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Eigenvalues of Laplacian with constant magnetic field on noncompact hyperbolic surfaces with finite area

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Abstract

We consider a magnetic Laplacian $-\Delta_A = (id + A)^*(id + A)$ on a noncompact hyperbolic surface \mathbf{M} with finite area. A is a real one-form and the magnetic field dA is constant in each cusp. When the harmonic component of A satisfies some quantified condition, the spectrum of $-\Delta_A$ is discrete. In this case we prove that the counting function of the eigenvalues of $-\Delta_A$ satisfies the classical Weyl formula, even when $dA = 0$.¹

1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface (\mathbf{M}, g) and a smooth, real one-form A on \mathbf{M} . We define the magnetic Laplacian, the Bochner Laplacian

$$-\Delta_A = (i d + A)^*(i d + A), \quad (1.1)$$

$$((i d + A)u = i du + uA, \forall u \in C_0^\infty(\mathbf{M}; \mathbb{C})).$$

The magnetic field is the exact two-form $\rho_B = dA$.

If dm is the Riemannian measure on \mathbf{M} , then

$$\rho_B = \tilde{\mathbf{b}} dm, \quad \text{with } \tilde{\mathbf{b}} \in C^\infty(\mathbf{M}; \mathbb{R}). \quad (1.2)$$

¹ *Keywords* : spectral asymptotics, magnetic field, Aharanov-Bohm, hyperbolic surface.

The magnetic intensity is $\mathbf{b} = |\tilde{\mathbf{b}}|$.

It is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(\mathbf{M})$, containing in its domain $C_0^\infty(\mathbf{M}; \mathbb{C})$, the space of smooth and compactly supported functions. The spectrum of $-\Delta_A$ is gauge invariant : for any $f \in C^1(\mathbf{M}; \mathbb{R})$, $-\Delta_A$ and $-\Delta_{A+df}$ are unitarily equivalent, hence they have the same spectrum.

We are interested in constant magnetic fields on \mathbf{M} in the case when (\mathbf{M}, g) is a non-compact geometrically finite hyperbolic surface of finite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$\mathbf{M} = \bigcup_{j=0}^J M_j \quad (1.3)$$

where the M_j are open sets of \mathbf{M} , such that the closure of M_0 is compact, and (when $J \geq 1$) the other M_j are cuspidal ends of \mathbf{M} .

This means that, for any j , $1 \leq j \leq J$, there exist strictly positive constants a_j and L_j such that M_j is isometric to $\mathbb{S} \times]a_j^2, +\infty[$, equipped with the metric

$$ds_j^2 = y^{-2}(L_j^2 d\theta^2 + dy^2); \quad (1.4)$$

($\mathbb{S} = \mathbb{S}^1$ is the unit circle and $M_j \cap M_k = \emptyset$ if $j \neq k$).

Let us choose some $z_0 \in M_0$ and let us define

$$d : \mathbf{M} \rightarrow \mathbb{R}_+; \quad d(z) = d_g(z, z_0); \quad (1.5)$$

$d_g(\cdot, \cdot)$ denotes the distance with respect to the metric g .

For any $b \in \mathbb{R}^J$, there exists a one-form A , such that the corresponding magnetic field dA satisfies

$$dA = \tilde{\mathbf{b}}(z)dm \quad \text{with} \quad \tilde{\mathbf{b}}(z) = b_j \forall z \in M_j. \quad (1.6)$$

The following statement on the essential spectrum is proven in [Mo-Tr1] :

Theorem 1.1 *Assume (1.3) and (1.6). Then for any j , $1 \leq j \leq J$ and for any $z \in M_j$ there exists a unique closed curve through z , $\mathcal{C}_{j,z}$ in (M_j, g) , not contractible and with zero g -curvature. ($\mathcal{C}_{j,z}$ is called an horocycle of M_j). The following limit exists and is finite:*

$$[A]_{M_j} = \lim_{d(z) \rightarrow +\infty} \int_{\mathcal{C}_{j,z}} A. \quad (1.7)$$

If $J^A = \{j \in \mathbb{N}, 1 \leq j \leq J \text{ s.t. } [A]_{M_j} \in 2\pi\mathbb{Z}\} \neq \emptyset$, then

$$\text{sp}_{ess}(-\Delta_A) = \left[\frac{1}{4} + \min_{j \in J^A} b_j^2, +\infty[. \quad (1.8)$$

If $J^A = \emptyset$, then $\text{sp}_{ess}(-\Delta_A) = \emptyset$:
 $-\Delta_A$ has purely discrete spectrum, (its resolvent is compact).

When the magnetic Laplacian $-\Delta_A$ has purely discrete spectrum, it is called a magnetic bottle, (see [Col2]).

If $A = df + A^H + A^\delta$ is the Hodge decomposition of A with A^H harmonic, ($dA^H = 0$ and $d^*A^H = 0$), then $\forall j$, $[A]_{M_j} = [A^H]_{M_j}$, so the discreteness of the spectrum of $-\Delta_A$ depends only on the harmonic component of A . So one can see the case $J^A = \emptyset$ as an Aharonov-Bohm phenomenon [Ah-Bo], a situation where the magnetic field dA is not sufficient to describe $-\Delta_A$ and the use of the magnetic potential A is essential : we can have magnetic bottle with null intensity.

2 The Weyl formula in the case of finite area with a non-integer class one-form

Here we are interested in the pure point part of the spectrum. We assume that $J^A = \emptyset$, then the spectrum of $-\Delta_A$ is discrete. In this case, we denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda, -\Delta_A) = \sum_{\lambda_j < \lambda} 1. \quad (2.1)$$

We will show that the asymptotic behavior of $N(\lambda)$ is given by the Weyl formula :

Theorem 2.1 *Consider a geometrically finite hyperbolic surface (\mathbf{M}, g) of finite area, and assume (1.6) with $J^A = \emptyset$, (see (1.7 for the definition).*

Then

$$N(\lambda, -\Delta_A) = \lambda \frac{|\mathbf{M}|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda). \quad (2.2)$$

Remark 2.2 As J^A depends only on the harmonic component of A , J^A is not empty when \mathbf{M} is simply connected. In [Go-Mo] there are some results close to Theorem 2.1, but for simply connected manifolds.

The cases where the magnetic field prevails were studied in [Mo-Tr1] and in [Mo-Tr2].

Proof of Theorem 2.1. Any constant depending only on the b_j and on $\min_{1 \leq j \leq J} \inf_{k \in \mathbb{Z}} |[A]_{M_j} - 2k\pi|$ will be denoted invariably C .

Consider a cusp $M = M_j = \mathbb{S} \times]\alpha^2, +\infty[$ equipped with the metric $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ for some $\alpha > 0$ and $L > 0$.

Let us denote by $-\Delta_A^M$ the Dirichlet operator on M , associated to $-\Delta_A$. The first step will be to prove that

$$N(\lambda, -\Delta_A^M) = \lambda \frac{|M|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda). \quad (2.3)$$

Since $-\Delta_A^M$ and $-\Delta_{A+d\varphi+kd\theta}^M$ are gauge equivalent for any $\varphi \in C^\infty(\overline{\mathbf{M}}; \mathbb{R})$ and any $k \in \mathbb{Z}$, we can assume that

$$-\Delta_A^M = L^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}, \quad \text{with } A_1 = -\xi \pm bL e^{-t}, \quad \xi \in]0, 1[,$$

($b = b_j$, $2\pi\xi - [A]_M \in 2\pi\mathbb{Z}$). Then we get that

$$\text{sp}(-\Delta_A^M) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell); \quad P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L} \pm b \right)^2,$$

for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha^2, +\infty[$. This implies that

$$N(\lambda, -\Delta_A^M) = \sum_{\ell \in \mathbb{Z}} N(\lambda, P_\ell) = \sum_{\ell \in X_\lambda} N(\lambda, P_\ell) \quad (2.4)$$

with $X_\lambda = \{ \ell / e^{\alpha^2} \frac{|\ell + \xi|}{L} < \sqrt{\lambda - 1/4} - b \}$.

Denoting by Q_ℓ the Dirichlet operator on I associated to

$$Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell + \xi)^2}{L^2} e^{2t},$$

we easily get that

$$Q_\ell - C\sqrt{Q_\ell} \leq P_\ell \leq Q_\ell + C\sqrt{Q_\ell}. \quad (2.5)$$

Therefore one can find a constant $C(b)$, depending only on b , such that, for any $\lambda \gg 1 + C(b)$,

$$N(\lambda - \sqrt{\lambda}C(b), Q_\ell) \leq N(\lambda, P_\ell) \leq N(\lambda + \sqrt{\lambda}C(b), Q_\ell). \quad (2.6)$$

Following Titchmarsh's method ([Tit], Theorem 7.4) we establish the following bounds

Lemma 2.3 *There exists $C > 1$ so that for any $\mu \gg 1$ and any $\ell \in X_\mu$,*

$$w_\ell(\mu) - \pi \leq \pi N(\mu - \frac{1}{4}, Q_\ell) \leq w_\ell(\mu) + \frac{1}{12} \ln \mu + C, \quad (2.7)$$

with

$$\begin{aligned} w_\ell(\mu) &= \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dt \\ &= \int_{\alpha^2}^{T_{\mu, L}} \left[\mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dt; \end{aligned} \quad (2.8)$$

($e^{T_{\mu, L}} = L\sqrt{\mu}/(\inf_{k \in \mathbb{Z}} |\xi - k|)$).

Proof of Lemma 2.3

The lower bound is easily obtained (see [Tit], Formula 7.1.2 p 143) so we focus on the upper bound.

Let us define $V_\ell = \frac{(\ell + \xi)^2}{L^2} e^{2t}$ and denote by ϕ_μ^ℓ a solution of $Q_\ell \phi = (\mu - \frac{1}{4})\phi$. Consider x_ℓ and y_ℓ so that $V_\ell(x_\ell) = \mu$ and $V_\ell(y_\ell) = \nu$, for a given $0 < \nu < \mu$ to be determined later. We denote by m the number of zeros of ϕ_μ^ℓ on $] \alpha^2, y_\ell[$. Recall that the number n of zeros of ϕ_μ^ℓ on $] \alpha^2, x_\ell[$ is equal to $N(\mu - \frac{1}{4}, Q_\ell)$. Applying Lemma 7.3 p 146 in [Tit] we deduce that

$$m\pi = \int_{\alpha^2}^{y_\ell} [\mu - V_\ell]^{1/2} dt + R_\ell$$

with $R_\ell = \frac{1}{4} \ln(\mu - V_\ell(\alpha^2)) - \frac{1}{4} \ln(\mu - V_\ell(y_\ell)) + \pi$, hence

$$|n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2} dt| \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2} + R_\ell + (n - m)\pi$$

According to the Sturm comparison theorem ([Tit], p 107-108), we have

$$(n - m)\pi \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2}$$

and

$$|n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2} dt| \leq \ln\left(\frac{\mu}{\nu}\right)(\mu - \nu)^{1/2} + \frac{1}{4} \ln \mu - \frac{1}{4} \ln(\mu - \nu) + 2\pi$$

Now taking $\nu = \mu - \mu^{2/3}$ we get the desired estimate.

In view of (2.4) we now compute $\sum_{\ell \in \mathbb{Z}} w_\ell(\mu)$. We first get the following

Lemma 2.4 *There exists $C > 1$ such that, for any $\mu \gg 1$ and any $t \in [\alpha^2, T_{\mu,L}]$,*

$$\left| \int_{\mathbb{R}} \left[\mu - \frac{(x + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dx - \sum_{\ell \in \mathbb{Z}} \left[\mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} \right| \leq C(\sqrt{\mu} + \frac{e^t}{L}).$$

This leads to

Lemma 2.5 *There exists $C > 1$ such that, for any $\mu \gg 1$,*

$$\left| \int_{\alpha^2}^{T_{\mu,L}} \int_{\mathbb{R}} \left[\mu - \frac{(x + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dx dt - \sum_{\ell \in \mathbb{Z}} w_\ell(\mu) \right| \leq C\sqrt{\mu} \ln \mu.$$

We now compute the integral in the left-hand side.

Making the change of variables $y^2 = \frac{(x + \xi)^2}{L^2} e^{2t}$ we obtain that it is equal to $\mu L \int_{\alpha^2}^{T_{\mu,L}} e^{-t} dt \int_{\mathbb{R}} [1 - x^2]_+^{1/2} dx$, so we get

Lemma 2.6 *There exists $C > 1$ such that, for any $\mu \gg 1$,*

$$\left| \int_{\alpha^2}^{T_{\mu,L}} \int_{\mathbb{R}} \left[\mu - \frac{(x + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dx dt - \mu L e^{-\alpha^2} \int_{\mathbb{R}} [1 - x^2]_+^{1/2} dx \right| \leq C\sqrt{\mu}.$$

Noticing that $|M| = 2\pi L e^{-\alpha^2}$ and using Lemmas 2.5 and 2.6 we have

Lemma 2.7

$$\frac{1}{\pi} \sum_{\ell} \in w_\ell(\mu) = \frac{|M|}{4\pi} \mu + \mathbf{O}(\sqrt{\mu} \ln \mu), \quad \text{as } \mu \rightarrow +\infty.$$

In view of (2.4),(2.6) and (2.7) Lemma 2.7 ends the proof of formula (2.3).
Now it remains to consider the whole surface \mathbf{M} .

$$\text{We have : } \mathbf{M} = \left(\bigcup_{j=0}^J M_j \right)$$

where the M_j are open sets of \mathbf{M} , such that the closure of M_0 is compact,
and the other M_j are cuspidal ends of \mathbf{M} and

$M_j \cap M_k = \emptyset$, if $j \neq k$. We denote $M_0^0 = \mathbf{M} \setminus \left(\bigcup_{j=1}^J \overline{M_j} \right)$, then

$$\mathbf{M} = \overline{M_0^0} \cup \left(\bigcup_{j=1}^J \overline{M_j} \right). \quad (2.9)$$

Let us denote respectively by $-\Delta_{A,D}^\Omega$ and by $-\Delta_{A,N}^\Omega$ the Dirichlet operator
and the Neumann-like operator on an open set Ω of \mathbf{M} associated to $-\Delta_A$.
The minimax principle and (2.9) imply that

$$\begin{aligned} N(\lambda, -\Delta_{A,D}^{M_0^0}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_{A,D}^{M_j}) &\leq N(\lambda, -\Delta_A) \\ &\leq N(\lambda, -\Delta_{A,N}^{M_0^0}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_{A,N}^{M_j}) \end{aligned} \quad (2.10)$$

The Weyl formula with remainder, (see [Hor] for Dirichlet boundary condition
and [Sa-Va] p. 9 for Neumann-like boundary condition), gives that

$$\left\{ \begin{array}{l} N(\lambda, -\Delta_{A,D}^{M_0^0}) = (4\pi)^{-1} |M_0^0| \lambda + \mathbf{O}(\sqrt{\lambda}) \\ N(\lambda, -\Delta_{A,N}^{M_0^0}) = (4\pi)^{-1} |M_0^0| \lambda + \mathbf{O}(\sqrt{\lambda}) \end{array} \right\}. \quad (2.11)$$

The asymptotic formula for $N(\lambda, -\Delta_{A,N}^{M_j})$,

$$N(\lambda, -\Delta_{A,N}^{M_j}) = \lambda \frac{|M_j|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda), \quad (2.12)$$

is obtained as for the Dirichlet case (2.3) (with $M = M_j$), by noticing that
 $N(\lambda, P_{\ell,D}) \leq N(\lambda, P_{\ell,N}) \leq N(\lambda, P_{\ell,D}) + 1$, where $P_{\ell,D}$ and $P_{\ell,N}$ are Dirichlet
and Neumann operators on a half-line $I =]\alpha^2, +\infty[$, associated to the same
differential Schödinger operator $P_\ell = D_t^2 + \frac{1}{4} + (e^t \frac{(\ell + \xi)}{L} \pm b)^2$.

We get (2.2) from (2.3) with $M = M_j$, (2.12), (for any $j = 1, \dots, J$),
(2.10) and (2.11). \square

Remark 2.8 *Theorem 2.1 still holds if the metric of \mathbf{M} is modified in a compact set.*

When $A = 0$, $-\Delta = -\Delta_0$ has embedded eigenvalues in its essential spectrum, ($sp_{ess}(-\Delta) = [\frac{1}{4}, +\infty[$). If $N_{ess}(\lambda, -\Delta)$ denotes the number of these eigenvalues in $[\frac{1}{4}, \lambda[$, then it is well known that one has an upper bound $N_{ess}(\lambda, -\Delta) \leq \lambda \frac{|\mathbf{M}|}{4\pi}$; see [Col1] and [Hej] for the history and related improvement of the upper bound.

Recently [Mul] established a sharp asymptotic formula, similar to our case,

$$N_{ess}(\lambda, -\Delta) = \lambda \frac{|\mathbf{M}|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda),$$

for some particular \mathbf{M} .

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References

- [Ah-Bo] Y. Aharonov, D. Bohm : Significance of electromagnetic potentials in quantum theory, Phys. Rev., 115, (1959), p. 485-491.
- [Bor] D. Borthwick : Spectral Theory of Infinite-Area Hyperbolic Surfaces, Birkhäuser, Boston, 2007.
- [Col1] Y. Colin de Verdière : Pseudo-laplaciens II, Ann. Institut Fourier, 33 (2), (1983), p. 87-113.
- [Col2] Y. Colin de Verdière : L'asymptotique de Weyl pour les bouteilles magnétiques, Comm. Math. Phys., 105, (1986), p. 327-335.
- [Go-Mo] S. Golénia, S. Moroianu : Spectral Analysis of Magnetic Laplacians on Conformally Cusp Manifolds, Ann. Henri Poincaré, 9, (2008), p. 131-179.
- [Hej] D. Hejhal : The Selberg trace formula for $PSL(2, \mathbb{R})$, II, Lecture Notes in Math. 1001, Springer-verlag, Berlin, 1983.

- [Hor] Lars Hörmander : The Analysis of Linear P.D.O. IV , Springer-verlag, Berlin, 1985.
- [Ivr] V. J. Ivrii : Microlocal Analysis and Precise Spectral Asymptotics, Springer-verlag, Berlin, 1998.
- [Mo-Tr1] A. Morame, F. Truc : Magnetic bottles on geometrically finite hypersurfaces, J. of Geometry and Physics, 59-7, (2009), p 1095-1105.
- [Mo-Tr2] A. Morame, F. Truc : Magnetic bottles on the Poincaré half-plane: spectral asymptotics, J. Math. Kyoto Univ., 48-3, (2008), p 597-616.
- [Mul] W. Müller : Weyl's law in the theory of automorphic forms, London Math. Soc. Lecture Notes, 354,(2008), p. 133-163.
- [Per] P. Perry : The Spectral Geometry of Geometrically Finite Hyperbolic Manifolds, Proc. Sympos. Pure Math., 76, Part1, AMS, (2007), p. 289-327.
- [Sa-Va] Y. Safarov, D. Vassiliev : The Asymptotic distribution of Eigenvalues of Partial Differential Operators, AMS Trans.155, 1996.
- [Shu] M. Shubin : The essential Self-adjointness for Semi-bounded Magnetic Schrödinger operators on Non-compact Manifolds, J. Func. Anal., 186, (2001), p. 92-116.
- [Tit] E. C. Titchmarsh : Eigenfunction Expansions associated with Second-order Differential equations I, second edition, Oxford at the Clarendon Press, 1962.