

$$x = t(\tilde{\xi}_1, \dots, \tilde{\xi}_n). \quad (4.4)$$

in $\mathcal{R}_V(\Lambda)$. Any subset of Λ containing $\{\xi_1, \dots, \xi_n\}$ is called a *support* of x . In particular, every element of $\mathcal{R}_V(\Lambda)$ has a finite support, and a subset S is a support of x if and only if x belongs to the range of e_S^{Λ} .

Lemma 4.2. *Let A and B be chains and let f be a morphism from A to B in **Ch**. Let $x \in \mathcal{R}_V(A)$ and let S be a support of x . Then $f(S) \setminus \{0^b, 1^b\}$ is a support of $\mathcal{R}_V(f)(x)$.*

Proof. There is a representation of x as in (4.4) in $\mathcal{R}_V(A)$, with $\xi_1, \dots, \xi_n \in S$. As $\mathcal{R}_V(f)$ is a $\Sigma_{\mathbb{F}}$ -homomorphism, we obtain

$$\mathcal{R}_V(f)(x) = t(\widetilde{f(\xi_1)}, \dots, \widetilde{f(\xi_n)}) \quad \text{in } \mathcal{R}_V(B).$$

As $\widetilde{f(\xi_i)}$ belongs to $f(S) \cup \{0, 1\}$ for each i and both elements 0 and 1 of $\mathcal{R}_V(B)$ are interpretations of symbols of constant, the conclusion follows. \square

The following result implies immediately that all maps $e_A^B: \mathcal{R}_V(A) \rightarrow \mathcal{R}_V(B)$, for A a subchain of a chain B , are $\Sigma_{\mathbb{F}}$ -embeddings.

Proposition 4.3. *Let A and B be chains and let $f: A \rightarrow B$ be an isotone map. If f is one-to-one, then so is $\mathcal{R}_V(f)$.*

Proof. It suffices to prove that $\mathcal{R}_V(f)(x) = 0$ implies that $x = 0$, for each $x \in \mathcal{R}_V(A)$. There is a representation of x as in (4.4) in $\mathcal{R}_V(A)$. Put $S := \{\xi_1, \dots, \xi_n\}$ and $u := t(\tilde{\xi}_1^S, \dots, \tilde{\xi}_n^S)$. Let $g: B \rightarrow S \sqcup \{0^b\}$ be the map defined by the rule

$$g(\beta) := \begin{cases} \text{largest } \xi \in S \text{ such that } f(\xi) \leq \beta, & \text{if such a } \xi \text{ exists,} \\ 0^b, & \text{otherwise,} \end{cases} \quad \text{for each } \beta \in B.$$

It is obvious that g is isotone. Furthermore, as f is one-to-one and isotone, we obtain $g \circ f \circ e_S^A = \text{id}_S$, so $\mathcal{R}_V(g) \circ \mathcal{R}_V(f) \circ e_S^A = \text{id}_{\mathcal{R}_V(S)}$, and so, using the equality $\mathcal{R}_V(f)(x) = 0$,

$$u = \mathcal{R}_V(g) \circ \mathcal{R}_V(f) \circ e_S^A(u) = \mathcal{R}_V(g) \circ \mathcal{R}_V(f)(x) = 0,$$

and therefore $x = e_S^A(u) = 0$. \square

Now we shall put more conditions on the variety \mathbf{V} of \mathbb{F} -algebras with quasi-inversion. We fix a *countable* field \mathbb{F} , and we consider the following elements in the matrix ring $\mathbb{F}^{3 \times 3}$:

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe that $A^2 = A$, $B^2 = B$, and $A = BA \neq AB$.

Denote by $\mathbb{F}[M]$ the \mathbb{F} -subalgebra of $\mathbb{F}^{3 \times 3}$ generated by $\{M\}$, for any $M \in \mathbb{F}^{3 \times 3}$. In particular, both maps from $\mathbb{F} \times \mathbb{F}$ to $\mathbb{F}^{3 \times 3}$ defined by $(x, y) \mapsto xA + y(I - A)$ and $(x, y) \mapsto xB + y(I - B)$ are isomorphisms of \mathbb{F} -algebras onto $\mathbb{F}[A]$ and $\mathbb{F}[B]$, respectively, and $\mathbb{F}[A] \cap \mathbb{F}[B] = \mathbb{F} \cdot I$. For each $X \in \mathbb{F}^{3 \times 3}$, let X' be a quasi-inverse of X in the smallest member of $\{\mathbb{F} \cdot I, \mathbb{F}[A], \mathbb{F}[B], \mathbb{F}^{3 \times 3}\}$ containing X as an element. Endowing each of the algebras $\mathbb{F} \cdot I$, $\mathbb{F}[A]$, $\mathbb{F}[B]$, and $\mathbb{F}^{3 \times 3}$ with this quasi-inversion, we obtain a commutative diagram in $\mathbf{Reg}_{\mathbb{F}}$, represented in Figure 1. We denote

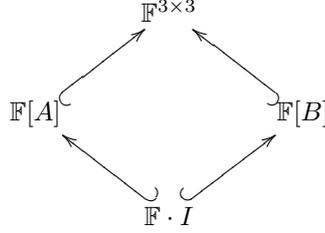


FIGURE 1. A commutative square in the variety $\mathbf{Reg}_{\mathbb{F}}$

by $R_{\mathbb{F}}$ the \mathbb{F} -algebra with quasi-inversion on $\mathbb{F}^{3 \times 3}$ just constructed, and we denote by $\mathbf{V}_{\mathbb{F}}$ the variety of \mathbb{F} -algebras with quasi-inversion generated by $R_{\mathbb{F}}$.

Proposition 4.4. *Let \mathbf{V} be any variety of \mathbb{F} -algebras with quasi-inversion such that $R_{\mathbb{F}} \in \mathbf{V}$. Then the following statements hold:*

- (i) *There exists no partial Banaschewski function on $\{\tilde{\xi} \mid \xi < \omega_1\}$ in the (unital, regular) ring $\mathcal{R}_{\mathbf{V}}(\omega_1)$. In particular, there is no Banaschewski function on the ring $\mathcal{R}_{\mathbf{V}}(\omega_1)$.*
- (ii) *There exists no partial Banaschewski function on $\{\tilde{\xi} \cdot \mathcal{R}_{\mathbf{V}}(\omega_1) \mid \xi < \omega_1\}$ in the (complemented, modular) lattice $\mathbb{L}(\mathcal{R}_{\mathbf{V}}(\omega_1))$. In particular, there is no Banaschewski function on the lattice $\mathbb{L}(\mathcal{R}_{\mathbf{V}}(\omega_1))$.*

Proof. A direct application of Lemma 3.3 shows that it is sufficient to establish the result of the first sentence of (i).

Set $X := \{\tilde{\xi} \mid \xi < \omega_1\}$ and suppose that there exists a partial Banaschewski function $\rho: X \rightarrow \text{Idemp } \mathcal{R}_{\mathbf{V}}(\Lambda)$. For each $\xi < \omega_1$, there exists $u_{\xi} \in \mathcal{R}_{\mathbf{V}}(\omega_1)$ such that

$$\tilde{\xi} = \tilde{\xi} \cdot u_{\xi} \cdot \tilde{\xi} \text{ and } \rho(\tilde{\xi}) = \tilde{\xi} \cdot u_{\xi} \text{ in } \mathcal{R}_{\mathbf{V}}(\Lambda). \quad (4.5)$$

Pick a finite support S_{ξ} of u_{ξ} containing $\{\xi\}$, for each $\xi < \omega_1$. By Lemma 2.2, there are a (finite) set Z and an uncountable subset W of ω_1 such that

$$S_{\xi} \cap S_{\eta} = Z \text{ and } Z < S_{\xi} \setminus Z < S_{\eta} \setminus Z \text{ for all } \xi < \eta \text{ in } W. \quad (4.6)$$

Put $S'_\xi := S_\xi \setminus Z$, for each $\xi \in W$. We define a map $f: \omega_1 \rightarrow W \sqcup \{0^b\}$ by the rule

$$f(\alpha) := \begin{cases} \text{least } \xi \in W \text{ such that } \alpha \in \omega_1 \downarrow S'_\xi, & \text{if } \alpha \in \omega_1 \uparrow S'_0, \\ 0^b, & \text{otherwise,} \end{cases} \quad \text{for each } \alpha < \omega_1.$$

The precaution to separate the case where $\alpha \in \omega_1 \downarrow S'_\xi$ is put there in order to ensure, using (4.6), that $f(\alpha) = 0^b$ for each $\alpha \in Z$. Observe that f is isotone and (using (4.6) again) that the restriction of f to S'_ξ is the constant map with value ξ , for each $\xi \in W$. In particular, $f \upharpoonright_W = \text{id}_W$.

Set $v_\xi := \mathcal{R}_{\mathbf{V}}(f)(u_\xi)$ and $e_\xi := \mathcal{R}_{\mathbf{V}}(f)(\rho(\tilde{\xi}))$, for each $\xi \in W$. By applying the morphism $\mathcal{R}_{\mathbf{V}}(f)$ to (4.5), we thus obtain that

$$\tilde{\xi} = \tilde{\xi} \cdot v_\xi \cdot \tilde{\xi} \text{ and } e_\xi = \tilde{\xi} \cdot v_\xi \text{ in } \mathcal{R}_{\mathbf{V}}(W), \quad \text{for each } \xi \in W. \quad (4.7)$$

Furthermore, by applying $\mathcal{R}_{\mathbf{V}}(f)$ to the relation $\rho(\tilde{\xi}) \leq \rho(\tilde{\eta})$, we obtain the system of relations

$$e_\xi \leq e_\eta \text{ in } \mathcal{R}_{\mathbf{V}}(W), \quad \text{for all } \xi \leq \eta \text{ in } W. \quad (4.8)$$

Furthermore, as u_ξ has support S_ξ and $f(S_\xi) = f(Z) \cup f(S'_\xi) \subseteq \{0^b, \xi\}$, it follows from Lemma 4.2 that $\{\xi\}$ is a support of v_ξ , so $v_\xi = \mathbf{t}_\xi(\tilde{\xi})$ for some term \mathbf{t}_ξ of $\Sigma_{\mathbb{F}}$. As \mathbb{F} is countable, there are only countably many terms in $\Sigma_{\mathbb{F}}$, thus, as W is uncountable, we may trim W further in order to ensure that there exists a term \mathbf{t} of $\Sigma_{\mathbb{F}}$ such that $\mathbf{t}_\xi = \mathbf{t}$ for each $\xi \in W$. Therefore, we have obtained that

$$v_\xi = \mathbf{t}(\tilde{\xi}) \text{ in } \mathcal{R}_{\mathbf{V}}(W), \quad \text{for each } \xi \in W. \quad (4.9)$$

Denote by \mathbf{e} the term of $\Sigma_{\mathbb{F}}$ defined by $\mathbf{e}(x) = x \cdot \mathbf{t}(x)$. In particular, from (4.7) and (4.9) it follows that $e_\xi = \mathbf{e}(\tilde{\xi})$ for each $\alpha \in W$.

From now on until the end of the proof, we shall fix $\alpha < \beta$ in W . As the \mathbb{F} -algebra with quasi-inversion $R_{\mathbb{F}}$ (with underlying ring $\mathbb{F}^{3 \times 3}$) belongs to the variety \mathbf{V} , as both A and B are idempotent with $A = BA$, and by the definition of $\mathcal{R}_{\mathbf{V}}(W)$, there exists a unique $\Sigma_{\mathbb{F}}$ -homomorphism $\varphi: \mathcal{R}_{\mathbf{V}}(W) \rightarrow R_{\mathbb{F}}$ such that

$$\varphi(\tilde{\xi}) = \begin{cases} A, & \text{if } \xi \leq \alpha, \\ B, & \text{otherwise,} \end{cases} \quad \text{for each } \xi \in W.$$

By applying the homomorphism φ to the equation $v_\alpha = \mathbf{t}(\tilde{\alpha})$, we obtain that $\varphi(v_\alpha) = \mathbf{t}(A)$ belongs to $\mathbb{F}[A]$ (because $\mathbb{F}[A]$ is a $\Sigma_{\mathbb{F}}$ -substructure of $R_{\mathbb{F}}$). Similarly, $\varphi(v_\beta) = \mathbf{t}(B)$ belongs to $\mathbb{F}[B]$. Using (4.7), it follows that

$$\varphi(e_\alpha) = \mathbf{e}(A), \quad \varphi(e_\beta) = \mathbf{e}(B), \quad A = A \cdot \mathbf{t}(A) \cdot A, \quad B = B \cdot \mathbf{t}(B) \cdot B. \quad (4.10)$$

From the third equation in (4.10) it follows that $A \cdot \mathbb{F}[A] = (A \cdot \mathbf{t}(A)) \cdot \mathbb{F}[A] = \mathbf{e}(A) \cdot \mathbb{F}[A]$. As the only non-trivial idempotent elements of $\mathbb{F}[A]$ are A and $I - A$, this leaves the only possibility $\mathbf{e}(A) = A$. Similarly, $\mathbf{e}(B) = B$.

However, by applying the homomorphism φ to the relation (4.8), we obtain that $\mathbf{e}(A) \leq \mathbf{e}(B)$ in $R_{\mathbb{F}}$ (*it is here that we really need the countability of \mathbb{F} , for we need $\mathbf{t}_\alpha = \mathbf{t}_\beta!$), so $A \leq B$. In particular, $A = AB$, a contradiction. \square*

Proposition 4.4 applies in particular to the case where \mathbf{V} is the variety $\mathbf{V}_{\mathbb{F}}$ generated by the algebra $R_{\mathbb{F}}$, that is, the class of all $\Sigma_{\mathbb{F}}$ -structures satisfying all the identities (in the similarity type $\Sigma_{\mathbb{F}}$) satisfied by $R_{\mathbb{F}}$.

The following result shows an additional property of the algebras $\mathcal{R}_{\mathbf{V}_{\mathbb{F}}}(\Lambda) := \mathcal{R}_{\mathbf{V}_{\mathbb{F}}}(\Lambda)$. Recall that the *index of nilpotence* of a nilpotent element a in a ring T is

the least positive integer n such that $a^n = 0$, and the *index* of T is the supremum of the indices of all elements of T .

Proposition 4.5. *Every member of the variety $\mathbf{V}_{\mathbb{F}}$ has index at most 3. In particular, the algebra $\mathcal{R}_{\mathbb{F}}(\Lambda)$ has index at most 3, for every chain Λ .*

Proof. By Birkhoff's **HSP** Theorem in Universal Algebra (see, for example, Theorems 9.5 and 11.9 in Burris and Sankappanavar [4]), every member T of $\mathbf{V}_{\mathbb{F}}$ is a $\Sigma_{\mathbb{F}}$ -homomorphic image of a $\Sigma_{\mathbb{F}}$ -substructure of a power of $R_{\mathbb{F}}$. As the underlying \mathbb{F} -algebra of $R_{\mathbb{F}}$ is $\mathbb{F}^{3 \times 3}$, it has index 3 (cf. [10, Theorem 7.2]), thus so does every power of $R_{\mathbb{F}}$, and thus also every subalgebra of every power of $R_{\mathbb{F}}$. As taking homomorphic images does not increase the index of regular rings (cf. [10, Proposition 7.7]), T has index at most 3. \square

Remark 4.6. It follows from Proposition 4.5 that $\mathcal{R}_{\mathbb{F}}(\omega_1)$ has index at most 3 (it is not hard to see that it is exactly 3). In particular, by [10, Corollary 7.11], $\mathcal{R}_{\mathbb{F}}(\omega_1)$ is unit-regular.

If \mathbb{F} is finite, then more can be said. Set $R := R_{\mathbb{F}}$ for brevity. It follows from one of the proofs of Birkhoff's **HSP** Theorem that the free algebra F_n on n generators in the variety $\mathbf{V}_{\mathbb{F}}$ is isomorphic to the $\Sigma_{\mathbb{F}}$ -substructure of R^{R^n} generated by the n canonical projections from R^n onto R . In particular, F_n is finite. It follows that the \mathbb{F} -algebra with quasi-inversion $\mathcal{R}_{\mathbb{F}}(\Lambda)$ is *locally finite*.

To summarize, we have obtained that *If \mathbb{F} is a finite field, then $\mathcal{R}_{\mathbb{F}}(\omega_1)$ is a locally finite regular \mathbb{F} -algebra with index 3, but without a Banaschewski function.*

Remark 4.7. Part (a) of [10, Proposition 2.13] implies that for every increasing sequence (indexed by the non-negative integers) $(I_n \mid n < \omega)$ of principal right ideals in a unital regular ring R , there exists a \leq -increasing sequence $(e_n \mid n < \omega)$ of idempotents of R such that $I_n = e_n R$ for each $n < \omega$. The origin of this argument can be traced back to Kaplansky's proof that every countably generated right ideal in a regular ring is projective [21, Lemma 1].

Proposition 4.4 implies that the result above cannot be extended to ω_1 -sequences of principal right ideals, even if the ring R has bounded index by Proposition 4.5.

Observe that Kaplansky finds in [21] a non-projective (uncountable) right ideal in a regular ring. Another example, suggested to the author by Luca Giudici, runs as follows. Let X be a locally compact, Hausdorff, non paracompact zero-dimensional space. A classical example of such a space is given by the closed subspace of Dieudonné's long ray consisting of the first uncountable ordinal ω_1 endowed with its order topology (all intervals of the form either $\omega_1 \downarrow \alpha$ or $\omega_1 \uparrow \alpha$, for $\alpha < \omega_1$, form a basis of closed sets of the topology). Now let Y be the one-point compactification of X . Denote by B the Boolean algebra of all clopen subsets of Y , and by I the ideal of B consisting of all the clopen subsets of X . Then B is a commutative regular ring and I is a non-projective ideal of B (cf. Bkouche [3], Finney and Rotman [5]). In the particular case where X is the example above, I is the union of the increasing chain of principal ideals corresponding to the intervals $[0, \alpha]$, for $\alpha < \omega_1$.

However, we do not know any relation, beyond the formal analogy outlined above, between projectivity of ideals and existence of Banaschewski functions. In particular, while Kaplansky's construction in [21] is given as an algebra over *any* field \mathbb{F} , the construction of our counterexample in Section 4 requires \mathbb{F} be *countable*. Moreover, in Giudici's example above, the identity function on B is a Banaschewski function on (the ring) B .

5. BANASCHEWSKI MEASURES ON SUBSETS OF LATTICES WITH ZERO

In order to reach our final coordinatization failure result (Theorem 7.5) we need the following variant of Banaschewski functions, introduced in [28, Definition 5.5].

Definition 5.1. Let X be a subset in a lattice L with zero. A L -valued *Banaschewski measure on X* is a map $\ominus: X^{[2]} \rightarrow L$, $(x, y) \mapsto y \ominus x$, isotone in y and antitone in x , such that $y = x \oplus (y \ominus x)$ for all $x \leq y$ in X .

The following lemma gives us an equivalent definition in case L is *modular*.

Lemma 5.2. *Let X be a subset in a modular lattice L with zero. Then a map $\ominus: X^{[2]} \rightarrow L$ is a Banaschewski measure if and only if*

$$y = x \oplus (y \ominus x) \quad \text{and} \quad z \ominus x = (z \ominus y) \oplus (y \ominus x), \quad \text{for all } x \leq y \leq z \text{ in } X. \quad (5.1)$$

Furthermore, if this holds, then

$$y \ominus x = y \wedge (z \ominus x), \quad \text{for all } x \leq y \leq z \text{ in } X. \quad (5.2)$$

Proof. Condition (5.1) trivially implies that \ominus is a Banaschewski measure on X . Conversely, assume that \ominus is a Banaschewski measure on X , and let $x \leq y \leq z$ in X . The equality $y = x \oplus (y \ominus x)$ follows from the definition of a Banaschewski measure. As, in addition, $z = y \oplus (z \ominus y)$ and from the associativity of the partial operation \oplus (which follows from the modularity of L), it follows that $z = x \oplus u$ where $u := (z \ominus y) \oplus (y \ominus x)$. Hence both u and $z \ominus x$ are sectional complements of x in z with $u \leq z \ominus x$, whence, by the modularity of L , $u = z \ominus x$. This concludes the proof of the first equivalence.

Now assume that \ominus is a Banaschewski measure on X , let $x \leq y \leq z$ in X , and set $v := y \wedge (z \ominus x)$. Trivially, $x \wedge v = 0$. Furthermore, as $x \leq y$ and by the modularity of L ,

$$x \vee v = y \wedge (x \vee (z \ominus x)) = y \wedge z = y.$$

Therefore, $x \oplus (y \ominus x) = y = x \oplus v$, thus, as $y \ominus x \leq v$ and L is modular, $v = y \ominus x$. \square

Lemma 5.3. *Let L be a modular lattice with zero, let $e, b \in L$ such that $e \oplus b = 1$, and let $X \subseteq L \downarrow b$. If there exists an L -valued Banaschewski function on $e \oplus X := \{e \oplus x \mid x \in X\}$, then there exists a $(L \downarrow b)$ -valued Banaschewski function on X .*

Proof. By assumption, there exists an L -valued Banaschewski measure \ominus on $e \oplus X$. We set

$$y \ominus' x := b \wedge [e \vee ((e \oplus y) \ominus (e \oplus x))], \quad \text{for all } x \leq y \text{ in } X.$$

Clearly, the map \ominus' thus defined is $(L \downarrow b)$ -valued, and isotone in y while antitone in x . For all $x \leq y$ in X , it follows from the equation $e \oplus y = e \oplus x \oplus ((e \oplus y) \ominus (e \oplus x))$ and the modularity of L that

$$x \wedge [e \vee ((e \oplus y) \ominus (e \oplus x))] = 0,$$

so, as $x \leq b$, we get $x \wedge (y \ominus' x) = 0$. On the other hand,

$$\begin{aligned} x \vee (y \ominus' x) &= b \wedge [x \vee e \vee ((e \oplus y) \ominus (e \oplus x))] \quad (\text{because } x \leq b \text{ and } L \text{ is modular}) \\ &= b \wedge (e \vee y) \\ &= (b \wedge e) \vee y \quad (\text{because } y \leq b \text{ and } L \text{ is modular}) \\ &= y, \end{aligned}$$

so $x \oplus (y \ominus' x) = y$. \square

6. AN ω_1 -SEQUENCE WITHOUT A BANASCHEWSKI MEASURE

Throughout this section we shall use the notation of Section 4. A term \mathbf{t} of a similarity type containing $\Sigma := (0, -, \cdot, ')$ is *strongly idempotent* if either $\mathbf{t} = \mathbf{u} \cdot \mathbf{u}'$ or $\mathbf{t} = \mathbf{u}' \cdot \mathbf{u}$ for some term \mathbf{u} of Σ . We define strongly idempotent terms \mathbf{k} and \mathbf{m} of Σ by

$$\mathbf{k}(x, y) := (yy' - xx'yy')' \cdot (yy' - xx'yy'), \quad (6.1)$$

$$\mathbf{m}(x, y) := (yy' - yy'\mathbf{k}(x, y)) \cdot (yy' - yy'\mathbf{k}(x, y))'. \quad (6.2)$$

We shall need the following lemma, that follows immediately from the trivial fact that $xx'R = xR$ for any element x with quasi-inverse x' in a regular ring R , together with [6, Section 3.2].

Lemma 6.1. *The equality $xR \cap yR = \mathbf{m}(x, y)R$ holds, for any elements x and y in a regular ring R with quasi-inversion.*

Until the statement of Theorem 6.4 we shall fix a countable field \mathbb{F} and a variety \mathbf{V} of regular \mathbb{F} -algebras with quasi-inversion. We shall denote by $\mathcal{L}_{\mathbf{V}} := \mathbb{L} \circ \mathcal{R}_{\mathbf{V}}$ the composite functor (from \mathbf{Ch} to the category of all sectionally complemented modular lattices with 0-lattice homomorphisms).

A subset S in a chain Λ is a *support* of an element $I \in \mathcal{L}_{\mathbf{V}}(\Lambda)$ if I belongs to the range of $\mathcal{L}_{\mathbf{V}}(e_S^\Lambda)$. Equivalently, $I = x \cdot \mathcal{R}_{\mathbf{V}}(\Lambda)$ for some $x \in \mathcal{R}_{\mathbf{V}}(\Lambda)$ with support S .

Lemma 6.2. *Let Λ be a chain, let $I \in \mathcal{L}_{\mathbf{V}}(\Lambda)$, let $X \subseteq \Lambda$, and let $\xi \in \Lambda$. If both X and $\Lambda \setminus \{\xi\}$ support I , then $X \setminus \{\xi\}$ supports I .*

Proof. As some finite subset of X is a support of I , we may assume that X is finite. Moreover, the conclusion is trivial in case $\xi \notin X$, so we may assume that $\xi \in X$. Let $f: \Lambda \rightarrow \Lambda \sqcup \{0^b, 1^b\}$ defined by

$$f(\eta) := \begin{cases} \xi, & \text{if } \eta = \xi, \\ \eta^X, & \text{if } \eta > \xi \text{ and } \eta \in \Lambda \downarrow X, \\ 1^b, & \text{if } \eta > \xi \text{ and } \eta \notin \Lambda \downarrow X, \\ \eta_X, & \text{if } \eta < \xi \text{ and } \eta \in \Lambda \uparrow X, \\ 0^b, & \text{if } \eta < \xi \text{ and } \eta \notin \Lambda \uparrow X \end{cases}$$

(we refer the reader to Section 2.1 for the notations η^X, η_X). Evidently, f is isotone. In particular, $\mathcal{L}_{\mathbf{V}}(f)$ is an endomorphism of $\mathcal{L}_{\mathbf{V}}(\Lambda)$.

From $f \upharpoonright_X = \text{id}_X$ and the assumption that X is a support of I it follows that $\mathcal{L}_{\mathbf{V}}(f)(I) = I$. On the other hand, as $\Lambda \setminus \{\xi\}$ is a support of I and $f(\Lambda \setminus \{\xi\})$ is contained in $(X \setminus \{\xi\}) \cup \{0^b, 1^b\}$, $X \setminus \{\xi\}$ is a support of $\mathcal{L}_{\mathbf{V}}(f)(I)$ (as in the proof of Lemma 4.2). The conclusion follows. \square

As every element of $\mathcal{L}_{\mathbf{V}}(\Lambda)$ has a finite support, we obtain immediately the following.

Corollary 6.3. *Let Λ be a chain. Then every element $I \in \mathcal{L}_{\mathbf{V}}(\Lambda)$ has a smallest (for containment) support, that we shall denote by $\text{supp } I$ and call the support of I . Furthermore, $\text{supp } I$ is finite.*

We can now prove the main result of this section. The \mathbb{F} -algebra with quasi-inversion $R_{\mathbb{F}}$ is defined in Section 4 (cf. Figure 1).

Theorem 6.4. *Let \mathbb{F} be a countable field and let \mathbf{V} be a variety of \mathbb{F} -algebras with quasi-inversion containing $R_{\mathbb{F}}$ as an element. Then there exists no $\mathcal{L}_{\mathbf{V}}(\omega_1)$ -valued Banaschewski measure on the subset $\mathcal{X}_{\mathbb{F}} := \{\tilde{\xi} \cdot \mathcal{R}_{\mathbf{V}}(\omega_1) \mid \xi < \omega_1\}$.*

Proof. The structure $T := \mathcal{R}_{\mathbf{V}}(\omega_1)$ is a regular \mathbb{F} -algebra with quasi-inversion. Let \mathbf{t} be a term of $\Sigma_{\mathbb{F}}$ with arity n , let Λ be a chain, and let $X = \{\xi_1, \dots, \xi_n\}$ with all $\xi_i \in \Lambda$ and $\xi_1 < \dots < \xi_n$. We shall write

$$\mathbf{t}[X] := \mathbf{t}(\tilde{\xi}_1, \dots, \tilde{\xi}_n) \quad \text{evaluated in } \mathcal{R}_{\mathbf{V}}(\Lambda).$$

Similarly, if $n = k + l$, $X = \{\xi_1, \dots, \xi_k\}$ with $\xi_1 < \dots < \xi_k$, and $Y = \{\eta_1, \dots, \eta_l\}$ with $\eta_1 < \dots < \eta_l$, we shall write

$$\mathbf{t}[X; Y] := \mathbf{t}(\tilde{\xi}_1, \dots, \tilde{\xi}_k, \tilde{\eta}_1, \dots, \tilde{\eta}_l) \quad \text{evaluated in } \mathcal{R}_{\mathbf{V}}(\Lambda).$$

If $Y = \{\eta_1, \dots, \eta_n\}$ with $\eta_1 < \dots < \eta_n$ and $a \in \mathcal{R}_{\mathbf{V}}(\Lambda)$, we shall write

$$\mathbf{t}[a; Y] := \mathbf{t}(a, \tilde{\eta}_1, \dots, \tilde{\eta}_n) \quad \text{evaluated in } \mathcal{R}_{\mathbf{V}}(\Lambda).$$

Now let \ominus be an $\mathcal{L}_{\mathbf{V}}(\omega_1)$ -valued Banaschewski measure on \mathcal{X} .

For all $\alpha \leq \beta < \omega_1$, there are a finite subset $S_{\alpha, \beta}$ of ω_1 and a term $\mathbf{t}_{\alpha, \beta}$ of $\Sigma_{\mathbb{F}}$ such that

$$\tilde{\beta} \cdot T \ominus \tilde{\alpha} \cdot T = \mathbf{t}_{\alpha, \beta}[S_{\alpha, \beta}] \cdot T. \quad (6.3)$$

As $x \cdot T = (xx') \cdot T$ for each $x \in T$, we may assume that the term $\mathbf{t}_{\alpha, \beta}$ is *strongly idempotent*. By Lemma 2.2, for each $\alpha < \omega_1$, there are an uncountable subset W_{α} and a finite subset Z_{α} of ω_1 such that, setting $S'_{\alpha, \beta} := S_{\alpha, \beta} \setminus Z_{\alpha}$,

$$S_{\alpha, \beta} \cap S_{\alpha, \gamma} = Z_{\alpha} \text{ and } Z_{\alpha} < S'_{\alpha, \beta} < S'_{\alpha, \gamma}, \quad \text{for all } \beta < \gamma \text{ in } W_{\alpha}. \quad (6.4)$$

As the similarity type $\Sigma_{\mathbb{F}}$ is countable, we may refine further the uncountable subset W_{α} in such a way that $\mathbf{t}_{\alpha, \beta} = \mathbf{t}_{\alpha} = \text{constant}$, for all $\beta \in W_{\alpha}$.

Now let $\alpha \leq \beta < \omega_1$. Pick $\gamma, \delta \in W_{\alpha}$ such that $\beta < \gamma < \delta$. We compute

$$\begin{aligned} \tilde{\beta} \cdot T \ominus \tilde{\alpha} \cdot T &= \tilde{\beta} \cdot T \cap (\tilde{\gamma} \cdot T \ominus \tilde{\alpha} \cdot T) \quad (\text{by the second part of Lemma 5.2}) \\ &= \tilde{\beta} \cdot T \cap \mathbf{t}_{\alpha}[S_{\alpha, \gamma}] \cdot T, \end{aligned}$$

so, by using Lemma 6.1,

$$\tilde{\beta} \cdot T \ominus \tilde{\alpha} \cdot T = \mathbf{m}(\tilde{\beta}, \mathbf{t}_{\alpha}[S_{\alpha, \gamma}]) \cdot T. \quad (6.5)$$

In particular, the support of $\tilde{\beta} \cdot T \ominus \tilde{\alpha} \cdot T$ (cf. Corollary 6.3) is contained in $S_{\alpha, \gamma} \cup \{\beta\}$. Similarly, this support is contained in $S_{\alpha, \delta} \cup \{\beta\}$, and so, by (6.4),

$$\text{supp}(\tilde{\beta} \cdot T \ominus \tilde{\alpha} \cdot T) \subseteq Z_{\alpha} \cup \{\beta\}. \quad (6.6)$$

Now set $k_{\alpha} := \text{card } Z_{\alpha}$, for each $\alpha < \omega_1$, and define a new term \mathbf{u}_{α} by

$$\mathbf{u}_{\alpha}(x, y_1, \dots, y_{k_{\alpha}}) := \mathbf{m}(x, \mathbf{t}_{\alpha}(y_1, \dots, y_{k_{\alpha}}, 1, \dots, 1)), \quad (6.7)$$

where the number of occurrences of the constant 1 in the right hand side of (6.7) is equal to $\text{arity}(\mathbf{t}_{\alpha}) - k_{\alpha}$. As \mathbf{m} is strongly idempotent, so is \mathbf{u}_{α} .

Claim 1. *The equality $\tilde{\beta} \cdot T \ominus \tilde{\alpha} \cdot T = \mathbf{u}_{\alpha}[\tilde{\beta}; Z_{\alpha}] \cdot T$ holds for all $\alpha \leq \beta < \omega_1$ such that $Z_{\alpha} \subseteq \beta + 1$.*

Proof of Claim. Pick $\gamma \in W_\alpha$ such that $\beta < S'_{\alpha, \gamma}$ (by (6.4), this is possible) and define the isotone map $f: \omega_1 \rightarrow \omega_1 \sqcup \{1^b\}$ by the rule

$$f(\xi) := \begin{cases} \xi & (\text{if } \xi \leq \beta) \\ 1^b & (\text{if } \xi > \beta) \end{cases}, \quad \text{for each } \xi < \omega_1.$$

Every element of $Z_\alpha \cup \{\beta\}$ lies below β , thus it is fixed by f , while f sends each element of $S'_{\alpha, \gamma}$ to 1^b . Hence, by applying the morphism $\mathcal{L}_\mathbf{V}(f)$ to each side of (6.5) and by using the definition (6.7), we obtain

$$\mathcal{L}_\mathbf{V}(f)(\tilde{\beta} \cdot T \oplus \tilde{\alpha} \cdot T) = \mathbf{u}_\alpha[\tilde{\beta}; Z_\alpha] \cdot T.$$

On the other hand, as every element of $Z_\alpha \cup \{\beta\}$ is fixed by f , it follows from (6.6) that $\tilde{\beta} \cdot T \oplus \tilde{\alpha} \cdot T$ is fixed under $\mathcal{L}_\mathbf{V}(f)$. The conclusion follows. \square Claim 1.

As \mathbf{u}_α is a strongly idempotent term, the element $e_\alpha := \mathbf{u}_\alpha[1; Z_\alpha]$ is idempotent in T .

Claim 2. *The relation $T = \tilde{\alpha} \cdot T \oplus e_\alpha \cdot T$ holds for each $\alpha < \omega_1$.*

Proof of Claim. Let $\beta < \omega_1$ with $\alpha < \beta$ and $Z_\alpha < \beta$, and define an isotone map $g: \omega_1 \rightarrow \omega_1 \sqcup \{1^b\}$ by the rule

$$g(\xi) := \begin{cases} \xi & (\text{if } \xi < \beta) \\ 1^b & (\text{if } \xi \geq \beta) \end{cases}, \quad \text{for each } \xi < \omega_1.$$

From Claim 1 it follows that $\tilde{\beta} \cdot T = \tilde{\alpha} \cdot T \oplus \mathbf{u}_\alpha[\tilde{\beta}; Z_\alpha] \cdot T$, thus, applying the 0-lattice homomorphism $\mathcal{L}_\mathbf{V}(g)$, we obtain

$$T = \tilde{\alpha} \cdot T \oplus \mathbf{u}_\alpha[1; Z_\alpha] \cdot T = \tilde{\alpha} \cdot T \oplus e_\alpha \cdot T. \quad \square \text{ Claim 2.}$$

Claim 3. *The containment $e_\beta \cdot T \subseteq e_\alpha \cdot T$ holds, for all $\alpha \leq \beta < \omega_1$.*

Proof of Claim. Pick $\gamma < \omega_1$ such that $\beta < \gamma$ and $Z_\alpha \cup Z_\beta < \gamma$. We compute

$$\begin{aligned} \mathbf{u}_\beta[\tilde{\gamma}; Z_\beta] \cdot T &= \tilde{\gamma} \cdot T \oplus \tilde{\beta} \cdot T && (\text{by Claim 1}) \\ &\subseteq \tilde{\gamma} \cdot T \oplus \tilde{\alpha} \cdot T && (\text{by the monotonicity assumption on } \ominus) \\ &= \mathbf{u}_\alpha[\tilde{\gamma}; Z_\alpha] \cdot T && (\text{by Claim 1}), \end{aligned}$$

thus, as $\mathbf{u}_\alpha[\tilde{\gamma}; Z_\alpha]$ is idempotent,

$$\mathbf{u}_\beta[\tilde{\gamma}; Z_\beta] = \mathbf{u}_\alpha[\tilde{\gamma}; Z_\alpha] \cdot \mathbf{u}_\beta[\tilde{\gamma}; Z_\beta]. \quad (6.8)$$

Now define an isotone map $h: \omega_1 \rightarrow \omega_1 \sqcup \{1^b\}$ by the rule

$$h(\xi) := \begin{cases} \xi & (\text{if } \xi < \gamma) \\ 1^b & (\text{if } \xi \geq \gamma) \end{cases}, \quad \text{for each } \xi < \omega_1.$$

By applying $\mathcal{R}_\mathbf{V}(h)$ to the equation (6.8), we obtain that $e_\beta = e_\alpha \cdot e_\beta$. The conclusion follows. \square Claim 3.

By Claims 2 and 3, the family $(e_\alpha \cdot T \mid \alpha < \omega_1)$ defines a partial Banaschewski function on $\{\tilde{\alpha} \cdot T \mid \alpha < \omega_1\}$ in $\mathcal{L}_\mathbf{V}(\omega_1) = \mathbb{L}(\mathcal{R}_\mathbf{V}(\omega_1))$. This contradicts the result of Proposition 4.4(ii). \square

7. A NON-COORDINATIZABLE LATTICE WITH A LARGE 4-FRAME

A weaker variant of Jónsson’s Problem, of finding a non-coordinatizable sectionally complemented modular lattice with a large 4-frame, asks for a *diagram counterexample* instead of an *object counterexample*. In order to solve the full problem, we shall first settle the weaker version, by finding an ω_1 -indexed diagram of 4/5-entire countable sectionally complemented modular lattices that cannot be lifted with respect to the \mathbb{L} functor (cf. Lemma 7.4).

The full solution of Jónsson’s Problem will then be achieved by invoking a tool from *category theory*, introduced in Gillibert and Wehrung [9], designed to turn diagram counterexamples to object counterexamples. This tool is called there the “Condensate Lifting Lemma” (CLL). The general context of CLL is the following. We are given *categories* \mathcal{A} , \mathcal{B} , \mathcal{S} together with *functors* $\Phi: \mathcal{A} \rightarrow \mathcal{S}$ and $\Psi: \mathcal{B} \rightarrow \mathcal{S}$, such that for “many” objects $A \in \mathcal{A}$, there exists an object $B \in \mathcal{B}$ such that $\Phi(A) \cong \Psi(B)$. We are trying to find an assignment $\Gamma: \mathcal{A} \rightarrow \mathcal{B}$, “as functorial as possible”, such that $\Phi \cong \Psi\Gamma$ on a “large” subcategory of \mathcal{A} . Roughly speaking, CLL states that if the initial categorical data can be augmented by subcategories $\mathcal{A}^\dagger \subseteq \mathcal{A}$ and $\mathcal{B}^\dagger \subseteq \mathcal{B}$ (the “small objects”) together with $\mathcal{S}^\Rightarrow \subseteq \mathcal{S}$ (the “double arrows”) such that $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \Phi, \Psi, \mathcal{A}^\dagger, \mathcal{B}^\dagger, \mathcal{S}^\Rightarrow)$ forms a *projectable larder*, then this can be done. Checking larderhood, although somehow tedious, is a relatively easy matter, the least trivial point, already checked in [9], being the verification of the Löwenheim-Skolem Property $\text{LS}_{\aleph_1}^r(B)$ (cf. the proof of Lemma 7.2).

Besides an infinite combinatorial lemma by Gillibert, namely [8, Proposition 4.6], we shall need only a small part of [9]; basically, referring to the numbering used in version 1 of [9] (which is the current version as to the present writing),

- The definition of a projectability witness (Definition 1-5.1 in [9]).
- The definition of a projectable larder (Definition 3-4.1 in [9]). Strong larders will not be used.
- The statement of CLL (Lemma 3-4.2 in [9]), for $\lambda = \mu = \aleph_1$. This statement involves the category \mathbf{Bool}_P (Definition 2-2.3 in [9]), here for $P := \omega_1$, and the definition of $\mathbf{B} \otimes \vec{A}$ for $\mathbf{B} \in \mathbf{Bool}_P$ and a P -indexed diagram \vec{A} . These constructions are rather easy and only a few of their properties, recorded in Chapter 2 of [9], will be used. A full understanding of *lifters*, or of the P -scaled Boolean algebra $\mathbf{F}(X)$ involved in the statement of CLL, is not needed.
- Parts of Chapter 6 in [9], that are, essentially, easy categorical statements about regular rings.

We shall consider the similarity type $\Gamma := (0, \vee, \wedge, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{l})$, where 0, 1, the \mathbf{a}_i s, and the \mathbf{c}_i s are symbols of constant, both \vee and \wedge are symbols of binary operations, and \mathbf{l} is a (unary) predicate symbol. Furthermore, we consider the axiom system \mathcal{T} in Γ that states the following:

- (LAT) $(0, \vee, \wedge)$ defines a sectionally complemented modular lattice structure;
- (HOM) $(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is independent and $\mathbf{a}_0 \sim_{\mathbf{c}_i} \mathbf{a}_i$ for each $i \in \{1, 2, 3\}$;
- (ID) \mathbf{l} is an ideal;
- (REM) every element of \mathbf{l} is subspective to \mathbf{a}_0 and disjoint from $\bigoplus_{i=0}^3 \mathbf{a}_i$;
- (BASE) every element lies below $x \oplus \bigoplus_{i=0}^3 \mathbf{a}_i$ for some $x \in \mathbf{l}$.

In particular, (the underlying lattice of) every model for \mathcal{T} is 4/5-entire (cf. Definition 2.1), so it has a large 4-frame.

Observe that every axiom of \mathcal{T} has the form $(\forall \vec{x})(\varphi(\vec{x}) \Rightarrow (\exists \vec{y})\psi(\vec{x}, \vec{y}))$ for finite conjunctions of atomic formulas φ and ψ . For example, the axiom (REM) can be written

$$(\forall x)\left(\mathbb{1}(x) \Rightarrow (x \wedge (a_0 \vee a_1 \vee a_2 \vee a_3) = 0 \text{ and } (\exists y)(x \wedge y = a_0 \wedge y = 0 \text{ and } x \leq a_0 \vee y))\right).$$

It follows that the category \mathcal{A} of all models of \mathcal{T} , with their homomorphisms, is closed under arbitrary products and direct limits (i.e., directed colimits) of models.

Denote by \mathcal{S} the category of all sectionally complemented modular lattices with 0-lattice homomorphisms, and denote by Φ the forgetful functor from \mathcal{A} to \mathcal{S} .

Denote by \mathcal{B} the category of all von Neumann regular rings with ring homomorphisms, and take $\Psi := \mathbb{L}$, which is indeed a functor from \mathcal{B} to \mathcal{S} .

Denote by \mathcal{A}^\dagger (respectively, \mathcal{B}^\dagger) the full subcategory of \mathcal{A} (respectively, \mathcal{B}) consisting of all *countable* structures.

Denote by \mathcal{S}^\Rightarrow the category of all sectionally complemented modular lattices with *surjective* 0-lattice homomorphisms. The morphisms in \mathcal{S}^\Rightarrow will be called the *double arrows* of \mathcal{S} .

Our first categorical statement about the data just introduced involves the *left ladders* developed in [9, Section 3.8].

Lemma 7.1. *The quadruple $(\mathcal{A}, \mathcal{S}, \mathcal{S}^\Rightarrow, \Phi)$ is a left ladder.*

Proof. We recall that left ladders are defined by the following properties:

- (CLOS(\mathcal{A})) \mathcal{A} has all small directed colimits;
- (PROD(\mathcal{A})) \mathcal{A} has all finite nonempty products;
- (CONT(Φ)) Φ preserves all small directed colimits;
- (PROJ($\Phi, \mathcal{S}^\Rightarrow$)) Φ sends any extended projection of \mathcal{A} (i.e., a direct limit $p = \varinjlim_{i \in I} p_i$ for projections $p_i: X_i \times Y_i \rightarrow X_i$ in \mathcal{A}) to a double arrow in \mathcal{S} .

All the corresponding verifications are straightforward (e.g., every extended projection f is surjective, thus $\Phi(f)$ is a double arrow). \square

Our second categorical statement states something about the more involved notion, defined in [9, Section 3.8], of a *right λ -ladder*. We shall also use the notions, introduced in that paper, of *projectability* of right ladders. The following result is a particular case, for $\lambda = \aleph_1$, of Theorem 6-2.2 in (version 1 of) [9].

Lemma 7.2. *Denote by \mathcal{S}^\dagger the class of all countable sectionally complemented modular lattices. Then the 6-uple $(\mathcal{B}, \mathcal{B}^\dagger, \mathcal{S}, \mathcal{S}^\dagger, \mathcal{S}^\Rightarrow, \mathbb{L})$ is a projectable right \aleph_1 -ladder.*

Proof. Right ladderhood amounts here to the conjunction of the two following statements:

- $\text{PRES}_{\aleph_1}(\mathcal{B}^\dagger, \mathbb{L})$: The lattice $\mathbb{L}(B)$ is “weakly \aleph_1 -presented” in \mathcal{S} (which means, here, *countable*), for each $B \in \mathcal{B}^\dagger$.
- $\text{LS}_{\aleph_1}^\dagger(B)$ (for every object B of \mathcal{B}): For every countable sectionally complemented modular lattice S , every surjective lattice homomorphism $\psi: \mathbb{L}(B) \rightarrow S$, and every sequence $(u_n: U_n \rightarrow B \mid n < \omega)$ of monomorphisms in \mathcal{B} with all U_n countable, there exists a monomorphism $u: U \rightarrow B$ in \mathcal{B} , lying above all u_n in the subobject ordering, such that U is countable and $\psi \circ \mathbb{L}(u)$ is surjective.

Both statements are verified in [9, Chapter 6]. \square

Now bringing together Lemmas 7.1 and 7.2 is a trivial matter:

Corollary 7.3. *The 8-uple $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{A}^\dagger, \mathcal{B}^\dagger, \mathcal{S}^\Rightarrow, \Phi, \mathbb{L})$ is a projectable \aleph_1 -larder.*

The following crucial result makes an essential use of our work on Banaschewski functions in Section 4.

Lemma 7.4. *There are increasing ω_1 -chains $\vec{A} = (A_\xi \mid \xi < \omega_1)$ and $\vec{A}' = (A'_\xi \mid \xi < \omega_1)$ of countable models in \mathcal{A} , all with a unit, such that the following statements hold:*

- (i) $\Phi \vec{A}$ cannot be lifted, with respect to the \mathbb{L} functor, by any diagram in \mathcal{B} .
- (ii) A_ξ is a principal ideal of A'_ξ , for each $\xi < \omega_1$.
- (iii) All the models A'_ξ share the same spanning 5-frame.

Proof. We fix a countable field \mathbb{F} and we define regular \mathbb{F} -algebras with quasi-inversion by $R_\xi := \mathcal{R}_{\mathbb{F}}(\xi)$ (as defined in the comments just before Proposition 4.5) and $S_\xi := R_\xi^{5 \times 5}$, for any ordinal ξ . We set $R := R_{\omega_1}$ and $S := S_{\omega_1}$, and we identify R_ξ with its canonical image in R , for each $\xi < \omega_1$ (this requires Proposition 4.3). We denote by $(e_{i,j} \mid 0 \leq i, j \leq 4)$ the canonical system of matrix units of S , so $\sum_{0 \leq i \leq 4} e_{i,i} = 1$ and $e_{i,j} e_{k,l} = \delta_{j,k} e_{i,l}$ (where δ denotes the Kronecker symbol) in S , for all $i, j, k, l \in \{0, 1, 2, 3, 4\}$.

We denote by $\psi := ((e_{i,i} S \mid 0 \leq i \leq 4), ((e_{i,i} - e_{0,i}) S \mid 1 \leq i \leq 4))$ the canonical spanning 5-frame of $\mathbb{L}(S)$. Furthermore, we set $e := \sum_{0 \leq i \leq 3} e_{i,i}$, $b := e_{4,4}$, and $b_\xi := \tilde{\xi} \cdot b$ for each $\xi < \omega_1$. Observe that e , b , and all b_ξ are idempotent, and that $1 = e \oplus b$ and $b_\xi \triangleleft b$ in S . We set $U_\xi := (e + b_\xi) S$, for each $\xi < \omega_1$, and

$$\begin{aligned} A'_\xi &:= \text{canonical copy of } \mathbb{L}((R_{\xi+1})^{5 \times 5}) \text{ in } \mathbb{L}(R^{5 \times 5}), \\ A_\xi &:= \text{ideal of } A'_\xi \text{ generated by } U_\xi, \end{aligned}$$

for each $\xi < \omega_1$. In particular, A'_ξ is a countable complemented sublattice of $\mathbb{L}(S)$ containing ψ while A_ξ contains $\phi := ((e_{i,i} S \mid 0 \leq i \leq 3), ((e_{i,i} - e_{0,i}) S \mid 1 \leq i \leq 3))$, the canonical spanning 4-frame of the principal ideal $\mathbb{L}(S) \downarrow eS$.

In each A_ξ , we interpret the constant \mathbf{a}_i by $e_{i,i} S$, for $0 \leq i \leq 3$, and the constant \mathbf{c}_i by $(e_{i,i} - e_{0,i}) S$, for $1 \leq i \leq 3$. Furthermore, we interpret the predicate symbol \perp of Γ in each A'_ξ by $A'_\xi \downarrow bS$, and in each A_ξ by $A_\xi \downarrow b_\xi S$. It is straightforward to verify that we thus obtain increasing ω_1 -chains \vec{A} and \vec{A}' of countable models in \mathcal{A} .

We claim that there is no $\mathbb{L}(S)$ -valued Banaschewski measure on $\{U_\xi \mid \xi < \omega_1\}$. Suppose otherwise. As $U_\xi = eS \oplus b_\xi S$ and $b_\xi S \subseteq bS$, with $eS \oplus bS = S$ in $\mathbb{L}(S)$, there exists, by Lemma 5.3, an $(\mathbb{L}(S) \downarrow bS)$ -valued Banaschewski measure on $\{b_\xi S \mid \xi < \omega_1\}$. However, it follows from [20, Lemma 10.2] that $\mathbb{L}(S) \downarrow bS$ is isomorphic to $\mathbb{L}(R)$, via an isomorphism that sends $b_\xi S$ to $\tilde{\xi} R$, for each $\xi < \omega_1$. Thus there exists an $\mathbb{L}(R)$ -valued Banaschewski measure on $\{\tilde{\xi} R \mid \xi < \omega_1\}$. This contradicts Theorem 6.4.

Any lifting of \vec{A} , with respect to the functor \mathbb{L} , in \mathcal{B} arises from an ω_1 -chain

$$B_0 \subset B_1 \subset \cdots \subset B_\xi \subset \cdots$$

of regular rings, and it can be represented by the commutative diagram of Figure 2, for a system $(\varepsilon_\xi \mid \xi < \omega_1)$ of isomorphisms. It follows from Lemma 2.4 that B_ξ is unital, for each $\xi < \omega_1$. Denote by 1_ξ the unit of B_ξ , and set

$$U_\beta \ominus U_\alpha := \varepsilon_\beta((1_\beta - 1_\alpha) \cdot B_\beta), \quad \text{for all } \alpha \leq \beta < \omega_1.$$

$$\begin{array}{ccccccc}
A_0 & \hookrightarrow & A_1 & \hookrightarrow & \dots & \hookrightarrow & A_\xi & \hookrightarrow & \dots \\
\varepsilon_0 \uparrow \cong & & \varepsilon_1 \uparrow \cong & & & & \varepsilon_\xi \uparrow \cong & & \\
\mathbb{L}(B_0) & \hookrightarrow & \mathbb{L}(B_1) & \hookrightarrow & \dots & \hookrightarrow & \mathbb{L}(B_\xi) & \hookrightarrow & \dots
\end{array}$$

FIGURE 2. A lifting of $\Phi\vec{A}$ with respect to \mathbb{L}

Let $\alpha \leq \beta \leq \gamma < \omega_1$. From the commutativity of the diagram in Figure 2 it follows that $U_\alpha = \varepsilon_\beta(1_\alpha \cdot B_\beta)$. Hence, by applying the lattice isomorphism ε_β to the relation $B_\beta = 1_\alpha \cdot B_\beta \oplus (1_\beta - 1_\alpha) \cdot B_\beta$, we obtain the relation $U_\beta = U_\alpha \oplus (U_\beta \ominus U_\alpha)$. Furthermore, from $1_\alpha \leq 1_\beta \leq 1_\gamma$ it follows that $1_\gamma - 1_\alpha = (1_\gamma - 1_\beta) \oplus (1_\beta - 1_\alpha)$ in $\text{Idemp } B_\gamma$, thus $(1_\gamma - 1_\alpha) \cdot B_\gamma = (1_\gamma - 1_\beta) \cdot B_\gamma \oplus (1_\beta - 1_\alpha) \cdot B_\gamma$ in $\mathbb{L}(B_\gamma)$, thus, applying ε_γ to each side of that relation, we obtain

$$\begin{aligned}
U_\gamma \ominus U_\alpha &= (U_\gamma \ominus U_\beta) \oplus \varepsilon_\gamma((1_\beta - 1_\alpha) \cdot B_\gamma) \\
&= (U_\gamma \ominus U_\beta) \oplus \varepsilon_\beta((1_\beta - 1_\alpha) \cdot B_\beta) && \text{(see Figure 2)} \\
&= (U_\gamma \ominus U_\beta) \oplus (U_\beta \ominus U_\alpha).
\end{aligned}$$

Therefore, \ominus defines an $\mathbb{L}(S)$ -valued Banaschewski measure on $\{U_\xi \mid \xi < \omega_1\}$, which we just proved impossible. \square

Observe that all the A'_ξ s share the same unit, while the ω_1 -sequence formed with all the units of the A'_ξ s is increasing.

Theorem 7.5. *There exists a non-coordinatizable, 4/5-entire sectionally complemented modular lattice L of cardinality \aleph_1 , which is in addition isomorphic to an ideal in a complemented modular lattice L' with a spanning 5-frame (so L' is coordinatizable).*

Proof. We use the notation and terminology of Gillibert and Wehrung [9]. It follows from Gillibert [8, Proposition 4.6] that there exists an \aleph_1 -lifter (X, \mathbf{X}) of the chain ω_1 such that $\text{card } X = \aleph_1$.

Consider the diagrams \vec{A} and \vec{A}' of Lemma 7.4, and observe that both A_ξ and A'_ξ belong to \mathcal{A}^\dagger (i.e., they are countable), for each $\xi < \omega_1$. We form the *condensates*

$$L := \Phi(\mathbf{F}(X) \otimes \vec{A}) \quad \text{and} \quad L' := \Phi(\mathbf{F}(X) \otimes \vec{A}').$$

From $\text{card } X \leq \aleph_1$ it follows that the ω_1 -scaled Boolean algebra $\mathbf{F}(X)$ is the directed colimit of a direct system of at most \aleph_1 finitely presented objects in the category \mathbf{Bool}_{ω_1} . It follows that $\text{card } L \leq \aleph_1$ and $\text{card } L' \leq \aleph_1$. We shall prove that L is not coordinatizable; in particular, by [20, Theorem 10.3], $\text{card } L = \aleph_1$.

Suppose that there exists an isomorphism $\chi: \mathbb{L}(B) \rightarrow L$, for some regular ring B . By CLL (cf. [9, Lemma 3-4.2]) together with Corollary 7.3, there exists an ω_1 -indexed diagram \vec{B} in \mathcal{B} such that $\mathbb{L}\vec{B} \cong \Phi\vec{A}$. This contradicts Lemma 7.4. Therefore, L is not coordinatizable.

Furthermore, $\mathbf{F}(X) \otimes \vec{A}$ is a direct limit of finite direct products of the form $\prod_{i=1}^n A_{\xi_i}$, where the shape of the indexing system depends only on X . As A_ξ is an ideal of A'_ξ for each $\xi < \omega_1$, $\prod_{i=1}^n A_{\xi_i}$ is an ideal of $\prod_{i=1}^n A'_{\xi_i}$ at each of those places. Therefore, taking direct limits, we obtain that $\mathbf{F}(X) \otimes \vec{A}$ is isomorphic to an ideal

of $\mathbf{F}(X) \otimes \vec{A}'$, so L is an ideal of L' . As the class of all lattices with a spanning 5-frame is closed under finite products and directed colimits and as all A'_ξ s have a spanning 5-frame, L' also has a spanning 5-frame. \square

Theorem 7.5 provides us with a non-coordinatizable ideal in a coordinatizable complemented modular lattice of cardinality \aleph_1 . We do not know whether an ideal in a *countable* coordinatizable sectionally complemented modular lattice is coordinatizable.

As the lattice L of Theorem 7.5 is 4/5-entire and sectionally complemented, it has a large 4-frame. Hence it solves negatively the problem, left open in Jónsson [20], whether a sectionally complemented modular lattice with a large 4-frame is coordinatizable.

Remark 7.6. As the lattice L of Theorem 7.5 has a large 4-frame, every principal ideal of L is coordinatizable. Indeed, fix a large 4-frame $\alpha = (a_0, a_1, a_2, a_3, c_1, c_2, c_3)$ in L and put $a := \bigoplus_{i=0}^3 a_i$. Every principal ideal I of L is contained in $L \downarrow b$ for some $b \in L$ such that $a \leq b$. As α is a large 4-frame of the complemented modular lattice $L \downarrow b$ and by [19, Theorem 8.2], $L \downarrow b$ is coordinatizable. As I is a principal ideal of $L \downarrow b$, it is also coordinatizable (cf. [20, Lemma 10.2]).

Remark 7.7. It is proved in Wehrung [27] that the union of a chain of coordinatizable lattices may not be coordinatizable. The lattices considered there are 2-distributive with unit. Theorem 7.5 extends this negative result to lattices (without unit) with a large 4-frame. Furthermore, it also shows that an ideal in a coordinatizable lattice L' may not be coordinatizable, even in case L' has a spanning 5-frame. By contrast, it follows from [20, Lemma 10.2] that any *principal* ideal of a coordinatizable lattice is coordinatizable. It is also observed in [27, Proposition 3.5] that the class of coordinatizable lattices is closed under homomorphic images, reduced products, and taking neutral ideals.

It is proved in Wehrung [27] that the class of all coordinatizable lattices *with unit* is not first-order. The lattices considered there are 2-distributive (thus without non-trivial homogeneous sequences) with unit. The following result extends this negative result to the class of all lattices (without unit) admitting a large 4-frame.

Corollary 7.8. *The class of all coordinatizable sectionally complemented modular lattices with a large 4-frame is not first-order definable.*

Proof. Fix a large 4-frame $\alpha = ((a_0, a_1, a_2, a_3), (c_1, c_2, c_3))$ in the lattice L of Theorem 7.5, and put $a := a_0 \oplus a_1 \oplus a_2 \oplus a_3$. As L is 4/5-entire, it satisfies the first-order statement, with parameters from $\{a_0, a\}$,

$$(\forall x)(\exists y)(x \leq a \oplus y \text{ and } y \lesssim a_0). \quad (7.1)$$

Let K be a countable elementary sublattice of L containing all the seven entries of α . As L satisfies (7.1), so does K , thus α is a large 4-frame in K . It follows from [20, Theorem 10.3] that K is coordinatizable. On the other hand, L is not coordinatizable and K is an elementary sublattice of L . \square

The following definition is introduced in [28, Definition 5.1].

Definition 7.9. A *Banaschewski trace* on a lattice L with zero is a family $(a_i^j \mid i \leq j \text{ in } \Lambda)$ of elements in L , where Λ is an upward directed partially ordered set with zero, such that

- (i) $a_i^k = a_i^j \oplus a_j^k$ for all $i \leq j \leq k$ in Λ ;
- (ii) $\{a_0^i \mid i \in \Lambda\}$ is cofinal in L .

We proved in [28, Theorem 6.6] that *A sectionally complemented modular lattice with a large 4-frame is coordinatizable iff it has a Banaschewski trace*. Hence we obtain the following result.

Corollary 7.10. *There exists a 4/5-entire sectionally complemented modular lattice of cardinality \aleph_1 without a Banaschewski trace.*

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