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A class of implicit variational inequalities and applications to frictional contact

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Abstract

This paper deals with the mathematical and numerical analysis of a class of abstract implicit evolution variational inequalities. The results obtained here can be applied to a large variety of quasistatic contact problems in linear elasticity, including unilateral contact or normal compliance conditions with friction. In particular, a quasistatic unilateral contact problem with nonlocal friction is considered. An algorithm is derived and some numerical examples are presented.

1 Introduction

This paper is concerned with the mathematical analysis and the approximation of a class of abstract implicit evolution variational inequalities which

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constitutes a generalization of variational inequalities related to various quasistatic contact problems and of some parabolic variational inequalities of second kind, see, e.g., [1]-[5] and references therein.

The static contact problems have been extensively studied and their analysis has contributed to considerably developing the mathematical and numerical analysis of elliptic variational inequalities. As is well known, in the case of frictional contact, this static formulation does not have any mechanical significance, except in some very special cases.

Nevertheless, the incremental formulation obtained by performing time discretization of quasistatic or dynamic problems is similar to a sequence of static problems, which facilitates the mathematical and numerical analysis.

In the present work, a new system of evolution variational inequalities, representing a unified approach to quasistatic contact problems in elasticity, is analyzed. In Section 2, using an implicit time discretization scheme and some estimates, strong convergence is obtained and an existence result is established.

In Section 3, numerical analysis is carried out on the general problem. Convergence results are proved using a method based on an semi-discrete internal approximation and an implicit time discretization scheme.

In Section 4, we apply the results of Sections 2 and 3 to a quasistatic unilateral contact problem with nonlocal friction in linear elasticity.

This particular problem concerns the contact between an elastic body and a rigid support but a large variety of contact conditions can be easily analyzed using our general results, as, for example, the unilateral or bilateral contact with nonlocal friction between two elastic bodies or the frictional contact with normal compliance conditions as well as the corresponding frictionless cases.

In Section 5, some algorithms and numerical results are presented and the effects of space and time discretizations are described on a test example.

2 A general system of evolution inequalities

Let $(V, (\cdot, \cdot))$ be a real Hilbert space with the associated norm $\|\cdot\|$ and let $(H, (\cdot, \cdot)_H)$ be a real Hilbert space with the associated norm $\|\cdot\|_H$. Let $a : V \times V \rightarrow \mathbb{R}$ be a bilinear and symmetric form on V . We assume that a is V -elliptic and continuous i.e.

$$a(v, v) \geq \alpha \|v\|^2, \quad |a(v, w)| \leq M \|v\| \|w\| \quad \forall v, w \in V, \quad (2.1)$$

where α, M are positive constants. Let K be a closed convex cone contained in V with its vertex at 0 and let $(K(g))_{g \in V}$ be a family of nonempty closed

convex subsets of K satisfying the following conditions: $0 \in K(0)$ and

$$\text{if } g_n \rightarrow g \text{ in } V, v_n \in K(g_n) \text{ and } v_n \rightarrow v \text{ in } V \text{ then } v \in K(g). \quad (2.2)$$

We assume that for all $g \in V$ there exists an operator $\beta(g, \cdot) : K(g) \rightarrow H$ such that

$$\begin{aligned} &\text{if } g_n \rightarrow g \text{ in } V, v_n \in K(g_n) \text{ and } v_n \rightarrow v \text{ in } V \\ &\text{then } \beta(g_n, v_n) \rightarrow \beta(g, v) \text{ in } H \end{aligned} \quad (2.3)$$

and for all $g \in V, v \in K(g)$,

$$\|\beta(g, v)\|_H \leq k_1(\|g\| + \|v\|), \quad (2.4)$$

where k_1 is a positive constant.

For all $g \in V$, let $j(g, \cdot, \cdot) : K(g) \times V \rightarrow \mathbb{R}$ be a functional satisfying the following conditions:

$$j(g, v, \cdot) \text{ is sub-additive for all } g \in V, v \in K(g), \text{ that is} \quad (2.5)$$

$$j(g, v, w_1 + w_2) \leq j(g, v, w_1) + j(g, v, w_2) \quad \forall g, w_{1,2} \in V, v \in K(g),$$

$$j(g, v, \cdot) \text{ is positively homogeneous for all } g \in V, v \in K(g), \quad (2.6)$$

that is $j(g, v, \theta w) = \theta j(g, v, w) \quad \forall g, w \in V, v \in K(g), \theta \geq 0$,

$$j(0, 0, w) = 0 \quad \forall w \in V, \quad (2.7)$$

and

$$\begin{aligned} &|j(g_1, v_1, w_2) + j(g_2, v_2, w_1) - j(g_1, v_1, w_1) - j(g_2, v_2, w_2)| \\ &\leq k_2(\|g_1 - g_2\| + \|\beta(g_1 - g_2, v_1 - v_2)\|_H)\|w_1 - w_2\| \\ &\quad \forall g_i, w_i \in V, v_i \in K(g_i), i = 1, 2, \end{aligned} \quad (2.8)$$

where k_2 is a positive constant.

We assume that k_1 and k_2 satisfy the following condition:

$$k_1 k_2 < \alpha. \quad (2.9)$$

For all $g \in V$, we consider a functional $b(g, \cdot, \cdot) : K(g) \times V \rightarrow \mathbb{R}$ which satisfies the following conditions:

$$\forall g \in V, v \in K(g), b(g, v, \cdot) \text{ is linear on } V, \quad (2.10)$$

$$\begin{aligned} &\text{if } g_n \rightarrow g \text{ in } V, v_n \in K(g_n), v_n \rightarrow v \text{ in } V \text{ and } w_n \rightarrow w \text{ in } V \\ &\text{then } b(g_n, v_n, w_n) \rightarrow b(g, v, w), \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} |b(g_1, v_1, w) - b(g_2, v_2, w)| &\leq k_3(\|g_1 - g_2\| + \|v_1 - v_2\|)\|w\| \\ \forall g_i, w \in V, v_i \in K(g_i), i = 1, 2, \end{aligned} \quad (2.12)$$

where k_3 is a positive constant. From the above properties of a , j and K and by a classical argument, it follows that for all $g \in V$, $d \in K$, $w \in K(g)$ the elliptic variational inequality

$$u \in K \quad a(u, v - u) + j(g, w, v - d) - j(g, w, u - d) \geq 0 \quad \forall v \in K$$

has a unique solution, so that we can define the mapping $S_{g,d} : K(g) \rightarrow K$ as $S_{g,d}(w) = u$. We assume that for all $g \in V$, $d \in K$

$$K(g) \text{ is stable under } S_{g,d} \text{ i.e. } S_{g,d}(K(g)) \subset K(g). \quad (2.13)$$

For all $g \in V$, $d \in K$, we consider the following problems:

$$(\tilde{P}) \quad \begin{cases} u \in K(g) & a(u, v - u) + j(g, u, v - d) - j(g, u, u - d) \\ & \geq b(g, u, v - u) \quad \forall v \in V, \\ b(g, u, z - u) \geq 0 & \forall z \in K, \end{cases}$$

and

$$(\tilde{Q}) \quad u \in K(g) \quad a(u, v - u) + j(g, u, v - d) - j(g, u, u - d) \geq 0 \quad \forall v \in K.$$

We assume that

$$\text{if } u \text{ is a solution of } (\tilde{Q}), \text{ then } u \text{ is a solution of } (\tilde{P}). \quad (2.14)$$

Remark 2.1 *If u satisfies (\tilde{P}) , then u obviously satisfies (\tilde{Q}) .*

Let $f \in W^{1,2}(0, T; V)$ be given and $u_0 \in K(f(0))$ be the unique solution of the following implicit elliptic variational inequality (see, e.g., [6]):

$$a(u_0, w - u_0) + j(f(0), u_0, w) - j(f(0), u_0, u_0) \geq 0 \quad \forall w \in K. \quad (2.15)$$

Indeed, using the hypotheses (2.1), (2.4), (2.8) and (2.13), it follows that $S_{f(0),0} : K(f(0)) \rightarrow K(f(0))$ is a contraction if the condition (2.9) holds. Therefore the variational inequality (2.15) has a unique solution.

Remark 2.2 *i) Since, by (2.6) $j(\cdot, \cdot, 0) = 0$, from (2.8) for $w_2 = 0$, $w_1 = w$ and from (2.4) it follows that*

$$\begin{aligned}
& |j(g_2, v_2, w) - j(g_1, v_1, w)| \\
& \leq k_2(\|g_1 - g_2\| + \|\beta(g_1 - g_2, v_1 - v_2)\|_H)\|w\| \\
& \leq ((k_1 + 1)k_2\|g_1 - g_2\| + k_1k_2\|v_1 - v_2\|)\|w\| \\
& \quad \forall g_i, w \in V, v_i \in K(g_i), i = 1, 2.
\end{aligned} \tag{2.16}$$

ii) As j satisfies (2.5), (2.6), $j(g, v, \cdot)$ is convex and from (2.5), (2.7), (2.16) for $g_2 = g$, $v_2 = v$, $g_1 = v_1 = 0$, it results that $j(g, v, \cdot)$ is Lipschitz continuous on V for all $g \in V$ and for all $v \in K(g)$.

iii) Using (2.3) and (2.16), as by (2.4) $\beta(0, 0) = 0$, we have the following property

$$\begin{aligned}
& \text{if } g_n \rightarrow g \text{ in } V, v_n \in K(g_n) \text{ and } v_n \rightarrow v \text{ in } V \\
& \text{then } j(g_n, v_n, w) \rightarrow j(g, v, w) \quad \forall w \in V.
\end{aligned} \tag{2.17}$$

We consider the following evolution system of coupled variational inequalities.

Problem P: Find $u \in W^{1,2}(0, T; V)$ such that

$$(P) \quad \begin{cases} u(0) = u_0, u(t) \in K(f(t)) \quad \forall t \in]0, T[, \\ a(u(t), v - \dot{u}(t)) + j(f(t), u(t), v) - j(f(t), u(t), \dot{u}(t)) \\ \quad \geq b(f(t), u(t), v - \dot{u}(t)) \quad \forall v \in V \text{ a.e. on }]0, T[, \\ b(f(t), u(t), z - u(t)) \geq 0 \quad \forall z \in K, \forall t \in]0, T[. \end{cases}$$

In order to prove that there exists a solution to problem P, we consider an incremental formulation obtained by using an implicit time discretization scheme.

For $n \in \mathbb{N}^*$, we set $\Delta t := T/n$ and $t_i := i \Delta t$, $i = 0, 1, \dots, n$. If θ is a continuous function of $t \in [0, T]$ valued in some vector space, we use the notations $\theta^i := \theta(t_i)$ unless $\theta = u$, and if η^i , $\forall i \in \{0, 1, \dots, n\}$, are elements of some vector space, then we set

$$\partial \eta^i := \frac{\eta^{i+1} - \eta^i}{\Delta t} \quad \forall i \in \{0, 1, \dots, n-1\}.$$

Let us also denote $K^i := K(f(t_i))$, $\forall i \in \{0, 1, \dots, n\}$, and $u^0 := u_0$. We then approximate (P) using the following sequence of incremental problems $(P_n^i)_{i=0,1,\dots,n-1}$.

Problem P_n^i : Find $u^{i+1} \in K^{i+1}$ such that

$$(P_n^i) \begin{cases} a(u^{i+1}, v - \partial u^i) + j(f^{i+1}, u^{i+1}, v) - j(f^{i+1}, u^{i+1}, \partial u^i) \\ \geq b(f^{i+1}, u^{i+1}, v - \partial u^i) \quad \forall v \in V, \\ b(f^{i+1}, u^{i+1}, z - u^{i+1}) \geq 0 \quad \forall z \in K. \end{cases}$$

Remark 2.3 *i)* Since K is a cone, the solutions u of (P) and u^{i+1} of (P_n^i) obviously satisfy

$$b(f(t), u(t), u(t)) = 0 \quad \text{on } [0, T]$$

and

$$b(f^{i+1}, u^{i+1}, u^{i+1}) = 0, \quad \forall i \in \{0, 1, \dots, n-1\}.$$

ii) It follows also that

$$b(f(t), u(t), \dot{u}(t)) = 0 \quad \text{a.e. on } [0, T], \quad (2.18)$$

which can be obtained simply by the fact that since u satisfies the second inequality of (P) we have, for all $t \in]0, T[$ and for all $\Delta t > 0$, sufficiently small,

$$b(f(t), u(t), \frac{u(t + \Delta t) - u(t)}{\Delta t}) \geq 0,$$

and

$$b(f(t), u(t), \frac{u(t - \Delta t) - u(t)}{-\Delta t}) \leq 0.$$

From (2.14) and Remark 2.1 for $g = f^{i+1}$ and $d = u^i$, it follows that the problem P_n^i is equivalent to the following variational inequality: find $u^{i+1} \in K^{i+1}$ such that

$$(Q_n^i) \begin{cases} a(u^{i+1}, w - u^{i+1}) + j(f^{i+1}, u^{i+1}, w - u^i) \\ -j(f^{i+1}, u^{i+1}, u^{i+1} - u^i) \geq 0 \quad \forall w \in K. \end{cases}$$

From the hypotheses (2.1), (2.4), (2.8), (2.9) and (2.13), it follows that $S_{f^{i+1}, u^i} : K^{i+1} \rightarrow K^{i+1}$ is a contraction. Therefore (Q_n^i) has a unique solution.

A straightforward computation and the fact that f is absolutely continuous give us the following estimates.

Lemma 2.1 *Let $u^{i+1} \in K^{i+1}$ be the solution of (Q_n^i) , $i \in \{0, 1, \dots, n-1\}$. Then*

$$\|u^0\| \leq C_0, \quad \|u^{i+1}\| \leq C_0, \quad (2.19)$$

$$\|u^{i+1} - u^i\| \leq C_1 \int_{t_i}^{t_{i+1}} \|\dot{f}(\tau)\| d\tau, \quad (2.20)$$

$$\sum_{i=0}^{n-1} \|u^{i+1} - u^i\|^2 \leq C_1^2 \Delta t \int_0^T \|\dot{f}(\tau)\|^2 d\tau, \quad (2.21)$$

where

$$C_0 = \frac{(k_1 + 1)k_2 \|f\|_{C([0,T];V)}}{\alpha - k_1 k_2}, \quad C_1 = \frac{(k_1 + 1)k_2}{\alpha - k_1 k_2}. \quad (2.22)$$

Now, if we define

$$\left\{ \begin{array}{l} u_n(0) = \hat{u}_n(0) = u^0, \quad f_n(0) = f^0 \quad \text{and} \\ \forall i \in \{0, 1, \dots, n-1\}, \quad \forall t \in]t_i, t_{i+1}], \\ u_n(t) = u^{i+1}, \\ \hat{u}_n(t) = u^i + (t - t_i) \partial u^i, \\ f_n(t) = f^{i+1}, \end{array} \right. \quad (2.23)$$

then the functions $u_n, f_n \in L^2(0, T; V)$ and $\hat{u}_n \in W^{1,2}(0, T; V)$ satisfy the following incremental formulation for all $t \in [0, T]$:

$$(P_n) \left\{ \begin{array}{l} u_n(t) \in K(f_n(t)), \quad a(u_n(t), v - \frac{d}{dt} \hat{u}_n(t)) + j(f_n(t), u_n(t), v) \\ -j(f_n(t), u_n(t), \frac{d}{dt} \hat{u}_n(t)) \geq b(f_n(t), u_n(t), v - \frac{d}{dt} \hat{u}_n(t)) \quad \forall v \in V, \\ b(f_n(t), u_n(t), z - u_n(t)) \geq 0 \quad \forall z \in K. \end{array} \right.$$

Remark 2.4 From (2.23) and Remark 2.3 i), it follows that

$$\begin{aligned} b(f_n(t), u_n(t), v - \frac{d}{dt} \hat{u}_n(t)) &= b(f^{i+1}, u^{i+1}, v - \partial u^i) \\ &\geq b(f_n(t), u_n(t), v) \quad \forall t \in]t_i, t_{i+1}[, \quad \forall v \in V. \end{aligned}$$

Lemma 2.2 For all $t \in [0, T]$ we have

$$\|u_n(t)\| \leq C_0, \quad (2.24)$$

and for all $s, t \in [0, T]$, $s < t$ we have

$$\|u_n(s) - u_n(t)\| \leq C_1 \int_s^{\min\{t+\Delta t, T\}} \|\dot{f}(\tau)\| d\tau, \quad (2.25)$$

$$\|u_n - \hat{u}_n\|_{L^2(0,T;V)} \leq C_1 \|\dot{f}\|_{L^2(0,T;V)} \frac{\Delta t}{\sqrt{3}}, \quad (2.26)$$

$$\left\| \frac{d}{dt} \hat{u}_n \right\|_{L^2(0,T;V)} \leq C_1 \|\dot{f}\|_{L^2(0,T;V)}. \quad (2.27)$$

Proof.

Inequalities (2.24), (2.25) follow from definitions (2.23) and estimates (2.19), (2.20). In order to obtain (2.26) and (2.27), we use the relations

$$\|u_n - \hat{u}_n\|_{L^2(0,T;V)} = \frac{\Delta t}{\sqrt{3}} \left\| \frac{d}{dt} \hat{u}_n \right\|_{L^2(0,T;V)} = \sqrt{\frac{\Delta t}{3} \sum_{i=0}^{n-1} \|u^{i+1} - u^i\|^2}$$

and estimate (2.21). \square

Lemma 2.3 *There exists a subsequence of $(u_n)_n$, denoted by $(u_{n_p})_p$, and an element $u \in W^{1,2}(0, T; V)$ such that*

$$u_{n_p}(t) \rightharpoonup u(t) \text{ in } V \quad \forall t \in [0, T], \quad (2.28)$$

$$\hat{u}_{n_p} \rightharpoonup u \text{ in } W^{1,2}(0, T; V). \quad (2.29)$$

In addition, for all $s \in [0, T]$, we have

$$\liminf_{p \rightarrow \infty} \int_0^s a(u_{n_p}(t), \frac{d}{dt} \hat{u}_{n_p}(t)) dt \geq \int_0^s a(u(t), \dot{u}(t)) dt, \quad (2.30)$$

$$\liminf_{p \rightarrow \infty} \int_0^s j(f_{n_p}(t), u_{n_p}(t), \frac{d}{dt} \hat{u}_{n_p}(t)) dt \geq \int_0^s j(f(t), u(t), \dot{u}(t)) dt. \quad (2.31)$$

Proof.

Applying a diagonal process (see, e.g., [5]), it follows from (2.24) and (2.25) that we can extract a subsequence $(u_{n_k})_k$ of $(u_n)_n$ such that $u_{n_k}(t) \rightharpoonup u(t)$ in $V \quad \forall t \in [0, T]$, hence $u_{n_k} \rightharpoonup u$ in $L^2(0, T; V)$. Thus from (2.26) and (2.27) we deduce that $u \in W^{1,2}(0, T; V)$ and there exists a subsequence of $(\hat{u}_n)_n$ such that (2.29) holds. Since $f, u \in W^{1,2}(0, T; V) \subset C([0, T]; V)$ and $f_n(t) \rightarrow f(t)$ in $V \quad \forall t \in [0, T]$, from (2.2) it results that $u(t) \in K(f(t)) \quad \forall t \in [0, T]$.

We shall omit the subscript p from now on. In order to prove (2.30) and (2.31), let $s \in]0, T]$. Using the definitions (2.23) and the properties of a , we

obtain

$$\begin{aligned} \int_0^s a(u_n(t), \frac{d}{dt} \hat{u}_n(t)) dt &\geq \frac{1}{2} a(u_n(s), u_n(s)) - \frac{1}{2} a(u_0, u_0) \\ &\quad - \int_s^{t_{i+1}} a(u_n(t), \frac{d}{dt} \hat{u}_n(t)) dt, \end{aligned} \quad (2.32)$$

where $i \in \{0, 1, \dots, n-1\}$ is such that $s \in [t_i, t_{i+1}]$. Therefore, by going to the limit, one obtains (2.30).

Next, since by Remark 2.2 ii) $j(f(t), u(t), \cdot)$ is convex lower semicontinuous for all $t \in [0, T]$, the mapping $v \mapsto \int_0^s j(f(t), u(t), v(t)) dt$ is convex and lower semicontinuous on $L^2(0, T; V)$ (see, e.g., [7]). Thus

$$\liminf_{n \rightarrow \infty} \int_0^s j(f(t), u(t), \frac{d}{dt} \hat{u}_n(t)) dt \geq \int_0^s j(f(t), u(t), \dot{u}(t)) dt. \quad (2.33)$$

Using (2.16) and (2.27) we have

$$\begin{aligned} &| \int_0^s (j(f_n(t), u_n(t), \frac{d}{dt} \hat{u}_n(t)) - j(f(t), u(t), \frac{d}{dt} \hat{u}_n(t))) dt | \\ &\leq C_1 \|\dot{f}\|_{L^2(0, T; V)} \left(\int_0^s (k_2 \|\beta(f_n(t) - f(t), u_n(t) - u(t))\|_H \right. \\ &\quad \left. + k_2 \|f_n(t) - f(t)\|^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

from which by using (2.3) it follows that

$$\lim_{n \rightarrow \infty} \int_0^s (j(f_n(t), u_n(t), \frac{d}{dt} \hat{u}_n(t)) - j(f(t), u(t), \frac{d}{dt} \hat{u}_n(t))) dt = 0. \quad (2.34)$$

Relations (2.33) and (2.34) imply (2.31). \square

We now prove the following strong convergence and existence result.

Theorem 2.1 *Under the assumptions (2.1)-(2.15) there exists a subsequence of $(u_n)_n$ such that*

$$u_n(t) \rightarrow u(t) \quad \text{in } V \quad \forall t \in [0, T], \quad (2.35)$$

$$\hat{u}_n \rightarrow u \quad \text{in } L^2(0, T; V), \quad (2.36)$$

$$\frac{d}{dt}\hat{u}_n \rightharpoonup \dot{u} \quad \text{in } L^2(0, T; V), \quad (2.37)$$

where $u \in W^{1,2}(0, T; V)$ is a solution of problem P.

Proof.

Let $(u_n)_n$ be the subsequence given by lemma 2.3. We shall prove that its limit u is a solution of problem P. Let $s \in [0, T]$. Integrating both sides in the first inequality of (P_n) over $[0, s]$ and passing to the limit, by the relations (2.1), (2.17), Remark 2.4 and lemma 2.3 it follows that for all $v \in L^2(0, T; V)$

$$\begin{aligned} & \int_0^s a(u(t), v(t) - \dot{u}(t)) dt + \int_0^s j(f(t), u(t), v(t)) dt \\ & - \int_0^s j(f(t), u(t), \dot{u}(t)) dt \geq \int_0^s b(f(t), u(t), v(t)) dt. \end{aligned} \quad (2.38)$$

Now, by writing the inequality corresponding to (Q_n^i) satisfied by u_n and passing to the limit, we obtain that for all $t \in [0, T]$

$$a(u(t), v - u(t)) + j(f(t), u(t), v - u(t)) \geq 0 \quad \forall v \in K, \quad (2.39)$$

where we have used that $u_n(t - \Delta t) \rightharpoonup u(t)$ in $V \quad \forall t \in]0, T]$ and (2.15).

From (2.14), taking $d = u$, it follows that u satisfies

$$b(f(t), u(t), z - u(t)) \geq 0 \quad \forall z \in K, \quad \forall t \in [0, T]. \quad (2.40)$$

Finally, by Remark 2.3 and Lebesgue's theorem, we obtain that u is a solution of problem P.

In order to show the strong convergences (2.35), (2.36) (see also [8]), we first prove that

$$\lim_{n \rightarrow \infty} \int_0^s a(u_n(t), \frac{d}{dt}\hat{u}_n(t)) dt = \int_0^s a(u(t), \dot{u}(t)) dt \quad \forall s \in [0, T]. \quad (2.41)$$

On one hand, taking $v = 0$ in (P_n) , $v = 0$, $v = 2\dot{u}(t)$ in (2.38) and using (2.17), (2.11) and (2.18), we obtain

$$\liminf_{n \rightarrow \infty} \int_0^s a(u_n(t), \frac{d}{dt}\hat{u}_n(t)) dt = \int_0^s a(u(t), \dot{u}(t)) dt \quad (2.42)$$

and

$$\liminf_{n \rightarrow \infty} \int_0^s j(f_n(t), u_n(t), \frac{d}{dt} \hat{u}_n(t)) dt = \int_0^s j(f(t), u(t), \dot{u}(t)) dt. \quad (2.43)$$

On the other hand, taking $v = \dot{u}(t)$ in (P_n) we get

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_0^s a(u_n(t), \frac{d}{dt} \hat{u}_n(t)) dt \leq \lim_{n \rightarrow \infty} \int_0^s a(u_n(t), \dot{u}(t)) dt \\ & + \lim_{n \rightarrow \infty} \int_0^s j(f_n(t), u_n(t), \dot{u}(t)) dt - \liminf_{n \rightarrow \infty} \int_0^s j(f_n(t), u_n(t), \frac{d}{dt} \hat{u}_n(t)) dt \\ & - \lim_{n \rightarrow \infty} \int_0^s b(f_n(t), u_n(t), \dot{u}(t)) dt = \int_0^s a(u(t), \dot{u}(t)) dt, \end{aligned}$$

so that from (2.42), (2.43) the convergence (2.41) follows, which implies that

$$\lim_{n \rightarrow \infty} a(u_n(s), u_n(s)) = a(u(s), u(s)) \quad \forall s \in [0, T].$$

Therefore, using the V-ellipticity of a and (2.26), we obtain the strong convergences (2.35), (2.36) and the theorem is proved. \square

3 Approximation and numerical analysis

This section deals with the discretization of problem P. We shall prove a convergence result for a method based on an internal approximation and a backward difference scheme.

First we consider a semi-discrete approximation of (P) . For a positive parameter h converging to 0, let $(V_h)_h$ be a family of finite-dimensional subspaces of V and let $(K_h)_h$ be a family of closed convex cones with their vertices at 0 such that $K_h \subset K$ and $(K_h)_h$ is an internal approximation of K (see, e.g., [1]), i.e.

$$\forall v \in K, \exists v_h \in K_h \text{ such that } v_h \rightarrow v \text{ in } V. \quad (3.1)$$

From the properties of a , j and K_h , it follows that for all $g \in V$, $d_h \in K_h$, $w_h \in K_h \cap K(g)$, the elliptic variational inequality $u_h \in K_h$

$$a(u_h, v_h - u_h) + j(g, w_h, v_h - d_h) - j(g, w_h, u_h - d_h) \geq 0 \quad \forall v_h \in K_h$$

has a unique solution. Hence we can define a mapping $S_{g,d_h}^h : K_h \cap K(g) \rightarrow K_h$ by $S_{g,d_h}^h(w_h) = u_h$. We suppose that for all $g \in V$, $d_h \in K_h$

$$S_{g,d_h}^h(K_h \cap K(g)) \subset K_h \cap K(g). \quad (3.2)$$

For all $g \in V$, $d_h \in K_h$, we consider the following problems:

$$(\tilde{P}_h) \begin{cases} u_h \in K_h \cap K(g), & a(u_h, v_h - u_h) + j(g, u_h, v_h - d_h) \\ & -j(g, u_h, u_h - d_h) \geq b(g, u_h, v_h - u_h) \quad \forall v_h \in V_h, \\ & b(g, u_h, z_h - u_h) \geq 0 \quad \forall z_h \in K_h, \end{cases}$$

and

$$(\tilde{Q}_h) \begin{cases} u_h \in K_h \cap K(g), & a(u_h, v_h - u_h) + j(g, u_h, v_h - d_h) \\ & -j(g, u_h, u_h - d_h) \geq 0 \quad \forall v_h \in K_h. \end{cases}$$

We assume that

$$\text{if } u_h \text{ is a solution of } (\tilde{Q}_h), \text{ then } u_h \text{ is a solution of } (\tilde{P}_h). \quad (3.3)$$

Remark 3.1 *If u_h satisfies (\tilde{P}_h) , then u_h obviously satisfies (\tilde{Q}_h) .*

Let us now consider the following semi-discrete problem.

Problem P_h : Find $u_h \in W^{1,2}(0, T; V_h)$ such that

$$(P_h) \begin{cases} u_h(0) = r_h u_0, & u_h(t) \in K_h \cap K(f(t)) \quad \forall t \in]0, T[, \\ & a(u_h(t), v_h - \dot{u}_h(t)) + j(f(t), u_h(t), v_h) - j(f(t), u_h(t), \dot{u}_h(t)) \\ & \geq b(f(t), u_h(t), v_h - \dot{u}_h(t)) \quad \forall v_h \in V_h \text{ a.e. on }]0, T[, \\ & b(f(t), u_h(t), z_h - u_h(t)) \geq 0 \quad \forall z_h \in K_h, \quad \forall t \in]0, T[, \end{cases}$$

where $r_h u_0 \in K_h \cap K(f(0))$ is the unique solution of the variational inequality

$$\begin{aligned} & a(r_h u_0, w_h - r_h u_0) + j(f(0), r_h u_0, w_h) \\ & -j(f(0), r_h u_0, r_h u_0) \geq 0 \quad \forall w_h \in K_h. \end{aligned} \quad (3.4)$$

The full discretization of (P_h) is obtained by using an implicit scheme as in Section 2 for (P) . For $u_h^0 := r_h u_0$ and $i \in \{0, 1, \dots, n-1\}$, we define u_h^{i+1} as the solution of the following problem.

Problem P_{hn}^i : Find $u_h^{i+1} \in K_h^{i+1}$ such that

$$(P_{hn}^i) \begin{cases} a(u_h^{i+1}, v_h - \partial u_h^i) + j(f^{i+1}, u_h^{i+1}, v_h) - j(f^{i+1}, u_h^{i+1}, \partial u_h^i) \\ \geq b(f^{i+1}, u_h^{i+1}, v_h - \partial u_h^i) \quad \forall v_h \in V_h, \\ b(f^{i+1}, u_h^{i+1}, z_h - u_h^{i+1}) \geq 0 \quad \forall z_h \in K_h, \end{cases}$$

where $K_h^{i+1} := K_h \cap K(f^{i+1})$.

By (3.3) and Remark 3.1 it is easily seen that (P_{hn}^i) is equivalent to the following implicit variational inequality: find $u_h^{i+1} \in K_h^{i+1}$ such that

$$(Q_{hn}^i) \begin{cases} a(u_h^{i+1}, w_h - u_h^i) + j(f^{i+1}, u_h^{i+1}, w_h - u_h^i) \\ - j(f^{i+1}, u_h^{i+1}, u_h^{i+1} - u_h^i) \geq 0 \quad \forall w_h \in K_h. \end{cases}$$

From (2.1), (2.4), (2.8), (2.9) and (3.2) it follows that $S_{f^{i+1}, u_h^i}^h : K_h^{i+1} \rightarrow K_h^{i+1}$ is a contraction, so that (Q_{hn}^i) has a unique solution.

If we define the functions

$$\begin{cases} u_{hn}(0) = \hat{u}_{hn}(0) = r_h u_0 \quad \text{and} \\ \forall i \in \{0, 1, \dots, n-1\}, \quad \forall t \in]t_i, t_{i+1}], \\ u_{hn}(t) = u_h^{i+1}, \\ \hat{u}_{hn}(t) = u_h^i + (t - t_i) \partial u_h^i, \end{cases} \quad (3.5)$$

then we have the analogue to theorem 2.1 in the finite-dimensional case.

Theorem 3.1 *Assume that (2.1)-(2.12), (3.2)-(3.4) hold. Then there exists a subsequence of $(u_{hn})_n$, still denoted by $(u_{hn})_n$, such that*

$$u_{hn}(t) \rightarrow u_h(t) \quad \text{in } V \quad \forall t \in [0, T], \quad (3.6)$$

$$\hat{u}_{hn} \rightarrow u_h \quad \text{in } W^{1,2}(0, T; V), \quad (3.7)$$

where $u_h \in W^{1,2}(0, T; V_h)$ is a solution of (P_h) .

We now proceed to find a priori estimates for the solutions of (P_h) which are limits of subsequences of $(u_{hn})_n$.

Lemma 3.1 *Let u_h be a solution of (P_h) given by theorem 3.1. Then*

$$\|u_h(t)\| \leq C_0 \quad \forall t \in [0, T], \quad (3.8)$$

$$\|u_h(s) - u_h(t)\| \leq C_1 \int_s^t \|\dot{f}(\tau)\| d\tau \quad \forall s, t \in [0, T], \quad s < t, \quad (3.9)$$

$$\|u_h\|_{W^{1,2}(0, T; V)} \leq \sqrt{C_0^2 T + C_1^2 \|\dot{f}\|_{L^2(0, T; V)}^2}, \quad (3.10)$$

C_0, C_1 being defined by (2.22).

Proof.

Using the same argument as in the proof of lemma 2.2, we obtain

$$\begin{aligned} \|u_{hn}(t)\| &\leq C_0 \quad \forall t \in [0, T], \\ \|u_{hn}(s) - u_{hn}(t)\| &\leq C_1 \int_s^{\min\{t+\Delta t, T\}} \|\dot{f}(\tau)\| d\tau \quad \forall s, t \in [0, T], \quad s < t, \\ \|\hat{u}_{hn}\|_{W^{1,2}(0,T;V)}^2 &\leq C_0^2 T + C_1^2 \|\dot{f}\|_{L^2(0,T;V)}^2. \end{aligned}$$

Combining these results with (3.6) and (3.7), the estimates (3.8)-(3.10) follow. \square

We have the following convergence and existence result.

Theorem 3.2 *Under the assumptions (2.1)-(2.15), (3.1)-(3.4), there exists a subsequence of $(u_h)_h$ such that*

$$u_h(t) \rightarrow u(t) \quad \text{in } V \quad \forall t \in [0, T], \quad (3.11)$$

$$\dot{u}_h \rightharpoonup \dot{u} \quad \text{in } L^2(0, T; V), \quad (3.12)$$

where $u \in W^{1,2}(0, T; V)$ is a solution of (P).

Proof.

From lemma 3.1 it follows that there exists a subsequence of $(u_h)_h$ and an element $u \in W^{1,2}(0, T; V)$ such that

$$u_h(t) \rightharpoonup u(t) \quad \text{in } V \quad \forall t \in [0, T], \quad (3.13)$$

$$u_h \rightharpoonup u \quad \text{in } W^{1,2}(0, T; V). \quad (3.14)$$

As one can prove that

$$r_h u_0 \rightarrow u_0 \quad \text{in } V, \quad (3.15)$$

from (3.13) we obtain

$$\begin{aligned} \liminf_{h \rightarrow 0} \int_0^s a(u_h(t), \dot{u}_h(t)) dt &\geq \frac{1}{2} (\liminf_{h \rightarrow 0} a(u_h(s), u_h(s))) \\ &- \lim_{h \rightarrow 0} a(r_h u_0, r_h u_0) \geq \frac{1}{2} (a(u(s), u(s)) - a(u_0, u_0)) \\ &= \int_0^s a(u(t), \dot{u}(t)) dt \quad \forall s \in [0, T]. \end{aligned} \quad (3.16)$$

Following the argument in the proof of lemma 2.3, by (3.13) and (3.14) we have that for all $s \in [0, T]$

$$\liminf_{h \rightarrow 0} \int_0^s j(f(t), u_h(t), \dot{u}_h(t)) dt \geq \int_0^s j(f(t), u(t), \dot{u}(t)) dt. \quad (3.17)$$

Next we prove that u satisfies (2.38). In order to pass to the limit in (P_h) , for $v \in V$ we set $v_h = \pi_h v$, where $\pi_h v$ is the projection of v on V_h defined by $a(\pi_h v, w_h) = a(v, w_h) \quad \forall w_h \in V_h$. Thus if $v \in L^2(0, T; V)$ then $\pi_h v \in L^2(0, T; V_h)$ and $\pi_h v(t) \rightarrow v(t)$ in V a.e. on $[0, T]$, which imply

$$\lim_{h \rightarrow 0} \int_0^s j(f(t), u_h(t), \pi_h v(t)) dt = \int_0^s j(f(t), u(t), v(t)) dt \quad \forall s \in [0, T]$$

and

$$\lim_{h \rightarrow 0} \int_0^s b(f(t), u_h(t), \pi_h v(t)) dt = \int_0^s b(f(t), u(t), v(t)) dt \quad \forall s \in [0, T].$$

Since $b(f(t), u_h(t), \dot{u}_h(t)) = 0$ a.e. on $[0, T]$, by integrating (P_h) over $[0, s]$ for $v_h = \pi_h v$ and passing to the limit, we obtain that u satisfies (2.38).

Now, we prove the strong convergence (3.11). Using the same argument in the proof of theorem 2.1, by taking $v = 0$, $v = 2\dot{u}$ in (2.38), $v_h = 0$, $v_h = 2\dot{u}_h(t)$ in (P_h) and using (3.16), (3.17), we have that for all $s \in [0, T]$

$$\liminf_{h \rightarrow 0} \int_0^s a(u_h(t), \dot{u}_h(t)) dt = \int_0^s a(u(t), \dot{u}(t)) dt, \quad (3.18)$$

$$\liminf_{h \rightarrow 0} \int_0^s j(f(t), u_h(t), \dot{u}_h(t)) dt = \int_0^s j(f(t), u(t), \dot{u}(t)) dt. \quad (3.19)$$

and, by taking $v_h = \pi_h \dot{u}(t)$ in (P_h) , we obtain

$$\limsup_{h \rightarrow 0} \int_0^s a(u_h(t), \dot{u}_h(t)) dt \leq \int_0^s a(u(t), \dot{u}(t)) dt \quad \forall s \in [0, T].$$

From the last relation, (3.15) and (3.18), the convergence (3.11) follows.

Finally we prove that u satisfies (2.40). From (P_h) we deduce that for all $t \in [0, T]$

$$a(u_h(t), v_h - u_h(t)) + j(f(t), u_h(t), v_h - u_h(t)) \geq 0 \quad \forall v_h \in K_h. \quad (3.20)$$

Let $v \in K$ and, by (3.1), let $v_h \in K_h$ be such that $v_h \rightarrow v$ in V . By passing to the limit in (3.20) and using (3.11), (2.14) we obtain that $u \in K(f)$ and satisfies (2.40). \square

Using theorems 3.1 and 3.2, we obtain the following main approximation result.

Theorem 3.3 *Under the assumptions of theorem 3.2, there exists a subsequence of $(u_{h_n})_{h_n}$ such that*

$$u_{h_n}(t) \rightarrow u(t) \quad \text{in } V \quad \forall t \in [0, T], \quad (3.21)$$

$$\dot{u}_{h_n} \rightharpoonup \dot{u} \quad \text{in } L^2(0, T; V), \quad (3.22)$$

where $u \in W^{1,2}(0, T; V)$ is a solution of (P) .

4 Applications to contact mechanics

4.1 Nonlocal friction laws and regularizations: a short overview

The classical formulation of the unilateral contact problem in linear elasticity, in the static frictionless case, was given by A. Signorini, who proved also the uniqueness of solution, and the existence of solutions was studied by G. Fichera, J.L. Lions and G. Stampacchia.

If the unilateral (or Signorini) contact conditions with local Coulomb's friction law are considered, some serious physical and mathematical difficulties appear. From the physical point of view, the application of Coulomb's law to continuum mechanics is not obvious, see, e.g., [9]-[12] and references therein. Also, the static formulation is not realistic even if it can be useful in solving some intermediate problems like, for example, the incremental ones. Hence the evolution problem, in the quasistatic or the dynamic case, should be considered. From the mathematical point of view, as the normal component of the stress vector on the contact boundary is generally defined in the dual space $H^{-\frac{1}{2}}$, this lack of smoothness does not guarantee good convergence properties for the friction functional that appears in the variational formulation of the problem.

Thus, there were physical and mathematical arguments for considering nonlocal friction laws by using appropriate regularizations of the pointwise Coulomb friction conditions that preserve the multivalued character of the unilateral contact conditions and of the friction law.

In elastostatics, for a class of frictional contact problems, G. Duvaut [13], L. Demkowicz, J.T. Oden [14] and M. Cocou [15] considered a mollification of the normal stresses on the contact boundary that enable to prove the existence, and uniqueness if the coefficient of friction is sufficiently small, of the variational solutions.

For the quasistatic unilateral contact problem, the same regularization technique was extended by M. Cocou et al. [5] in order to prove, for a sufficiently small coefficient of friction, the existence of a solution. In that case, the simultaneous presence of the displacement and of the velocity fields is necessary to describe both the Signorini conditions and the physically consistent friction law. As was shown by P. Ballard [16], the uniqueness of the quasistatic solution does not hold, in general, even for an arbitrarily small coefficient of friction.

Another classical approach consisted in a regularization of the unilateral contact conditions by using a normal compliance model, see [11], [12] for its description and [8], [2] for existence results in the quasistatic case. The contact law is there described by a single-valued mapping instead a multi-valued one and small penetrations are allowed.

4.2 A quasistatic contact problem with nonlocal friction

Let us consider a linearly elastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a Lipschitz boundary $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are open and disjoint parts of Γ with $\text{meas}(\Gamma_1) > 0$.

We assume that in Ω a body force φ_1 is prescribed, on Γ_1 the displacement vector equals zero, a traction φ_2 is applied on Γ_2 and that the inertial effects are negligible.

The body is in contact with a rigid fixed support on Γ_3 and the displacements on this part of the boundary satisfy Signorini's contact conditions and are subjected to nonlocal friction when contact occurs.

Let us denote by $\boldsymbol{\sigma}$ the stress tensor, with the components $\sigma = (\sigma_{ij})$, by $\boldsymbol{\varepsilon}$ the infinitesimal strain tensor, with the components $\varepsilon = (\varepsilon_{ij})$, by \mathbf{u} the displacement field, with the components $u = (u_i)$, and by $\boldsymbol{\mathcal{E}}$ the elasticity tensor, with the components $\mathcal{E} = (a_{ijkl})$ satisfying the usual properties of symmetry and ellipticity.

We shall use the classical decomposition into the normal and tangential components of the displacement vector and stress vector $\mathbf{u} = u_N \mathbf{n} + \mathbf{u}_T$ with $u_N = \mathbf{u} \cdot \mathbf{n}$, $\boldsymbol{\sigma} \mathbf{n} = \sigma_N \mathbf{n} + \boldsymbol{\sigma}_T$ with $\sigma_N = (\boldsymbol{\sigma} \mathbf{n}) \cdot \mathbf{n}$, where \mathbf{n} is the outward normal unit vector to Γ with the components $n = (n_i)$.

The classical formulation of the quasistatic problem is as follows.

Problem \mathbf{P}_c : Find a displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ which satisfies the initial condition $\mathbf{u}(0) = \mathbf{u}_0$ in Ω and for all $t \in]0, T[$, the following equations and boundary conditions:

$$(P_c) \quad \left\{ \begin{array}{l} \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) = -\boldsymbol{\varphi}_1 \quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) = \boldsymbol{\mathcal{E}} \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \\ \boldsymbol{\sigma} \mathbf{n} = \boldsymbol{\varphi}_2 \quad \text{on } \Gamma_2, \\ u_N \leq 0, \quad \sigma_N \leq 0, \quad u_N \sigma_N = 0 \quad \text{on } \Gamma_3, \\ |\boldsymbol{\sigma}_T| \leq \mu |\mathcal{R} \sigma_N| \quad \text{on } \Gamma_3 \\ \text{and} \quad \left\{ \begin{array}{l} |\boldsymbol{\sigma}_T| < \mu |\mathcal{R} \sigma_N| \Rightarrow \dot{\mathbf{u}}_T = \mathbf{0}, \\ |\boldsymbol{\sigma}_T| = \mu |\mathcal{R} \sigma_N| \Rightarrow \exists \lambda \geq 0, \dot{\mathbf{u}}_T = -\lambda \boldsymbol{\sigma}_T, \end{array} \right. \end{array} \right.$$

where μ is the coefficient of friction and $\mathcal{R} \sigma_N$ is a regularization of the normal contact force.

In order to obtain a variational formulation for this problem, we adopt the following hypotheses:

$$\begin{aligned} \boldsymbol{\varphi}_1 &\in W^{1,2}(0, T; [L^2(\Omega)]^d), \quad \boldsymbol{\varphi}_2 \in W^{1,2}(0, T; [L^2(\Gamma_2)]^d), \\ a_{ijkl} &\in L^\infty(\Omega), \quad i, j, k, l = 1, \dots, d, \quad \mu \in L^\infty(\Gamma_3), \quad \mu \geq 0 \text{ a.e. on } \Gamma_3. \end{aligned}$$

We use the following notations:

$$\begin{aligned} V_0 &:= \{\mathbf{v} \in [H^1(\Omega)]^d; \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_1\}, \quad K_0 := \{\mathbf{v} \in V_0; v_N \leq 0 \text{ a.e. on } \Gamma_3\}, \\ H^{\frac{1}{2}}(\Gamma_3) &:= \{w : \Gamma_3 \rightarrow \mathbb{R}; w \in H^{\frac{1}{2}}(\Gamma), w = 0 \text{ a.e. on } \Gamma_1\}, \quad (\cdot, \cdot) = (\cdot, \cdot)_{[H^1(\Omega)]^d}, \\ \forall \mathbf{L} \in V_0 \quad S_{\mathbf{L}} &:= \{\mathbf{w} \in V_0; \int_{\Omega} \boldsymbol{\sigma}(\mathbf{w}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\psi}) dx = (\mathbf{L}, \boldsymbol{\psi}) \quad \forall \boldsymbol{\psi} \in V_0 \text{ such} \\ &\quad \text{that } \boldsymbol{\psi} = \mathbf{0} \text{ a.e. on } \Gamma_3\}. \end{aligned}$$

For all $\mathbf{L} \in V_0$ and $\mathbf{v} \in S_{\mathbf{L}}$ we define the stress vector $\boldsymbol{\sigma}(\mathbf{v}) \mathbf{n} \in ([H^{\frac{1}{2}}(\Gamma_3)]^d)'$ by

$$\forall \mathbf{w} \in [H^{\frac{1}{2}}(\Gamma_3)]^d \quad \langle \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n}, \mathbf{w} \rangle = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\bar{\mathbf{w}}) dx - (\mathbf{L}, \bar{\mathbf{w}}), \quad (4.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing on $([H^{\frac{1}{2}}(\Gamma_3)]^d)' \times [H^{\frac{1}{2}}(\Gamma_3)]^d$, $\bar{\mathbf{w}} \in V_0$ satisfies $\bar{\mathbf{w}} = \mathbf{w}$ a.e. on Γ_3 , and we define the normal component of the stress vector $\sigma_N(\mathbf{v}) \in (H^{\frac{1}{2}}(\Gamma_3))'$ by

$$\forall w \in H^{\frac{1}{2}}(\Gamma_3) \quad \langle \sigma_N(\mathbf{v}), w \rangle = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\bar{\mathbf{w}}) dx - (\mathbf{L}, \bar{\mathbf{w}}), \quad (4.2)$$

where $\langle \cdot, \cdot \rangle$ denotes also the duality pairing on $(H^{\frac{1}{2}}(\Gamma_3))' \times H^{\frac{1}{2}}(\Gamma_3)$, $\bar{\mathbf{w}} \in V_0$ satisfies $\bar{\mathbf{w}}_T = \mathbf{0}$ a.e. on Γ_3 , $\bar{\mathbf{w}}_N = w$ a.e. on Γ_3 .

It is easy to verify that for all $\mathbf{v} \in S_L$ the definitions of $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ and of $\sigma_N(\mathbf{v})$ do not depend on the choices of $\bar{\mathbf{w}}$ having the above properties, respectively.

For all $\mathbf{L} \in V_0$ we introduce the functional $J_L : S_L \times V_0 \rightarrow \mathbb{R}$ by

$$J_L(\mathbf{v}, \mathbf{w}) = \int_{\Gamma_3} \mu |\mathcal{R}\sigma_N(\mathbf{v})| |\mathbf{w}_T| ds \quad \forall \mathbf{v} \in S_L, \mathbf{w} \in V_0, \quad (4.3)$$

where $\mathcal{R} : (H^{\frac{1}{2}}(\Gamma_3))' \rightarrow L^2(\Gamma_3)$ is a linear and compact mapping.

Now, let $\mathbf{L} \in V_0$ be given by the relation

$$(\mathbf{L}, \mathbf{v}) = (\boldsymbol{\varphi}_1, \mathbf{v})_{[L^2(\Omega)]^d} + (\boldsymbol{\varphi}_2, \mathbf{v})_{[L^2(\Gamma_2)]^d} \quad \forall \mathbf{v} \in V_0 \quad (4.4)$$

and let $\mathbf{u}_0 \in K_0$ satisfying the following compatibility condition:

$$\begin{aligned} \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}_0) \cdot \boldsymbol{\varepsilon}(\mathbf{w} - \mathbf{u}_0) dx + J_{\mathbf{L}(0)}(\mathbf{u}_0, \mathbf{w}) - J_{\mathbf{L}(0)}(\mathbf{u}_0, \mathbf{u}_0) \\ \geq (\mathbf{L}(0), \mathbf{w} - \mathbf{u}_0) \quad \forall \mathbf{w} \in K_0. \end{aligned} \quad (4.5)$$

A primal variational formulation of P_c is as follows, see, e.g., [5].

Problem P_0 : Find $\mathbf{u} \in W^{1,2}(0, T; V_0)$ such that

$$(P_0) \begin{cases} \mathbf{u}(0) = \mathbf{u}_0, \mathbf{u}(t) \in K_0 \quad \forall t \in]0, T[, \\ \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}(t)) \cdot \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)) dx + J_{\mathbf{L}(t)}(\mathbf{u}(t), \mathbf{v}) - J_{\mathbf{L}(t)}(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ \geq (\mathbf{L}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + \langle \sigma_N(\mathbf{u}(t)), v_N - \dot{u}_N(t) \rangle \quad \forall \mathbf{v} \in V_0 \text{ a.e. on }]0, T[, \\ \langle \sigma_N(\mathbf{u}(t)), z_N - u_N(t) \rangle \geq 0 \quad \forall \mathbf{z} \in K_0, \forall t \in]0, T[. \end{cases}$$

Let us define $a : V_0 \times V_0 \rightarrow \mathbb{R}$ by

$$a(\mathbf{v}, \mathbf{w}) = \int_{\Omega} a_{ijkl} \varepsilon_{ij}(\mathbf{v}) \varepsilon_{kl}(\mathbf{w}) dx = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{w}) dx \quad \forall \mathbf{v}, \mathbf{w} \in V_0. \quad (4.6)$$

The bilinear form $a(\cdot, \cdot)$ is continuous on $V_0 \times V_0$ and since $\text{meas}(\Gamma_1) > 0$, by Korn's inequality is also V_0 -elliptic i.e. satisfies

$$\begin{aligned} \exists M > 0 \text{ such that } |a(\mathbf{v}, \mathbf{w})| &\leq M \|\mathbf{v}\| \|\mathbf{w}\| \quad \forall \mathbf{v}, \mathbf{w} \in V_0, \\ \exists \alpha > 0 \text{ such that } a(\mathbf{v}, \mathbf{v}) &\geq \alpha \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in V_0, \end{aligned}$$

where $\|\cdot\| = \|\cdot\|_{[H^1(\Omega)]^d}$.

Let $\mathbf{G}_1, \mathbf{G}_2 \in V_0$ and $\mathbf{v}_1, \mathbf{v}_2$ be such that $\mathbf{v}_1 \in S_{\mathbf{G}_1}, \mathbf{v}_2 \in S_{\mathbf{G}_2}$. Then from the properties of σ_N, \mathcal{R} and a it follows that the mapping J has the following property: $\exists C, C' > 0$ such that

$$\begin{aligned} &|J_{\mathbf{G}_1}(\mathbf{v}_1, \mathbf{w}_2) + J_{\mathbf{G}_2}(\mathbf{v}_2, \mathbf{w}_1) - J_{\mathbf{G}_1}(\mathbf{v}_1, \mathbf{w}_1) - J_{\mathbf{G}_2}(\mathbf{v}_2, \mathbf{w}_2)| \\ &\leq C \bar{\mu} \int_{\Gamma_3} |\mathcal{R}\sigma_N(\mathbf{v}_1) - \mathcal{R}\sigma_N(\mathbf{v}_2)| |\mathbf{w}_1 - \mathbf{w}_2| ds \\ &\leq C' \bar{\mu} (\|\mathbf{G}_1 - \mathbf{G}_2\| + M \|\mathbf{v}_1 - \mathbf{v}_2\|) \|\mathbf{w}_1 - \mathbf{w}_2\| \end{aligned} \quad (4.7)$$

for all $\mathbf{G}_i, \mathbf{w}_i \in V_0, \mathbf{v}_i \in S_{\mathbf{G}_i}, i = 1, 2$, where $\bar{\mu} = \|\mu\|_{L^\infty(\Gamma_3)}$.

An incremental formulation can be written by using a time discretization of (P_0) as in Section 2. Therefore we obtain the following sequence of incremental problems $(P_{0,n}^i)_{i=0,1,\dots,n-1}$.

Problem $\mathbf{P}_{0,n}^i$: Find $\mathbf{u}^{i+1} \in K_0$ such that

$$(P_{0,n}^i) \begin{cases} a(\mathbf{u}^{i+1}, \mathbf{v} - \partial \mathbf{u}^i) + J_{\mathbf{L}^{i+1}}(\mathbf{u}^{i+1}, \mathbf{v}) - J_{\mathbf{L}^{i+1}}(\mathbf{u}^{i+1}, \partial \mathbf{u}^i) \\ \geq (\mathbf{L}^{i+1}, \mathbf{v} - \partial \mathbf{u}^i) + \langle \sigma_N(\mathbf{u}^{i+1}), v_N - \partial u_N^i \rangle \quad \forall \mathbf{v} \in V_0, \\ \langle \sigma_N(\mathbf{u}^{i+1}), z_N - u_N^{i+1} \rangle \geq 0 \quad \forall \mathbf{z} \in K_0. \end{cases}$$

Let us now use similar notations to (2.23). Then $\mathbf{u}_n \in L^2(0, T; V_0)$ and $\hat{\mathbf{u}}_n \in W^{1,2}(0, T; V_0)$ satisfy the following incremental problem:

$$(P_{0,n}) \begin{cases} a(\mathbf{u}_n(t), \mathbf{v} - \frac{d}{dt} \hat{\mathbf{u}}_n(t)) + J_{\mathbf{L}_n(t)}(\mathbf{u}_n(t), \mathbf{v}) \\ - J_{\mathbf{L}_n(t)}(\mathbf{u}_n(t), \frac{d}{dt} \hat{\mathbf{u}}_n(t)) \geq (\mathbf{L}_n(t), \mathbf{v} - \frac{d}{dt} \hat{\mathbf{u}}_n(t)) \\ + \langle \sigma_N(\mathbf{u}_n(t)), v_N - \frac{d}{dt} \hat{u}_{nN}(t) \rangle \quad \forall \mathbf{v} \in V_0, \forall t \in [0, T], \\ \langle \sigma_N(\mathbf{u}_n(t)), z_N - u_{nN}(t) \rangle \geq 0 \quad \forall \mathbf{z} \in K_0, \forall t \in [0, T]. \end{cases}$$

We have the following existence and approximation result.

Theorem 4.1 *Under the above assumptions and if $\bar{\mu} < \frac{\alpha}{C'}$ there exists a subsequence $(\mathbf{u}_{n_p})_p$ of $(\mathbf{u}_n)_n$ such that $\mathbf{u}_{n_p}(t) \rightarrow \mathbf{u}(t)$ in $V_0 \forall t \in [0, T]$,*

$\hat{\mathbf{u}}_{n_p} \rightarrow \mathbf{u}$ in $L^2(0, T; V_0)$ and $\frac{d}{dt} \hat{\mathbf{u}}_{n_p} \rightarrow \dot{\mathbf{u}}$ in $L^2(0, T; V_0)$, as $p \rightarrow \infty$, where \mathbf{u} is a solution of (P_0) .

Proof.

Taking $V = V_0$, $K = K_0$, $H = L^2(\Gamma_3)$ and

$$j(\mathbf{L}, \mathbf{v}, \mathbf{w}) = J_{\mathbf{L}}(\mathbf{v}, \mathbf{w}) - (\mathbf{L}, \mathbf{w}), \quad b(\mathbf{L}, \mathbf{v}, \mathbf{w}) = \langle \sigma_N(\mathbf{v}), w_N \rangle,$$

$$K(\mathbf{L}) = K_0 \cap S_{\mathbf{L}}, \quad \beta(\mathbf{L}, \mathbf{v}) = \mu |\mathcal{R}\sigma_N(\mathbf{v})| \quad \forall \mathbf{v} \in S_{\mathbf{L}}, \mathbf{w} \in V_0,$$

we see that (P_0) can be written in the form (P) with $f = \mathbf{L}$, where \mathbf{L} is defined by (4.4).

Using the properties of J and Green's formula, it can be easily seen that the hypotheses of theorem 2.1 are satisfied and the theorem therefore follows. \square

We can also apply theorems 3.1, 3.2 and 3.3 by choosing an internal approximation as in [6].

Using a similar approach that will not be repeated here, one can study the existence and approximation of solutions to the quasistatic unilateral contact problem with nonlocal friction between two linearly elastic bodies.

Clearly, the results presented in Sections 2 and 3 can be equally applied to frictional contact with normal compliance or to bilateral contact problems, see, e.g., [2], [3] for various formulations of these problems. In particular, the set of admissible displacement fields K is a whole vector subspace and $b = 0$ in all these cases so that their variational formulation consists in a single evolution implicit inequality.

5 Algorithms and numerical examples

5.1 Description of the algorithm

Let $(\mathcal{T}_h)_h$ be a set of regular triangulations of $\bar{\Omega}$ such that $\mathcal{T}_h = (\Omega_j)_{j \in \mathcal{J}_h}$ satisfies the relation $\bar{\Omega} = \cup_{j \in \mathcal{J}_h} \bar{\Omega}_j$ with $\Omega_k \cap \Omega_l = \emptyset$ for all $k, l \in \mathcal{J}_h$, $k \neq l$. We introduce the following sets:

$$V_{0h} := \{\mathbf{v}_h \in [C(\bar{\Omega})]^d; \mathbf{v}_h|_{\Omega_j} \in [P_1(\Omega_j)]^d, \forall j \in \mathcal{J}_h, \mathbf{v}_h = \mathbf{0} \text{ on } \Gamma_1\},$$

$$K_{0h} := \{\mathbf{v}_h \in V_{0h}; v_{hN} \leq 0 \text{ a.e. on } \Gamma_3\},$$

$$S_{0h} := \{\tau_h \in L^2(\Gamma_3); \tau_h|_{\Gamma_{3,j}} \in P_0(\Gamma_{3,j}), \forall j \in \mathcal{J}_h \text{ such that } \Gamma_{3,j} \neq \emptyset\},$$

where $P_k(\omega)$ denotes the space of polynomials of degree lower or equal to k on ω and $\Gamma_{3,j} = \Gamma_3 \cap \bar{\Omega}_j$.

When the classical finite element approximations (V_{0h}, K_{0h}) of (V_0, K_0) are used for the problem $(P_{0,n}^i)$, it can be shown, by using a similar argument as in [5] and [17], where the continuous case was considered, that the fully discrete problem can be set under the form of the following implicit variational inequality: find $\mathbf{u}_h^{i+1} \in K_{0h}^{i+1}$ such that

$$(\bar{Q}_{hn}^i) \quad \begin{cases} a(\mathbf{u}_h^{i+1}, \mathbf{w}_h - \mathbf{u}_h^{i+1}) + J_{\mathbf{L}^{i+1}}(\mathbf{u}_h^{i+1}, \mathbf{w}_h - \mathbf{u}_h^i) \\ - J_{\mathbf{L}^{i+1}}(\mathbf{u}_h^{i+1}, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i) \geq (\mathbf{L}^{i+1}, \mathbf{w}_h - \mathbf{u}_h^{i+1}) \quad \forall \mathbf{w}_h \in K_{0h}. \end{cases}$$

In the following μ is supposed to be constant. We choose as a regularization mapping \mathcal{R} the projection on the finite-dimensional space S_{0h_0} for a given h_0 (see [18]). So within finite element formulation the regularization can be considered as a natural consequence of the discretization. A fixed point method is then introduced on the sliding threshold and (\bar{Q}_{hn}^i) is found to be equivalent to the following problem:

$$(P_{inc}^1) \quad \begin{cases} \text{at each time step, find the fixed point of the mapping} \\ \Phi : S_{0h} \rightarrow S_{0h} \text{ defined by } \Phi(\mathbf{g}) = -\mu \mathcal{R} \sigma_N(\mathbf{u}_{\mathbf{g}}^{i+1}) \quad \forall \mathbf{g} \in S_{0h}, \end{cases}$$

where $\mathbf{u}_{\mathbf{g}}^{i+1}$ is the solution of the problem

$$(P_{inc}^2) \quad \begin{cases} \text{for a given } \mathbf{g} \in S_{0h}, \text{ find } \mathbf{u}_{\mathbf{g}}^{i+1} \in K_{0h} \text{ such that} \\ a(\mathbf{u}_{\mathbf{g}}^{i+1}, \mathbf{v}_h - \mathbf{u}_{\mathbf{g}}^{i+1}) + p_{\mathbf{g}}(\mathbf{v}_h - \mathbf{u}_h^i) - p_{\mathbf{g}}(\mathbf{u}_{\mathbf{g}}^{i+1} - \mathbf{u}_h^i) \\ \geq (\mathbf{L}^{i+1}, \mathbf{v}_h - \mathbf{u}_{\mathbf{g}}^{i+1}) \quad \forall \mathbf{v}_h \in K_{0h} \end{cases}$$

with

$$p_{\mathbf{g}}(\mathbf{w}) := \int_{\Gamma_3} \mathbf{g} |\mathbf{w}_T| ds \quad \forall \mathbf{w} \in V_0.$$

Now, (P_{inc}^2) is equivalent to the following minimization problem under constraint:

$$(P_{opt}) \quad \begin{cases} \text{For a given } \mathbf{g} \in S_{0h}, \text{ find } \mathbf{u}_{\mathbf{g}}^{i+1} \in K_{0h} \text{ such that} \\ F(\mathbf{u}_{\mathbf{g}}^{i+1}) \leq F(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in K_{0h} \end{cases}$$

with

$$F(\mathbf{v}) := \frac{1}{2} a(\mathbf{v}, \mathbf{v}) + p_{\mathbf{g}}(\mathbf{v} - \mathbf{u}_h^i) - (\mathbf{L}^{i+1}, \mathbf{v}) \quad \forall \mathbf{v} \in V_0.$$

This problem is very similar to a static problem apart from the fact that the known solution \mathbf{u}_h^i of the previous step is now present in the friction term. The influence of the loading history, due to the velocity formulation of the friction condition, is characterized by this extra term. The convex set K_{0h} remains unchanged from one step to the next. A minimization method is used

to solve (P_{opt}). We consider here the numerical solvers for two-dimensional problems. A Gauss-Seidel method with projection is robust and very easy to implement when dealing with the non differentiable part related to the friction term. The convergence of the algorithm is improved by using an Aitken acceleration procedure (see [17]) which is as efficient as an overrelaxation method but does not require an optimum coefficient. Other methods can be found in [17]: mathematical programming (using a complementarity formulation of the problem), conjugate gradient method, etc. The Gauss-Seidel algorithm with Aitken acceleration has been used in the following computations illustrating the behaviour of the solution regarding the space and time discretizations. For three-dimensional problems, the conjugate gradient method with projection and preconditioning given in [17] can be used but the friction law needs to be regularized in that case.

5.2 A test example: a block sliding on a plane

The previous algorithm has been successfully used on many problems, including industrial ones (like, e.g., pressure vessel, composite material, friction and squeal). Here, our aim is to stress the points concerning the convergence regarding the space and the time discretizations that can be observed numerically. The influence of the coefficient of friction is presented and the convergence of the fixed point method is studied. For that purpose, a very simple example has been chosen, the solution of which is sufficiently complex to illustrate this influence of the friction and of the fully discrete schemes on the accuracy of the numerical solution.

5.2.1 The example

This example is a part of a study on the instability phenomena which can be involved in friction problems [19]. A Polyurethane block ($L = 80mm$, $h = 40mm$, $E = 5daN/m^2$, $\nu = 0.48$) is firstly pressed on a rigid plane with a normal prescribed displacement ($u_N^0 = -0.5 mm$) on the top of the block, and a tangential displacement is then imposed on the plane at a constant velocity (see Figure 1 and Figure 2). A plane stress hypothesis is assumed here (the thickness is $9.6 mm$). The block being elastic, the time variable can be considered as a loading parameter.

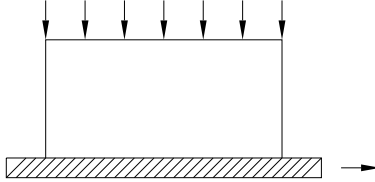


Figure 1: Geometry and prescribed displacements

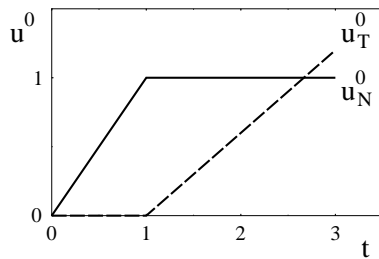


Figure 2: Changes in the prescribed displacements with time

5.2.2 The solution

It is worth noting that the solution is strongly affected by the friction and, depending on the coefficient of friction, the shape and the regularity of solutions can be different. To illustrate the influence of the coefficient of friction on the convergence, the changes in the contact forces σ_N and σ_T/μ on the contact boundary, occurring when the prescribed displacement u_T^0 of the plate increases, are shown in Figure 3 and Figure 4 for $\mu = 0.5$ and $\mu = 1$. It can be noted that at $\mu = 0.5$, the sliding zone spreads out from the center of the zone, while at $\mu = 1$ it spreads out from the right edge. In both cases, before complete total sliding occurs, the solution shows the existence of three zones: a separate part (where the forces are zero), a sliding one (where the plot of σ_N overlaps that of σ_T/μ) and a sticking one (where the plots of σ_N and σ_T/μ are separate).

5.2.3 Convergence of the algorithm

Fixed point method on the sliding limit

The convergence of the fixed point method described by P_{inc}^1 on the sliding threshold is generally observed after less than 10 iterations and it is not very sensitive regarding the mesh size. A diagonal process is used, which means that for the first iterations the prescribed accuracy of the Gauss-

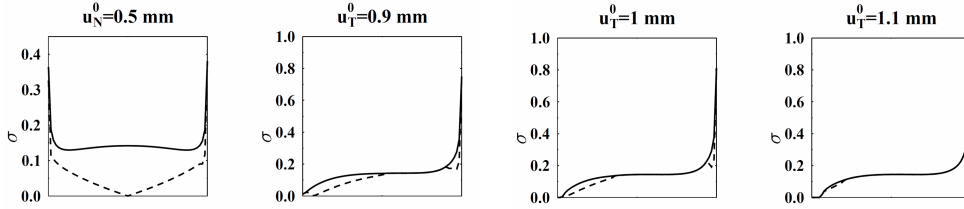


Figure 3: Changes in time in the contact forces σ_N (solid line) and σ_T/μ (broken line) at $\mu = 0.5$ (sticking, sliding and separation zones)

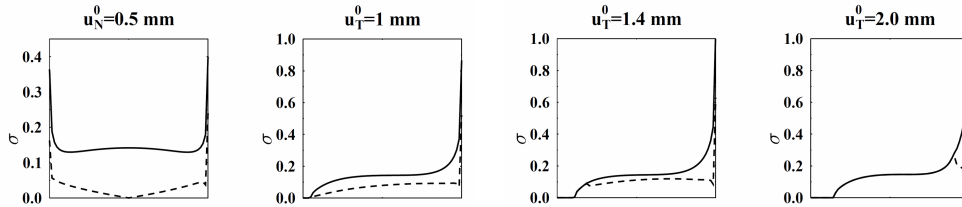


Figure 4: Changes in time in the contact forces σ_N (solid line) and σ_T/μ (broken line) at $\mu = 1$ (sticking, sliding and separation zones)

Seidel method is low (a rough solution is sufficient to get the contact force) and increase progressively up to $\epsilon = 5.10^{-6}$, a convenient convergence test value. This test value saves about 50% of Gauss-Seidel iterations.

Convergence of Gauss-Seidel algorithm with Aitken acceleration

The Aitken acceleration is an heuristic process based on considering 3 terms of the Projected Gauss-Seidel sequence and computing a new sequence based on an assumption of geometrical sequence property. It is combined with the Gauss-Seidel iterations in order to compute a good initial value to a regular Gauss-Seidel process, the convergence of which has been established on the unilateral contact problem with friction.

In that way, the number of iterations needed to treat one loading step and including the fixed point method on the sliding threshold \mathbf{g} , is of the same order of magnitude as the number of iterations required when an overrelaxation method with projection is used. This avoids the determination of an optimal relaxation parameter which needs a lot of preliminary computations.

In order to illustrate these comments on the algorithm convergence, we

shall present some numerical results. For our example, by considering a mesh with 4193 nodes, including 65 contact nodes, for the step $t = 2$ where both u_T^0 and u_N^0 are applied, we have the following results:

- convergence on the sliding threshold is obtained within 8 iterations;
- convergence of the Gauss-Seidel method with Aitken acceleration (including the previous fixed point iterations) is obtained after 654 iterations, while using an overrelaxation method, which requires a costly preliminary computation of the optimal overrelaxation parameter, the convergence is obtained within 621 iterations.

5.2.4 Effects of space and time discretizations

Various structured meshes were used here and the same P_1 elements (linear interpolation) were used in order to obtain an internal approximation of the convex K_0 . The characteristics of the various meshes are given in Table 1 and the mesh M_2 is given in Figure 5 whereas the other meshes can be easily deduced by cutting every triangle in 3 parts.

Mesh name	Total number of nodes	Number of contact nodes
M_0	6	3
M_1	15	5
M_2	45	9
M_3	153	17
M_4	561	33
M_5	2145	65
M_6	8385	129
M_7	33153	257

Table 1: Characteristics of the various meshes

The comparisons between the various solutions have been conducted on the values of the tangential displacement at the central node of the contact zone. In order to obtain a sliding condition for this node in every case, the solutions have been compared at $u_T^0 = -1mm$ when $\mu = 0.5$, $u_T^0 = -2mm$ when $\mu = 1$ and $u_T^0 = -3mm$ when $\mu = 1.5$.

Effects of time discretization

When the loading is piecewise proportional and monotone, the step size can be theoretically as large as a whole interval. This is globally confirmed on

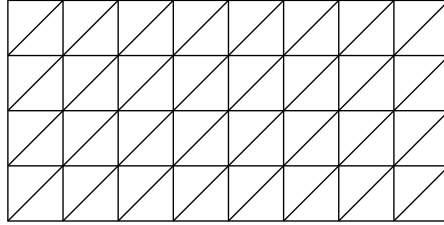


Figure 5: The medium size mesh M_2

tables 2 and 3 in the present example including two intervals. When at least two and respectively three increments are used on each interval the error is less than 0.4 per cent. This accuracy is only slightly improved when a larger number of increments is used. This is not surprising since a more accurate solution is obtained when the intermediate solutions (those obtained before total sliding occurs) are well evaluated, but the difference is slight. That is of great importance for the computations: it allows that small time steps are not required on the intervals where the loading is piecewise proportional and monotone.

The computations presented in tables 2 and 3 were carried out on the mesh M_4 (561 nodes, 33 contact nodes). The value chosen as reference to estimate the error was the solution computed with 24 increments for the normal loading and 36 for the tangential one.

Increment number		Relative error on the central node
normal	tangential	
1	1	2.1 %
2	3	0.35 %
4	6	0.28 %
12	18	0.01 %

Table 2: Effects of the number of increments ($\mu = 0.5$)

Effects of space discretization and dependency on the coefficient of friction

In Figure 6, the convergence with respect to the mesh size h is shown at various values of the coefficient of friction ($\mu = 0.5, 1, 1.5$). The solution chosen as reference was that computed using the mesh M_7 (33153 nodes, 257 contact nodes). The plotted relative error is a mean value of the errors

Increment number		Relative error on the central node
normal	tangential	
1	1	1.5 %
2	3	0.37 %
4	6	0.28 %
12	18	0.01%

Table 3: Effects of the number of increments ($\mu = 1$)

computed on the sliding displacements of several nodes located in the central zone for the various meshes. Meshes M_0 to M_5 (see Table 1) are considered in Figure 6 and L/h is the number of contact nodes minus one. It has to be noted that no significant influence of the magnitude of the coefficient of friction was observed. The differences observed between the convergence at $\mu = 0.5$ and at $\mu = 1$ or $\mu = 1.5$ were mostly due to the difference between the morphologies of the solutions (see Figure 3 and Figure 4): sliding zone spreading from a center zone when the coefficient of friction is small ($\mu = 0.5$) and from the left side when it is larger ($\mu = 1, \mu = 1.5$).

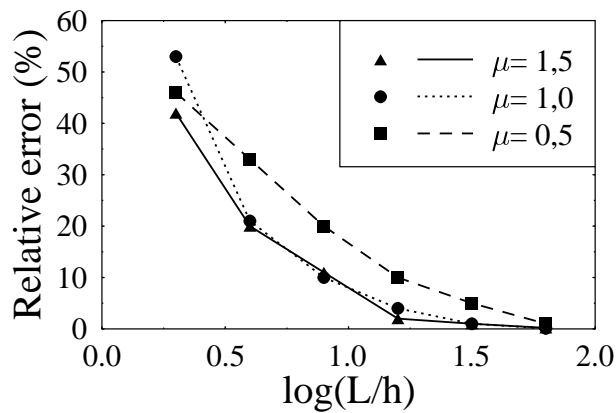


Figure 6: Error depending on the mesh size at $\mu = 0.5, 1, 1.5$

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