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A characterization of b-perfect graphs*

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Abstract

A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes, and the b-chromatic number of a graph G is the largest integer k such that G admits a b-coloring with k colors. A graph is b-perfect if the b-chromatic number is equal to the chromatic number for every induced subgraph of G . We prove that a graph is b-perfect if and only if it does not contain as an induced subgraph a member of a certain list of twenty-two graphs. This entails the existence of a polynomial-time recognition algorithm and of a polynomial-time algorithm for coloring exactly the vertices of every b-perfect graph.

Keywords: Coloration, b-coloring, a-chromatic number, b-chromatic number.

1 Introduction

A proper coloring of a graph G is a mapping c from the vertex-set $V(G)$ of G to the set $\{1, 2, \dots\}$ of positive integers (colors) such that any two adjacent vertices are mapped to different colors. Each set of vertices colored with one color is a stable set of vertices of G , so a coloring is a partition of $V(G)$

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into stable sets. The smallest number k for which G admits a coloring with k colors is the chromatic number $\chi(G)$ of G [1].

Many graph invariants related to colorings have been defined. Most of them try to minimize the number of colors used to color the vertices under some constraints. For some other invariants, it is meaningful to try to maximize this number. The b-chromatic number is such an example. When we try to color the vertices of a graph, we can start from a given coloring and try to decrease the number of colors by eliminating color classes. One possible such procedure consists in trying to reduce the number of colors by transferring every vertex from a fixed color class to a color class in which it has no neighbour, if any such class exists. A *b-coloring* is a proper coloring in which this is not possible, that is, every color class i contains at least one vertex that has a neighbor in all the other classes. Any such vertex will be called a *b-vertex* of color i . The *b-chromatic number* $b(G)$ is the largest integer k such that G admits a b-coloring with k colors. Irving and Manlove [11, 13] proved that deciding whether a graph G admits a b-coloring with a given number of colors is an NP-complete problem, even when it is restricted to the class of bipartite graphs [10]. On the other hand, they gave a polynomial-time algorithm that solves this problem for trees. The NP-completeness results has incited researchers to establish bounds on the b-chromatic number in general or to find its exact values for subclasses of graphs.

Clearly every $\chi(G)$ -coloring of a graph G is a b-coloring, and so every graph G satisfies $\chi(G) \leq b(G)$. As usual with such an inequality, it may be interesting to look at the graphs that satisfy it with equality. However, graphs such that $\chi(G) = b(G)$ do not have a specific structure; to see this, we can take any arbitrary graph G and add a component that consists of a clique of size $b(G)$; we obtain a graph G' that satisfies $\chi(G') = b(G') = b(G)$. This led Hoàng and Kouider [8] to introduce the class of *b-perfect* graphs: a graph G is called b-perfect if every induced subgraph H of G satisfies $\chi(H) = b(H)$. Hoàng and Kouider [8] proved the b-perfectness of some classes of graphs, and asked for a good characterization of the whole class of b-perfect graphs. Hoàng, Linhares Sales and Maffray [9] proposed the conjecture below. Here we solve the problem by establishing the validity of the conjecture. For a fixed graph F , we say that a graph G is *F-free* if it does not have an induced subgraph that is isomorphic to F . For a set \mathcal{F} of graphs, we say that a graph G is *\mathcal{F} -free* if it does not have an induced subgraph that is isomorphic to a member of \mathcal{F} . Let $\mathcal{F} = \{F_1, \dots, F_{22}\}$ be the set of graphs depicted in Figure 1.

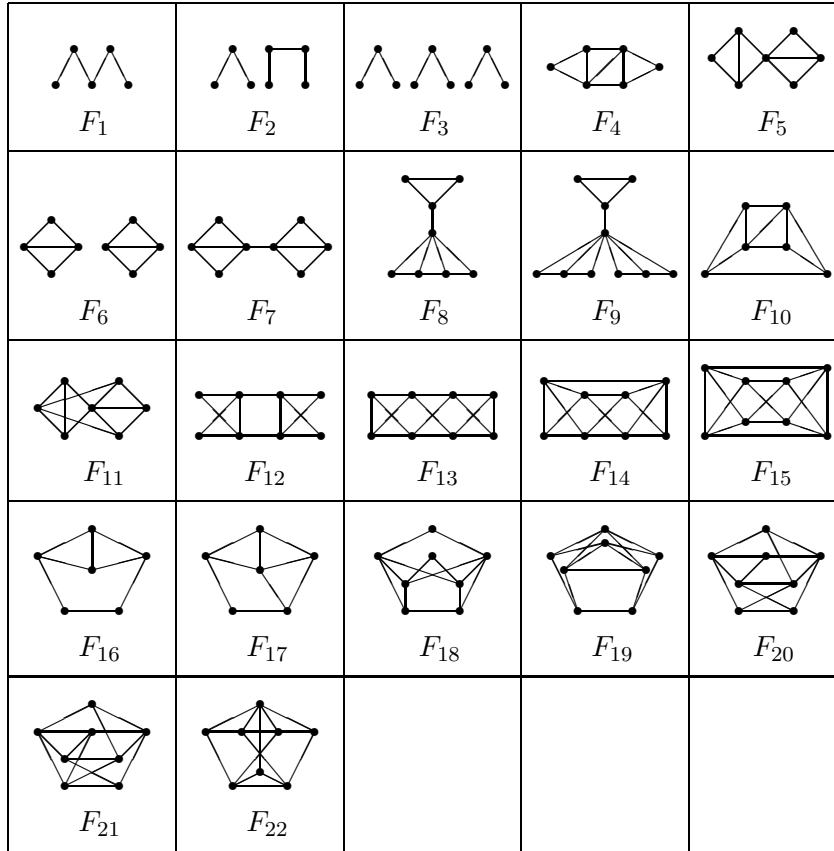


Figure 1: Class $\mathcal{F} = \{F_1, \dots, F_{22}\}$

Conjecture 1 (Hoàng, Linhares Sales, Maffray [9]) *A graph is b-perfect if and only if it is \mathcal{F} -free.*

Our main result is the following.

Theorem 1.1 *Conjecture 1 is true.*

The following theorem was proved before Conjecture 1 was formulated, but it can be seen as evidence for its validity.

Theorem 1.2 (Hoàng and Kouider [8])

A bipartite graph is b-perfect if and only if it contains no F_1, F_2 or F_3 .

A P_4 -free graph is b-perfect if and only if it contains no F_3 or F_6 .

Moreover, some other partial results were obtained.

Theorem 1.3 (Hoàng, Linhares Sales, Maffray [9]) *Conjecture 1 holds for 3-colorable graphs and for diamond-free graphs.*

Theorem 1.4 (Maffray, Mechebbek [12]) *Conjecture 1 holds for chordal graphs.*

In the remainder of this section, we introduce some definitions and notation. For any vertex v of a graph G , the *neighborhood* of v is the set $N(v) = \{u \in V(G) \mid uv \in E\}$ and the *degree* of v is $\deg(v) = |N(v)|$. If a vertex x is adjacent to a vertex y , then we will say that x *sees* y , otherwise we will say x *misses* y . Let us say that a set A of vertices is *complete* (respectively, *anti-complete*) to a set B if every vertex of A sees (respectively, misses) every vertex of B .

A *component* of a graph G is a maximal connected subgraph of G , and a *co-component* of G is a component of its complementary graph. Two vertices x, y are *twins* if $N(x) - \{x, y\} = N(y) - \{x, y\}$; in addition, if x sees y then they are *true twins*, otherwise they are *false twins*. A vertex x *dominates* a vertex y if $N(y) \subseteq N(x) \cup \{x\}$; x and y are *comparable* if x dominates y , or vice versa.

For any integer $k \geq 1$, we denote by P_k the chordless path with k vertices. For integer $k \geq 3$, we denote by C_k the chordless cycle with k vertices. A *diamond* is a graph with four vertices that consists in a clique minus an edge.

2 Some lemmas

We say that a graph G is *b-imperfect* if it is not b-perfect, and *minimally b-imperfect* if G is b-imperfect and every proper induced subgraph of G is b-perfect. We say that a graph G is a *minimal counterexample* to Conjecture 1 if it is a counterexample (i.e., a b-imperfect \mathcal{F} -free graph) with the smallest number of vertices and, among all such graphs, with the smallest number of edges. Note that every minimal counterexample is minimally b-imperfect.

Let $\omega(G)$ denote the number of vertices in a largest clique of G .

Lemma 2.1 (Hoàng and Kouider [8]) *Let G be a minimally b -imperfect graph. Then no component of G is a clique.*

Lemma 2.2 ([9]) *Let G be graph and x be any simplicial vertex of G . Let c be any b -coloring of G with k colors, where $k > \omega(G)$. Then x is not a b -vertex for c .*

Proof. Suppose that x is a b -vertex for c . Then all k colors of c appear in the clique formed by x and its neighbours. Thus $\omega(G) \geq k$, a contradiction. \square

Lemma 2.3 ([9]) *Let G be a minimally b -imperfect graph, and let c be any b -coloring with $b(G)$ colors. Let u, v be two non-adjacent vertices of G such that $N(u) \subseteq N(v)$. Then $c(u) \neq c(v)$, and u is not a b -vertex. In particular, if $N(u) = N(v)$, then none of u, v is a b -vertex.*

Proof. Assume without loss of generality that $c(u) = c(v) = 1$. Consider the restriction of c to $G \setminus u$. Every b -vertex z of color $i \geq 2$ in G is still a b -vertex in $G \setminus u$, because it cannot be that u is the only neighbour of z of color 1. Moreover, it cannot be that u is the only b -vertex of G of color 1, because if it is a b -vertex then v is also a b -vertex. But then $b(G \setminus u) \geq b(G) > \chi(G) \geq \chi(G \setminus u)$, so $G \setminus u$ is b -imperfect, a contradiction. Thus $c(u) \neq c(v)$. This implies that u cannot be a b -vertex, because it has no neighbour of color $c(v)$. In particular, if $N(u) = N(v)$, then the preceding argument works both ways, which leads to the desired conclusion. \square

Lemma 2.4 *Let G be a minimally b -imperfect graph, and let u, v be two true twins of G . Then in any b -coloring of G , u is a b -vertex if and only if v is.*

Proof. Let c be any b -coloring of G with k colors. If u is a b -vertex for c , then $k - 2$ colors appear in $N(u) - \{v\}$. Since $N(u) - \{u, v\} = N(v) - \{u, v\}$, $k - 1$ colors (including the color of u) appear in $N(v)$. Thus, v is a b -vertex. \square

Lemma 2.5 ([9]) *Let G be a minimally b -imperfect \mathcal{F} -free graph. Then G is connected.*

Proof. Suppose that G has several components G_1, \dots, G_p , $p \geq 2$. By Lemma 2.1, each G_i has a subset S_i of three vertices that induce a chordless path. Then G is P_4 -free, for otherwise, since a P_4 is in one component of G , G contains an F_2 . But then Theorem 1.2 is contradicted. Thus the lemma holds. \square

Lemma 2.6 *Let G be a minimally b -imperfect graph, and let c be a b -coloring of G with $b(G)$ colors. If a vertex x is not a b -vertex, then there is a color $f(x)$ such that x is the only neighbor of color $c(x)$ of every b -vertex of color $f(x)$.*

Proof. The definition of G implies $b(G) > \chi(G) \geq \chi(G-x) = b(G-x)$. If the lemma is false, then the coloring c restricted to $G-x$ still has all b -vertices of all colors from 1 to $b(G)$, implying $b(G-x) = b(G)$, a contradiction. \square

Lemma 2.7 *Suppose that a graph G has a coloring (not necessarily a b -coloring) with k colors where $k > \omega(G)$. Then:*

- *If there is a b -vertex, then G contains a P_3 .*
- *If there are two b -vertices of different colors, then G contains as an induced subgraph a P_4 , a $2P_3$ or a diamond.*

Proof. To prove the first part, suppose on the contrary that G contains no P_3 . Then every component of G is a clique. Let G_0 be a component that contains a b -vertex. Then G_0 has vertices of all colors, so G has a clique of size k , a contradiction.

To prove the second part, let x_i be a b -vertex of color i for each $i = 1, 2$. Let $Q = N(x_1) \cap N(x_2)$.

First assume that x_1 sees x_2 . If Q contains non-adjacent vertices u, v , then x_1, x_2, u, v induce a diamond. So, Q is a clique. Then $Q \cup \{x_1, x_2\}$ is a clique, and since $k > \omega(G)$, there is a color j that does not appear in $Q \cup \{x_1, x_2\}$. Since x_1 is a b -vertex, it has a neighbor y of color j , and similarly x_2 has a neighbor z of color j . By the definition of j , vertices y, z are not in Q , so y misses x_2 , z misses x_1 , and $y \neq z$. Then y, x_1, x_2, z induce P_4 in G .

Now, we know x_1 misses x_2 . Suppose that Q contains non-adjacent vertices u, v . Since x_1 is a b -vertex, it has a neighbor y of color 2. Thus y misses x_2 and is not in Q . If y misses u , then y, x_1, u, x_2 induce a P_4 .

So, y sees u , and similarly v . Then y, x_1, u, v induce a diamond. So, Q is a (possibly empty) clique. By Lemma 2.2, for $i = 1, 2$, the vertex x_i is not simplicial, so it has non-adjacent neighbors u_i, v_i . If x_1, x_2 are not in the same component of G , then $x_1, u_1, v_1, x_2, u_2, v_2$ induce a $2P_3$. Thus, x_1, x_2 are in the same component of G , and so there is a chordless path between them. If this path has length at least three, then G contains a P_4 . So let this path be x_1-z-x_2 . We may assume that $z \neq u_1$ (or else, symmetrically, $z \neq v_1$). If u_1 misses z , then either u_1 sees x_2 , and then u_1, z are non-adjacent members of Q , a contradiction, or u_1 misses x_2 , and then $u_1-x_1-z-x_2$ is a P_4 . Thus we may assume that u_1 sees z . It follows that $z \neq v_1$, which restores the symmetry between u_1 and v_1 , and so v_1 too sees z , and u_1, x_1, v_1, z induce a diamond. Thus the lemma holds. \square

Lemma 2.8 *Let G be a P_4 -free graph with $V(G) = X \cup Y$, where $X = \{x_1, \dots, x_{j-1}\}$ and $Y = \{y_2, \dots, y_j\}$ are cliques of size $j-1 \geq 2$, vertices x_1 and y_j are adjacent, and for each $i = 2, \dots, j-1$, vertices x_i and y_i are either different and not adjacent or equal. Then G has a clique on j vertices that consists of x_1, y_j and one of x_i, y_i for each $i = 2, \dots, j-1$.*

Proof. We prove the lemma by induction on the number n of indices i such that $x_i \neq y_i$. If $n = 0$, then G itself is the desired clique. Now let $n > 0$, so $j \geq 3$ and, up to symmetry, we may assume that $x_2 \neq y_2$. Let G' be the graph obtained by contracting x_2 and y_2 , that is, replacing them by a vertex z adjacent to all other vertices. It is easy to see that G' satisfies the conditions of the lemma; in particular, G' is P_4 -free because $G' \setminus z$ is equal to $G \setminus \{x_2, y_2\}$ and z is adjacent to every vertex of $G' \setminus z$. Thus, by the induction hypothesis, G' contains a clique $K' = \{x_1, z, z_3, \dots, z_{j-1}, y_j\}$, where, for $i = 3, \dots, j-1$, vertex z_i is either x_i or y_i . Let $K_x = (K' \setminus z) \cup \{x_2\}$ and $K_y = (K' \setminus z) \cup \{y_2\}$. If none of K_x, K_y is a clique, then x_2 misses some vertex y_h of K' and y_2 misses some vertex x_g of K' , but then $\{x_2, x_g, y_h, y_2\}$ induces a P_4 in G . Thus one of K_x, K_y is the desired clique. \square

A set H of vertices of G is *homogeneous* if every vertex in $G \setminus H$ either sees all or misses all vertices of H . We say that a homogeneous set H of G is *proper* if $H \neq V(G)$.

Lemma 2.9 *Let G be a minimal counterexample to Conjecture 1. If H is a proper homogeneous set in G , then H is a clique or a stable set.*

Proof. We prove this lemma by induction on the size of H . Suppose that H is not a clique or a stable set. Let T be the set of vertices of $G \setminus H$ that

see all of H , and Z be the set of vertices of $G \setminus H$ that miss all of H . So H, T, Z form a partition of $V(G)$. Note that $T \cup Z \neq \emptyset$ by the definition of a proper homogeneous set. Now, by Lemma 2.5, we have $T \neq \emptyset$.

Let c be any b-coloring of G with $k > \chi(G)$ colors. We may assume that the colors that appear in H are $1, \dots, h$ and that those that have a b-vertex in H are colors $1, \dots, h_b$. Clearly we have $h_b \leq h$ and $\chi(H) \leq h$. We claim that

$$h_b \leq \chi(H). \quad (1)$$

For suppose that $\chi(H) < h_b$. Let H' be the subgraph of G induced by the vertices of H that have colors $1, \dots, h_b$, and let c' be the restriction of c to H' . Then the graph H' has strictly fewer vertices than G (because $T \cup Z \neq \emptyset$), it is \mathcal{F} -free, and c' is a b-coloring of H' with h_b colors, where $h_b > \chi(H) \geq \chi(H')$, so H' contradicts the minimality of G . Thus (1) holds.

$$h_b > 0. \quad (2)$$

For suppose that $h_b = 0$. Let G' be the graph obtained from G by replacing H with a stable set S of size h (so that all vertices of S see all of T and none of Z in G'). We establish four properties (i)–(iv) of G' .

(i) G' is \mathcal{F} -free. For suppose that G' contains a member F of \mathcal{F} . If F has three or more vertices of S , then these vertices are pairwise twins in F ; but no member of F has three pairwise twins. So F has at most two vertices of S . Then, since H is not a clique, these vertices can be replaced by the same number of non-adjacent vertices of H so that we obtain a copy of F that is an induced subgraph of G , a contradiction.

(ii) Consider the coloring c' of G' that is obtained by setting $c'(x) = c(x)$ for every $x \in T \cup Z$ and by giving colors $1, \dots, h$ to the vertices of S . Then c' is a b-coloring of G' with k colors, because every b-vertex in G is still a b-vertex in G' .

(iii) $\chi(G') \leq \chi(G)$, because every coloring of G with $\chi(G)$ colors can be transformed into a $\chi(G)$ coloring of G' by maintaining the color of the vertices in $V \setminus H$ and giving to all vertices of S the color of a fixed vertex of H .

(iv) $|V(G')| \leq |V(G)|$; and if $|V(G')| = |V(G)|$ then, since H is not a stable set, we have $|E(G')| < |E(G)|$.

It follows from Properties (i)–(iv) that G' contradicts the minimality of G . Thus (2) holds.

Note that, since $T \neq \emptyset$, we have $k \geq h + 1$. By (2), all colors $h + 1, \dots, k$ appear in T . We claim that

$$h > \chi(H). \quad (3)$$

For suppose on the contrary that $h = \chi(H)$. Let G' be the graph obtained from G by replacing H with a clique K of size h (so that all vertices of K see all of T and none of Z in G'). We establish four properties (i)–(iv) of G' .

(i) G' is \mathcal{F} -free. For suppose that G' contains a member F of \mathcal{F} . If F has three or more vertices of K , then these vertices are pairwise adjacent twins in F ; but no member of F has three pairwise adjacent twins. So F has at most two vertices of K . Then these vertices can be replaced by the same number of adjacent vertices of H , so that we obtain a copy of F that is an induced subgraph of G , a contradiction.

(ii) Consider the coloring c' of G' obtained by setting $c'(x) = c(x)$ for every $x \in T \cup Z$ and by giving colors $1, \dots, h$ to the vertices of K . Then c' is a b-coloring of G' with k colors, because every b-vertex of color $> h$ in G is still a b-vertex in G' , and every vertex of K is a b-vertex in G' since all colors $h + 1, \dots, k$ appear in T .

(iii) $\chi(G') \leq \chi(G)$, because every coloring of G with $\chi(G)$ colors must use at least $\chi(H) = h$ colors on H and can be turned into a $\chi(G)$ coloring of G' by giving colors $1, \dots, h$ to the vertices of K .

(iv) Since H is not a clique, we have $|H| > \chi(H)$, so G' has strictly fewer vertices than G .

It follows from Properties (i)–(iv) that G' contradicts the minimality of G . Thus (3) holds.

Now, we can apply the first part of Lemma 2.7 to H and to the restriction of c to H , which implies that H contains a P_3 . Note that (1) and (3) imply $h_b < h$.

Now we claim that:

$$H \text{ contains no } P_4 \text{ and no } 2P_3. \quad (4)$$

For suppose that H contains a P_4 or a $2P_3$, with vertex-set X . We distinguish between two cases.

Case 1: $h_b = 1$. Let x be a b-vertex of H with $c(x) = 1$. Let G' be the graph obtained from G by removing every edge whose two endvertices are in $H \setminus \{x\}$. We establish three properties (i)–(iii) of G' .

(i) G' is \mathcal{F} -free. For suppose that G' contains a member F of \mathcal{F} . Then $F \cap H$

is a homogeneous set of F ; and since H (in G') contains no P_4 , no $2P_3$ and no diamond, this is possible only if $F \cap H$ either is a P_3 or has at most two vertices; and in either case it is possible to replace $F \cap H$ by a subgraph of H in G that is isomorphic to $F \cap H$, so that we obtain a copy of F that is an induced subgraph of G , a contradiction.

(ii) c is a b-coloring of G' with k colors (because every b-vertex in G is still a b-vertex in G').

(iii) $\chi(G') \leq \chi(G)$, clearly.

It follows from Properties (i)–(iii) that if G' has strictly fewer edges than G , then G' contradicts the minimality of G . So it must be that $G' = G$. This means that every edge in H is adjacent to x . Thus H contains no P_4 and no $2P_3$ as desired.

Case 2: $h_b \geq 2$. So $h \geq 3$. Let u be any b-vertex of color h . Since $h_b < h$, vertex u is not in H , and since color h appears in H it is not in T , so we have $u \in Z$. Note that Z contains no P_3 , for otherwise, if X' is the vertex-set of a P_3 in Z , then $X \cup X'$ induces an F_2 or F_3 . Therefore every component of Z is a clique. Let Y be the component of Z that contains u . Since u is a b-vertex, all colors $1, \dots, h$ must appear in Y . So $|Y| \geq h \geq 3$. Suppose that Y is not homogeneous. Then there are vertices $y_1, y_2 \in Y$ and a vertex t that sees y_1 and misses y_2 . Clearly $t \in T$. Let y_3 be a vertex of $Y \setminus \{y_1, y_2\}$. If t misses y_3 , then $X \cup \{t, y_1, y_2, y_3\}$ induces an F_8 or F_9 . If t sees y_3 then, letting X'' be the vertex-set of a P_3 in H , we obtain that $X'' \cup \{t, y_1, y_2, y_3\}$ induces an F_5 . Therefore Y is a homogeneous set. It follows that all vertices of Y are b-vertices, and so Y contains b-vertices of colors $1, \dots, h$. Let G' be the graph obtained from G by removing every edge whose two endvertices are in H . We establish four properties (i)–(iv) of G' .

(i) G' is \mathcal{F} -free. For suppose that G' contains a member F of \mathcal{F} . If F has three or more vertices of H , then these vertices are pairwise non-adjacent twins in F ; but no member of F has three pairwise non-adjacent twins. So F has at most two vertices of H . Then these vertices can be replaced by the same number of non-adjacent vertices of H in G , so that we obtain a copy of F that is an induced subgraph of G , a contradiction.

(ii) c is a b-coloring of G' with k colors (because every b-vertex of color $> h$ in G is still a b-vertex in G' , and Y contains b-vertices of colors $1, \dots, h$).

(iii) $\chi(G') \leq \chi(G)$, obvious.

(iv) Since H is not a stable set, G' has strictly fewer edges than G .

It follows from Properties (i)–(iv) that G' contradicts the minimality of G . Thus (4) holds.

Suppose that H has at least two components. Since H contains no $2P_3$,

one of these components K is a clique. Let x be a b-vertex in H . If $x \in K$, then all colors $1, \dots, h$ appear in K and all vertices of K are b-vertices; thus $h_b = h$, a contradiction. So K contains no b-vertex, and x is in another component of H . It follows that all colors that appear in K also appear in $H \setminus K$. Consider the graph $G \setminus K$ and the restriction c' of c to that graph. Then c' is a b-coloring of $G \setminus K$ with k colors, and $k > \chi(G) \geq \chi(G \setminus K)$, so $G \setminus K$ contradicts the minimality of G . So H is connected.

Since H is P_4 -free, connected, and has more than one vertex, a classical theorem of Seinsche [16] states that H can be partitioned into two non-empty sets Q, S such that every vertex of Q is adjacent to every vertex of S . Now, each of Q, S is a homogeneous set that is strictly smaller than H . By the induction hypothesis, each of Q, S is a clique or a stable set. If Q and S are two cliques, then H is a clique as desired. If Q and S are two stable sets (of size at least two), then Lemma 2.3 implies that no vertex of H is a b-vertex, which contradicts $h_b > 0$. Therefore we may assume up to symmetry that Q is a clique and S is a stable set of size at least two. Let $\ell = |S|$ and $S = \{s_1, \dots, s_\ell\}$. By Lemma 2.3, all vertices of S have different colors and are not b-vertices. Up to renaming colors, we will assume that s_i has color i for each $i = 1, \dots, \ell$. Since H is complete to T , the set T contains no vertex of color $1, \dots, \ell$; thus we know that:

$$Z \text{ contains b-vertices of all colors } 1, \dots, \ell. \quad (5)$$

We claim that:

$$\text{Each vertex of } Q \text{ is the only b-vertex of its color.} \quad (6)$$

First note that, since $h_b > 0$ and the vertices of S are not b-vertices, some vertex of Q is a b-vertex; and since the vertices of Q are pairwise adjacent twins, by Lemma 2.4 they are all b-vertices. Now suppose that some vertex q of Q is not the only b-vertex of its color, say color $\ell + 1$. Let G' be the graph obtained from G by removing every edge between q and S . Note that the subgraph $G'[H]$ contains no P_4 or $2P_3$. We establish four properties (i)–(iv) of G' .

(i) G' is \mathcal{F} -free. For suppose that G' contains a member F of \mathcal{F} . If F has at most two vertices of H , then these vertices can be replaced by the same number of vertices of H in G , so that we obtain a copy of F that is an induced subgraph of G , a contradiction. If F has three or more vertices of H , then, since $G'[H]$ contains no P_4 or $2P_3$, it must be (by examination of the list \mathcal{F}) that $F \cap H$ is either a P_3 or diamond; but this implies that there

is a P_3 or diamond in H , and so, G contains a copy of F .

(ii) c is a b-coloring of G' with k colors, because every b-vertex of color $\neq \ell + 1$ in G is still a b-vertex in G' and there is a b-vertex of color $\ell + 1$ different from q .

(iii) $\chi(G') \leq \chi(G)$, clearly.

(iv) G' has strictly fewer edges than G , clearly.

It follows from Properties (i)–(iv) that G' contradicts the minimality of G . Thus (6) holds.

Pick a vertex $q \in Q$, and assume that its color is $\ell + 1$. We note that:

$$Z \text{ contains no } P_4 \text{ and no } 2P_3. \quad (7)$$

For otherwise the union of the vertices of such a subgraph of Z with vertices q, s_1, s_2 induces an F_2 or F_3 . Thus (7) holds.

Let C be a clique in Z and let i be a color not in C such that every vertex of C has a neighbor of color i . Then there is a vertex of color i that is adjacent to all of C . (8)

Pick a vertex u of color i that has the most neighbors in C . Suppose that u has a non-neighbor y in C . By the hypothesis we know that y has a neighbor v of color i . By the choice of u , there exists a vertex x of C that sees u and misses v . So $\{u, v, x, y\}$ induce a P_4 . By (7), one of u, v is not in Z and thus is in T . If exactly one of u, v is in T , then $\{u, v, x, y, q\}$ induces an F_1 ; if both u, v are in T , then $\{u, v, x, y, q, s_1\}$ induces an F_{16} , a contradiction. So u is adjacent to all of C . Thus (8) holds.

For every set $J \subseteq \{1, \dots, \ell\}$, Z contains a clique of b-vertices of all colors from J . (9)

We prove (9) by induction on $|J|$. The assertion holds when $|J| = 1$ by (5). Let us now assume that $|J| \geq 2$. To simplify notation put $J = \{1, 2, \dots, j\}$. By the induction hypothesis, Z contains a clique $X = \{x_1, \dots, x_{j-1}\}$ where each $x_i \in X$ is a b-vertex of color i . For each $i = 1, \dots, j - 1$, vertex x_i has a neighbor of color j , so, by (8), there exists a vertex u_j of color j that is adjacent to all of X ; moreover u_j is in Z since color j does not appear in T . If u_j is a b-vertex, then the desired conclusion holds with clique $X \cup \{u_j\}$. So let us assume that there is a color $g \neq j$ such that u_j has no neighbor of color g . Clearly $g \notin \{1, \dots, j - 1\}$. Every member of X has a neighbor of

color g , so, by (8), there is a vertex v_g of color g that is adjacent to all of X . Note that v_g is in $Z \cup T$. Similarly, by the induction hypothesis, Z contains a clique $Y = \{y_2, \dots, y_j\}$ where each $y_i \in Y$ is a b-vertex of color i . By the same argument as for X , there exists in Z a vertex u_1 that is adjacent to all of Y , vertex u_1 is not a b-vertex, so there is a color $h \notin \{1, \dots, j\}$ such that u_1 has no neighbor of color h , and there is a vertex v_h of color h in $Z \cup T$ that is adjacent to all of Y .

If x_1 sees y_j , then X and Y satisfy the hypothesis of Lemma 2.8, so there exists a clique that contains x_1, y_j and one of x_i, y_i for each $i = 2, \dots, j - 1$, which is the desired clique for (9). Let us now assume that x_1 misses y_j . Then u_1 misses u_j , for otherwise $\{u_1, x_1, u_j, y_j\}$ induces a P_4 in Z , which contradicts (7). We have $v_g \neq v_h$, for otherwise $\{u_j, x_1, u_1, y_j, v_g\}$ induces an F_1 . Suppose that v_g, v_h are both in Z . Then v_g misses u_1 , for otherwise $\{u_j, x_1, v_g, u_1\}$ induces a P_4 in Z . Similarly, v_g misses y_j , and v_h misses both u_j, x_1 . Then v_g misses v_h , for otherwise $\{x_1, v_g, y_j, v_h\}$ induces a P_4 in Z . But then $\{u_j, x_1, v_g, u_1, y_j, v_h\}$ induces a $2P_3$ in Z , which contradicts (7). Therefore we may assume up to symmetry that v_g is in T . If $j \geq 3$, then $\{s_1, s_2, q, v_g, x_1, x_2, u_j\}$ induces an F_5 . So $j = 2$.

Suppose that u_1, u_2 are in the same component of Z . So there is a chordless path in Z between them, and since Z contains no P_4 , there is a vertex $z \in Z$ that sees both u_1, u_2 . Then z sees x_1 , for otherwise $\{x_1, u_2, z, u_1\}$ induces a P_4 , and similarly z sees y_2 . Then v_g misses z , for otherwise $\{q, s_1, s_2, v_g, z, x_1, u_2\}$ induces an F_5 . Then v_g sees u_1 , for otherwise $\{q, v_g, x_1, z, u_1\}$ induces an F_1 , and similarly v_g sees y_2 . But then $\{q, s_1, s_2, v_g, u_1, y_2, z\}$ induces an F_5 . Therefore u_1 and u_2 are in different components of Z .

Let X' be the component of Z that contains x_1 and u_2 , and let Y' be the component that contains y_2 and u_1 . If both X' and Y' contain a P_3 , then each of H, X', Y' contains a P_3 , so G contains an F_3 , a contradiction. Thus one of X', Y' is a clique.

Since x_1 is a b-vertex, it has a neighbor r of color $\ell + 1$ (the color of q), and this neighbor is in X' because color $\ell + 1$ does not appear in T . Note that if v_g has a neighbor in Y' , then it is Y' -complete, for otherwise $\{u_2, x_1, v_g, z, z'\}$ induces an F_1 for some adjacent vertices $z, z' \in Y'$.

Suppose that Y' is not a clique (and thus X' is a clique); so it contains a P_3 with vertices y, y', y'' . Then v_g sees at least one of y, y', y'' , for otherwise $\{q, v_g, x_1, u_2, y, y', y''\}$ induces an F_2 ; and so v_g is Y' -complete. Now, v_g misses r , for otherwise $\{s_1, q, s_2, v_g, x_1, r, u_2\}$ induces an F_5 . But then $\{s_1, q, s_2, v_g, y, y', y'', x_1, u_2, r\}$ induces an F_9 . Therefore Y' is a clique, and thus $v_h \in T$. It follows by symmetry that X' is also a clique, and if v_h

has a neighbor in X' then it is X' -complete. Suppose that v_g and v_h are not adjacent. If v_h has no neighbor in X' , then $\{u_2, x_1, v_g, q, v_h\}$ induces an F_1 . So v_h is X' -complete, and similarly v_g is Y' -complete. But then $\{u_1, u_2, v_g, v_h, q\}$ induces an F_1 . Therefore v_g and v_h are adjacent. Suppose v_g is Y' -complete. If v_h is X' -anti-complete, then $\{q, v_h, y_2, u_1, v_g, x_1, u_2, r\}$ induces an F_8 (note that v_g misses r , for otherwise $\{s_1, q, s_2, v_g, x_1, r, u_2\}$ induces an F_5). If v_h is X' -complete, then $\{u_2, x_1, v_g, v_h, y_2, u_1\}$ induces a F_4 . So v_g has no neighbor in X' , and similarly v_h has no neighbor in Y' . But then $\{u_2, x_1, v_g, v_h, y_2\}$ induces an F_1 , a contradiction. Thus (9) holds.

By (9), we know that Z contains a clique of b-vertices b_1, \dots, b_ℓ , which we call D_ℓ . By (8) there is a vertex $u_{\ell+1}$ of color $\ell + 1$ (the color of q) that is adjacent to all of D_ℓ , and $u_{\ell+1}$ is in Z because color $\ell + 1$ does not appear in T . By (6), q is the only b-vertex of color $\ell + 1$, so there exists a color m such that $u_{\ell+1}$ has no neighbor of color m . Clearly, $m > \ell$. By (8) there is a vertex u_m of color m that is adjacent to all of D_ℓ . If u_m is in T , then $\{q, s_1, s_2, u_m, b_1, b_2, u_{\ell+1}\}$ induces an F_5 . So u_m is in Z . If $|Q| \geq 2$, pick any vertex $q' \in Q$ with $q' \neq q$; then $\{q, q', s_1, s_2, b_1, b_2, u_{\ell+1}, u_m\}$ induces an F_6 . So $Q = \{q\}$. Since q has a neighbor of color m , and $m > \ell$, there is a vertex t_m of color m in T . Then t_m misses one of b_1, b_2 , for otherwise $\{q, s_1, s_2, t_m, b_1, b_2, u_m\}$ induces an F_5 . Say t_m misses b_1 . Recall that $u_{\ell+1}$ has no neighbor of color m . Then $\{q, s_1, s_2, t_m, b_1, b_2, u_{\ell+1}, u_m\}$ induces an F_6 (if t_m misses b_2) or F_7 (if t_m sees b_2), a contradiction. This concludes the proof of Lemma 2.9. \square

3 Graphs that contain a C_5

The following lemma was proved in [9].

Lemma 3.1 ([9]) *Let G be an \mathcal{F} -free graph that contains a C_5 . Then $V(G)$ can be partitioned into sets X_1, \dots, X_6, T, Z such that:*

1. *Each of X_1, \dots, X_5 is not empty.*
2. *For every j modulo 5, X_j is complete to X_{j+1} .*
3. *For every j modulo 5 and $j \neq 4$, X_j is anti-complete to X_{j+2} , and some vertex of X_1 misses a vertex of X_4 .*
4. *X_6 is complete to $X_2 \cup X_3 \cup X_5$ and anti-complete to $X_1 \cup X_4$.*

5. X_2, X_3, X_5 are stable sets.
6. The sets $X'_1 = \{x \in X_1 \mid x \text{ has a non-neighbour in } X_4\}$ and $X'_4 = \{x \in X_4 \mid x \text{ has a non-neighbour in } X_1\}$ are stable sets, and there is no edge between X'_1 and $X_1 \setminus X'_1$ and no edge between X'_4 and $X_4 \setminus X'_4$.
7. At least one of $X_1 \setminus X'_1, X_4 \setminus X'_4, X_6$ is empty.
8. Any two non-adjacent vertices of X_1 have inclusionwise comparable neighbourhoods in $V(G) \setminus X_1$, and the same holds for X_4 and X_6 .
9. T is complete to $X_1 \cup \dots \cup X_6$.
10. Z is anti-complete to $X'_1 \cup X_2 \cup X_3 \cup X'_4 \cup X_5$; and if $X_6 \neq \emptyset$, then Z is anti-complete to $X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$.
11. Every component of Z is a clique and is a homogeneous set in $G \setminus T$.

Theorem 3.2 *Let G be a minimal counterexample to Conjecture 1. Then G contains no C_5 .*

Proof. Suppose on the contrary that G contains a C_5 . So $\chi(G) \geq 3$ and consequently $b(G) \geq 4$. Let c be a b-coloring of G with $b(G)$ colors. For each $i \in \{1, \dots, b(G)\}$, let b_i be a b-vertex of color i . Since G contains a C_5 , it has a partition into sets X_1, \dots, X_6, T, Z with the notation and properties given in Lemma 3.1. For each $j \in \{2, 3, 5\}$ let a_j be an arbitrary vertex in X_j , and let a_1, a_4 be non-adjacent vertices of X_1 and X_4 respectively. Such vertices exist by item 3 of Lemma 3.1. For each $j \in \{1, 4\}$, let $X''_j = X_j \setminus X'_j$. So every vertex of X''_j is X_{5-j} -complete. Lemmas 3.1 and 2.3 imply easily the following two facts:

For each $j \in \{2, 3, 5\}$, any two vertices in X_j are twins and have different colors. If X_j contains a b-vertex, then $|X_j| = 1$. (10)

For each $j \in \{1, 4\}$, if $u, v \in X'_j$, then either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$. If $u \in X'_j$ and $v \in X''_j$, then $N(u) \subset N(v)$. If X'_j contains a b-vertex u , then $X''_j = \emptyset$ and any vertex v of $X'_j \setminus \{u\}$ satisfies $N(v) \subset N(u)$ and is not a b-vertex. (11)

Let $X' = X'_1 \cup X_2 \cup X_3 \cup X'_4 \cup X_5$. Note that X' contains a_1, \dots, a_5 , which induce a C_5 ; and that every vertex of Z misses every vertex of X' , by Lemma 3.1.

If a component Y of Z has at least three vertices, then Y is a homogeneous set and a clique. (12)

Recall that Y is a clique by item 11 of Lemma 3.1. Suppose that Y is not homogeneous. Then, by item 11 again and by the definition of Z , there is a vertex $t \in T$ and vertices $y_1, y_2 \in Y$ such that t sees y_1 and misses y_2 . Consider a vertex $y_3 \in Y \setminus \{y_1, y_2\}$. If t sees y_3 , then $\{t, y_1, y_2, y_3, a_1, a_2, a_3\}$ induces an F_5 ; if t misses y_3 , then $\{t, y_1, y_2, y_3, a_1, a_2, a_3, a_4\}$ induces an F_8 , a contradiction, so (12) holds.

If a component Y of Z contains a b-vertex, then Y is a homogeneous set and a clique, $|Y| \geq 3$, every vertex of Y is a b-vertex, and every color that appears in X' appears in Y . (13)

Let $M = N(Y)$. By Lemma 3.1 we have $M \subseteq X_1'' \cup X_4'' \cup X_6 \cup T$. Since Y contains a b-vertex, all colors appear in $Y \cup M$.

Suppose that $M \cap X_6 \neq \emptyset$. So there is an edge yu_6 with $y \in Y$ and $u_6 \in X_6$. Then there is no edge x_1x_4 with $x_1 \in X_1$ and $x_4 \in X_4$, for otherwise, by item 10 of Lemma 3.1, $\{y, u_6, a_2, x_1, x_4\}$ induces an F_1 . It follows that $X_1 = X_1'$ and $X_4 = X_4'$, and so these are stable sets. Now, by items 4 and 10 of Lemma 3.1, every vertex $x \in X_1 \cup X_4$ satisfies $N(x) \subseteq N(x_6)$ for every $x_6 \in X_6$, and so, by Lemma 2.3, the color of x is different from all colors in X_6 . Clearly the color of every vertex in $X_2 \cup X_3 \cup X_5$ is different from all colors in X_6 ; and the color of every vertex in X' is also different from all colors in T . Thus every color that appears in X' does not appear in M , and consequently it appears in Y . So $|Y| \geq 3$, since at least three colors appear in X' because X' contains a C_5 . By (12), Y is a homogeneous set and a clique, and so every vertex of Y is a b-vertex. Thus in this case (13) holds. Now suppose that $M \cap (X_1 \cup X_4) \neq \emptyset$. So, up to symmetry, there exist adjacent vertices $y \in Y$ and $x_1 \in X_1$. By item 10 of Lemma 3.1, x_1 sees all of X_4 (that is, $x_1 \in X_1''$), and X_6 is empty. Now every vertex x in $X_1' \cup X_3$ satisfies $N(x) \subseteq N(v_1)$ for every $v_1 \in X_1''$, and so, by Lemma 2.3, the color of x is different from all colors in X_1'' . Clearly the color of every vertex in $X_2 \cup X_4 \cup X_5$ is different from all colors in X_1'' ; and the color of every vertex in X' is also different from all colors in T . Thus every color that appears in X' does not appear in M , and consequently it appears in Y . As above, this implies that $|Y| \geq 3$, Y is a homogeneous set and a clique, and every vertex of Y is a b-vertex. Thus in this case too (13) holds.

Finally suppose that $M \subset T$. Then every vertex of M is adjacent to every vertex of X' , so we conclude immediately as above. Thus (13) holds.

If some component Y of Z contains a b-vertex, then there does not exist vertices $a \in X_3 \cup X_4, b_h \in T \cup X_4''$ such that $c(b_h) = h$, a is not a b-vertex but b_h is, and $f(a) = h$, that is, every b-vertex of color h has a as its unique neighbor of color $c(a)$. (14)

Suppose Y, a, b_h exist. Let y, y' be two vertices of Y . For simplicity, assume $c(a) = 1$. So Y satisfies the properties given in (13). By (13), we may assume that $b_1 \in Y$. Let $M = N(Y)$. By Lemma 3.1 we have $M \subseteq X_1'' \cup X_4'' \cup X_6 \cup T$. By (13), the vertices of Y must have a neighbor u_h of color h , so $u_h \in M$. If $u_h \in X_6$, then $b_h \in X_4''$, and so b_1, u_h, a_3, b_h, a_1 induces an F_1 , a contradiction. If $u_h \in X_1$, then it sees b_h , a contradiction. So u_h is in $T \cup X_4''$. In other words, b_h and u_h are in the same co-component K of $T \cup X_4''$ (with either $K \subset T$ or $K \subseteq X_4''$). Since u_h is adjacent to b_1 , it is not a b-vertex, and so it has no neighbor of color i for some $i \neq h$. Since u_h is Y -complete, color i does not appear in Y . The vertices of Y must have a neighbor u_i of color i , so $u_i \in M$. If $u_i \in X_6$, then $u_h, b_h \in X_4''$, and so b_1, u_i, a_3, b_h, a_1 induce an F_1 . It follows that $u_i \in K$. Then b_h is not adjacent to u_i , for otherwise $\{a_3, a_5, b_h, u_i, u_h, y, y'\}$ induces an F_{11} . Vertex b_h must have a neighbor v_i of color i . If v_i is in M , then it can play the role of u_i (in particular, $v_i \notin X_1$ and v_i sees x_3, x_5) and we find an F_{11} again. So $v_i \notin M$. If v_i is in Z , then $\{v_i, b_h, a_3, u_i, y\}$ induces an F_1 . So $v_i \notin Z$. If v_i is in $X_1 \cup X_2 \cup X_3 \cup X_5$, then it is either K -complete or K -anticomplete, which is impossible (as v_i sees b_h and misses u_i). So v_i is in $T \cup X_4''$ (that is, in K). But then $\{a_3, a_5, b_h, u_i, u_h, v_i, y, y'\}$ induces an F_{13} . We have established (14).

Z contains no b-vertex. (15)

For suppose that Y is a component of Z that contains a b-vertex. So Y satisfies the properties given in (13). Let $M = N(Y)$. By Lemma 3.1 we have $M \subseteq X_1'' \cup X_4'' \cup X_6 \cup T$. There must be an edge xz with $x \in X_1 \cup \dots \cup X_6$ and $z \in Y$, for otherwise $X_1 \cup \dots \cup X_6$ is a homogeneous set, which contradicts Lemma 2.9. By item 10 of Lemma 3.1, x is in $X_1'' \cup X_4'' \cup X_6$. Up to symmetry we distinguish two cases.

Case 1: $x \in X_1''$. Then item 10 of Lemma 3.1 implies $X_6 = \emptyset$. So $M \subseteq X_1'' \cup X_4'' \cup T$. Recall from item 9 of Lemma 3.1 and the definition of X_1'', X_4''

that the three sets X_1'', X_4'' and T are complete to each other. Since $X_6 = \emptyset$, we have $N(a_3) \subset N(x)$, so by Lemma 2.3, a_3 is not a b-vertex and we can apply Lemma 2.6. Let $c(a_3) = 1$ and $f(a_3) = h > 1$. By (13), we may assume that $b_1 \in Y$. The definition of $f(a_3)$ means that every b-vertex of color h is in $N(a_3)$ and not in $N(b_1)$. Recall that $N(a_3) = X_2 \cup X_4 \cup T$. Moreover b_h is not in $X_2 \cup X_4'$, for otherwise, by (13) there would be a b-vertex of color h in Y , a contradiction. Thus b_h is in $T \cup X_4''$. But, the existence of a_3 and b_h is contradicted by (14).

Case 2: $x \in X_6$. Then item 10 of Lemma 3.1 implies that there is no edge between Z and $X_1 \cup X_4$. So $M \subseteq X_6 \cup T$. By item 9 of Lemma 3.1, X_6 and T are complete to each other. If there is an edge between two vertices $x_1 \in X_1$ and $x_4 \in X_4$, then the five vertices z, x, a_3, x_4, x_1 induce an F_1 . Item 6 of Lemma 3.1 implies X_4 is a stable set. Pick any $a \in X_4$; so a has no neighbor in X_1 . We have $N(a) \subset N(x)$, so by Lemma 2.3, a is not a b-vertex and we can apply Lemma 2.6. Let $c(a) = 1$ and $f(a) = h > 1$. By (13), we may assume that $b_1 \in Y$. The definition of $f(a)$ means that every b-vertex of color h is in $N(a)$ and not in $N(b_1)$. Recall that $N(a) = X_3 \cup X_5 \cup T$. Moreover, b_h is not in $X_3 \cup X_5$, for otherwise, by (13) there would be a b-vertex of color h in Y , a contradiction. Thus, b_h is in T . But, the existence of a and b_h is contradicted by (14).

$$X_6 = \emptyset. \tag{16}$$

For suppose the contrary. Pick any $x_6 \in X_6$. By item 10 of Lemma 3.1, there is no edge between Z and $X_1 \cup \dots \cup X_5$. By item 7, we may assume that $X_1'' = \emptyset$; so X_1 is a stable set. Suppose that every vertex of X_4' has a neighbor in X_1 . Consider a vertex u of X_1 for which $N(u) \cap X_4'$ is maximal; then, the first line of (11) implies $X_4' \subset N(u)$, and so $u \in X_1''$, a contradiction. Therefore some vertex $a \in X_4$ has no neighbor in X_1 . Let $c(a) = 1$. We have $N(a) \subset N(x)$ for every $x \in X_6 \cup X_4 \setminus \{a\}$, so Lemma 2.3 implies that color 1 does not appear in $X_6 \cup X_4 \setminus \{a\}$ and that a is not a b-vertex. By (15), we have $b_1 \in X_1 \cup X_2$. Let $f(a) = 3$. So b_3 is in $N(a)$ and not in $N(b_1) \cup Z$. Note that $N(a) = X_3 \cup X_5 \cup T$.

Suppose that $b_1 \in X_1$. So $b_3 \in X_3$, and $X_3 = \{b_3\}$. We may assume that $c(x_6) = 2$. Vertex b_1 must have a neighbor u_2 of color 2, and necessarily $u_2 \in X_4$. Then there is no edge xz with $x \in X_6$ and $z \in Z$, for otherwise $\{z, x, a_2, b_1, u_2\}$ induces an F_1 . Thus, vertices of X_6 are not b-vertices as they cannot have any neighbor of color 1. It follows that b_2 is in $X_1 \cup X_4$; in fact it cannot be in X_1 by (11) (X_1' cannot contain two b-vertices); so b_2 is in

X_4 . Vertex b_1 must have a neighbor u_3 of color 3, and necessarily $u_3 \in X_5$. The definition of $f(a)$ implies that u_3 is not a b-vertex, so it has no neighbor of some color $h \neq 3$. So color h does not appear in $X_1 \cup X_4 \cup X_6 \cup T$, and $h \geq 4$. Vertex b_2 must have a neighbor u_h of color h , and (since $|X_3| = 1$) we have $u_h \in X_5$. But then $\{b_1, a_2, b_3, u_2, a, x_6, u_3, u_h\}$ induces an F_{21} , a contradiction.

Therefore $b_1 \in X_2$, so $X_2 = \{b_1\}$. Then (because b_3 sees a and not b_1) $b_3 \in X_5$, so $X_5 = \{b_3\}$. Vertex b_1 must have a neighbor of color 3, so we may assume that $c(a_3) = 3$. The definition of $f(a)$ implies that a_3 is not a b-vertex, so we may assume that it misses color 4. So color 4 does not appear in $X_6 \cup T$, and we may assume $c(x_6) = 2$. Vertex b_1 must have a neighbor u_4 of color 4, and necessarily $u_4 \in X_1 \cup X_3$. If $u_4 \in X_3$, then $|X_3| \geq 2$, so u_4 is not a b-vertex by (10), and b_4 can only be in X_1 . Then b_4 must have a neighbor u_2 of color 2, and u_2 can only be in X_4 . But then $\{b_1, b_3, b_4, a, u_2, a_3, u_4, x_6\}$ induces an F_{21} , a contradiction. So color 4 does not appear in X_3 , and consequently $u_4 \in X_1$ and $b_4 \in X_1 \cup X_4$.

If b_4 is in X_4 , then it has no neighbor of color 1, a contradiction. So, b_4 is in X_1 . Vertex b_4 must have a neighbor u_2 of color 2, and (because of x_6) u_2 can only be in X_4 . Then there is no edge xz with $x \in X_6$ and $z \in Z$, for otherwise $\{z, x, b_1, b_4, u_2\}$ induces an F_1 . Thus, vertices of X_6 are not b-vertices as they have neither color 4 nor a neighbor of color 4. It follows that b_2 is in $X_1 \cup X_4$; in fact it cannot be in X_1 by (11) (X_1' cannot contain two b-vertices); so b_2 is in X_4 . Recall that the definition of $f(a)$ implies that a is the only neighbor of color 1 of $a_5 = b_3$. But then b_2 cannot have any neighbor of color 1. Thus (16) holds.

$$X_1'' = \emptyset \text{ and } X_4'' = \emptyset. \quad (17)$$

For suppose on the contrary and up to symmetry that there is a vertex x in X_1'' . Let $c(a_1) = 1$. Every vertex $u \in X_1' \cup X_3$ satisfies $N(u) \subset N(x)$, so by Lemma 2.3, $X_1' \cup X_3$ contains no b-vertex. In particular a_1 is not a b-vertex. By (11) and Lemma 2.3, no vertex of $X_1 \setminus \{a_1\}$ has color 1. Thus, and by (15), b_1 is in X_4 , and since it misses a_1 it is in X_4' and has no neighbor in Z . Since a_1 is not a b-vertex, color $f(a_1)$ exists, say $f(a_1) = 2$. By (15) and the definition of $f(a_1)$, all b-vertices of color 2 must be in $X_2 \cup X_4$. In fact if b_2 is in X_4 , then by (11) we have $N(b_1) \subset N(b_2)$, which contradicts Lemma 2.3. So, X_4 contains no b-vertex of color 2, and, by (10), we have $X_2 = \{b_2\}$. Clearly color 2 does not appear in X_3 , and by the definition of $f(a_1)$, color 1 does not appear in X_3 . Let $c(a_5) = h$. Then h must appear in X_3 , for otherwise b_2 cannot have any neighbor of color h . So we may

assume that $c(a_5) = c(a_3) = 3$. Therefore a_3 and a_5 are the only vertices of color 3 in $V \setminus Z$, and since X_3 contains no b-vertices and by (15), we have $b_3 = a_5$, and, by (10), $|X_5| = 1$. But now b_1 cannot have any neighbor of color 2. Thus (17) holds.

$$Z = \emptyset \text{ and } T = \emptyset. \tag{18}$$

By item 10 of Lemma 3.1 and (17) there is no edge between $X_1 \cup \dots \cup X_5$ and Z . So if Z or T is not empty, then $X_1 \cup \dots \cup X_5$ is a homogeneous set that contradicts Lemma 2.9. Thus (18) holds.

By the preceding points, each X_i ($i = 1, \dots, 5$) is a stable set and contains at most one b-vertex, and $T \cup Z \cup X_6 = \emptyset$. So $b(G) \leq 5$. Moreover, if a_2, a_3, a_5 are b-vertices of three different colors, then, by (10), a_2 and a_3 cannot have a neighbor of color $c(a_5)$, a contradiction. It follows that $b(G) = 4$ and that we may assume up to symmetry that $b_1 \in X_1$, $b_4 \in X_4$, and $X_2 = \{b_2\}$. Vertex b_4 must have a neighbor u_2 of color 2, and u_2 can only be in X_5 . Then, by (10), b_3 must be in X_3 , so $X_3 = \{b_3\}$. Vertex b_1 must have a neighbor u_3 of color 3, and necessarily, $u_3 \in X_5$. Vertex b_2 must have a neighbor u_4 of color 4, and u_4 can only be in X_1 . Vertex b_3 must have a neighbor u_1 of color 1, and u_1 can only be in X_4 . If both b_1b_4 and u_1u_4 are edges, then $\{b_1, b_2, b_4, u_1, u_4\}$ induces an F_1 . If exactly one is an edge, then $\{b_1, b_2, b_3, b_4, u_1, u_2, u_3, u_4\}$ induces an F_{19} . So both are non-edges. Then b_1 must have a neighbor v_4 of color 4, which can only be in X_4 ; and $\{b_1, b_2, b_3, b_4, v_4, u_2, u_3, u_4\}$ induces an F_{19} again. This completes the proof of Theorem 3.2. \square

4 Graphs that contain a boat

Let us call *boat* any graph whose vertex-set can be partitioned into sets $A_0, A_1, \dots, A_q, B_0, B_1, \dots, B_q$ that satisfy the following properties, where $A = A_0 \cup A_1 \cup \dots \cup A_q$ and $B = B_0 \cup B_1 \cup \dots \cup B_q$:

- $q \geq 2$ and each of $A_1, \dots, A_q, B_1, \dots, B_q$ is not empty;
- If $q = 2$ then also A_0 and B_0 are not empty;
- A_0, A_1, \dots, A_q are pairwise complete to each other, and B_0, B_1, \dots, B_q are pairwise complete to each other;

- For $j = 1, \dots, q$, A_j is complete to B_j and anticomplete to $B \setminus B_j$;
- A_0 is anticomplete to B_0 .

Note that the smallest boats have six vertices: these are the boat with $q = 2$ where each of $A_0, A_1, A_2, B_0, B_1, B_2$ has size one; and the boat with $q = 3$ where each of $A_1, A_2, A_3, B_1, B_2, B_3$ has size one and $A_0 = B_0 = \emptyset$. We call these two graphs the *small boats*.

Lemma 4.1 *Let G be a graph that contains no F_1, F_4, F_{10} or C_5 . If G contains a boat, then $V(G)$ can be partitioned into sets M, T, Z such that the subgraph induced by M is a boat and every vertex of M is complete to T and anticomplete to Z .*

Proof. Since G contains a boat, there is a set M of vertices of G that induces a boat and is maximal with this property. We use the same notation $A_0, A_1, \dots, A_q, B_0, B_1, \dots, B_q$ and properties as in the definition of a boat. Let then T be the set of M -complete vertices and Z be the set of M -anticomplete vertices. In order to prove the lemma, we need only establish that $V(G) = M \cup T \cup Z$. Assume the contrary. Let x be a vertex of G that is not in $M \cup T \cup Z$. Let us fix some notation. Let $I = \{i \mid 0 \leq i \leq q \text{ and } A_i \neq \emptyset\}$. So I is equal to either $\{0, 1, \dots, q\}$ or $\{1, \dots, q\}$. Likewise, let $J = \{j \mid 0 \leq j \leq q \text{ and } B_j \neq \emptyset\}$. Note that we have $|I \cap J| \geq 3$ (even when $q = 2$) by the definition of a boat. For each $i \in I$, let a_i be an arbitrary vertex in A_i and u_i be a neighbor of x in A_i (if any). Likewise, for each $j \in J$, let b_j be an arbitrary vertex in B_j and v_j be a neighbor of x in B_j (if any). We claim that:

If there is a pair of integers $i \in I, j \in J, i \neq j$ such that x has a neighbor in each of A_i, B_j , then x is complete to either $A \setminus (A_i \cup A_j)$ or $B \setminus (B_i \cup B_j)$. (19)

Note that vertices u_i and v_j exist. If x has non-neighbors $u' \in A \setminus (A_i \cup A_j)$ and $v' \in B \setminus (B_i \cup B_j)$, then $\{u', u_i, x, v_j, v'\}$ induces an F_1 or a C_5 , a contradiction. Thus (19) holds.

There is no pair of integers $i, j \in I \cap J, i \neq j$ such that x has a neighbor in each of A_i, A_j, B_i, B_j . (20)

For suppose the contrary; so vertices u_i, u_j, v_i, v_j exist. Since i, j play symmetric roles, we may assume that $i \neq 0$, so u_i and v_i are adjacent. By (19)

and up to symmetry, we may assume that x is complete to $A \setminus (A_i \cup A_j)$. Consider any $h \in (I \cap J) \setminus \{i, j\}$, which is not empty. We know that x is complete to A_h , and $B_h \neq \emptyset$. Then x is adjacent to b_h , for otherwise $\{a_h, u_i, x, v_i, v_j, b_h\}$ induces an F_4 (if $h = 0$) or F_{10} (if $h \neq 0$). So x is complete to B_h . We can repeat this argument for every pair of integers from $I \cap J$, and it follows that x is complete to $A_\ell \cup B_\ell$ for every $\ell \in I \cap J$, in particular for $1 \leq \ell \leq q$. Thus, if $0 \in I \cap J$, we obtain $x \in T$, a contradiction. So $0 \notin I \cap J$, say $B_0 = \emptyset$, and so $q \geq 3$. Since x is not in T , it has a non-neighbor w in $A \cup B$, and it must be that $w \in A_0$. But then $\{w, a_1, a_2, x, b_1, b_3\}$ induces an F_4 . Thus (20) holds.

There do not exist two pairs of integers $g, h \in I$, $g \neq h$
and $i, j \in J$, $i \neq j$ such that x has a neighbor in each of
 A_g, A_h, B_i, B_j . (21)

For suppose the contrary; so vertices u_g, u_h, v_i, v_j exist. First suppose that $\{g, h\} \cap \{i, j\} \neq \emptyset$, say $g = i$. By (20) we may assume that $h \neq j$. If $g = i = 0$, then A_j and B_h are not empty, and by (20), x has no neighbor in those two sets; but then $\{a_j, u_0, x, v_0, b_h\}$ induces an F_1 . So $g = i \neq 0$, and u_g sees v_i . One of h, j is not equal to 0, say $j \neq 0$. So $A_j \neq \emptyset$ and, by (20), x has no neighbor in that set; but then $\{a_j, u_g, u_h, x, v_i, v_j\}$ induces an F_{10} . Now we may assume that the four integers g, h, i, j are different. We may assume that none of h, i, j is equal to 0; so A_j and B_h are not empty. If x has no neighbor in those two sets, then $\{a_j, u_g, x, v_i, b_h\}$ induces an F_1 . Now, by (20), x has a neighbor in A_j , or in B_h , but not in both. If x has a neighbor in B_h , then $\{a_j, u_h, u_g, x, v_h, v_j\}$ induces an F_{10} . If x has a neighbor in A_j , then $\{b_h, v_j, v_i, x, u_h, u_j\}$ induces an F_{10} . Thus, (21) holds.

There is no pair of integers $i \in I$, $j \in J$, $i \neq j$ such that x
has a neighbor in each of A_i, B_j . (22)

For suppose the contrary; so vertices u_i and v_j exist. By (19) and up to symmetry, we may assume that x is complete to $A \setminus (A_i \cup A_j)$. Thus x has neighbors in at least two of the sets A_0, \dots, A_q , and so, by (21), it has no neighbor in $B \setminus B_j$. Consider any $h \in I \cap J \setminus \{i, j\}$. So A_h and B_h are not empty, and x is complete to A_h and anticomplete to B_h . We claim that there is a non-neighbor w of x in $B \setminus (B_j \cup B_h)$; indeed, if $B_i \neq \emptyset$, then we can take any $w \in B_i$; and if $B_i = \emptyset$, then $i = 0$, $q \geq 3$, so there is an integer $g \in (I \cap J) \setminus \{h, j\}$, and we can take any $w \in B_g$. Now we can

apply (19) to h and j , and the existence of w implies that x is complete to $A \setminus (A_j \cup A_h)$. In summary, we have established that x is complete to $A \setminus A_j$ and anti-complete to $B \setminus B_j$. If x is complete to B_j , then we can add x to A_j and obtain a boat (with sets $A_0, \dots, A_j \cup \{x\}, \dots, A_q, B_0, \dots, B_q$) that contradicts the maximality of M . Therefore x has a non-neighbor w_j in B_j . Suppose that $j \neq 0$; so $A_j \neq \emptyset$ and a_j sees both v_j, w_j . Then w_j sees v_j , for otherwise $\{w_j, b_h, v_j, x, a_i\}$ induces an F_1 ; and x sees a_j , for otherwise $\{b_h, w_j, a_j, a_i, x\}$ induces an F_1 ; but then $\{a_h, a_j, x, v_j, w_j, b_h\}$ induces an F_4 (if $h = 0$) or F_{10} (if $h \neq 0$), a contradiction. So $j = 0$. If x has a non-neighbor $y \in A_0$, then $\{y, a_i, x, v_0, b_h\}$ induces an F_1 . Thus x is complete to A_0 . Set $A_{q+1} = \{x\}$, $B_{q+1} = B_0 \cap N(x)$ and $B'_0 = B_0 \setminus N(x)$. Note that $v_j \in B_{q+1}$ and $w_j \in B'_0$. Moreover, every $v \in B_{q+1}$ sees every $w \in B'_0$, for otherwise $\{a_i, x, v, b_h, w\}$ induces an F_1 . But now we find a larger boat, with sets $A_0, \dots, A_q, A_{q+1}, B'_0, B_1, \dots, B_q, B_{q+1}$, which contradicts the maximality of M . Thus (22) holds.

Since $x \notin Z$, up to symmetry we may assume that x has a neighbor $u_h \in A_h$ for some h with $0 \leq h \leq q$. By (22), x has no neighbor in $B \setminus B_h$. Consider any $i \in (I \cap J) \setminus \{h\}, i \neq 0$, and suppose that x has a non-neighbor w_i in A_i . Pick any $j \in J \setminus \{h, i\}$. Then $\{x, u_h, w_i, b_i, b_j\}$ induces an F_1 , a contradiction. Thus x is complete to A_i . By repeating this argument with i instead of h , we obtain that x is complete to $A \setminus A_0$, and by (22) it is anti-complete to B . But now we find a larger boat, with sets $A_0 \cup \{x\}, A_1, \dots, A_q, B_0, B_1, \dots, B_q$, which contradicts the maximality of M . This completes the proof of Lemma 4.1. \square

Theorem 4.2 *Let G be a minimal counterexample to Conjecture 1. Then G contains no boat.*

Proof. By Theorem 3.2, G contains no C_5 . Suppose that G contains a boat. Then, by Lemma 4.1, the vertex-set of G can be partitioned into sets M, T, Z such that M induces a boat and every vertex of M is T -complete and Z -anticomplete. Then T and Z are empty, for otherwise M is a homogeneous set that contradicts Lemma 2.9. Thus $V(G) = M$. We use the same notation $A_0, A_1, \dots, A_q, B_0, B_1, \dots, B_q$ and properties as in the definition of a boat. As in Lemma 4.1, let $I = \{i \mid 0 \leq i \leq q \text{ and } A_i \neq \emptyset\}$, $J = \{j \mid 0 \leq j \leq q \text{ and } B_j \neq \emptyset\}$, and note that $|I \cap J| \geq 3$. For each $i \in I$, let a_i be an arbitrary vertex in A_i , and for each $j \in J$, let b_j be an arbitrary vertex in B_j .

Let c be a b-coloring of G with $k = b(G) > \chi(G)$ colors. For each color $\ell \in \{1, \dots, k\}$ let d_ℓ be a b-vertex of color ℓ , and let $D = \{d_1, \dots, d_k\}$.

By the definition of a boat each of $A_0, A_1, \dots, A_q, B_0, B_1, \dots, B_q$ is a homogeneous set, so it satisfies the properties described in Lemma 2.9. It follows (recall Lemma 2.3) that:

Each A_i ($i \in I$) is a clique or a stable set. All vertices of A have different colors. Any two non-adjacent vertices of A are not b-vertices. The same holds for B . (23)

We will now prove

If two of the sets A_1, \dots, A_q contain a member of D , then A contains vertices of all colors. The same holds for B . (24)

For suppose up to symmetry that $d_1 \in A_1$ and $d_2 \in A_2$. Consider any color ℓ that appears in B . If ℓ appears in $B \setminus B_1$, then it does not appear in B_1 , and since d_1 must have a neighbor of color ℓ , such a neighbor must be in A . If ℓ appears in B_1 , then it does not appear in B_2 , and since d_2 must have a neighbor of color ℓ , such a neighbor must be in A . So we have established that all colors that appear in B also appear in A ; and so, all colors appear in A . Thus (24) holds.

If A_i is not a clique for some $i \in \{1, \dots, q\}$, then every B_j with $j \in \{1, \dots, q\} \setminus \{i\}$ is a stable set. The same holds with A and B interchanged. (25)

For suppose on the contrary, and up to symmetry, that there are two non-adjacent vertices u, v in A_1 and two adjacent vertices x, y in B_2 . Let $h = 3$ if $q \geq 3$ and $h = 0$ if $q = 2$. Then $\{a_h, u, v, a_2, b_1, x, y\}$ induces an F_{11} , a contradiction. Thus (25) holds.

For each $i \in \{0, \dots, q\}$, one of A_i, B_i is a clique. (26)

For suppose on the contrary that there exist non-adjacent vertices $u, v \in A_i$ and non-adjacent vertices $x, y \in B_i$. Pick two integers h, j in $(I \cap J) \setminus \{i\}$. Then $\{u, v, x, y, a_h, a_j, b_h, b_j\}$ induces an F_{12} (if $i = 0$) or F_{14} (if $h = 0$ or $j = 0$) or F_{15} (if $h, i, j > 0$). Thus (26) holds.

One of A, B is a clique. (27)

For suppose the contrary. So one of the A_i 's ($i \in I$) is not a clique and one of the B_j 's ($j \in J$) is not a clique. By (26) and up to symmetry, we may assume that A_1 is not a clique and one of B_0, B_2 is not a clique. By (25) and (26), B_1 is a clique and each of B_2, \dots, B_q is a stable set. Note that this implies, by (23), that each of B_2, \dots, B_q contains at most one b-vertex (and if it contains one, then it has size one). Let u, v be two non-adjacent vertices in A_1 , and let x, y be two non-adjacent vertices in B_0 or B_2 . By (23), u and v are not b-vertices, and we may assume that they have color 1 and 2. By (23) again, d_1, d_2 are in B , each of them is adjacent to all other vertices of B , and clearly they are not in B_1 . Suppose that B_0 is not a clique, say x, y are in B_0 . If $q \geq 3$, then $\{u, v, a_2, a_3, b_2, b_3, x, y\}$ induces an F_{12} . So $q = 2$, and so $A_0 \neq \emptyset$. Note that d_1, d_2 are not both in B_2 (because B_2 can contain at most one b-vertex). So we may assume that $d_1 \in B_0$. But d_1 is adjacent to x , a contradiction to (23). Therefore B_0 is a clique. So x, y are in B_2 , which restores the symmetry between A and B . Thus A_2 is a clique, each of A_1, A_3, \dots, A_q is a stable set and contains at most one b-vertex, and A_0 is a clique. If $q \geq 4$, then $\{u, v, a_3, a_4, b_3, b_4, x, y\}$ induces an F_{12} . So $q \leq 3$. None of d_1, d_2 is in B_2 (because B_2 is now a stable set of size at least two, so it does not contain any b-vertex), so they are in $B_0 \cup B_3$. Since B_3 is a stable set, it contains at most one of d_1, d_2 , and so at least one of these is in B_0 , say $d_2 \in B_0$. By symmetry, we may assume that x, y have color 3 and 4 respectively, vertices d_3, d_4 are in $A_0 \cup A_3$, vertex d_4 is in A_0 , and each of d_3, d_4 is adjacent to all other vertices of A . But then $\{u, v, d_3, d_4, x, y, d_1, d_2\}$ induces an F_7 (if $d_3 \in A_3$ and $d_1 \in B_3$) or an F_6 (else). Thus (27) holds.

$$\text{Both } A, B \text{ are cliques.} \tag{28}$$

For suppose the contrary. By (27), we may assume that A is not a clique and B is a clique. The vertices of B_0 are simplicial, so they are not b-vertices by Lemma 2.2. Moreover, if two of the sets B_1, \dots, B_q contain a b-vertex, then, by (24), B contains vertices of all colors, and so G has a clique of size k , a contradiction. Therefore we may assume up to symmetry that $B \cap D \subseteq B_1$. Let u, v be two non-adjacent vertices in A . By (23), u, v are not b-vertices and we may assume that they have color 1 and 2. Then vertices d_1, d_2 are not in A , so they are in B_1 . If $u, v \in A_i, i \neq 0, i \neq 1$, then by (24), B_1 is a stable set of size at least two, a contradiction to the assumption that B is a clique. So, u, v are not in $A \setminus A_0$, and this argument shows that $A \setminus A_0$ is a clique. Thus u, v are in A_0 . Now $A_1 \cup B_1$ is a clique, so we may assume that it does not contain any vertex of color 3. Since d_1, d_2 are b-vertices, they must have a neighbor x_3 of color 3, which must be in $B \setminus B_1$. Vertex

x_3 is not a b-vertex (because $B \cap D \subseteq B_1$), so d_3 is in A , and $d_3 \notin A_1$ by the choice of color 3. By (23), d_3 is adjacent to all of $A \setminus \{d_3\}$. But then $\{u, v, d_3, a_1, d_1, d_2, x_3\}$ induces an F_5 . Thus (28) holds.

Since A is a clique, there is a color, say color 1, that does not appear in A . So d_1 is in B and is the unique vertex of color 1. Likewise, there is a color, say color 2, that does not appear in B , and so d_2 is in A and is the unique vertex of color 2. Since d_1 must have a neighbor of color 2, vertices d_1, d_2 are adjacent, and so we may assume that $d_1 \in B_1$ and $d_2 \in A_1$. By (24) we have $A \cap D \subseteq A_1$ and $B \cap D \subseteq B_1$. But then $A_1 \cup B_1$ is a clique that contains vertices of all colors, a contradiction. This concludes the proof of the theorem. \square

5 Proof of the main result

Suppose that Theorem 1.1 fails. So there is a minimal counterexample G to Conjecture 1. By Theorems 3.2 and 4.2, G contains no C_5 and no boat. Since G contains no C_5 , no F_1 ($= P_5$) and no F_{10} ($= \overline{P}_6$), it contains no odd hole and no odd antihole, so it is *perfect* [2], that is, G satisfies $\chi(H) = \omega(H)$ for every induced subgraph H of G . (Actually, since G also contains no \overline{C}_6 (which is a boat), it is *weakly chordal* (i.e., it contains no hole and no antihole of length at least five), and reference [6] implies the perfectness of G more simply than [2].)

Let c be a b-coloring of G with $k = b(G) > \chi(G)$ colors. For each color $i \in \{1, \dots, k\}$, let d_i be a b-vertex of color i , and let $D = \{d_1, \dots, d_k\}$. Note that G contains no clique of size k , for otherwise we would have $k \leq \omega(G) \leq \chi(G) < b(G) = k$, which is impossible. In particular D is not a clique.

We observe that:

$$G \text{ contains a } 2K_2. \tag{29}$$

For suppose that G contains no $2K_2$. Since D is not a clique, we may assume without loss of generality that d_1, d_2 are not adjacent. Since d_1 is a b-vertex, it has a neighbor x_2 of color 2. Since d_2 is a b-vertex, it has a neighbor x_1 of color 1. Then x_1, x_2 are adjacent, for otherwise $\{d_1, d_2, x_1, x_2\}$ induces a $2K_2$. Since d_1 is a b-vertex, by Lemma 2.3 we cannot have $N(d_1) \subseteq N(x_1)$; so there exists a vertex u that is adjacent to d_1 and not to x_1 . Likewise, there exists a vertex v that is adjacent to d_2 and not to x_2 . Then u is adjacent to d_2 , for otherwise $\{u, d_1, d_2, x_1\}$ induces a $2K_2$; and u is adjacent to x_2 , for

otherwise $\{u, d_1, d_2, x_1, x_2\}$ induces a C_5 . Likewise, v is adjacent to d_1 and x_1 . But now $\{u, v, d_1, d_2, x_1, x_2\}$ induces a boat (if u, v are not adjacent) or an F_{10} (if u, v are adjacent), a contradiction. Thus (29) holds.

Since G contains a $2K_2$, there is a subset B of $V(G)$ such that the subgraph induced by B has at least two components, each component of B has at least two vertices, and B is maximal with this property. Let B_1, \dots, B_r be the components of B , with $r \geq 2$. Let S be the set of vertices of $V \setminus B$ that are B -anticomplete. Note that S is a stable set, for otherwise two adjacent vertices of S could be added to B , which would contradict the maximality of B . Let $A = V \setminus (B \cup S)$. We claim that:

$r = 2$ and there is a component of B , say B_2 , such that B_2 is a clique and every vertex of A is B_2 -complete and has a neighbor in B_1 . (30)

Consider any vertex $a \in A$. By the definition of S , a has a neighbor in B . If a has no neighbor in some component B_j of B , then $B \cup \{a\}$ contradicts the maximality of B (as every component of $B \cup \{a\}$ has size at least two and B has at least two components, one that contains a and one that includes B_j). So a has a neighbor in each component of B . If every vertex of A is B -complete, then B is a homogeneous set, which contradicts Lemma 2.9. So there is a vertex a_0 of A that is not B -complete, say a_0 has a non-neighbor in B_1 . Since B_1 is connected, there are adjacent vertices u_1, v_1 in B_1 such that a_0 is adjacent to u_1 and not to v_1 . If a_0 also has a non-neighbor in another component B_i of B ($i \neq 1$), then there are adjacent vertices u_i, v_i in B_i such that a_0 is adjacent to u_i and not to v_i , and $\{a_0, u_1, v_1, u_i, v_i\}$ induces an F_1 . Therefore only component B_1 of B contains a non-neighbor of a_0 , that is, a_0 is $B \setminus B_1$ -complete. Consider any other vertex a'_0 in A that is not B -complete. Just like for a_0 , there is a component B_i and adjacent vertices u_i, v_i in B_i such that a'_0 is adjacent to u_i and not to v_i and a'_0 is $B \setminus B_i$ -complete. If $i \neq 1$, then $\{a_0, a'_0, u_1, v_1, u_i, v_i\}$ induces an F_4 or a boat, a contradiction. So $i = 1$ for each a'_0 . This implies that every vertex of A is $B \setminus B_1$ -complete, so $B \setminus B_1$ is homogeneous, and Lemma 2.9 implies that $r = 2$ and B_2 is a clique. Thus (30) holds.

For each $a \in A$ and each component C of $B_1 \setminus N(a)$, every vertex in $B_1 \setminus C$ is either C -complete or C -anticomplete. Furthermore, there is a vertex in $B_1 \setminus C$ that is $(C \cup \{a\})$ -complete. (31)

Suppose that some vertex z in $B_1 \setminus C$ has a neighbour and a non-neighbour in C . Then z is adjacent to a by the definition of C . Since C is connected,

there are adjacent vertices $y, y' \in C$ such that z sees y and misses y' . Let t be any vertex in B_2 . Then $\{t, a, z, y, y'\}$ induces an F_1 , a contradiction. Furthermore, since B_1 is connected, some vertex in $B_1 \cap N(a)$ must have a neighbour in C , and so is $(C \cup \{a\})$ -complete. Thus (31) holds.

For every $a \in A$, each component of $B_1 \setminus N(a)$ is a homogeneous set and a clique. (32)

Let C be any component of $B_1 \setminus N(a)$. Suppose that C is not homogeneous. So there are adjacent vertices x, y in C and a vertex u not in C that sees x and misses y . By (31), we have $u \notin B_1$ and so $u \in A$. Let t be any vertex in B_2 . Then u sees a , for otherwise $\{a, t, u, x, y\}$ induces an F_1 . By (31), there is a vertex z in $B_1 \cap N(a)$ that is $(C \cup \{a\})$ -complete. But now $\{a, t, u, x, y, z\}$ induces a boat or an F_4 , a contradiction. So C is homogeneous. Then Lemma 2.9 implies that C is a clique. Thus (32) holds.

S contains no b-vertex. (33)

Indeed, if x is any vertex in S and t is any vertex in B_2 , then $N(x) \subset N(t)$ and Lemma 2.3 implies that x cannot be a b-vertex. Thus (33) holds.

Let G' be the graph obtained from G by removing all edges whose two endvertices are in B_2 .

G' does not contain any C_5 , boat, or member of \mathcal{F} . (34)

For suppose that G' has an induced subgraph F that is either a C_5 , a boat, or a member of \mathcal{F} . If F is a boat, we may assume that it is a small boat, since every boat contains a small boat. If F contains at most one vertex of B_2 , then F is an induced subgraph of G , a contradiction. So F must contain at least two vertices of B_2 . Then these vertices are pairwise non-adjacent twins in F , which implies that F is not a C_5 or a small boat (since such graphs do not have any pair of non-adjacent twins); more precisely F is one of $F_2, F_3, F_5, F_6, F_7, F_9, F_{11}, F_{12}, F_{13}, F_{14}, F_{15}, F_{19}, F_{21}$ and it has exactly two vertices of B_2 . In fact F cannot be any of F_{19}, F_{21} , for that would imply that G' contains a C_5 , which we have already excluded. Likewise, F cannot be any of F_{12}, F_{14}, F_{15} , since that would imply that G' contains a boat, which is also excluded. Therefore F is one $F_2, F_3, F_5, F_6, F_7, F_9, F_{11}, F_{13}$.

Suppose that F is either F_2 or F_3 . So F has vertices x, y, a, z_1, \dots, z_p , with $x, y \in B_2$, and edges xa, ya , and either (if F is F_2) $p = 4$ and $\{z_1, \dots, z_4\}$ induces a P_4 , or (if F is F_3) $p = 6$ and $\{z_1, \dots, z_6\}$ induces a $2P_3$. Since x, y

are in B_2 , vertices z_1, \dots, z_p must be in B_1 and a must be in A . But then a, z_1, \dots, z_p contradict Claim (32).

Suppose that F is either F_5 or F_9 . So F has vertices $x, y, a, b, z_1, \dots, z_p$, with $x, y \in B_2$, and edges $xa, xb, ya, yb, ab, az_1, z_1z_2, z_1z_3, z_2z_3$ and either (if F is F_5) $p = 3$ and az_2 is an edge, or (if F is F_9) $p = 6$ and vertices z_4, z_5, z_6 induce a P_3 and are adjacent to a . Since x, y are in B_2 , vertices z_1, \dots, z_p must be in B_1 and a, b must be in A . But then b, z_1, z_2, z_3 contradict Claim (32).

Suppose that F is either F_6 or F_7 . So F has vertices $x, y, a, b, z_1, \dots, z_4$, with $x, y \in B_2$, and edges $xa, xb, ya, yb, ab, z_1z_2, z_1z_3, z_1z_4, z_2z_3, z_2z_4$ and (if F is F_7) the edge az_1 . Since x, y are in B_2 , vertices z_1, \dots, z_4 must be in B_1 and a, b must be in A . But then b, z_2, z_3, z_4 contradict Claim (32).

Suppose that F is F_{11} . So F has vertices x, y, a, b, z_1, z_2, w , with $x, y \in B_2$, and edges $xa, xb, ya, yb, ab, az_1, az_2, z_1z_2, z_1w, z_2w, xw, yw$. Since x, y are in B_2 , vertices z_1, z_2 must be in B_1 and a, b, w must be in A . By Claim (31), there is a vertex z in B_1 that is adjacent to b, z_1, z_2 . Then z sees w , for otherwise $\{z, b, x, w, z_1\}$ induces a C_5 . But then $\{a, b, x, w, z, z_1\}$ induces a small boat (\overline{C}_6) or F_{10} in G , a contradiction.

Finally suppose that F is an F_{13} . So F has vertices $x, y, a, b, u, v, z_1, z_2$, with $x, y \in B_2$, and edges $ab, ax, ay, bx, by, xu, xv, yu, yv, uz_1, uz_2, vz_1, vz_2, z_1z_2$. Since x, y are in B_2 , vertices z_1, z_2 must be in B_1 and a, b, u, v must be in A . By Claim (31), there is a vertex z in B_1 that is adjacent to b, z_1, z_2 . Vertex z sees u , for otherwise $\{z, b, x, u, z_1\}$ induces a C_5 . Similarly z sees v . Then z sees a , for otherwise $\{a, b, x, u, z, z_1\}$ induces a boat. But then $\{a, b, x, z, u, v, z_1\}$ induces an F_{11} in G , a contradiction. Thus (34) holds.

Some vertex d in B_2 is the unique b-vertex of G of color $c(d)$. (35)

Suppose for each b-vertex d in B_2 , there is a b-vertex d' of the same color. Then $d' \notin B_2$ (B_2 is a clique); so, c is a b-coloring of G' with k colors, and G' is a smaller counterexample than G , a contradiction. Thus, (35) holds.

Since the vertices of B_2 are pairwise adjacent twins, (35) implies that they are all b-vertices, and we may assume that $B_2 = \{d_1, \dots, d_\ell\}$, with $\ell \geq 2$.

Next, we will prove

B_1 contains a vertex x_i of color i , $1 \leq i \leq \ell$, that is not a b-vertex of G . (36)

Let S_1, \dots, S_k be the color classes of the b-coloring c , with $d_i \in S_i$ for

each $i = 1, \dots, \ell$. We have $k = b(G) > \chi(G) = \omega(G)$ since G is perfect. So $k - 1 \geq \omega(G)$. Note that S_2, \dots, S_k form a b-coloring of $G \setminus S_1$; so $b(G \setminus S_1) \geq k - 1$. Since G is minimally b-imperfect, we have $b(G \setminus S_1) = \chi(G \setminus S_1) = \omega(G \setminus S_1)$. Combining the above inequalities, we get $k - 1 \geq \omega(G) \geq \omega(G \setminus S_1) = \chi(G \setminus S_1) = b(G \setminus S_1) \geq k - 1$. So, equality must hold throughout, in particular we have $\omega(G \setminus S_1) = \omega(G) = k - 1$. So $G \setminus d_1$ contains a clique K of size $\omega(G) = k - 1$. If K contains a vertex x of S , then $(K \setminus x) \cup \{d_1, d_2\}$ is a clique of size k in G , a contradiction. If K contains no vertex of B_1 , then we have $K \subseteq A \cup B_2 \setminus d_1$, and then $K \cup \{d_1\}$ is a clique of size k in G , again a contradiction. So K contains a vertex of B_1 and $K \subseteq B_1 \cup A$. Then we have $|K \cap A| \leq k - 1 - \ell$, for otherwise $(K \cap A) \cup B_2$ would be a clique of size at least k . Consequently, $K \cap B_1$ has size at least ℓ and at least $\ell - 1$ of the colors $1, \dots, \ell$ appear in $K \cap B_1$. So we may assume up to symmetry that B_1 contains a vertex x_i of color i , $1 \leq i \leq \ell$. If x_i is not a b-vertex, then we are done. Suppose x_i is a b-vertex. So, it must have neighbors x_j of color j , for all $j \in \{1, \dots, \ell\} \setminus \{i\}$. The vertices x_j are in B_1 necessarily. By (35), some such x_j is not a b-vertex. So, (36) holds.

For simplicity, let x_1 be the vertex of color 1 in B_1 that is not a b-vertex. There must be a color m such that x_1 is the only neighbor of color 1 of every b-vertex of color m . Thus d_m is not adjacent to d_1 , so it is not in A ; therefore, it is in B_1 . Moreover $m > \ell$. Vertices d_1, \dots, d_ℓ must have a neighbor u_m of color m , and clearly u_m is in A . Moreover u_m is not a b-vertex (because it is adjacent to d_1 and by the property of x_1), so there is a color $n \neq 1$ such that u_m has no neighbor of color n . Thus $n > \ell$. Vertices d_1, \dots, d_ℓ must have a neighbor u_n of color n , and clearly u_n is in A . Let C be the component of $B_1 \setminus N(u_m)$ that contains d_m . By (32), C is a homogeneous set and a clique.

$$N(d_m) \cap B_1 \text{ is a clique.} \quad (37)$$

Suppose on the contrary that d_m has two neighbors x, y in B_1 that are not adjacent. Since C is a homogeneous set and a clique, vertices x, y are in $B_1 \setminus C$ and so they are both adjacent to u_m . Then u_n sees x , for otherwise $\{u_n, d_1, u_m, x, d_m\}$ induces an F_1 or C_5 . Likewise u_n sees y . Then u_n misses d_m , for otherwise $\{d_1, d_2, u_n, u_m, d_m, x, y\}$ induces an F_{11} . Since this holds for every vertex u_n of color n in A , and d_m must have a neighbor z_n of color n , it must be that such a vertex z_n is in B_1 . Then z_n sees x since C is a homogeneous set. Likewise z_n sees y . But then $\{d_m, z_n, x, y, u_n, u_m, d_1, d_2\}$ induces an F_{13} . So Claim (37) holds.

Every neighbor of d_m in A is adjacent to all of $N(d_m) \cap B_1$. (38)

For suppose that some neighbor a of d_m in A is not adjacent to some vertex y in $N(d_m) \cap B_1$. Vertex y is not in C since C is homogeneous. So y is adjacent to u_m . Then a sees u_m , for otherwise $\{d_1, a, u_m, d_m, y\}$ induces a C_5 . Thus $a \neq u_n$. Then u_n sees y , for otherwise $\{u_n, d_1, u_m, y, d_m\}$ induces an F_1 or C_5 . Then u_n misses d_m , for otherwise $\{d_1, u_m, u_n, d_m, y, a\}$ induces a boat (\overline{C}_6) or an F_{10} . Since this holds for every vertex u_n of color n in A , and d_m must have a neighbor z_n of color n , it must be that such a vertex z_n is in B_1 . Since u_m has no neighbor of color n , we have $z_n \in C$. Since C is homogeneous, a sees z_n . Note that a sees u_n , for otherwise $\{d_1, a, u_n, d_m, y\}$ induces a C_5 . Recall that z_n sees y by (37) and misses u_m by the definition of color n . But then $\{d_1, a, u_m, u_n, d_m, y, z_n\}$ induces an F_{11} , a contradiction. So (38) holds.

Vertex d_m must have a neighbor z_i of color i for each $i \in \{1, \dots, \ell\}$, and clearly z_i is in B_1 since it is not adjacent to d_i . By (37) and (38), every neighbor of d_m (other than z_i) is adjacent to z_i . It follows that z_1, \dots, z_ℓ are b-vertices, a contradiction to (35). This completes the proof of the main theorem. \square

6 Optimizing b-perfect graphs

In this section we describe polynomial time algorithms to find an optimal coloring and a largest clique of a b-perfect graph.

Suppose a graph G is assigned an arbitrary coloring. We want to find a way to reduce the number of colors used to hopefully obtain a better coloring of G . The notion of b-vertices can be used for this purpose. If there is a color c with no b-vertex then we can eliminate c from our coloring as follows. For each vertex x of color c , give x a color that is missing in the neighborhood of x . We may repeat this process until every color has a b-vertex, thus obtaining a b-coloring of G . We will call the above algorithm the *b-greedy* (coloring) algorithm. It is easy to see that the b-greedy algorithm can be implemented in polynomial time. If G is a b-perfect graph, then the b-greedy algorithm will deliver an optimal coloring since $b(G) = \chi(G)$.

Our notion of b-perfect graph is thus analogous to Chvátal's notion of perfectly orderable graph [3]. On a perfectly ordered graph, the greedy

algorithm delivers an optimal coloring. The recognition of perfectly orderable graphs is NP-complete [15]; in comparison, the recognition of b-perfect graphs can be done in polynomial time, since our main result above is that b-perfect graphs are characterized by forbidding as induced subgraphs twenty-two graphs, which have at most eight vertices.

Now, we consider the problem of finding a largest clique of a b-perfect graph. First, we need establish some preliminary results.

Lemma 6.1 *Let G be a b-perfect graph that contains a C_5 . Then either*
(i) G has two non-adjacent comparable vertices, or
(ii) $X_6 = \emptyset$, and $|X_i| = 1$ for $i = 1, 2, 3, 4, 5$ (with the notation of Lemma 3.1).

Proof of Lemma 6.1. We know that G has the structure described in Lemma 3.1, and we use the same notation. Write $X_1'' = X_1 \setminus X_1'$, $X_2'' = X_2 \setminus X_2'$. Let a_1 be any vertex in X_1' . Suppose that X_1'' contains a vertex x_1 . By items 6 and 8 of Lemma 3.1, x_1 dominates a_1 and we obtain conclusion (i). Now let us assume that X_1'' is empty, and similarly X_4'' is empty. Any two vertices of X_1' are non-adjacent (by item 6 of Lemma 3.1) and comparable (by item 8); so if $|X_1| \geq 2$ or $|X_4| \geq 2$ we obtain (i) again. So let $|X_1| = 1$ and $|X_4| = 1$. Suppose that X_6 contains a vertex x_6 . Since $|X_4| = 1$, a_1 has no neighbor in X_4 and so is dominated by x_6 and we obtain (i). Thus, $X_6 = \emptyset$. Now, if $|X_i| \geq 2$ ($i = 2, 3, 5$), then X_i contains two non-adjacent comparable vertices, and we have (i) again. Thus the lemma holds. \square

A *special boat* is a boat (with the same notation as in Section 4) such that all A_i 's and all B_i 's are cliques.

Lemma 6.2 *Let G be a C_5 -free b-perfect graph that contains a boat. Then either*
(i) G has a proper homogeneous set that is not a clique, or
(ii) G is a special boat.

Proof of Lemma 6.2. Consider a set M that induces a largest boat in G . By Lemma 4.1, M is a homogeneous set of G . If $M \neq V(G)$, we obtain conclusion (i). So let $M = V(G)$. Since each A_i and B_i with at least two vertices is a homogeneous set of G , either one of them is not a clique, and we obtain (i), or all are cliques, and we obtain (ii). Thus the Lemma holds. \square

Lemma 6.3 *Let G be a b-perfect graph. Then one of the following holds:*

1. G has two non-adjacent comparable vertices.
2. G has a proper homogeneous set that is not a clique.
3. G is a C_5 .
4. G is weakly chordal.
5. G is a special boat.

Proof of Lemma 6.3. Let G be a b-perfect graph, and suppose that it does not have two non-adjacent comparable vertices. First suppose that G contains a C_5 , and let us use the same notation as in Lemma 3.1. By Lemma 6.1, we have $X_6 = \emptyset$, and $|X_i| = 1$ for $i = 1, 2, 3, 4, 5$. Thus the set $X = X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$ induces a C_5 . By Lemma 3.1, X is a homogeneous set. If $X \neq V(G)$, we obtain conclusion 2. If $X = V(G)$, we obtain conclusion 3. Now let G be C_5 -free. If G contains a \overline{C}_6 (which is a boat), then, by Lemma 6.2, we obtain conclusion 2 or 5. If G does not contain a \overline{C}_6 , then it does not contain any hole or antihole of length at least five (because G contains no $F_1 = P_5$ and $F_{10} = \overline{P}_6$), and we obtain conclusion 4. \square

We are now in position to describe a polynomial-time algorithm to find a largest clique in a b-perfect graph G . First, if G has two non-adjacent comparable vertices x, y , where x dominates y , then $\omega(G) = \omega(G - y)$; thus, we can remove y from consideration and recursively find a largest clique in $G - y$. We apply this reduction as long as possible. Second, if G has a proper homogeneous set H that is not a clique, then we recursively find a largest clique K of H . Let G' be the graph $G - (H - K)$. Clearly, every maximal clique of G either is disjoint from H or contains a maximal clique of H ; so $\omega(G) = \omega(G')$. Note that G' does not have two non-adjacent comparable vertices x, y (for otherwise, it is easy to see that x, y would have the same property in G , a situation which we have already dealt with); so we do not need to go back to the first step with G' . Finally, if G does not have the above two properties, then by Lemma 6.3, G is either a C_5 , a special boat, or weakly chordal. It is easy to determine $\omega(G)$ when G is a special boat (which is the complement of a bipartite graph), and there are efficient algorithms to find a largest clique of a weakly chordal graph [7]. We can formalize our algorithm as follows.

Algorithm CLIQUE(G)**Input:** A b -perfect graph G **Output:** A largest clique of G

1. If G has two non-adjacent comparable vertices x, y with x dominating y , then return the clique produced by CLIQUE($G - y$);
2. If G has a proper homogeneous set H that is not a clique then
Let K be the clique returned by CLIQUE(H)
Return the clique produced by CLIQUE($G - (H - K)$);
3. If G is a C_5 then return a clique of size two of G ;
4. If G is weakly chordal then return a largest clique of G produced by the algorithm in [7];
5. /* Now is G is a special boat */
Return (as described below) a largest clique of G .

In Step 5, the special boat is the complement of a bipartite graph. There are well-known algorithms (see, for example, [4]) for finding a largest clique of the complement of a bipartite graph. However, we can directly find a largest clique in a special boat given the sets A_i, B_i (with the same notation as in Section 4). Indeed, the largest clique of G is, clearly, one of the sets $A, B, A_1 \cup B_1, \dots, A_q \cup B_q$. So the question is how to find the sets A_i, B_i . When the algorithm reaches Step 5, we know that G must contain a \overline{C}_6 because G contains no P_5, C_5, \overline{P}_6 and is not weakly chordal. By Lemmas 4.1 and 6.2, every boat extends into a special boat containing all vertices of G . Thus, starting with the \overline{C}_6 , we can extend it into a special boat with sets $A_0, A_1, \dots, A_q, B_0, B_1, \dots, B_q$ as desired. Clearly, this can be done in polynomial time.

Now, we show that our algorithm can be implemented in polynomial time. Clearly, Step 1 can be performed in polynomial time. Considering Step 2, there are many efficient algorithms to find a homogeneous set in a graph G . The most efficient ones are based on the theory of *modular decomposition*. This theory is rich and complex, and we recall here the relevant facts only. A *module* is defined to be any homogeneous set M such that every homogeneous set H satisfies either $H \subseteq M, M \subseteq H$, or $M \cap H = \emptyset$. Note that $V(G)$ and each singleton $\{v\} \subseteq V(G)$ is a module. For every module M of size at least two, let M_1, \dots, M_h be those modules of G that are properly included in M and are maximal with that property;

then M_1, \dots, M_h form a partition of M ; they are called the *children* of M . The child relation defines a tree, which is called the *modular decomposition tree* of G . Note that the root of the tree is the module $V(G)$ and the leaves of the tree are the singleton modules. Here is an important property of every module M . (Let $G[M]$ denotes the subgraph of G induced by M .)

Property (*): If $G[M]$ is not connected, then the children of M are the vertex-sets of the components of $G[M]$; if $\overline{G}[M]$ is not connected, then the children of M are the vertex-sets of the components of $\overline{G}[M]$; if $G[M]$ and $\overline{G}[M]$ are connected, and H is any homogeneous set of G that is properly included in M , then H is included in a child of M .

Now Algorithm CLIQUE can be implemented so as to return a maximum clique K_M of every module M of G , starting from the leaves up to the root, as follows. If M is a leaf, let the algorithm return $K_M = M$. Now suppose that M is not a leaf, that its children are M_1, \dots, M_h , and that the algorithm has already produced cliques K_{M_1}, \dots, K_{M_h} . If $G[M]$ is not connected, then let K_M be the largest of K_{M_1}, \dots, K_{M_h} . If $\overline{G}[M]$ is not connected, then let K_M be the union of K_{M_1}, \dots, K_{M_h} . If $G[M]$ and $\overline{G}[M]$ are connected, then consider the graph $G'_M = G[K_{M_1} \cup \dots \cup K_{M_h}]$. By Property (*), graph G'_M does not have a homogeneous set that is not a clique. Moreover, as observed earlier, G'_M does not have two non-adjacent comparable vertices. Thus, by Lemma 6.3, G'_M is either a C_5 , a boat or a weakly chordal graph. So we can compute a maximum clique of G'_M in polynomial time, and the algorithm returns such a clique as K_M .

There are efficient (but conceptually complex) algorithms that compute the modular decomposition tree of any graph with n vertices and m edges in time $O(n + m)$, see [14, 17]. Moreover, the modular decomposition tree has at most $2n$ nodes (including the leaf nodes). This ensures that Step 2 of the Algorithm is performed at most n times. Consequently, Steps 3, 4 and 5 are performed at most n times and Algorithm CLIQUE runs in polynomial time.

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