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CONTINUOUS ANALYSIS OF THE ADDITIVE SCHWARZ METHOD: A STABLE DECOMPOSITION IN H^1

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Abstract. The classical convergence result for the additive Schwarz preconditioner with coarse grid is based on a stable decomposition. The result holds for discrete versions of the Schwarz preconditioner, and states that the preconditioned operator has a uniformly bounded condition number that depends only on the number of colors of the domain decomposition, and the ratio between the average diameter of the subdomains and the overlap width.

The classical Schwarz method was however defined at the continuous level, and similarly, the additive Schwarz preconditioner can also be defined at the continuous level. We present in this paper a continuous analysis of the additive Schwarz preconditioned operator with coarse grid in two dimensions. We show that the classical condition number estimate also holds for the continuous formulation, and as in the discrete case, the result is based on a stable decomposition, but now of the Sobolev space H^1. The advantage of such a continuous result is that it is independent of the type of fine grid discretization, and thus does the more natural continuous formulation of the Schwarz method justice.

1. Introduction. With the generalization of parallelism in today's computers, parallelizable mathematical algorithms are of increasing importance. Domain decomposition methods make it possible to perform numerical simulations in parallel, see for example the books [29, 27, 31], or the monographs [32, 7], and references therein. Consider a PDE to be solved on a big domain Ω. In domain decomposition methods, an iterative approach introduced by Schwarz [28] is to decompose the big domain Ω into several smaller overlapping subdomains Ω_i, $\Omega = \bigcup_{i=1}^n \Omega_i$, and then to compute approximations u_i^k defined by

$$\begin{array}{rcll} \mathcal{L}u_i^k & = & f & \text{in } \Omega_i, \\ u_i^k & = & u_j^{k-1} & \text{on } \Gamma_{ij}, \end{array} \tag{1.1}$$

where Γ_{ij} denote the interfaces. In practice, it is more efficient to use the general algorithm (1.1) as a preconditioner for a Krylov subspace method, like GMRES or conjugate gradients, see for example [17, 18] for a more detailed explanation. The Additive Schwarz operator defines one such preconditioned operator, related to (1.1). For a domain decomposition with both an overlap and a coarse mesh, Dryja and Widlund [12] proved that the condition number of the discrete Additive Schwarz operator is uniformly bounded, *i.e.* it does not depend on the number of subdomains. However it depends on the number of colors of the domain decomposition and on the ratio between the diameter of the subdomain and the thickness of the overlaps, see also Toselli and Widlund [31, Chap. 2]. Schwarz preconditioners have then mostly been analyzed at the discrete level, see for example [6, 25, 26, 19] for spectral discretizations, [2] for the non-selfadjoint case, [3] for parabolic problems, [5] for some non-symmetric and indefinite problems, [4] for multiplicative versions of the algorithm, [9] for discretizations on unstructured meshes, [8] when also the coarse grid is non-matching, [14, 10, 23, 24] for mixed finite element discretizations, [11] for mortar finite element problems, [15] for discontinuous Galerkin discretizations, and [16] for numerical linear algebra techniques.

Schwarz domain decomposition methods are however naturally defined and analyzed at the continuous level, like in (1.1), see for example [20, 21, 22]. Schwarz methods were also invented by Schwarz at the continuous level [28], and the more recent class of optimized Schwarz methods was formulated and analyzed at the continuous level, for an introduction see [17] and references therein. The purpose of our

paper is to analyze a continuous formulation of the Additive Schwarz operator, and to prove that its condition number is bounded independently of the number of subdomains, when there is a coarse mesh (the coarse space remains a discrete space). The proof succeeds by establishing the existence of a stable decomposition of every function in $H_0^1(\Omega)$ as a sum of functions belonging to the $H_0^1(\Omega_i)$ plus a coarse function belonging to the space of continuous piecewise linear functions $P_1(\mathcal{T})$ where $\mathcal{T}$ is our coarse triangular mesh.

First, we recall in section 2 the definition of the preconditioned additive Schwarz operator, and the abstract results giving an estimate of the condition number of the Additive Schwarz operator as soon as three assumptions hold. The rest of the paper is then devoted to showing that these assumptions hold for a decomposition at the continuous level, the key assumption being the existence of a stable decomposition. After specifying in section 3 the geometric parameters of the domain decomposition, we prove in section 4 the existence of a stable decomposition in the continuous case in the absence of a coarse mesh albeit with a constant that depends on the number of subdomains. Section 5 is dedicated to establishing that in the presence of a coarse mesh there exists a uniformly stable decomposition where the constant does not depend on the number of subdomains. Using this result, we prove in section 6 that the condition number of the additive Schwarz operator has a uniformly bounded condition number in the continuous case when there is a coarse P_1 mesh.

2. The Additive Schwarz operator. In this section, we recall the abstract results in Toselli and Widlund [31, chap. 2]. Let $(V_i)_{0\leq i\leq N}$ be Hilbert spaces, with V_0 being a coarse space. Let $V = \sum_{i=0}^{n} R_i^T V_i$, where the R_i^T are linear extension operators. Let $a(\cdot,\cdot)$ be a positive definite bilinear form on V. We wish to find the unique u in V satisfying

$$a(u,v) = (f,v) \text{ for all } v \text{ in } V.$$

Let $\widetilde{a}_i(\cdot,\cdot)$ be a positive definite bilinear forms on the V_i. We define $\widetilde{P}_i : V \to V_i$ by

$$\widetilde{a}_i(\widetilde{P}_i u, v) = a(u, R_i^T v) \qquad \text{for all } v \text{ in } V_i.$$

Let $P_i = R_i^T \widetilde{P}_i$. The additive Schwarz operator is defined by

$$P_{ad} := \sum_{i=0}^{N} P_i. \tag{2.1}$$

This is an a-symmetric a-positive operator. We are interested in bounding the condition number (with respect to the bilinear form a) of the preconditioned operator P_{ad}.

DEFINITION 2.1. *Let a be a positive bilinear form on a vector space H. Let P be a continuous linear application from H to H. We call*

$$\kappa(P_{ad}) = \frac{\max\limits_{\substack{\boldsymbol{u}\in H \\ a(\boldsymbol{u},\boldsymbol{u})=1}} a(P_{ad}u,u)}{\min\limits_{\substack{\boldsymbol{u}\in H \\ a(\boldsymbol{u},\boldsymbol{u})=1}} a(P_{ad}u,u)}$$

the a-condition number of P_{ad}.

ASSUMPTION 2.2 (Stable decomposition). *There exists a constant C_0 such that all u in V admit the decomposition*

$$u = \sum_{i=0}^{N} R_i^T u_i, \quad \textit{with} \quad \{u_i \in V_i\}, \qquad \textit{and} \quad \sum_{i=0}^{N} \widetilde{a}_i(u_i, u_i) \leq C_0^2 a(u,u). \tag{2.2}$$

ASSUMPTION 2.3 (Strengthened Cauchy Schwarz inequality). *There exist constants $0 \leq \varepsilon_{ij} \leq 1$ such that*

$$|a(R_i^T u_i, R_j^T u_j)| \leq \varepsilon_{ij} a(R_i^T u_i, R_i^T u_i)^{\frac{1}{2}} a(R_j^T u_j, R_j^T u_j)^{\frac{1}{2}}, \tag{2.3}$$

for all $i, j \geq 1$. We denote by $\rho(\mathcal{E})$ the spectral radius of the matrix $\mathcal{E} = \{\varepsilon_{ij}\}$.

ASSUMPTION 2.4 (Local stability). *There exists $\omega > 0$ such that $\forall i \geq 1$ and $\forall u_i \in range(\widetilde{P}_i)$ we have*

$$a(R_i^T u_i, R_i^T u_i) \leq \omega \widetilde{a}_i(u_i, u_i). \tag{2.4}$$

The following fundamental result can be found in Toselli and Widlund [31], see Theorem 2.7.

THEOREM 2.5. *Let Assumptions 2.2, 2.3 and 2.4, be satisfied. Then the a-condition number κ of P_{ad} satisfies*

$$\kappa(P_{ad}) \leq C_0^2 \omega (\rho(\mathcal{E}) + 1).$$

Proof. The proof of Theorem 2.7 in Toselli and Widlund [31] also holds if the V_i have infinite dimension. □

While the local stability and the strengthened Cauchy Schwarz inequality can naturally be extended to the continuous case, the stable decomposition result is traditionaly shown using properties of the fine discretization of the problem, see for example Toselli and Widlund [31]. For a continuous formulation, we need to use different techniques, which is the purpose of this paper.

3. Geometry and decomposition into subdomains. First, we recall the defintion of a domain:

DEFINITION 3.1. *A domain of $\mathbb{R}^2$ is an open connected set of $\mathbb{R}^2$ whose boundary $\partial\Omega$ is of null Lebesgue measure*[1]*. We denote by $|\Omega|$ the Lebesgue measure of the domain Ω.*

We recall the definition of a non overlapping and an overlapping domain decomposition:

DEFINITION 3.2 (Non Overlapping Decomposition). *Let Ω be a bounded domain of $\mathbb{R}^2$. A collection of domains $(U_i)_{1 \leq i \leq N}$, is a non overlapping domain decomposition of Ω if*

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{U}_i, \quad U_i \cap U_j = \emptyset \qquad \textit{for all } i \neq j. \tag{3.1}$$

[1] It is possible for a pathological open connected set of $\mathbb{R}^2$ to have a boundary with strictly positive measure. For example $(0,1) \times (1/4, 3/4) \cup \bigcup_{j=1}^{\infty} \bigcup_{k=0}^{2^{j-1}-1} (\frac{2k+1}{2^j} - 2^{-4j}, \frac{2k+1}{2^j} + 2^{-4j}) \times (0,1)$ is open, connected and dense in $(0,1) \times (0,1)$ but has a measure smaller than $9/14$.

DEFINITION 3.3 (Overlapping Decomposition). *Let Ω be a bounded domain of $\mathbb{R}^2$. A collection of domains $(\Omega_i)_{1\leq i\leq N}$ is an overlapping domain decomposition of Ω if*

$$\Omega = \bigcup_{i=1}^{N} \Omega_i.$$

In this article, we use the parameter H to represent the average size of subdomains. For the definition of H, we use the concept of diameter, which we recall here:

DEFINITION 3.4 (Domain Diameter). *Let U be a bounded subset of $\mathbb{R}^2$. We define the diameter of U to be*

$$\operatorname{diam}(U) = \sup_{\substack{\boldsymbol{x}\in U\\ \boldsymbol{y}\in U}} \|\boldsymbol{x}-\boldsymbol{y}\|.$$

The concept of an overlapping domain decomposition raises the question on how to define the overlap width of the decomposition. We use the following definition:

DEFINITION 3.5 (Overlap of the Decomposition). *A domain decomposition $(\Omega_i)_{1\leq i\leq N}$ is said to have overlap width $\delta > 0$, if there exists a non overlapping domain decomposition $(U_i)_{1\leq i\leq N}$ of Ω such that for all i, $1 \leq i \leq N$, $U_i \subset \Omega_i$ and*

$$\{\boldsymbol{x} \in \Omega \mid \operatorname{dist}(\boldsymbol{x}, U_i) < \delta\} \subset {\Omega_i}^2.$$

In practice, it is easier to start with a non overlapping domain decomposition, and then to build from it an overlapping one. If $(U_i)_{1\leq i\leq N}$ is a non overlapping domain decomposition of Ω, then the $(\Omega_i)_{1\leq i\leq N}$ defined by

$$\Omega_i = \{\boldsymbol{x} \in \Omega \mid \operatorname{dist}(\boldsymbol{x}, U_i) < \delta\} \tag{3.2}$$

forms an overlapping domain decomposition of Ω with overlap width δ. We denote by $((U_i)_{1\leq i\leq N}, (\Omega_i)_{1\leq i\leq N})$ such a decomposition.

DEFINITION 3.6 (Colors of the Decomposition). *The number of colors of an overlapping domain decomposition $(\Omega_i)_{1\leq i\leq N}$ of domain Ω is the smallest integer N_c such that there exists a partition of $\{1,\dots,N\}$ into N_c sets $(I_k)_{1\leq k\leq N_c}$ such that*

$$\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset,$$

whenever $i \neq j$ and i, j both belong to the same color I_k. In order to determine the number of colors of the decomposition, it is easiest to consider the nonoverlapping decomposition $(U_i)_{1\leq i\leq N}$. Clearly the number of colors can only increase with δ. However for δ small enough it remains constant: there exists δ_0 and $N_c > 0$ that depend only on the (U_i) such that for all δ, $0 < \delta < \delta_0$, the overlapping domain decomposition $(\Omega_i)_{1\leq i\leq N}$ with overlap width δ derived from the $(U_i)_{1\leq i\leq N}$ has a number of colors equal to N_c.

4. Stable decomposition without a coarse mesh. To understand the importance of the coarse mesh, we begin by proving the existence of a stable decomposition without a coarse mesh. In that case, the constant C_0 of the stable decomposition

[2] Geometrically, the parameter δ corresponds to half the overlap of the subdomains

depends on the number of subdomains. We consider a bounded domain Ω being decomposed into N overlapping subdomains Ω_i with overlap width δ. We make the following assumptions on the domain decomposition:

ASSUMPTION 4.1. *The domain decomposition $(\Omega_i)_{1\leq i\leq N}$ is derived from a non overlapping one by formula* (3.2)*, and we refer to it by $((U_i)_{1\leq i\leq N}, (\Omega_i)_{1\leq i\leq N})$.*

ASSUMPTION 4.2. *Let H be the smallest diameter among the diameters of the subdomains U_i. We suppose there exist uniform parameters $C_d > 0$, $c_a > 0$ and $C_a > 0$ such that for all i in $\{1, \ldots, N\}$*

$$H \leq \operatorname{diam}(U_i) \leq C_d H, \quad c_a H^2 \leq |U_i| \leq C_a H^2. \tag{4.1}$$

To construct the stable decomposition, we use a partition of unity.

LEMMA 4.3 (Partition of Unity). *Let Ω be an open domain of $\mathbb{R}^2$, $N > 0$ be the number of subdomains, and $(U_i)_{1\leq i\leq N}$ be domains of $\mathbb{R}^2$ satisfying* (3.1)*. With $\delta > 0$ the overlap width, we define*

$$\widetilde{\Omega}_i = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \operatorname{dist}(\boldsymbol{x}, U_i) < \delta\},$$

and denote by N_c the number of colors of this domain decomposition[3]*. Then, there exists a universal*[4] *constant $\lambda_2 > 0$ and $(\psi_i)_{1\leq i\leq N}$ in $\mathcal{C}^\infty(\mathbb{R}^2)$ having the following properties:*

1. *For all i in $\{1, \ldots, N\}$, ψ_i vanishes outside of $\widetilde{\Omega}_i$.*
2. *For all $\boldsymbol{x}$ in $\mathbb{R}^2$, $0 \leq \psi_i(\boldsymbol{x}) \leq 1$.*
3. *We have $1 = \sum_i \psi_i(\boldsymbol{x})$ when $\boldsymbol{x}$ belongs to Ω.*
4. *We have the estimate $\|\nabla \psi_i\|_\infty \leq \lambda_2 \frac{N_c - 1}{\delta}$.*

Proof. The result is classical and well known, see [1, Th. 3.15], we only show how to obtain the constant in 4. We start with a function ρ in $\mathcal{C}^\infty(\mathbb{R}^2)$ which vanishes outside the unit ball, and satisfies for all $\boldsymbol{x}$ in $\mathbb{R}^2$ that $0 \leq \rho(\boldsymbol{x}) \leq 1$, and the integral $\int_{\mathbb{R}^2} \rho(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 1$. For all $\varepsilon > 0$, we then set $\rho_\varepsilon(\boldsymbol{x}) = \frac{1}{\varepsilon^2}\rho(\frac{\boldsymbol{x}}{\varepsilon})$, and we define for all i in $\{1, \ldots, N\}$ the function

$$h_i(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \operatorname{dist}(\boldsymbol{x}, U_i) < \frac{\delta}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

We now regularize the functions h_i using a convolution,

$$\phi_i := \rho_{\delta/2} * h_i.$$

The functions ϕ_i vanish outside of $\widetilde{\Omega}_i$, are identically equal to 1 in $\overline{U}_i$, and for all $\boldsymbol{x}$ in $\mathbb{R}^2$ we have $0 \leq \phi_i \leq 1$. Moreover, since $\|\nabla \phi_i\|_{L^\infty(\mathbb{R}^2)} \leq \|\nabla \rho_{\frac{\delta}{2}}\|_{L^1(\mathbb{R}^2)} \|h_i\|_{L^\infty(\mathbb{R}^2)}$, we obtain

$$\|\nabla \phi_i\|_{L^\infty(\mathbb{R}^2)} \leq \frac{2\|\nabla \rho\|_{L^1(\mathbb{R}^2)}}{\delta}.$$

We then set

$$\psi_i = \phi_i \prod_{k=1}^{i-1} (1 - \phi_k),$$

[3]The $\tilde{\Omega}_i$ can extend beyond the domain Ω, in contrast to the Ω_i defined earlier

[4]It depends only on the dimension but we have restricted ourselves to two-dimensional domains

and the $(\psi_i)_{1\le i\le N}$ are then a partition of unity. Moreover

$$\nabla\psi_i = \nabla\phi_i \prod_{k=1}^{i-1}(1-\phi_k) - \sum_{j=1}^{i-1}\phi_i\nabla\phi_j \prod_{\substack{k=1\\k\neq j}}^{i-1}(1-\phi_k).$$

At a given point $\boldsymbol{x}$, at most N_c-1 terms of the above sum may be non zero. Moreover each term is bounded by $\max_{1\le j\le i}\|\nabla\phi_j\|_\infty$, and hence

$$\|\nabla\psi_i\|_{L^\infty(\mathbb{R}^2)} \le \frac{2(N_c-1)\|\nabla\rho\|_{L^1(\mathbb{R}^2)}}{\delta}.$$

Setting $\lambda_2 := 2\|\nabla\rho\|_{L^1(\mathbb{R}^2)}$, the result follows. ∎

THEOREM 4.4 (Stable Decomposition without Coarse Grid). *Let Ω be a domain of $\mathbb{R}^2$, and $((U_i)_{1\le i\le N},(\Omega_i)_{1\le i\le N})$ be an overlapping domain decomposition of Ω with overlap width $\delta>0$. If u is in $H^1_0(\Omega)$, there exist $(u_i)_{1\le i\le N}$ such that for all i, $1\le i\le N$, u_i is in $H^1_0(\Omega_i)$ and*

$$u = \sum_{i=1}^N u_i, \tag{4.2}$$

$$\sum_{i=1}^N \|u_i\|^2_{L^2(\Omega_i)} \le \|u\|^2_{L^2(\Omega)}, \tag{4.3}$$

$$\sum_{i=1}^N \|\nabla u_i\|^2_{L^2(\Omega_i)} \le 2\|\nabla u\|^2_{L^2(\Omega)} + \frac{2\lambda_2^2 N_c(N_c-1)^2}{\delta^2}\|u\|^2_{L^2(\Omega)}, \tag{4.4}$$

where λ_2 is the universal constant of Lemma 4.3.

Proof. We just use Lemma 4.3 and set $u_i := \psi_i u$, which satisfies already (4.2). We then estimate

$$\sum_{i=1}^N \int_\Omega |u_i(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = \int_\Omega \sum_{i=1}^N |\psi_i(\boldsymbol{x})u(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = \int_\Omega |u(\boldsymbol{x})|^2 \sum_{i=1}^N (\psi_i(\boldsymbol{x}))^2 \mathrm{d}\boldsymbol{x} \le \int_\Omega |u(\boldsymbol{x})|^2,$$

since $\sum_{i=1}^N(\psi_i(\boldsymbol{x}))^2 \le 1$, which shows (4.3). We finally need to estimate the derivative term. We have $\nabla u_i = \psi_i\nabla u + u\nabla\psi_i$, and therefore

$$\sum_{i=1}^N \int_\Omega |\nabla u_i(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \le 2\int_\Omega |\nabla u(\boldsymbol{x})|^2 \sum_{i=1}^N |\psi_i|^2 \mathrm{d}\boldsymbol{x} + 2\int_\Omega |u(\boldsymbol{x})|^2 \sum_{i=1}^N |\nabla\psi_i|^2 \mathrm{d}\boldsymbol{x}. \tag{4.5}$$

The first term on the right in (4.5) can be bounded as above, and to bound the second term, we estimate

$$\int_\Omega |u(\boldsymbol{x})|^2 \sum_{i=1}^N |\nabla\psi_i(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \le \|\sum_{i=1}^N |\nabla\psi_i|^2\|_{L^\infty(\mathbb{R}^2)} \int_\Omega |u(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \le \frac{\lambda_2^2 N_c(N_c-1)^2}{\delta^2}\|u\|^2_{L^2(\Omega)}, \tag{4.6}$$

because $\|\nabla\psi_i\|_{L^\infty(\mathbb{R}^2)} \le \frac{\lambda_2(N_c-1)}{\delta}$, and because at no point $\boldsymbol{x}$ more than N_c functions $\nabla\psi_i$ may be non zero. Combining these estimates leads to (4.4) and concludes the proof. ∎

The lone $1/\delta^2$ factor in estimate (4.4) can further be treated using the Poincaré inequality on Ω, see [1, Th. 6.30], which then explicitly reveals the dependence on the number of subdomains:

COROLLARY 4.5. *Let Ω be a bounded domain. Let U_i, Ω_i and $(u_i)_{1\leq i\leq N}$ be as in Theorem 4.4. Then we have*

$$\sum_{i=1}^{N}\|\nabla u_i\|^2_{L^2(\Omega_i)} \leq \left(2+\frac{3}{8}\lambda_2^2 C_a N_c(N_c-1)^2 N\frac{\mathrm{diam}^2(\Omega)}{|\Omega|}\frac{H^2}{\delta^2}\right)\|\nabla u\|^2_{L^2(\Omega)}, \qquad (4.7)$$

where C_a is the constant of Assumption 4.2.

Proof. We start with (4.4), and use Poincaré's inequality on $H^1_0(\Omega)$,

$$\|u\|^2_{L^2(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega)},$$

but we need an estimate of the constant C. Since Ω is a subset of $B(\boldsymbol{x},\mathrm{diam}(\Omega))$ for any $\boldsymbol{x}$ in Ω, the constant C is bounded by the Poincaré constant for the ball of radius $\mathrm{diam}(\Omega)$, and a direct calculation shows that C is smaller than $\frac{3}{16}\mathrm{diam}^2(\Omega)$. We therefore obtain

$$\sum_{i=1}^{N}\|\nabla u_i\|^2_{L^2(\Omega_i)} \leq \left(2+\frac{3\lambda_2^2 N_c(N_c-1)^2}{8}\frac{\mathrm{diam}^2(\Omega)}{\delta^2}\right)\|\nabla u\|^2_{L^2(\Omega)}.$$

But we also have

$$\frac{\mathrm{diam}^2(\Omega)}{\delta^2} = \frac{\mathrm{diam}^2(\Omega)}{|\Omega|}\frac{|\Omega|}{H^2}\frac{H^2}{\delta^2} \leq C_a N\frac{\mathrm{diam}^2(\Omega)}{|\Omega|}\frac{H^2}{\delta^2},$$

because $\frac{|\Omega|}{H^2}\leq C_a N$ by Assumption 4.2, which concludes the proof. ∎

The dependence on the number of subdomains N in estimate (4.7) is undesirable for domain decomposition methods, since these methods should be scalable, which means their convergence behavior should not deteriorate as one uses more and more subdomains (which corresponds to more and more processors). In the next section, we show how to establish a better estimate with the use of a coarse mesh.

5. Stable decomposition with a coarse mesh. We now introduce a discrete structure into our continuous analysis, namely a coarse mesh over the entire domain, in order to remove the dependence on the number of subdomains in estimate (4.7). We present the general idea of the continuous proof in the presence of a discrete, coarse mesh first in subsection 5.1. We then show the details of the proof in the next three subsections. In subsection 5.2, we construct the coarse component of the stable decomposition. In subsection 5.3, we construct the non coarse components. Finally, we conclude by stating our main theorem in subsection 5.4.

5.1. General idea. The main idea is to use the following classical lemma [30, chap. II §1.4 pp. 51]:

LEMMA 5.1 (Generalized Poincaré's inequality). *Let O be a bounded open set satisfying the cone condition[5]. Let ℓ be a continuous linear form on $H^1(O)$ such that for all constant functions α , $\ell(\alpha)=0$ implies $\alpha=0$. Then, there exists a constant $C>0$ such that*

$$\|u\|^2_{L^2(O)} \leq C(\|\nabla u\|^2_{L^2(O)} + |\ell(u)|^2)$$

[5] See [1, §4.6].

for all u in $H^1(O)$.

Proof. The proof is by contradiction. Suppose this is not the case. Then, there exists a sequence $(u_n)_{n\in\mathbb{N}^*}$ such that for all $n \geq 1$, u_n in $H^1(O)$, $u_n \neq 0$ and

$$\|u_n\|^2_{L^2(O)} > n(\|\nabla u_n\|^2_{L^2(O)} + |\ell(u_n)|^2). \tag{5.1}$$

We normalize and replace u_n by $u_n/\|u_n\|_{L^2(O)}$, therefore we may also suppose $\|u_n\|_{L^2(O)} = 1$. We extract a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ that converges weakly to u in $H^1(O)$. We have by (5.1)

$$\|\nabla u\|_{L^2(O)} \leq \liminf_{k\to\infty} \|\nabla u_{n_k}\|_{L^2(O)} = 0.$$

Therefore u is a constant function. Moreover, since ℓ is continuous, it is weakly continuous, and by (5.1)

$$\ell(u) = \lim_{k\to\infty} \ell(u_{n_k}) = 0.$$

Therefore $u = 0$. By the compactness [1, Th. 6.3] of the inclusion $H^1(O) \subset L^2(O)$, we can take the limit in $\|u_n\|_{L^2(O)} = 1$ and obtain $\|u\|_{L^2(O)} = 1$. This is a contradiction. ∎

For our purposes, we need estimates for the constants. Unfortunately the previous proof by contradiction is not constructive and does not allow us to estimate the constant C when the domain O varies. However for convex and star shaped domains, the constants can be estimated, as we will show later in Lemma 5.10.

We return to the stable decomposition problem with a coarse mesh. How can we use the coarse mesh to prevent the constant to depend on the number of subdomains? The basic idea is to define N linear forms ℓ_i on $H^1(\Omega_i)$ such that for all u in $H^1_0(\Omega)$ there exists $(u_i)_{1\leq i\leq N}$, such that for all i, $1 \leq i \leq N$, u_i is in $H^1_0(\Omega_i)$ and

$$u = \sum_{i=1}^{N} u_i \tag{5.2}$$

$$\sum_{i=1}^{N} \|\nabla u_i\|^2_{L^2(\Omega_i)} \leq C(\frac{\delta}{H}, N_c)\|\nabla u\|^2_{L^2(\Omega)} + C(\frac{\delta}{H}, N)\sum_{i=1}^{N} |\ell_i(u)|^2, \tag{5.3}$$

where by extension $\ell_i(u)$ means $\ell_i(u_{|\Omega_i})$, effectively replacing the L^2 square norm in (4.4) with $\sum_{i=1}^N |\ell_i(u)|^2$. We propose here to take $\ell_i(u) := \frac{1}{|A_i|}\int_{A_i} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ with $A_i \subset \Omega_i$. We then search for u_0 in the space of continuous, piecewise linear functions $P_1(\mathcal{T})$, where $\mathcal{T}$ is a coarse triangular grid, such that $\ell_i(u_0) = \ell_i(u)$ for all i, $1 \leq i \leq N$ and $\|\nabla u_0\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$. Then, we apply (5.3) to $u - u_0$. The second term vanishes and the constant of the stable decomposition does not depend on the number of subdomains in the decomposition any longer. This idea implies that the coarse mesh should be able to control at least one constant in each subdomain, *i.e.*, for the coarse mesh to prevent the dependence of the condition number on the number of subdomains, it only needs to be able to subtract one constant per subdomain! Intuitively, this means that the coarse mesh must have at least one node in each subdomain.

5.2. Projection of H^1_0 into $\mathcal{P}_1(\mathcal{T})$. In this subsection, we will consider a family of triangular meshes $\mathcal{T}$ of domain Ω with the following uniform properties:

Assumption 5.2 (Geometric Properties of the Coarse Grid).

1. *All angles θ for all cells in the mesh $\mathcal{T}$ are bounded by $0 < \theta_{min} \leq \theta \leq \theta_{max} < \pi$ where θ_{min} and θ_{max} do not depend on H.*
2. *The length of any edge in mesh $\mathcal{T}$ lies between $c_p H$ and $C_P H$ where $c_p > 0$ and $C_p > 0$ depend neither on the cell nor on H.*
3. *No node has more than K neighbors.*

In order to simplify our analysis, we make the following assumption:

ASSUMPTION 5.3. *We assume that the coarse mesh $\mathcal{T}$ has precisely one node $\boldsymbol{x}_i$ per subdomain Ω_i with $\boldsymbol{x}_i \in \Omega_i$.* Even though it should be possible to derive mesh independent estimates for the norm of the coarse component without Assumption 5.3, this could be rather cumbersome, since it leads to a rectangular instead of a square matrix, see the analysis below. In addition, in practical situations, one node for the coarse mesh per subdomain is a common choice.

Given a mesh $\mathcal{T}$, and given $r > 0$, we introduce the linear forms

$$\begin{aligned} \ell_i : H_0^1(\Omega) &\to \mathbb{R}, \\ u &\mapsto \frac{1}{\pi r^2} \int_{B(\boldsymbol{x}_i, r)} u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{5.4}$$

where i belongs to $\{1, \ldots, N\}$ and where $\boldsymbol{x}_i$ is the position of the i-th node in mesh $\mathcal{T}$. We also define

$$\begin{aligned} \ell : H_0^1(\Omega) &\to \mathbb{R}^N, \\ u &\mapsto (\ell_i(u))_{1 \leq i \leq N}. \end{aligned}$$

THEOREM 5.4. *Let Ω be a bounded domain of $\mathbb{R}^2$. Let $\mathcal{T}$ be a coarse mesh on Ω satisfying Assumption 5.2, with H_h the shortest height of all triangles in $\mathcal{T}$, K the maximum number of neighbors of any node in $\mathcal{T}$, and let r be smaller than $\frac{H_h}{4K+1}$. Then, for all u in $H_0^1(\Omega)$, there exists u_H in $P_1(\mathcal{T}) \cap H_0^1(\Omega)$ such that*

$$\begin{aligned} l_i(u_H) &= l_i(u) \qquad \text{for all } i \text{ in } \{1, \ldots, N\}, \\ \|\nabla u_H\|^2_{L^2(\Omega)} &\leq \frac{1}{\tan \theta_{min}} \frac{1 + 2r/H_h}{1 - \big((2K+1) + (4K+1) r/H_h\big) r/H_h} 2K \Big(\frac{C_p^2 H^2}{\pi r^2} + \pi\Big) \|\nabla u\|^2_{L^2(\Omega)}. \end{aligned}$$

Note here that $r \leq \frac{H_h}{4K+1}$ ensures that $1 - \big((2K+1) + (4K+1) r/H_h\big) r/H_h$ is positive. The remainder of this subsection is dedicated to the proof of this theorem.

5.2.1. An equivalent norm. Our goal is to construct a convenient equivalent norm to the $H_0^1(\Omega)$ norm for functions in $P_1(\mathcal{T})$. Let $\mathcal{T}$ be a mesh of Ω having N nodes. As a convention, nodes of mesh $\mathcal{T}$ located exactly on $\partial\Omega$ will be called exterior nodes and are not counted among the numbered nodes. This choice is motivated by the homogenous Dirichlet condition. We denote by $\mathcal{V}$ the set of all (i, j) in $\{1, \ldots, N\}^2$ that are indices of neighboring nodes. We also denote by $\mathcal{B}$ the set of all nodes i in $\{1, \ldots, N\}$ who are neighbor to an exterior node, see Figure 5.1.

DEFINITION 5.5. *Let $\mathcal{T}$ be a mesh of domain Ω. Let $\mathcal{V}$ and $\mathcal{B}$ be the neighbor and the boundary set of mesh $\mathcal{T}$. We define*

$$\begin{aligned} \|\cdot\|_{\mathcal{V},\mathcal{B}} : \mathbb{R}^N &\to \mathbb{R}^+, \\ \boldsymbol{y} &\mapsto \sqrt{\sum_{(i,j) \in \mathcal{V}} |y_i - y_j|^2 + \sum_{i \in \mathcal{B}} |y_i|^2}. \end{aligned}$$

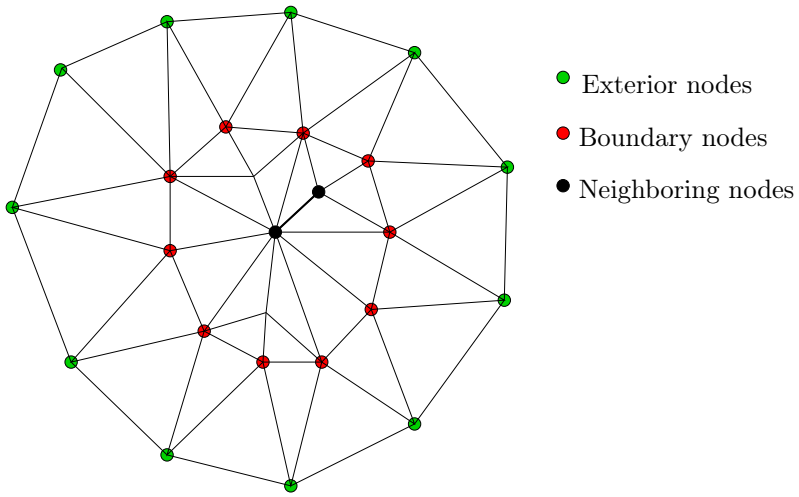

FIGURE 5.1. *Boundary and exterior nodes in mesh $\mathcal{T}$*

When u is in $P_1(\mathcal{T}) \cap H_0^1(\Omega)$, we define

$$\|u\|_{\mathcal{V},\mathcal{B}} := \|(u(\boldsymbol{x}_i))_{1\leq i\leq N}\|_{\mathcal{V},\mathcal{B}},$$

where the $\boldsymbol{x}_i$ are the interior nodes of mesh $\mathcal{T}$.

LEMMA 5.6. *Let u_H belong to $P_1(\mathcal{T})\cap H_0^1(\Omega)$, then the norms $u_H \mapsto \|\nabla u_H\|_{L^2(\Omega)}$ and $\|\cdot\|_{\mathcal{V},\mathcal{B}}$ are equivalent. Moreover, the equivalence constants depend only on the constants of Assumption 5.2,*

$$\frac{2}{3}\frac{\min_{ABC\in\mathcal{T}}|ABC|}{C_p^2 H^2}\|u_H\|_{\mathcal{V},\mathcal{B}}^2 \leq \|\nabla u_H\|_{L^2(\Omega)}^2 \leq \frac{1}{\tan\theta_{min}}\|u_H\|_{\mathcal{V},\mathcal{B}}^2. \tag{5.5}$$

Proof. It is easy to compute the norm, see Appendix A for details. For all u_H in $P_1(\mathcal{T}) \cap H_0^1(\Omega)$, we then have

$$\begin{aligned}\|\nabla u_H\|_{L^2(\Omega)}^2 = &\frac{1}{2}\sum_{(i,j)\in\mathcal{V}}\left(\frac{1}{\tan(\theta_{ij})}+\frac{1}{\tan(\theta_{ji})}\right)|u_i-u_j|^2\\ &+\frac{1}{2}\sum_{i\in\mathcal{B}}\sum_{i'\in\mathcal{V}_i'}\left(\frac{1}{\tan(\theta_{ii'})}+\frac{1}{\tan(\theta_{i'i})}\right)|u_i|^2,\end{aligned}$$

where θ_{ij} and θ_{ji} are the angles opposite to edge $[\boldsymbol{x}_i\boldsymbol{x}_j]$, see Figure 5.2, and where $\mathcal{V}_i'$ is the set of all exterior nodes located on the boundary of Ω that are neighbors of node i. The problem is that the $\tan(\theta_{ij})$ can be negative when $\theta_{ij} > \frac{\pi}{2}$. This is not a problem for the right hand-side of inequality (5.5), but to establish the left hand-side

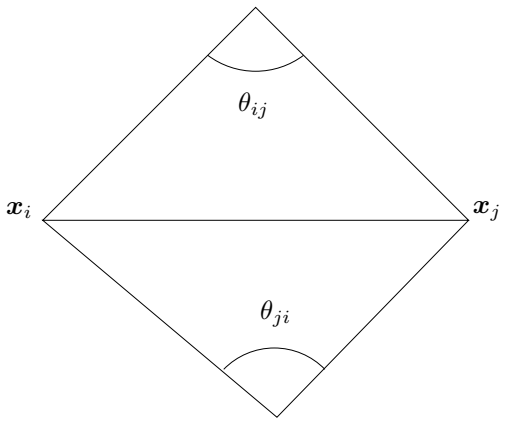

FIGURE 5.2. *Angles and gradient norm in* $P_1(\mathcal{T})$

of inequality (5.5), when there are obtuse angles in the mesh, we need to estimate

$$\begin{aligned}
\|u_H\|^2_{\mathcal{V},\mathcal{B}} &= \frac{1}{2}\sum_{ABC\in\mathcal{T}} \left(|u_H(A)-u_H(B)|^2 + |u_H(B)-u_H(C)|^2 + |u_H(C)-u_H(A)|^2\right)\\
&= \frac{1}{2}\sum_{ABC\in\mathcal{T}} \left(|\nabla u_H(ABC)\cdot(\boldsymbol{x}_A-\boldsymbol{x}_B)|^2 + |\nabla u_H(ABC)\cdot(\boldsymbol{x}_B-\boldsymbol{x}_C)|^2 + |\nabla u_H(ABC)\cdot(\boldsymbol{x}_C-\boldsymbol{x}_A)|^2\right)\\
&\leq \frac{1}{2}\sum_{ABC\in\mathcal{T}} \|\nabla u_H(ABC)\|^2_{\mathbb{R}^2}\left(\|\boldsymbol{x}_A-\boldsymbol{x}_B\|^2 + \|\boldsymbol{x}_B-\boldsymbol{x}_C\|^2 + \|\boldsymbol{x}_C-\boldsymbol{x}_A\|^2\right)\\
&\leq \frac{3}{2}C_p^2H^2\sum_{ABC\in\mathcal{T}} \|\nabla u_H(ABC)\|^2_{\mathbb{R}^2}\\
&\leq \frac{3}{2}C_p^2H^2\sum_{ABC\in\mathcal{T}} \frac{\|\nabla u_H\|^2_{L^2(ABC)}}{|ABC|}\\
&\leq \frac{3}{2}C_p^2H^2\sum_{ABC\in\mathcal{T}} \frac{\|\nabla u_H\|^2_{L^2(ABC)}}{|ABC|}\\
&\leq \frac{3}{2}\frac{C_p^2H^2}{\min_{ABC\in\mathcal{T}}|ABC|}\sum_{ABC\in\mathcal{T}} \|\nabla u_H\|^2_{L^2(ABC)}\\
&= \frac{3}{2}\frac{C_p^2H^2}{\min_{ABC\in\mathcal{T}}|ABC|}\sum_{ABC\in\mathcal{T}} \|\nabla u_H\|^2_{L^2(\Omega)},
\end{aligned}$$

where the sum is taken over all triangles ABC in mesh $\mathcal{T}$. □

5.2.2. Boundedness. Our goal now is to estimate $\|\ell(u)\|_{\mathcal{V},\mathcal{B}}$ as function of $\|\nabla u\|_{L^2(\Omega)}$ when u is in $H^1_0(\Omega)$.

LEMMA 5.7. *Let $\mathcal{T}$ be a coarse mesh on Ω, and let $r>0$ be such that $2r$ is smaller than the smallest height of any triangle in $\mathcal{T}$. Then, for all u in $H^1_0(\Omega)$, we have*

$$\sum_{(i,j)\in\mathcal{V}} |\ell_i(u)-\ell_j(u)|^2 + \sum_{i\in\mathcal{B}} |\ell_i(u)|^2 \leq 2(\frac{C_p^2H^2}{\pi r^2}+\pi)K\|\nabla u\|^2_{L^2(\Omega)}. \tag{5.6}$$

Proof. By density, we only need to prove the result for u in $\mathcal{C}_c^\infty$. Dealing with the second term of (5.6) is possible but cumbersome. It would be much easier to estimate this term if the sum was over the exterior nodes that are physically on the boundary of Ω. Let $\mathcal{B}'$ be the set of the indices of the exterior nodes of $\mathcal{T}$ located on the boundary: their indices are outside of $\{1,\dots,N\}$. Let $\mathcal{V}'$ be the set of all pairs of indices of neighboring nodes including exterior nodes (these nodes were excluded in $\mathcal{V}$). Note that i belongs to $\mathcal{B}$ if and only if there exists at least one index j in $\mathcal{B}'$ such that (i,j) belongs to $\mathcal{V}'$. We have

$$\begin{aligned}\sum_{i\in\mathcal{B}}|\ell_i(u)|^2 &\le \sum_{i\in\mathcal{B}}\sum_{\substack{j\in\mathcal{B}'\\(i,j)\in\mathcal{V}'}}|\ell_i(u)-\ell_j(u)+\ell_j(u)|^2\\ &\le \sum_{i\in\mathcal{B}}\sum_{\substack{j\in\mathcal{B}'\\(i,j)\in\mathcal{V}'}}\left(2\,|\ell_i(u)-\ell_j(u)|^2+2\,|\ell_j(u)|^2\right)\\ &\le 2\sum_{\substack{(i,j)\in\mathcal{V}'\\j\in\mathcal{B}'}}|\ell_i(u)-\ell_j(u)|^2+2K\sum_{j\in\mathcal{B}'}|\ell_j(u)|^2,\end{aligned}$$

where the first sum has been dropped, since the indices i can only vary in $\mathcal{B}$ due to the constraints on the second sum. We thus obtain

$$\sum_{(i,j)\in\mathcal{V}}|\ell_i(u)-\ell_j(u)|^2+\sum_{i\in\mathcal{B}}|\ell_i(u)|^2\le 2\sum_{(i,j)\in\mathcal{V}'}|\ell_i(u)-\ell_j(u)|^2+2K\sum_{i\in\mathcal{B}'}|\ell_i(u)|^2. \tag{5.7}$$

We start by estimating the first term. Let (i,j) be in $\mathcal{V}'$, *i.e.* be neighbor nodes. We have

$$\begin{aligned}&\frac{1}{\pi^2r^4}\left|\int_{B(\boldsymbol{x}_i,r)}u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}-\int_{B(\boldsymbol{x}_j,r)}u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2\\ &\quad=\frac{1}{\pi^2r^4}\left|\int_{B(0,r)}u(\boldsymbol{x}+\boldsymbol{x}_i)-u(\boldsymbol{x}+\boldsymbol{x}_j)\mathrm{d}\boldsymbol{x}\right|^2\\ &\quad=\frac{1}{\pi^2r^4}\left|\int_{B(0,r)}\int_0^1\nabla u(\boldsymbol{x}+(1-t)\boldsymbol{x}_j+t\boldsymbol{x}_i)\cdot(\boldsymbol{x}_i-\boldsymbol{x}_j)\mathrm{d}t\mathrm{d}\boldsymbol{x}\right|^2\\ &\quad\le\frac{1}{\pi r^2}\int_{B(0,r)}\int_0^1\|\nabla u(\boldsymbol{x}+(1-t)\boldsymbol{x}_j+t\boldsymbol{x}_i)\|^2\,\mathrm{d}t\mathrm{d}\boldsymbol{x}\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^2\\ &\quad\le\frac{C_p^2H^2}{\pi r^2}\int_{B(0,r)}\int_0^1\|\nabla u(\boldsymbol{x}+(1-t)\boldsymbol{x}_j+t\boldsymbol{x}_i)\|^2\,\mathrm{d}t\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Now we define $d:=\|\boldsymbol{x}_i-\boldsymbol{x}_j\|$, $\boldsymbol{v}:=\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{d}$, and let $\boldsymbol{w}$ be a unit vector orthogonal to $\boldsymbol{v}$. Then using the equality $\boldsymbol{x}_i-\boldsymbol{x}_j=d\boldsymbol{v}$ and the change of variables $\boldsymbol{x}=s\boldsymbol{v}+\sigma\boldsymbol{w}$,

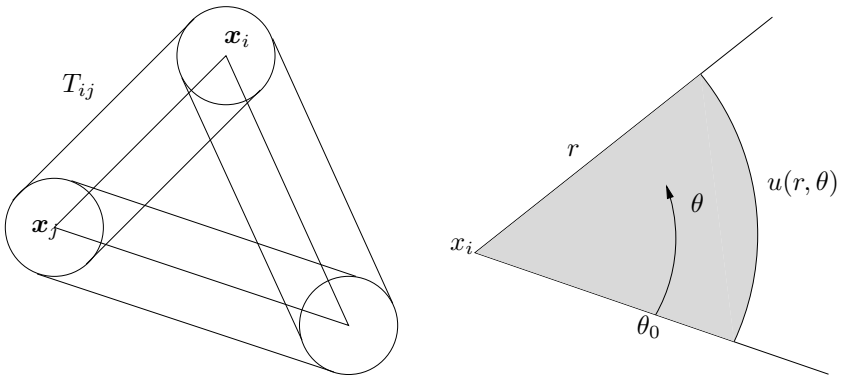

FIGURE 5.3. *Tubes and their overlaps on the left, and estimate of the mean on a ball centered on an exterior edge on the right*

we get

$$
\begin{aligned}
&\int_{B(0,r)} \int_0^1 \|\nabla u(\boldsymbol{x} + (1-t)\boldsymbol{x}_j + t\boldsymbol{x}_i)\|^2 \,\mathrm{d}t\mathrm{d}\boldsymbol{x} \\
&= \int_{-r}^{r} \int_{-\sqrt{r^2-\sigma^2}}^{+\sqrt{r^2-\sigma^2}} \int_0^1 \|\nabla u(\boldsymbol{x}_j + \sigma\boldsymbol{w} + (s+td)\boldsymbol{v})\|^2 \,\mathrm{d}t\mathrm{d}s\mathrm{d}\sigma \\
&= \int_{-r}^{r} \int_0^1 \int_{-\sqrt{r^2-\sigma^2}}^{+\sqrt{r^2-\sigma^2}} \|\nabla u(\boldsymbol{x}_j + \sigma\boldsymbol{w} + (s+td)\boldsymbol{v})\|^2 \,\mathrm{d}s\mathrm{d}t\mathrm{d}\sigma \\
&= \int_{-r}^{r} \int_0^1 \int_{-\sqrt{r^2-\sigma^2}+td}^{+\sqrt{r^2-\sigma^2}+td} \|\nabla u(\boldsymbol{x}_j + \sigma\boldsymbol{w} + \tilde{s}\boldsymbol{v})\|^2 \,\mathrm{d}\tilde{s}\mathrm{d}t\mathrm{d}\sigma \\
&= \int_{-r}^{r} \int_{-\sqrt{r^2-\sigma^2}}^{+\sqrt{r^2-\sigma^2}+d} \|\nabla u(\boldsymbol{x}_j + \sigma\boldsymbol{w} + \tilde{s}\boldsymbol{v})\|^2 \left(\int_0^1 \chi_{[\frac{\tilde{s}-\sqrt{r^2-\sigma^2}}{d}, \frac{\tilde{s}+\sqrt{r^2-\sigma^2}}{d}]}(t)\mathrm{d}t \right) \mathrm{d}\tilde{s}\mathrm{d}\sigma \\
&\leq \int_{-r}^{r} \int_{-\sqrt{r^2-\sigma^2}}^{+\sqrt{r^2-\sigma^2}+d} \|\nabla u(\boldsymbol{x}_j + \sigma\boldsymbol{w} + \tilde{s}\boldsymbol{v})\|^2 \,\mathrm{d}\tilde{s}\mathrm{d}\sigma,
\end{aligned}
$$

which leads to the estimate

$$
\frac{1}{\pi^2 r^4} \left| \int_{B(\boldsymbol{x}_i,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_{B(\boldsymbol{x}_j,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \right|^2 \leq \frac{C_p^2 H^2}{\pi r^2} \int_{T_{i,j}} \|\nabla u(\boldsymbol{x})\|^2 \mathrm{d}\boldsymbol{x},
$$

where $T_{i,j}$ is the set of all points $\boldsymbol{x}$ whose distance to the segment $[\boldsymbol{x}_i, \boldsymbol{x}_j]$ is smaller than r. Since $2r$ is smaller than the height of any triangle in the mesh, no point $\boldsymbol{x}$ may belong to more than K tubes $T_{i,j}$, see Figure 5.3 on the left. Therefore, we have

$$
\sum_{(i,j)\in\mathcal{V}'} \left| \frac{1}{\pi r^2} \int_{B(\boldsymbol{x}_i,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \frac{1}{\pi r^2} \int_{B(\boldsymbol{x}_j,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \right|^2 \leq K \frac{C_p^2 H^2}{\pi r^2} \int_{\Omega} \|\nabla u(\boldsymbol{x})\|^2 \mathrm{d}\boldsymbol{x}. \tag{5.8}
$$

We now estimate the second term of the right hand-side of (5.7). Let i be in $\mathcal{B}'$, *i.e.* i is the index of a node located exactly on the boundary of domain Ω, then u vanishes on at least two radii. Let θ_1 be the angle between the horizontal and one of

the radii on which u is zero, see Figure 5.3 on the right. With $\boldsymbol{e}_\rho(\theta) := (\cos\theta, \sin\theta)$ and $\boldsymbol{e}_\theta(\theta) := (-\sin\theta, \cos\theta)$, we obtain

$$
\begin{aligned}
&\frac{1}{\pi^2 r^4}\left|\int_{B(\boldsymbol{x}_i,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2 \\
&\leq \frac{1}{\pi r^2}\int_{B(\boldsymbol{x}_i,r)} |u(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \\
&= \frac{1}{\pi r^2}\int_0^r \int_{\theta_0-\pi}^{\theta_0+\pi} |u(\boldsymbol{x}_i+\rho\boldsymbol{e}_\rho(\theta))|^2\,\mathrm{d}\theta\rho\mathrm{d}\rho \\
&= \frac{1}{\pi r^2}\int_0^r \rho\int_{\theta_0-\pi}^{\theta_0+\pi} \left|\int_{\theta_0}^{\theta} \nabla u(\boldsymbol{x}_i+\rho\boldsymbol{e}_\rho(t))\cdot(\rho\boldsymbol{e}_\theta(t))\mathrm{d}t\right|^2 \mathrm{d}\theta\rho\mathrm{d}\rho \\
&\leq \frac{1}{\pi r^2}\int_0^r \rho^2\int_{\theta_0-\pi}^{\theta_0+\pi} |\theta-\theta_0|\int_{\min(\theta_0,\theta)}^{\max(\theta_0,\theta)} \|\nabla u(\boldsymbol{x}_i+\rho\boldsymbol{e}_\rho(t))\|^2\,\mathrm{d}t\mathrm{d}\theta\rho\mathrm{d}\rho \\
&\leq \int_0^r \int_{\theta_0-\pi}^{\theta_0+\pi}\int_{\min(\theta_0,\theta)}^{\max(\theta_0,\theta)} \|\nabla u(\boldsymbol{x}_i+\rho\boldsymbol{e}_\rho(t))\|^2\,\mathrm{d}t\mathrm{d}\theta\rho\mathrm{d}\rho \\
&= \int_0^r \int_{\theta_0-\pi}^{\theta_0+\pi} \|\nabla u(\boldsymbol{x}_i+\rho\boldsymbol{e}_\rho(t))\|^2 \left(\int_{\theta_0-\pi}^{\theta_0+\pi} \chi_{(\min(\theta_0,\theta),\max(\theta_0,\theta))}(t)\mathrm{d}\theta\right)\mathrm{d}t\rho\mathrm{d}\rho \\
&\leq \pi\int_0^r \int_{\theta_0-\pi}^{\theta_0+\pi} \|\nabla u(\boldsymbol{x}_i+\rho\boldsymbol{e}_\rho(t))\|^2\,\mathrm{d}t\rho\mathrm{d}\rho \\
&= \pi\int_{B(\boldsymbol{x}_i,r)} \|\nabla u(\boldsymbol{x})\|^2\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

No point[6] $\boldsymbol{x}$ in Ω can be in more than one ball $B(\boldsymbol{x}_i, r)$, therefore summing this inequality over i in $\mathcal{B}'$, we get

$$\sum_{i\in\mathcal{B}'} |\ell_i(u)|^2 \leq \pi\int_\Omega \|\nabla u(\boldsymbol{x})\|^2\mathrm{d}\boldsymbol{x}. \tag{5.9}$$

Combining (5.7) with Inequalities (5.8) and (5.9), we finally obtain

$$\|u\|^2_{\mathcal{V},\mathcal{B}} \leq 2\left(\frac{C_p^2 H^2}{\pi r^2}+\pi\right)K\int_\Omega \|\nabla u(\boldsymbol{x})\|^2\mathrm{d}\boldsymbol{x}.$$

□

5.2.3. Continuity of the ℓ^{-1} linear form. Let $\varepsilon \in \mathbb{R}$, with $0 < \varepsilon < \frac{1}{2}$, and choose $r := \varepsilon H_h$, where H_h is the smallest triangle height among all the triangles in the coarse mesh $\mathcal{T}$. Let $L := [l_{ij}]$ be the matrix associated with the linear function ℓ, *i.e.* the matrix such that $L\cdot(u(\boldsymbol{x}_i))_{1\leq i\leq N} = \ell(u)$ for all u_H in $P_1(\Omega)$. This is a square matrix, by Assumption 5.3, of size $N\times N$, and satisfies the following properties:

- For all i, j in $\{1,\ldots,N\}$, we have $l_{ij} \geq 0$.

[6] One can construct pathological meshes in non pathological cases where two exterior nodes A and B that are **not** neighbors are closer than H_h. However, in that case, one can easily avoid that problem by redefining $\ell_A(u)$ whenever A is in $\mathcal{B}'$ to be $\frac{1}{\pi r^2}\int_{V_A\cap B(\boldsymbol{x}_A,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ where V_A is the union of all triangles in mesh $\mathcal{T}$ that have node A as a vertex.

- For all i, j not belonging to $\mathcal{V}$, $l_{i,j} = 0$, which implies that for any given i, there are at most K integers j such that $l_{ij} \neq 0$.
- For all i in $\{1, \dots, N\}$, we have $l_{ii} \geq 1 - \varepsilon$.
- For all i in $\{1, \dots, N\}$, we have $\sum_{j=1}^{N} l_{ij} = 1$ if $i \notin \mathcal{B}$, and $\sum_{j=1}^{N} l_{ij} \leq 1$ if $i \in \mathcal{B}$.

LEMMA 5.8. *If $\varepsilon \leq \frac{1}{4K+1}$, then the matrix L is invertible, and for all $\boldsymbol{u}$ in $\mathbb{R}^n$, we have, with $1 - \big((2K+1) + (4K+1)\varepsilon\big)\varepsilon \geq 0$ that*

$$\frac{1 - \big((2K+1) + (4K+1)\varepsilon\big)\varepsilon}{1 + 2\varepsilon} \|\boldsymbol{u}\|_{\mathcal{V},\mathcal{B}} \leq \|L\boldsymbol{u}\|_{\mathcal{V},\mathcal{B}} \leq (1+(2K+3)\varepsilon+(4K+1)\varepsilon^2)\|\boldsymbol{u}\|_{\mathcal{V},\mathcal{B}}. \tag{5.10}$$

Proof. For all integers i, $1 \leq i \leq N$, we have $l_{ii} \geq 1 - \varepsilon$ and $\sum_j |l_{ij}| \leq 1$. Since $\varepsilon < \frac{1}{2}$, L is a strictly diagonally dominant matrix, hence invertible. For the remainder of this proof, we will denote by $\mathcal{V}_i$ the set of all integer j such that (i, j) belongs to $\mathcal{V}$. We also define $l_i^* := 1 - \sum_{j=1}^{n} l_{ij}$, and note that l_i^* is always non negative and smaller than ε, and it vanishes if i does not belong to $\mathcal{B}$.

We start by estimating the first term of the norm $\|\cdot\|_{\mathcal{V},\mathcal{B}}$, see Definition 5.5. We have

$$\begin{aligned}
&\sum_{(i,j)\in\mathcal{V}} |\sum_{k=1}^{N} (l_{ik} - l_{jk}) u_k|^2 \\
&= \sum_{(i,j)\in\mathcal{V}} \left| \sum_{\substack{k=1 \\ k\neq i}}^{N} l_{ik}(u_k - u_i) - \sum_{\substack{k=1 \\ k\neq j}}^{N} l_{jk}(u_k - u_j) + (u_i - u_j) - l_i^* u_i + l_j^* u_j \right|^2,
\end{aligned}$$

and using now the Cauchy-Schwarz inequality for $\sqrt{l_{ik}} \times \sqrt{l_{ik}}(u_k - u_i)$, we obtain

$$\begin{aligned}
&\leq \sum_{(i,j)\in\mathcal{V}} \left(\sum_{\substack{k=1 \\ k\neq i}}^{N} l_{ik} + \sum_{\substack{k=1 \\ k\neq j}}^{N} l_{jk} + 1 + l_i^* + l_j^* \right) \times \\
&\qquad \times \left(\sum_{\substack{k=1 \\ k\neq i}}^{N} l_{ik}|u_k - u_i|^2 + \sum_{\substack{k=1 \\ k\neq j}}^{N} l_{jk}|u_k - u_j|^2 + |u_i - u_j|^2 + l_i^*|u_i|^2 + l_j^*|u_j|^2 \right) \\
&\leq (1 + 2\varepsilon) \sum_{(i,j)\in\mathcal{V}} \left(\sum_{\substack{k=1 \\ k\neq i}}^{N} l_{ik}|u_k - u_i|^2 + \sum_{\substack{k=1 \\ k\neq j}}^{N} l_{jk}|u_k - u_j|^2 + |u_i - u_j|^2 + l_i^*|u_i|^2 + l_j^*|u_j|^2 \right) \\
&\leq (1 + 2\varepsilon) \left(\sum_{(i,j)\in\mathcal{V}} |u_i - u_j|^2 + 2K \max_{i\neq j} |l_{ij}| \sum_{(i,j)\in\mathcal{V}} |u_j - u_i|^2 + 2K \max_{i\in\mathcal{B}} |l_i^*| \sum_{i\in\mathcal{B}}^{N} |u_i|^2 \right) \\
&\leq (1 + 2\varepsilon) \left((1 + 2K\varepsilon) \sum_{(i,j)\in\mathcal{V}} |u_i - u_j|^2 + 2K\varepsilon \sum_{i\in\mathcal{B}}^{N} |u_i|^2 \right),
\end{aligned}$$

which yields the inequality

$$\sum_{(i,j)\in\mathcal{V}} |\sum_{k=1}^{N}(l_{ik}-l_{jk})u_k|^2 \leq (1+2\varepsilon)\left((1+2K\varepsilon)\sum_{(i,j)\in\mathcal{V}}|u_i-u_j|^2+2K\varepsilon\sum_{i\in\mathcal{B}}^{N}|u_i|^2\right). \tag{5.11}$$

We now estimate the second term of the norm in Definition 5.5,

$$\sum_{i\in\mathcal{B}}\left|\sum_{k=1}^{N}l_{ik}u_k\right|^2=\sum_{i\in\mathcal{B}}\left|\sum_{\substack{k=1\\k\neq i}}^{N}l_{ik}(u_k-u_i)+(1-l_i^*)u_i\right|^2$$

and again using the Cauchy-Schwarz inequality on $\sqrt{l_{ik}}\times\sqrt{l_{ik}}(u_k-u_i)$, we obtain

$$\begin{aligned}
&\leq \sum_{i\in\mathcal{B}}\left(\sum_{\substack{k=1\\k\neq i}}^{N}l_{ik}+(1-l_i^*)\right)\left(\sum_{k=1}^{N}l_{ik}|u_k-u_i|^2+(1-l_i^*)|u_i|^2\right),\\
&\leq (1+\varepsilon)\sum_{i\in\mathcal{B}}\left(\sum_{k=1}^{N}l_{ik}|u_k-u_i|^2+(1-l_i^*)|u_i|^2\right)\\
&\leq (1+\varepsilon)\max_{i\neq k}|l_{ik}|\sum_{(i,k)\in\mathcal{V}}|u_k-u_i|^2+(1+\varepsilon)\sum_{i\in\mathcal{B}}|u_i|^2\\
&\leq (1+\varepsilon)\varepsilon\sum_{(i,k)\in\mathcal{V}}|u_k-u_i|^2+(1+\varepsilon)\sum_{i\in\mathcal{B}}|u_i|^2,
\end{aligned}$$

which proves the inequality

$$\sum_{i\in\mathcal{B}}\left|\sum_{k=1}^{N}l_{ik}u_k\right|^2\leq(1+\varepsilon)\varepsilon\sum_{(i,k)\in\mathcal{V}}|u_k-u_i|^2+(1+\varepsilon)\sum_{i\in\mathcal{B}}|u_i|^2. \tag{5.12}$$

Now combining inequalities (5.11) and (5.12), we establish the right part of inequality (5.10).

Proving the left part of inequality (5.10) is a little more difficult. We start by estimating the first term of the norm $\|\cdot\|_{\mathcal{V},\mathcal{B}}$. To establish (5.11), we used the equality

$$\sum_{k=1}^{N}(l_{ik}-l_{jk})u_k=\sum_{\substack{k=1\\k\neq i}}^{N}l_{ik}(u_k-u_i)-\sum_{\substack{k=1\\k\neq j}}^{N}l_{jk}(u_k-u_j)+(u_i-u_j)-l_i^*u_i+l_j^*u_j.$$

Putting the $(u_i - u_j)$ term onto the left hand-side of the equation and all the other terms onto the right hand-side, we get

$$\begin{aligned}
&\sum_{(i,j)\in\mathcal{V}} |u_i - u_j|^2 \\
&= \sum_{(i,j)\in\mathcal{V}} \left| \sum_{k=1}^{N} (l_{ik} - l_{jk}) u_k - \sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}(u_k - u_i) + \sum_{\substack{k=1\\k\neq j}}^{N} l_{jk}(u_k - u_j) + l_i^* u_i - l_j^* u_j \right|^2 \\
&= \sum_{(i,j)\in\mathcal{V}} \left| \big((L\boldsymbol{u})_i - (L\boldsymbol{u})_j\big) - \sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}(u_k - u_i) + \sum_{\substack{k=1\\k\neq j}}^{N} l_{jk}(u_k - u_j) + l_i^* u_i - l_j^* u_j \right|^2,
\end{aligned}$$

and using again the Cauchy-Schwarz inequality, as we did earlier, we find

$$\begin{aligned}
&\leq \sum_{(i,j)\in\mathcal{V}} \big(1 + (1 - l_i^* - l_{ii}) + (1 - l_j^* - l_{jj}) + l_i^* + l_j^*\big) \times \\
&\qquad \times \left(|(L\boldsymbol{u})_i - (L\boldsymbol{u})_j|^2 + \sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}|u_k - u_i|^2 + \sum_{\substack{k=1\\k\neq j}}^{N} l_{jk}|u_k - u_j|^2 + l_i^*|u_i|^2 + l_j^*|u_j|^2 \right) \\
&\leq (1+2\varepsilon) \sum_{(i,j)\in\mathcal{V}} \Bigg(|(L\boldsymbol{u})_i - (L\boldsymbol{u})_j|^2 + \max_{k\neq i}|l_{ik}| \sum_{k\in\mathcal{V}_i} |u_k - u_i|^2 \\
&\qquad\qquad + \max_{k\neq j}|l_{jk}| \sum_{k\in\mathcal{V}_j} |u_k - u_j|^2 + l_i^*|u_i|^2 + l_j^*|u_j|^2 \Bigg) \\
&\leq (1+2\varepsilon) \sum_{(i,j)\in\mathcal{V}} \left(|(L\boldsymbol{u})_i - (L\boldsymbol{u})_j|^2 + \varepsilon \sum_{k\in\mathcal{V}_i} |u_k - u_i|^2 + \varepsilon \sum_{k\in\mathcal{V}_j} |u_k - u_j|^2 + l_i^*|u_i|^2 + l_j^*|u_j|^2 \right) \\
&\leq (1+2\varepsilon) \sum_{(i,j)\in\mathcal{V}} |(L\boldsymbol{u})_i - (L\boldsymbol{u})_j|^2 + 2(1+2\varepsilon)K\varepsilon \sum_{(i,j)\in\mathcal{V}} |u_j - u_i|^2 + 2(1+2\varepsilon)K\varepsilon \sum_{i\in\mathcal{B}} |u_i|^2.
\end{aligned}$$

The ε terms will be absorbed by the left hand-side, provided we choose ε small enough. To absorb the third term, we must first estimate the second term in norm $\|\cdot\|_{\mathcal{V},\mathcal{B}}$. To establish (5.12), we used

$$\sum_{k=1}^{N} l_{ik} u_k = \sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}(u_k - u_i) + (1 - l_i^*) u_i.$$

We put u_i onto the left hand-side of the equality and all the other terms onto the right hand-side to obtain

$$
\begin{aligned}
&\sum_{i\in\mathcal{B}}|u_i|^2\\
&=\sum_{i\in\mathcal{B}}|\sum_{k=1}^{N} l_{ik}u_k-\sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}(u_k-u_i)+l_i^*u_i|^2\\
&=\sum_{i\in\mathcal{B}}|(L\boldsymbol{u})_i-\sum_{k=1}^{N} l_{ik}(u_k-u_i)+l_i^*u_i|^2\\
&\leq\sum_{i\in\mathcal{B}}\left(1+\sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}+l_i^*\right)\left(|(L\boldsymbol{u})_i|^2+\sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}|u_k-u_i|^2+l_i^*|u_i|^2\right)\\
&\leq(1+\varepsilon)\sum_{i\in\mathcal{B}}\left(|(L\boldsymbol{u})_i|^2+\max_{k\neq i}|l_{ik}|\sum_{k\in\mathcal{V}_i}|u_k-u_i|^2+\max_{j\in\mathcal{B}}|l_j^*||u_i|^2\right)\\
&\leq(1+\varepsilon)\sum_{i\in\mathcal{B}}|(L\boldsymbol{u})_i|^2+(1+\varepsilon)\max_{i\neq j}|l_{ij}|\sum_{(i,j)\in\mathcal{V}}|u_j-u_i|^2+(1+\varepsilon)\max_{i\in\mathcal{B}}|l_i^*|\sum_{i\in\mathcal{B}}|u_i|^2\\
&\leq(1+\varepsilon)\sum_{i\in\mathcal{B}}|(L\boldsymbol{u})_i|^2+(1+\varepsilon)\varepsilon\sum_{(i,j)\in\mathcal{V}}|u_j-u_i|^2+(1+\varepsilon)\varepsilon\sum_{i\in\mathcal{B}}|u_i|^2.
\end{aligned}
$$

We add now the last two estimates to get

$$\|\boldsymbol{u}\|^2_{\mathcal{V},\mathcal{B}}\leq(1+2\varepsilon)\|L\boldsymbol{u}\|^2_{\mathcal{V},\mathcal{B}}+\big((2K+1)+(4K+1)\varepsilon\big)\varepsilon\|\boldsymbol{u}\|^2_{\mathcal{V},\mathcal{B}}.$$

If $\varepsilon\leq\frac{1}{4K+1}$ then $\big((2K+1)+(4K+1)\varepsilon\big)\varepsilon<1$, which concludes the proof. □

5.2.4. End of the proof of the Theorem 5.4. We just combine Lemma 5.6, Lemma 5.7, and Lemma 5.8, and we have successively the existence and uniqueness of u_H (since the matrix L is invertible), and the estimates

$$\|\nabla u_H\|^2_{L^2(\Omega)}\leq C_1\|u_H\|^2_{\mathcal{V},\mathcal{B}}\leq C_2C_1\|\ell(u)\|^2_{\mathcal{V},\mathcal{B}}\leq C_3C_2C_1\|\nabla u\|^2_{L^2(\Omega)},$$

where $C_1=\frac{1}{\tan\theta_{min}}$, $C_2=\frac{1+2r/H_h}{1-\big((2K+1)+(4K+1)r/H_h\big)r/H_h}$ and $C_3=2K(\frac{C_p^2H^2}{\pi r^2}+\pi)$. To apply these inequalities, it is sufficient for the ratio r/H_h to be smaller than $1/((4K+1))$, where H_h is the length of the shortest height of any triangle in the mesh $\mathcal{T}$.

5.3. Non coarse elements. In this subsection, we construct the non coarse elements of the stable decomposition. We make the following assumption on the U_i:

ASSUMPTION 5.9 (Star shape of U_i). *We assume that there exists a uniform ε such that for all the domain decompositions we consider for Ω, U_i is star shaped with respect to any point in the ball $B(\boldsymbol{x}_i,r)$, where $r=\varepsilon H_h$ and where the $\boldsymbol{x}_i$ are the nodes of the coarse mesh $\mathcal{T}$ and where H_h is the length of the shortest height of any triangle in mesh $\mathcal{T}$.*

First we improve Lemma 5.1 in order to obtain estimates for the constants involved.

LEMMA 5.10. *Let ω be an open domain of $\mathbb{R}^2$ with a diameter smaller than H. Let $r<H$. We suppose there exists $\boldsymbol{x}_O$ in ω such that*

- *The ball $B(\boldsymbol{x}_O, 2r)$ is included in ω.*
- *The set ω is star-shaped with respect to all $\boldsymbol{x}$ in the ball $B(\boldsymbol{x}, r)$.*

Then for all u in $H^1(\omega)$, we have the estimate

$$\int_\omega |u(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y} \leq \frac{2r^2}{3}\left(\left(\left(\frac{H^2}{r^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{H}{\sqrt[4]{2}r}\right)^4-\frac{1}{2}-\frac{H^2}{r^2}-\frac{H^4}{2r^4}\right)\int_\omega \|\nabla u(\boldsymbol{x})\|^2 \mathrm{d}\boldsymbol{x} + \frac{2H^2}{\pi r^4}\left|\int_{B(\boldsymbol{x}_O,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2.$$

Proof. Without loss of generality, we can suppose that $\boldsymbol{x}_O = \boldsymbol{0}$. We then compute

$$\begin{aligned}
&\int_\omega |u(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y}\\
&= \int_\omega \left|u(\boldsymbol{y}) - \frac{1}{\pi r^2}\int_{B(\boldsymbol{0},r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \frac{1}{\pi r^2}\int_{B(\boldsymbol{0},r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2 \mathrm{d}\boldsymbol{y}\\
&\leq \frac{2}{\pi^2 r^4}\int_\omega \left|\int_{B(\boldsymbol{0},r)} (u(\boldsymbol{y})-u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right|^2 \mathrm{d}\boldsymbol{y} + \frac{2|\omega|}{\pi^2 r^4}\left|\int_{B(\boldsymbol{0},r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2\\
&\leq \frac{2}{\pi^2 r^4}\int_\omega \left|\int_{B(\boldsymbol{0},r)} (u(\boldsymbol{y})-u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right|^2 \mathrm{d}\boldsymbol{y} + \frac{2H^2}{\pi r^4}\left|\int_{B(\boldsymbol{0},r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2,
\end{aligned}$$

and it remains to estimate the first term in the sum on the right,

$$\begin{aligned}
&\frac{1}{\pi^2 r^4}\int_\omega \left|\int_{B(\boldsymbol{0},r)} (u(\boldsymbol{y})-u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right|^2 \mathrm{d}\boldsymbol{y}\\
&= \frac{1}{\pi^2 r^4}\int_\omega \left|\int_{B(\boldsymbol{0},r)}\int_0^1 \nabla u((1-t)\boldsymbol{x}+t\boldsymbol{y})\cdot(\boldsymbol{y}-\boldsymbol{x})\mathrm{d}t\mathrm{d}\boldsymbol{x}\right|^2 \mathrm{d}\boldsymbol{y}\\
&\leq \frac{1}{\pi r^2}\int_\omega\int_{B(\boldsymbol{0},r)}\int_0^1 \|\nabla u((1-t)\boldsymbol{x}+t\boldsymbol{y})\|^2\|\boldsymbol{y}-\boldsymbol{x}\|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

Now using the change of variables $\boldsymbol{x}' = (1-t)\boldsymbol{x}+t\boldsymbol{y}$, we get

$$\begin{aligned}
&\leq \frac{1}{\pi r^2}\int_\omega\int_0^1\int_{B(t\boldsymbol{y},(1-t)r)} \|\nabla u(\boldsymbol{x}')\|^2\|\boldsymbol{y}-\boldsymbol{x}'\|^2\mathrm{d}\boldsymbol{x}'\frac{\mathrm{d}t}{(1-t)^4}\mathrm{d}\boldsymbol{y}\\
&= \frac{1}{\pi r^2}\int_\omega \|\nabla u(\boldsymbol{x}')\|^2\int_0^1\int_\omega \|\boldsymbol{y}-\boldsymbol{x}'\|^2\chi\{\|\boldsymbol{x}'-t\boldsymbol{y}\|\leq(1-t)r\}\mathrm{d}\boldsymbol{y}\frac{\mathrm{d}t}{(1-t)^4}\mathrm{d}\boldsymbol{x}'.
\end{aligned}$$

Using the further change of variables $\boldsymbol{y}' = \boldsymbol{y}-\boldsymbol{x}'$ yields

$$\begin{aligned}
&\frac{1}{\pi r^2}\int_\omega \|\nabla u(\boldsymbol{x}')\|^2\int_0^1\int_{\omega-\boldsymbol{x}'} \|\boldsymbol{y}'\|^2\chi\{\|\boldsymbol{x}'-\frac{t}{1-t}\boldsymbol{y}'\|\leq r\}\mathrm{d}\boldsymbol{y}'\frac{\mathrm{d}t}{(1-t)^4}\mathrm{d}\boldsymbol{x}'\\
&\leq \frac{1}{\pi r^2}\int_\omega \|\nabla u(\boldsymbol{x}')\|^2\int_0^1\int_{B(0,H)} \|\boldsymbol{y}'\|^2\chi\{\|\boldsymbol{x}'-\frac{t}{1-t}\boldsymbol{y}'\|\leq r\}\mathrm{d}\boldsymbol{y}'\frac{\mathrm{d}t}{(1-t)^4}\mathrm{d}\boldsymbol{x}'
\end{aligned}$$

and a final change of variables $\boldsymbol{y}'' = \frac{t}{1-t}\boldsymbol{y}'$ gives

$$
\begin{aligned}
&= \frac{1}{\pi r^2}\int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \int_0^1 \int_{B(0,\frac{tH}{1-t})\cap B(\boldsymbol{x}',r)} \|\boldsymbol{y}''\|^2 \mathrm{d}\boldsymbol{y}'' \frac{\mathrm{d}t}{t^4}\mathrm{d}\boldsymbol{x}' \\
&\quad \frac{1}{\pi r^2}\int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \int_0^1 \min\Big(\int_{B(0,\frac{tH}{1-t})} \|\boldsymbol{y}''\|^2 \mathrm{d}\boldsymbol{y}'', \int_{B(\boldsymbol{x}',r)} \|\boldsymbol{y}''\|^2 \mathrm{d}\boldsymbol{y}''\Big)\frac{\mathrm{d}t}{t^4}\mathrm{d}\boldsymbol{x}' \\
&= \frac{1}{r^2}\int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \int_0^1 \min\Big(\frac{t^4H^4}{2(1-t)^4}, r^2\big(\frac{r^2}{2}+\|\boldsymbol{x}'\|^2\big)\Big)\frac{\mathrm{d}t}{t^4}\mathrm{d}\boldsymbol{x}' \\
&\le \frac{1}{r^2}\left(\int_0^1 \min\Big(\frac{t^4H^4}{2(1-t)^4}, r^2\big(\frac{r^2}{2}+H^2\big)\Big)\frac{\mathrm{d}t}{t^4}\right)\int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \mathrm{d}\boldsymbol{x}' \\
&= \frac{r^2}{3}\left(\left(\left(\frac{H^2}{r^2}+\frac{1}{2}\right)^{\frac{1}{4}} + \frac{H}{\sqrt[4]{2}r}\right)^4 - \frac{1}{2} - \frac{H^2}{r^2} - \frac{H^4}{2r^4}\right)\int_\omega \|\nabla u(\boldsymbol{x})\|^2 \mathrm{d}\boldsymbol{x},
\end{aligned}
$$

which is the desired result. ∎

LEMMA 5.11. *Let Ω be a bounded domain of $\mathbb{R}^2$, and $(U_i,\Omega_i)_{1\le i\le N}$ be an associated domain decomposition with overlap width $\delta>0$. Let $\mathcal{T}$ be a coarse mesh on Ω, and assume that Assumptions 5.3, 5.2 and 5.9 are verified. Then for any u in $H_0^1(\Omega)$, there exists $(u_i)_{1\le i\le N}$ in $H_0^1(\Omega_i)$, such that for all i, $1\le i\le n$, u_i is in $H_0^1(\Omega_i)$, and*

$$
\begin{aligned}
u &= \sum_{i=1}^N u_i \\
\sum_{i=1}^N \|\nabla u_i\|^2_{L^2(\Omega_i)} &\le \left(2+\frac{4}{3}\lambda_2^2 N_c(N_c-1)^2\frac{r^2}{\delta^2}\left(\left(\left(\frac{C_d^2H^2}{r^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{C_dH}{\sqrt[4]{2}r}\right)^4 - \frac{C_d^4H^4}{2r^4} - \frac{1}{2} - \frac{C_d^2H^2}{r^2}\right)\right)\|\nabla u\|^2_{L^2(\Omega)} \\
&\quad + 4\lambda_2^2 N_c(N_c-1)^2\pi\frac{C_d^2H^2}{\delta^2}\sum_{i=1}^N |\ell_i(u)|^2,
\end{aligned}
$$

where $\ell_i(u) = \frac{1}{\pi r^2}\int_{B(\boldsymbol{x}_i,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$.

Proof. We use the same u_i as in the proof of Theorem 4.4. Since $\operatorname{diam}(U_i)\le C_dH$, we have

$$
\begin{aligned}
\|u\|^2_{L^2(\Omega)} &= \sum_{i=1}^N \|u\|^2_{L^2(U_i)} \\
&\le \frac{2r^2}{3}\left(\left(\left(\frac{C_d^2H^2}{r^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{C_dH}{\sqrt[4]{2}r}\right)^4 - \frac{C_d^4H^4}{2r^4} - \frac{1}{2} - \frac{C_d^2H^2}{r^2}\right)\sum_{i=1}^N \|\nabla u\|^2_{L^2(U_i)} \\
&\quad + 2\pi C_d^2H^2\sum_{i=1}^N |\ell_i(u)|^2 \\
&= \frac{2r^2}{3}\left(\left(\left(\frac{C_d^2H^2}{r^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{C_dH}{\sqrt[4]{2}r}\right)^4 - \frac{C_d^4H^4}{2r^4} - \frac{1}{2} - \frac{C_d^2H^2}{r^2}\right)\|\nabla u\|^2_{L^2(\Omega)} \\
&\quad + 2\pi C_d^2H^2\sum_{i=1}^N |\ell_i(u)|^2.
\end{aligned}
$$

Inserting this estimate into estimate (4.4) concludes the proof. ∎

5.4. Stable Decomposition with Coarse Mesh. Combining our previous results, we obtain now our main theorem on the existence of a stable decomposition with a coarse mesh. We provide this theorem with all assumptions in order for it to be self contained.

THEOREM 5.12 (Stable Decomposition of H_0^1 with Coarse Mesh). *Let Ω be a bounded domain of $\mathbb{R}^2$, and $((U_i)_{1\leq i\leq N}, (\Omega_i)_{1\leq i\leq N})$ be an associated overlapping domain decomposition of Ω with overlap width $\delta > 0$. Let H be the smallest diameter among all U_i, $C_d := \max_i \mathrm{diam} U_i/H$, and N_c be the number of colors of this decomposition.*

Let $\mathcal{T}$ be a triangular coarse mesh of the domain Ω with N nodes, and suppose that the i-th node $\boldsymbol{x}_i$ of $\mathcal{T}$ is in U_i. Let C_p be the ratio between the length of the longest edge in $\mathcal{T}$ and H, and H_h be the length of the shortest height of any triangle in $\mathcal{T}$. Let θ_{min} be the smallest angle in the mesh $\mathcal{T}$, and K be the maximum number of neighbors a node can have.

Let $r \leq \frac{H_h}{4K+1}$, such that for all i in $\{1,\dots,N\}$, the ball $B(\boldsymbol{x}_i, 2r)$ is a subset of U_i, and that U_i is star shaped with respect to any point in the ball $B(\boldsymbol{x}_i, r)$.

Then, there exists a stable decomposition of $H_0^1(\Omega)$ in $P_1(\mathcal{T})\cap H_0^1(\Omega)+\sum_{i=1}^N H_0^1(\Omega_i)$, i.e. for all u in $H_0^1(\Omega)$, there exists u_0 in $P_1(\mathcal{T})\cap H_0^1(\Omega)$ and $(u_i)_{1\leq i\leq N}$, $u_i \in H_0^1(\Omega_i)$, such that

$$u = \sum_{i=0}^{N} u_i,$$
$$\sum_{i=0}^{N} \|\nabla u_i\|^2_{L^2(\Omega_i)} \leq C\|\nabla u\|^2_{L^2(\Omega)},$$

where $C = C_1 + 2(1+C_1)C_2$ and[7]

$$C_1 = \frac{1}{\tan\theta_{min}} \frac{1+2r/H_h}{1-\big((2K+1)+(4K+1)r/H_h\big)r/H_h} 2K\left(\frac{C_p^2H^2}{\pi r^2}+\pi\right),$$
$$C_2 = \left(2+\frac{4}{3}\lambda_2^2 N_c(N_c-1)^2\frac{r^2}{\delta^2}\left(\left(\left(\frac{C_d^2H^2}{r^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{C_dH}{\sqrt[4]{2}r}\right)^4-\frac{C_d^4H^4}{2r^4}-\frac{1}{2}-\frac{C_d^2H^2}{r^2}\right)\right).$$

Proof. It suffices to take $u_0 = u_H$ from Theorem 5.4, and to apply Lemma 5.11 to $u - u_0$ in order to obtain the stable decomposition with the given constant. □

6. Bounded condition number of the additive Schwarz operator at the continuous level. We can now use the stable decomposition we established to bound the condition number of the continuous Additive Schwarz operator, which leads to the following result:

THEOREM 6.1 (Condition Number Estimate at the Continuous Level). *Let Ω be a bounded domain of $\mathbb{R}^2$. Let A be a continuous function from $\overline{\Omega}$ to the set of 2×2 symmetric positive definite matrices. We suppose that $A(\boldsymbol{x})$ is uniformly coercive and uniformly bounded: there exist $\alpha > 0$ and $\beta > 0$ such that for all $\boldsymbol{x}$ in Ω, and for all $\boldsymbol{\xi}$ in $\mathbb{R}^2$*

$$\alpha\|\boldsymbol{\xi}\|_2 \leq |\boldsymbol{\xi}^{\mathrm{T}} A(\boldsymbol{x})\boldsymbol{\xi}| \leq \beta\|\boldsymbol{\xi}\|_2.$$

[7] Note that $r \leq \frac{H_h}{4K+1}$ ensures that $1-\big((2K+1)+(4K+1)r/H_h\big)r/H_h$ is positive.

Let $a(\cdot,\cdot)$ be the continuous bilinear form on $H_0^1(\Omega)$ defined by

$$a(u,v) = \int_\Omega \nabla u(\boldsymbol{x}) \cdot A(\boldsymbol{x}) \nabla v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

We use the same notation and the same hypotheses as in Theorem 5.12 to define the U_i, the Ω_i, the mesh $\mathcal{T}$ and all the geometric parameters on which the constants depend. In addition, let N_K be the maximum number of neighboring subdomains a subdomain can have[8]*.*

Let $V_0 = P_1(\mathcal{T})$. Let $V_i = H_0^1(\Omega_i)$ for $1 \leq i \leq N$. Let R_i^T be defined by

$$R_i^T : H_0^1(\Omega_i) \to H_0^1(\Omega),$$
$$\boldsymbol{x} \mapsto \begin{cases} u(\boldsymbol{x}) & \text{if } \boldsymbol{x} \text{ belongs to } \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

For all $1 \leq i \leq N$, let $\widetilde{a}_i$ be the bilinear forms on $H_0^1(\Omega_i)$ defined by $\widetilde{a}_i(u,v) = a(R_i^T u, R_i^T v)$, i.e.

$$\widetilde{a}_i(u,v) = \int_{\Omega_i} \nabla u(\boldsymbol{x}) \cdot A(\boldsymbol{x}) \nabla v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

Let P_{ad} be the preconditioned Additive Schwarz operator defined by equation (2.1). Then the a-condition number $\kappa(P_{ad})$ of P_{ad} is bounded bounded by

$$\frac{\beta^2}{\alpha^2} C(N_K + 2),$$

where

$$C = C_1 + 2(1 + C_1)C_2$$
$$C_1 = \frac{1}{\tan\theta_{min}} \frac{1 + 2r/H_h}{1 - \big((2K+1) + (4K+1)r/H_h\big) r/H_h} 2K\left(\frac{C_p^2 H^2}{\pi r^2} + \pi\right)$$
$$C_2 = \left(2 + \frac{4}{3}\lambda_2^2 N_c (N_c - 1)^2 \frac{r^2}{\delta^2} \left(\left(\left(\frac{C_d^2 H^2}{r^2} + \frac{1}{4}\right)^{\frac{1}{4}} + \frac{C_d H}{\sqrt{2} r}\right)^4 - \frac{C_d^4 H^4}{4r^4} - \frac{1}{4} - \frac{C_d^2 H^2}{r^2}\right)\right).$$

Proof. Assumption 2.3 is satisfied with the matrix ε_{ij} set to 1, if $\Omega_i \cap \Omega_j \neq \emptyset$ and set to 0 otherwise. Therefore the spectral radius of the matrix $[\varepsilon_{ij}]$ is bounded by $N_K + 1$. Assumption 2.4 is satisfied by definition with the local stability parameter $\omega = 1$, and Assumption 2.2 is satisfied by Theorem 5.12, since A is uniformly coercive and uniformly bounded. Therefore, we have a stable decomposition whose constant is the C of Theorem 5.12 multiplied by $\frac{\beta^2}{\alpha^2}$. We apply then Theorem 2.5 to conclude. ∎

The bound of the condition does not depend on the number of subdomains and the lengths in the formulas always come in ratios, which means that the condition number stays bounded.

[8] It is possible to obtain a slightly sharper bound where $N_k + 1$ is replaced by N_c, see for example [13, Th. 4.1]

7. Conclusion. We have analyzed the Additive Schwarz preconditioned operator with a coarse mesh at the continuous level. We provided explicit estimates which show that the condition number is independent of the number of subdomains. This continuous estimate should be helpful to prove properties of the Additive Schwarz preconditioned operator when discretized by various consistent numerical methods for partial differential equations, as soon as the discretization error is small enough. In particular, the condition number estimate should not depend on the fine discretization.

Appendix A. The L^2 norm of the gradient in $P_1(\mathcal{T})$. Let ABC be a triangle, and let v_a, v_b, v_c in $\mathbb{R}$ be the values at the corners. There exists a unique affine mapping u defined over ABC, such that $u(A) = v_A$, $u(B) = v_B$ and $u(C) = v_C$. We want to compute $\int_{ABC} \|\nabla u\|^2$. Inside ABC, ∇u is a constant that satisfies the two equations

$$\nabla u \cdot (\boldsymbol{AB}) = v_B - v_A, \qquad \nabla u \cdot (\boldsymbol{AC}) = v_C - v_A.$$

Hence, in a matrix formulation, we have

$$\begin{bmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{bmatrix} \nabla u = \begin{bmatrix} v_B - v_A \\ v_C - v_A \end{bmatrix}.$$

The inverse of this matrix is readily computed, and we obtain

$$\begin{aligned} \nabla u &= \frac{\begin{bmatrix} y_C - y_A & -(y_B - y_A) \\ -(x_C - x_A) & x_B - x_A \end{bmatrix} \begin{bmatrix} v_B - v_A \\ v_C - v_A \end{bmatrix}}{\begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}} \\ &= \frac{1}{2\mathcal{S}(ABC)} \begin{bmatrix} (y_C - y_A)(v_B - v_A) - (y_B - y_A)(v_C - v_A) \\ -(x_C - x_A)(v_B - v_A) + (x_B - x_A)(v_C - v_A) \end{bmatrix}, \end{aligned}$$

where $\mathcal{S}(ABC)$ is the area of triangle ABC. Therefore, we obtain

$$\begin{aligned} \|\nabla u\|^2_{\mathbb{R}^2} &= \frac{\|\boldsymbol{AC}\|^2 (v_B - v_A)^2 + \|\boldsymbol{AB}\|^2 (v_C - v_A)^2 - 2(\boldsymbol{AB}, \boldsymbol{AC})(v_B - v_A)(v_C - v_A)}{4\mathcal{S}(ABC)^2} \\ &= \frac{(\boldsymbol{CB}, \boldsymbol{CA})(v_B - v_A)^2 + (\boldsymbol{BA}, \boldsymbol{BC})(v_C - v_A)^2 + (\boldsymbol{AB}, \boldsymbol{AC})(v_B - v_C)^2}{4\mathcal{S}(ABC)^2}, \end{aligned}$$

since $2(v_B - v_A)(v_c - v_A) = (v_B - v_A)^2 + (v_C - v_A)^2 - (v_B - v_C)^2$. We thus have

$$\|\nabla u\|^2_{L^2(ABC)} = \mathcal{S}(ABC) \|\nabla u\|^2_{\mathbb{R}^2} = \frac{(v_B - v_A)^2}{2\tan(\theta_C)} + \frac{(v_A - v_C)^2}{2\tan(\theta_B)} + \frac{(v_C - v_B)^2}{2\tan(\theta_A)}. \quad \text{(A.1)}$$

REFERENCES

[1] Robert A. Adams and John F. Fournie. *Sobolev Spaces*, volume 140 of *Pure and Applied Mathematics Series*. Academic Press, second edition, 2003.

[2] Xiao-Chuan Cai. An additive Schwarz algorithm for nonselfadjoint elliptic equations. In Tony Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 232–244. SIAM, Philadelphia, PA, 1990.

[3] Xiao-Chuan Cai. Additive Schwarz algorithms for parabolic convection-diffusion equations. *Numer. Math.*, 60(1):41–61, 1991.

[4] Xiao-Chuan Cai. Multiplicative Schwarz methods for parabolic problems. *SIAM J. Sci Comput.*, 15(3):587–603, 1994.

[5] Xiao-Chuan Cai and Olof Widlund. Multiplicative Schwarz algorithms for some nonsymmetric and indefinite problems. *SIAM J. Numer. Anal.*, 30(4):936–952, August 1993.

[6] Claudio Canuto and Daniele Funaro. The Schwarz algorithm for spectral methods. *SIAM J. Numer. Anal.*, 25(1):24–40, 1988.

[7] Tony F. Chan and Tarek P. Mathew. Domain decomposition algorithms. In *Acta Numerica 1994*, pages 61–143. Cambridge University Press, 1994.

[8] Tony F. Chan, Barry F. Smith, and Jun Zou. Overlapping Schwarz methods on unstructured meshes using non-matching coarse grids. *Numer. Math.*, 73(2):149–167, 1996.

[9] Tony F. Chan and Jun Zou. Additive Schwarz domain decomposition methods for elliptic problems on unstructured meshes. *Numer. Algorithms*, 8(2–4):329–346, 1994.

[10] Lawrence C. Cowsar. Dual variable Schwarz methods for mixed finite elements. Technical Report TR93-09, Department of Mathematical Sciences, Rice University, March 1993.

[11] Maksymilian Dryja. Additive Schwarz methods for elliptic mortar finite element problems. In K. Malanowski, Z. Nahorski, and M. Peszynska, editors, *Modeling and Optimization of Distributed Parameter Systems with Applications to Engineering*. IFIP, Chapman & Hall, London, 1996.

[12] Maksymilian Dryja and Olof B. Widlund. An additive variant of the Schwarz alternating method for the case of many subregions. Technical Report 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute, 1987.

[13] Maksymilian Dryja and Olof B. Widlund. Domain decomposition algorithms with small overlap. *SIAM J. Sci.Comput.*, 15(3):604–620, May 1994.

[14] Richard E. Ewing and Junping Wang. Analysis of the Schwarz algorithm for mixed finite element methods. *RAIRO Mathematical Modelling and Numerical Analysis*, 26(6):739–756, 1992.

[15] Xiaobing Feng and Ohannes A. Karakashian. Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.*, 39(4):1343–1365, 2002.

[16] Andreas Frommer and Daniel B. Szyld. Weighted max norms, splittings, and overlapping additive Schwarz iterations. *Numerische Mathematik*, 83:259–278, 1999.

[17] Martin J. Gander. Optimized Schwarz methods. *SIAM J. Numer. Anal.*, 44(2):699–731, 2006.

[18] Martin J. Gander. Schwarz methods over the course of time. *Electronic Transactions on Numerical Analysis*, 31:228–255, 2008.

[19] Mario Arlindo Casarin Jr. Quasi-optimal Schwarz methods for the conforming spectral element discretization. *SIAM J. Numer. Anal.*, 34(6):2482–2502, 1997.

[20] Pierre-Louis Lions. On the Schwarz alternating method. I. In Roland Glowinski, Gene H. Golub, Gérard A. Meurant, and Jacques Périaux, editors, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 1–42, Philadelphia, PA, 1988. SIAM.

[21] Pierre-Louis Lions. On the Schwarz alternating method. II. In Tony Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Domain Decomposition Methods*, pages 47–70, Philadelphia, PA, 1989. SIAM.

[22] Pierre-Louis Lions. On the Schwarz alternating method. III: a variant for nonoverlapping subdomains. In Tony F. Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations , held in Houston, Texas, March 20-22, 1989*, Philadelphia, PA, 1990. SIAM.

[23] Tarek P. Mathew. Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part I: Algorithms and numerical results. *Numer. Math.*, 65(4):445–468, 1993.

[24] Tarek P. Mathew. Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part II: Theory. *Numer. Math.*, 65(4):469–492, 1993.

[25] Luca F. Pavarino. Additive Schwarz methods for the p-version finite element method. *Numer. Math.*, 66(4):493–515, 1994.

[26] Luca F. Pavarino. Schwarz methods with local refinement for the p-version finite element method. *Numer. Math.*, 69(2):185–211, 1994.

[27] Alfio Quarteroni and Alberto Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford Science Publications, 1999.

[28] H. A. Schwarz. Über einen Grenzübergang durch alternierendes Verfahren. *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, 15:272–286, May 1870.

[29] Barry F. Smith, Petter E. Bjørstad, and William Gropp. *Domain Decomposition: Parallel*

Multilevel Methods for Elliptic Partial Differential Equations. Cambridge University Press, 1996.

[30] Roger Temam. *Infinite Dimensional Dynamical Systems*, volume 68 of *Applied Mathematical Sciences*. Springer Verlag, second edition, 1997.

[31] Andrea Toselli and Olof Widlund. *Domain Decomposition Methods - Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer, 2004.

[32] Jinchao Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review*, 34(4):581–613, December 1992.