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Eigenvalues of Laplacian with constant magnetic field on noncompact hyperbolic surfaces with finite area

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Abstract

We consider a magnetic Laplacian $-\Delta_A = (id + A)^\star(id + A)$ on a noncompact hyperbolic surface $\mathbf{M}$ with finite area. A is a real one-form and the magnetic field dA is constant in each cusp. When the harmonic component of A satifies some quantified condition, the spectrum of $-\Delta_A$ is discrete. In this case we prove that the counting function of the eigenvalues of $-\Delta_A$ satisfies the classical Weyl formula, even when $dA = 0$. [1]

1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface $(\mathbf{M}, g)$ and a smooth, real one-form A on $\mathbf{M}$. We define the magnetic Laplacian, the Bochner Laplacian

$$\begin{gathered} -\Delta_A \;=\; (i\ d + A)^\star(i\ d + A)\ , \\ (\ (i\ d + A)u = i\ du + uA\ ,\ \forall\ u\ \in\ C_0^\infty(\mathbf{M};\mathbb{C})\ )\ . \end{gathered} \tag{1.1}$$

The magnetic field is the exact two-form $\rho_B \;=\; dA$.

If dm is the Riemannian measure on $\mathbf{M}$, then

$$\rho_B \;=\; \widetilde{\mathbf{b}}\, dm\ , \quad \text{with} \quad \widetilde{\mathbf{b}}\ \in\ C^\infty(\mathbf{M};\mathbb{R})\ . \tag{1.2}$$

[1] *Keywords* : spectral asymptotics, magnetic field, Aharanov-Bohm, hyperbolic surface.

The magnetic intensity is $\mathbf{b} = |\widetilde{\mathbf{b}}|$.

It is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(\mathbf{M})$, containing in its domain $C_0^\infty(\mathbf{M};\mathbb{C})$, the space of smooth and compactly supported functions. The spectrum of $-\Delta_A$ is gauge invariant : for any $f \in C^1(\mathbf{M};\mathbb{R})$, $-\Delta_A$ and $-\Delta_{A+df}$ have the same spectrum.

We are interested in constant magnetic fields on $\mathbf{M}$ in the case when $(\mathbf{M},\ g)$ is a non-compact geometrically finite hyperbolic surface of finite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$\mathbf{M} = \bigcup_{j=0}^{J} M_j \tag{1.3}$$

where the M_j are open sets of $\mathbf{M}$, such that the closure of M_0 is compact, and $(J \geq 1)$ the other M_j are cuspidal ends of $\mathbf{M}$.

This means that, for any j, $1 \leq j \leq J$, there exist strictly positive constants a_j and L_j such that M_j is isometric to $\mathbb{S}\times]a_j^2, +\infty[$, equipped with the metric

$$ds_j^2 = y^{-2}(\ L_j^2\ d\theta^2 + dy^2\ )\ ; \tag{1.4}$$

($\mathbb{S} = \mathbb{S}^1$ is the unit circle and $M_j \cap M_k = \emptyset$ if $j \neq k$) .
Let us choose some $z_0 \in M_0$ and let us define

$$d\ :\ \mathbf{M} \to \mathbb{R}_+\ ; \quad d(z) = d_g(z, z_0)\ ; \tag{1.5}$$

$d_g(\ .\ ,\ .\ )$ denotes the distance with respect to the metric g.

It is not possible to have a constant magnetic field on $\mathbf{M}$, but for any $b \in \mathbb{R}^J$, there exists a one-form A , such that the corresponding magnetic field dA satisfies

$$dA = \widetilde{\mathbf{b}}(z)dm \quad \text{with} \quad \widetilde{\mathbf{b}}(z) = b_j\ \forall\, z \in M_j\ . \tag{1.6}$$

The following statement on the essential spectrum is proven in [Mo-Tr1] :

Theorem 1.1 *Assume (1.3) and (1.6). Then for any j , $1 \leq j \leq J$ and for any $z \in M_j$ there exists a unique closed curve through z , $\mathcal{C}_{j,z}$ in $(M_j,\ g)$, not contractible and with zero $g-$curvature. The following limit exists and is finite:*

$$[A]_{M_j} = \lim_{d(z)\to+\infty} \int_{\mathcal{C}_{j,z}} A\ . \tag{1.7}$$

If $J^A = \{j \in \mathbb{N}\ ,\ 1 \leq j \leq J \ s.t.\ [A]_{M_j} \in 2\pi\mathbb{Z}\ \} \neq \emptyset$, *then*

$$\mathrm{sp}_{ess}(-\Delta_A) = [\frac{1}{4} + \min_{j \in J^A} b_j^2\ ,\ +\infty[\ . \tag{1.8}$$

If $J^A = \emptyset$, *then* $\mathrm{sp}_{ess}(-\Delta_A) = \emptyset$:
$-\Delta_A$ *has purely discrete spectrum, (its resolvent is compact).*

When the magnetic Laplacian $-\Delta_A$ has purely discrete spectrum, it is called a magnetic bottle, (see [Col]).

If $A = df + A^H + A^\delta$ is the Hodge decomposition of A with A^H harmonic, ($dA^H = 0$ and $d^\star A^H = 0$) , then $\forall\ j$, $[A]_{M_j} = [A^H]_{M_j}$, so the discreteness of the spectrum of $-\Delta_A$ depends only on the harmonic component of A . So one can see the case $J^A = \emptyset$ as an Aharonov-Bohm phenomenon [Ah-Bo], a situation where the magnetic field dA is not sufficient to describe $-\Delta_A$ and the use of the magnetic potential A is essential : we can have magnetic bottle with null intensity.

2 The Weyl formula in the case of finite area with a non-integer class one-form

Here we are interested in the pure point part of the spectrum. We assume that $J^A = \emptyset$, then the spectrum of $-\Delta_A$ is discrete. In this case, we denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda, -\Delta_A) = \sum_{\lambda_j < \lambda} 1\ . \tag{2.1}$$

We will show that the asymptotic behavior of $N(\lambda)$ is given by the Weyl formula :

Theorem 2.1 *Consider a geometrically finite hyperbolic surface* $(\mathbf{M},\ g)$ *of finite area, and assume (1.6) with* $J^A = \emptyset$, *(see (1.7 for the definition).*

Then

$$N(\lambda, -\Delta_A) = \lambda \frac{|\mathbf{M}|}{4\pi} + \mathbf{O}(\frac{\lambda}{\ln \lambda})\ . \tag{2.2}$$

Remark 2.2 *As J^A depends only on the harmonic component of A , J^A is not empty when $\mathbf{M}$ is simply connected. In [Go-Mo] there are some results close to Theorem 2.1, but for simply connected manifolds.*

The cases where the magnetic field prevails were studied in [Mo-Tr1] and in [Mo-Tr2].

Proof of Theorem 2.1. Any constant depending only on the b_j and on $\min\limits_{1\leq j\leq J}\inf\limits_{k\in\mathbb{Z}}|[A]_{M_j}-2k\pi|$ will be denoted invariably C .

Consider a cusp $M=M_j=\ \mathbb{S}\times]\alpha^2,+\infty[$ equipped with the metric $ds^2\ =\ L^2e^{-2t}d\theta^2\ +\ dt^2$ for some $\alpha>0$ and $L>0$.

Let us denote by $-\Delta_A^M$ the Dirichlet operator on M , associated to $-\Delta_A$.

The first step will be to prove that

$$N(\lambda,-\Delta_A^M)\ =\ \lambda\frac{|M|}{4\pi}\ +\ \mathbf{O}(\frac{\lambda}{\ln\lambda})\ . \tag{2.3}$$

Since $-\Delta_A^M$ and $-\Delta_{A+d\varphi+kd\theta}^M$ are gauge equivalent for any $\varphi\ \in\ C^\infty(\overline{M};\mathbb{R})$ and any $k\ \in\ \mathbb{Z}$, we can assume that

$$-\Delta_A^M=L^{-2}e^{2t}(D_\theta-A_1)^2+D_t^2+\frac{1}{4}\ ,\quad \text{with}\quad A_1=-\xi\pm bLe^{-t}\ ,\ \xi\in]0,1[\ ,$$

$(b=b_j\ ,\ 2\pi\xi-[A]_M\ \in\ 2\pi\mathbb{Z})$. Then we get that

$$\mathrm{sp}(-\Delta_A^M)=\bigcup_{\ell\in\mathbb{Z}}\mathrm{sp}(P_\ell)\ ;\ P_\ell=D_t^2+\frac{1}{4}+\left(e^t\frac{(\ell+\xi)}{L}\pm b\right)^2\ ,$$

for the Dirichlet condition on $L^2(I;dt)$; $I=]\alpha^2,+\infty[$. This implies that

$$N(\lambda,-\Delta_A^M)\ =\ \sum_{\ell\in\mathbb{Z}}N(\lambda,P_\ell)=\sum_{\ell\in X_\lambda}N(\lambda,P_\ell) \tag{2.4}$$

with $X_\lambda=\{\ell\ /\ e^{\alpha^2}\dfrac{|\ell+\xi|}{L}<\sqrt{\lambda-1/4}-b\ \}$.

Denoting by Q_ℓ the Dirichlet operator on I associated to

$$Q_\ell=D_t^2+\frac{1}{4}+\frac{(\ell+\xi)^2}{L^2}e^{2t}\ ,$$

we easily get that

$$Q_\ell-C\sqrt{Q_\ell}\ \leq\ P_\ell\ \leq\ Q_\ell+C\sqrt{Q_\ell}\ . \tag{2.5}$$

Therefore one can find a constant $C(b)$, depending only on b , such that, for any $\lambda >> 1 + C(b)$,

$$N(\lambda - \sqrt{\lambda}C(b), Q_\ell) \ \leq \ N(\lambda, P_\ell) \ \leq \ N(\lambda + \sqrt{\lambda}C(b), Q_\ell) \ ; \tag{2.6}$$

Applying the Weyl formula we thus get the following

Lemma 2.3 *There exists $C_0 > 1$ such that, for any $\lambda >> 1$ and any $\ell \in X_\lambda$,*

$$w_\ell(\lambda - C_0\sqrt{\lambda}) \ \leq \ \pi N(\lambda, P_\ell) \ \leq \ w_\ell(\lambda + C_0\sqrt{\lambda}) \ ,$$

with

$$\begin{aligned} w_\ell(\mu) \ &= \ \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dt \\ &= \ \int_{\alpha^2}^{T(\mu,L)} \left[\mu - \frac{(\ell+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dt \ ; \end{aligned} \tag{2.7}$$

$(e^{T(\mu,L)} = L\sqrt{\mu}/(\inf_{k\in\mathbb{Z}} |\xi - k|) \ )$.

In view of (2.4) we now compute $\sum_{\ell\in\mathbb{Z}} w_\ell(\mu)$. We first get the following

Lemma 2.4 *There exists $C_0 > 1$ such that, for any $\mu >> 1$ and any $t \in [\alpha^2, T(\mu, L)]$,*

$$\left| \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dx \ - \ \sum_{\ell\in\mathbb{Z}} \left[\mu - \frac{(\ell+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} \right| \ \leq \ C_0(\sqrt{\mu} + \frac{e^t}{L}) \ .$$

This leads to

Lemma 2.5 *There exists $C_0 > 1$ such that, for any $\mu >> 1$,*

$$\left| \int_{\alpha^2}^{T(\mu,L)} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dxdt \ - \ \sum_{\ell\in\mathbb{Z}} w_\ell(\mu) \right| \ \leq \ C_0\sqrt{\mu}\ln\mu \ .$$

Changing variables in the integral in the right-hand side we get

Lemma 2.6 *There exists $C_0 > 1$ such that, for any $\mu >> 1$,*

$$\left| \int_{\alpha^2}^{T(\mu,L)} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dxdt \ - \ \mu L e^{-\alpha^2} \int_{\mathbb{R}} \left[1 - x^2\right]_+^{1/2} dx \right| \ \leq \ C_0 \frac{\mu}{\ln\mu} \ .$$

Noticing that $|M| = 2\pi L e^{-\alpha^2}$ we deduce from Lemmas 2.5 and 2.6 that

Lemma 2.7

$$\frac{1}{\pi}\sum_{\ell\in\mathbb{Z}} w_\ell(\mu) \;=\; \frac{|M|}{4\pi}\mu \;+\; \mathbf{O}(\frac{\mu}{\ln\mu})\;, \quad \text{as} \quad \mu\to+\infty\;.$$

In view of (2.4) this ends the proof of (2.3).

Now it remains to consider the whole surface $\mathbf{M}$.

We have : $\mathbf{M} \;=\; \left(\bigcup_{j=0}^{J} M_j\right)$

where the M_j are open sets of $\mathbf{M}$, such that the closure of M_0 is compact, and the other M_j are cuspidal ends of $\mathbf{M}$ and

$M_j \cap M_k = \emptyset$, if $j \neq k$. We denote $M_0^0 = \mathbf{M} \setminus (\bigcup_{j=1}^{J} \overline{M_j})$, then

$$\mathbf{M} \;=\; \overline{M_0^0} \bigcup \left(\bigcup_{j=1}^{J} \overline{M_j}\right)\;. \tag{2.8}$$

Let us denote by $-\Delta_A^\Omega$ the Dirichlet operator on an open set Ω of $\mathbf{M}$ associated to $-\Delta_A$.

The minimax principle and (2.8) imply that

$$N(\lambda, -\Delta_A^{M_0^0}) + \sum_{1\le j\le J} N(\lambda, -\Delta_A^{M_j}) \le N(\lambda, -\Delta_A) \tag{2.9}$$

To get an upper bound we use a partition of unity. Let us consider, for any $1 \le j \le J$ the diffeorphism

$$\Phi_j \;:\; \overline{M_j} \;\to\; \mathbb{S}\times[\alpha_j^2, +\infty[$$

and define the sets

$$O_j^\lambda \;=\; \Phi_j^{-1}(\mathbb{S}\times[\alpha_j^2, \alpha_j^2 + \frac{1}{\ln\lambda}[)\;.$$

Then we have the following covering of $\mathbf{M}$ with open sets

$$\mathbf{M} \;=\; M_0^\lambda \bigcup \left(\bigcup_{j=0}^{J} M_j\right)\;, \quad \text{with} \quad \mathrm{M}_0^\lambda \;=\; \mathrm{M}_0^0 \bigcup \left(\bigcup_{\mathrm{j}=0}^{\mathrm{J}} \mathrm{O}_\mathrm{j}^\lambda\right)\;.$$

We associate to this covering, a smooth partition of unity $(\psi_{j,\lambda})_{j=0,1,\dots,J}$, $\sum_{0\le j\le J} \psi_{j,\lambda}^2(x) = 1,\ \forall x \in \mathbf{M}$, such that

$$\begin{cases} \psi_{j,\lambda} = 1 \text{ on } M_j \setminus O_j^\lambda\ ,\ (1 \le k \le J) \\ \psi_{j,\lambda} = 0 \text{ on } M_k \text{ if } k \ne j, \\ \psi_{0,\lambda} = 1 \text{ on } M_0^0 \\ |\nabla \psi_{j,\lambda}(x)| \le C \ln \lambda \end{cases}$$

Using again minimax principle as in [Mo-Tr1], we get the following upper bound

$$N(\lambda, -\Delta_A) \le N(\lambda + C\ln^2\lambda, -\Delta_A^{M_0^\lambda}) + \sum_{1\le j\le J} N(\lambda + C\ln^2\lambda, -\Delta_A^{M_j}) \quad (2.10)$$

The Weyl formula with remainder, (see [Hor] for smooth boundary and [Ivr] for boundary with cone-like singularities), gives that

$$\left\{ \begin{array}{l} N(\lambda, -\Delta_A^{M_0^0}) = (4\pi)^{-1}|M_0^0|\lambda + \mathbf{O}(\sqrt{\lambda}) \\ N(\lambda + C\ln^2\lambda, -\Delta_A^{M_0^\lambda}) = (4\pi)^{-1}|M_0^\lambda|(\lambda + C\ln^2\lambda) + \mathbf{O}(\sqrt{\lambda}) \end{array} \right\} \quad (2.11)$$

Noticing that $|M_0^\lambda|(\lambda + C\ln^2\lambda) = |M_0^0|\lambda + \mathbf{O}(\lambda/\ln\lambda\ )$,
we get (2.2) from (2.3) (with $M = M_j$, $j = 1, \dots, J$), (2.9), (2.10) and (2.11) $\square$

Remark 2.8 *Theorem 2.1 still holds if the metric of* $\mathbf{M}$ *is modified in a compact set.*

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