Counting walks in a quadrant : a unified approach *via* boundary value problems

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Abstract

The aim of this article is to introduce a unified method for giving explicit integral representations of the trivariate generating function of the number of paths for walks with small steps confined in a quadrant. For a number of such walks, this yields for the first time an explicit expression of this counting function. Moreover, the nature of the integrand of the integral formulations obtained here is shown to be very directly dependent on the finiteness of a well-known and naturally attached group of birational transformations.

Keywords : lattice walks, generating function, Riemann-Carleman boundary value problem, conformal gluing function, Weierstrass elliptic function, Riemann surface, uniformization.

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1 Introduction and main results

The enumeration of planar lattice walks is a classical topic in combinatorics. For a given set $\mathcal{S}$ of steps, it is a matter of counting the number of paths with jumps in $\mathcal{S}$, starting and ending at some arbitrary points in a certain time and eventually restricted to some regions of the plane. A first natural question is simply : how many such paths exist ? A second question concerns the nature of the associated generating function : is it rational, algebraic, holonomic (*i.e.* solution of a linear differential equation with polynomial coefficients) or non-holonomic ?

For instance, if no restriction on the paths is made, it is well-known and easy to make explicit the counting generating function which is rational. As an other example, if the walks are supposed to remain in a half-plane, then the generating function can also be made explicit and is algebraic, see *e.g.* [BMP03].

It was next natural to consider the walks confined in the intersection of two half-planes, as the quadrant $\mathbb{Z}_+^2$. The situation seemed then more multifarious : some walks admit an algebraic generating function, see *e.g.* [FH84] and [Ges86] for the walk with

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steps set $\mathcal{S} = \{(-1,0),(1,1),(0,-1)\}$ and starting at $(0,0)$, while some others admit a counting function that is even not holonomic, see *e.g.* [BMP03] for the walk with steps set $\mathcal{S} = \{(-1,2),(2,-1)\}$ and starting at $(1,1)$. It appeared therefore interesting to focus on these walks staying in $\mathbb{Z}_+^2$.

This is how that M. Bousquet-Mélou and M. Mishna have very recently in [BMM09] initiated a systematic study of the walks confined in $\mathbb{Z}_+^2$, starting at the origin and having small steps, which means that the set of admissible steps $\mathcal{S}$ is included in $\{(i,j) : |i|,|j| \leq 1\} \setminus \{(0,0)\}$.

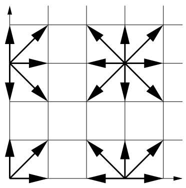

Figure 1: Walks with small steps confined in the quadrant $\mathbb{Z}_+^2$

There are obviously 2^8 such models. After eliminating the trivial models as well as the ones intrinsic to the half-plane and taking account of the fact that some models are obtained by symmetry starting from other ones, M. Bousquet-Mélou and M. Mishna show that it remains 79 inherently different problems to study – we will often refer to these 79 walks and to their tables stated in [BMM09].

A common starting point for the study of these 79 walks is the following : denoting by $q(i,j,k)$ the number of paths confined in $\mathbb{Z}_+^2$, starting at $(0,0)$ and ending at (i,j) in time k, their generating function

$$Q(x,y,z) = \sum_{i,j,k\geq 0} q(i,j,k)x^i y^j z^k \tag{1}$$

verifies the functional equation (proved in [BMM09])

$$xyz\left[\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z\right] Q(x,y,z) = c(x,z)Q(x,0,z) + \widetilde{c}(y,z)Q(0,y,z) - z\delta Q(0,0,z) - xy, \tag{2}$$

where we have noted $c(x,z) = zx\sum_{(i,-1)\in\mathcal{S}} x^i$, $\tilde{c}(y,z) = zy\sum_{(-1,j)\in\mathcal{S}} y^j$ and $\delta = 1$ if $(-1,-1) \in \mathcal{S}$, $\delta = 0$ otherwise. If $n = \#\mathcal{S}$ is the number of steps, then the equality (2) is valid at least on $\{|x| \leq 1, |y| \leq 1, |z| < 1/n\}$, since obviously $q(i,j,k) \leq n^k$.

In this way, for answering both questions stated at the very beginning of this article it is enough to solve (2) and to study the expression of (1) so obtained.

It turns out that a key idea for examining Equation (2) is to consider some group, introduced in a probabilistic context in [FIM99] and called there *the group of the walk*. This is a group of birational transformations leaving invariant the steps generating function $\sum_{(i,j)\in\mathcal{S}} x^i y^j$, and more precisely this is the group $W = \langle \Psi, \Phi \rangle$ generated by

$$\Psi(x,y) = \left(x, \frac{\sum_{(i,-1)\in\mathcal{S}} x^i}{\sum_{(i,+1)\in\mathcal{S}} x^i}\frac{1}{y}\right), \qquad \Phi(x,y) = \left(\frac{\sum_{(-1,j)\in\mathcal{S}} y^j}{\sum_{(+1,j)\in\mathcal{S}} y^j}\frac{1}{x}, y\right).$$

Obviously $\Psi \circ \Psi = \Phi \circ \Phi = \mathrm{id}$ and W is a dihedral group – of order even and larger than four. In [BMM09] is calculated this order for each of the 79 cases : 23 walks admit a finite group (and then of order four, six or eight) and the 56 others have an infinite group.

For the 23 walks with finite group, the answers to both questions (explicit expression and nature of the function (1)) in which we are interested have been given very recently. Indeed, in the rather voluminous paper [BMM09] are solved successfully 22 of the 23 models associated with a finite group : the series (1) is made explicit and is shown to be either algebraic or transcendental but holonomic. As for the 23th walk, known as Gessel's walk, A. Bostan and M. Kauers have given in [BK09] a computer-aided proof of the fact that the function (1) is algebraic. And even more, using a powerful computer algebra system, they have made explicit minimal polynomials. Thanks to these polynomials, M. van Hoeij has then managed to express the function (1) by radicals, see the appendix of [BK09]. Before the latter result was public, we have given in [KR09a] an explicit integral representation of the generating function (1), without computer help. Moreover, in the work in preparation [BCK+10], A. Bostan et al. obtain integral representations of the function (1) for the 23 walks having a finite group, by using a mathematical and algorithmic method, based on creative telescoping and on the resolution of differential equations of order two in terms of hypergeometric functions.

On the other hand, only two cases of the 56 walks having an infinite group have been solved : in [MR09] M. Mishna has considered the walks with steps sets $\mathcal{S} = \{(-1,1),(1,1),(1,-1)\}$ and $\mathcal{S} = \{(-1,1),(0,1),(1,-1)\}$, see at the left of Picture 2 below, has made explicit the series (1) and has shown that in both cases it is non-holonomic.

The aim of this article is to introduce a unified approach giving an explicit expression of the generating function (1) for any of the 79 walks.

For 54 of the 56 walks with an infinite group, this result is new – up to our knowledge. This article also gives rise, after the work [KR09a] on Gessel's walk, to write integral representations of the function (1) for the 22 other walks with a finite group. We will show, besides, that the finiteness of the group and the sign of the covariance of $\mathcal{S}$ act very directly on the nature (rational, algebraic, holonomic, non-holonomic) of the functions in these integral representations.

The approach that we are going to use here is not the same according to the walks under consideration are, or not, singular – by singular we mean here that none of the steps $(0,-1),(-1,-1),(-1,0)$ belongs to $\mathcal{S}$.

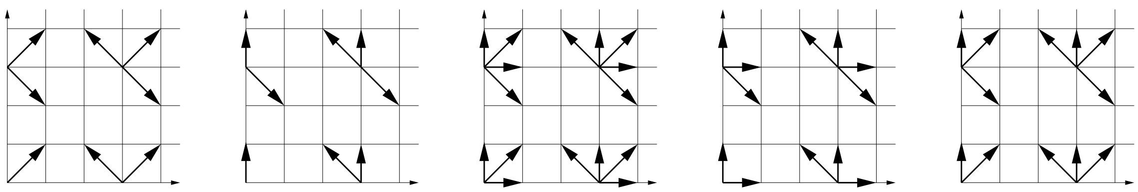

Figure 2: The five singular walks in the classification of [BMM09]

In order to achieve our aim for the non-singular walks – this concerns the 23 walks having a finite group and 51 of the 56 ones attached to an infinite group, see the tables stated in [BMM09] –, we will here generalize up to three variables x, y, z the profound analytic approach developed in [FIM99] by G. Fayolle, R. Iasnogorodski and V. Malyshev for two variables x, y. To summarize, starting from the functional equation (2) we will show that $c(x,z)Q(x,0,z)$ and $\tilde{c}(y,z)Q(0,y,z)$ verify boundary value problems of Riemann-Carleman type ; next, transforming them into boundary value problems of Riemann-Hilbert type by use of *conformal gluing functions* and solving these new problems, we will deduce the announced integral representation of (1). Moreover, we will here study in-depth these conformal gluing functions, their analysis being just sketched out in [FIM99]. The study of these functions, that will use a uniformization of the Riemann surface $\{(x,y) \in \mathbb{C}^2 : \sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z = 0\}$ given by the zeros of the kernel and fine properties of the Weierstrass elliptic functions, will imply that the finiteness of the group and the sign of the covariance of $\mathcal{S}$ are decisive quantities for the analysis of these walks.

For the 5 singular walks, the curves associated with the boundary value problems above become degenerated into a point and the previous arguments will not work anymore. But more easily, and once again starting from (2), we will be able in this case to make explicit a series representation of the function (1) .

We are now going to state the main results of this article. But first of all, let us have a close look to the kernel $xyz\big[\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z\big]$ that appears in (2) and let us take some notations.

This kernel can be written as $xyz\big[\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z\big] = \tilde{a}(y,z)x^2 + \tilde{b}(y,z)x + \tilde{c}(y,z) = a(x,z)y^2 + b(x,z)y + c(x,z)$, where $\tilde{c}(y,z), c(x,z)$ are defined just below (2), $\tilde{a}(y,z) = zy\sum_{(+1,j)\in\mathcal{S}} y^j$, $\tilde{b}(y,z) = -1 + zy\sum_{(0,j)\in\mathcal{S}} y^j$, $a(x,z) = zx\sum_{(i,+1)\in\mathcal{S}} x^i$ and $b(x,z) = -1 + zx\sum_{(i,0)\in\mathcal{S}} x^i$. Let us also define $\tilde{d}(y,z) = \tilde{b}(y,z)^2 - 4\tilde{a}(y,z)\tilde{c}(y,z)$ and $d(x,z) = b(x,z)^2 - 4a(x,z)c(x,z)$.

If the walk is non-singular then for any $z \in]0, 1/n[$, $\tilde{d}$ (resp. d) has three or four roots, that we call the $y_k(z)$ (resp. the $x_k(z)$). They are such that $|y_1(z)| < y_2(z) < 1 < y_3(z) < |y_4(z)|$ (resp. $|x_1(z)| < x_2(z) < 1 < x_3(z) < |x_4(z)|$), as shown in Part 2.3 of [FIM99].

If the walk is singular, the roots above become $y_1(z) = y_2(z) = 0 < 1 < y_3(z) < |y_4(z)|$ (resp. $x_1(z) = x_2(z) = 0 < 1 < x_3(z) < |x_4(z)|$), see Part 6.1 of [FIM99].

The behavior of the branch points $y_k(z)$ and $x_k(z)$ is not so simple for $z \notin]0, 1/n[$ and for this reason *we will suppose in the sequel that z is fixed in* $]0, 1/n[$.

Now remark that $\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z = 0$ if and only if $(\tilde{b}(y,z) + 2\tilde{a}(y,z)x)^2 = \tilde{d}(y,z)$ or $(b(x,z) + 2a(x,z)y)^2 = d(x,z)$. In particular, the algebraic functions $X(y,z)$ and $Y(x,z)$ defined by $\sum_{(i,j)\in\mathcal{S}} X(y,z)^i y^j - 1/z = 0$ and $\sum_{(i,j)\in\mathcal{S}} x^i Y(x,z)^j - 1/z = 0$ have two branches, meromorphic on $\mathbb{C} \setminus ([y_1(z), y_2(z)] \cup [y_3(z), y_4(z)])$ and $\mathbb{C} \setminus ([x_1(z), x_2(z)] \cup [x_3(z), x_4(z)])$ (resp. $\mathbb{C} \setminus [y_3(z), y_4(z)]$ and $\mathbb{C} \setminus [x_3(z), x_4(z)]$) in the non-degenerate (resp. degenerate) case.

The following straightforward result (see Part 5.3 of [FIM99]) gives some properties and settles the notations for the two branches of the algebraic functions $X(y,z)$ and $Y(x,z)$.

Lemma 1. *Call $X_0(y,z) = [-\tilde{b}(y,z) + \tilde{d}(y,z)^{1/2}]/[2\tilde{a}(y,z)]$ and $X_1(y,z) = [-\tilde{b}(y,z) - \tilde{d}(y,z)^{1/2}]/[2\tilde{a}(y,z)]$ the branches of $X(y,z)$. For all $y \in \mathbb{C}$ we have $|X_0(y,z)| \leq |X_1(y,z)|$.*

Now we call $Y_0(x,z) = [-b(x,z) + d(x,z)^{1/2}]/[2a(x,z)]$ *and* $Y_1(x,z) = [-b(x,z) - d(x,z)^{1/2}]/[2a(x,z)]$ *the branches of* $Y(x,z)$*. For all* $x \in \mathbb{C}$ *we have* $|Y_0(x,z)| \leq |Y_1(x,z)|$*.*

With these notations we can state the result concerning the explicit expression of the function (1) for the 5 singular walks – see Picture 2. Below, by $f^{\circ k}$ we mean $f \circ \cdots \circ f$ with k occurrences of f.

Theorem 2. *Suppose that the walk is singular. The following series representation holds :*

$$Q(x,0,z) = \frac{1}{zx^2} \sum_{k\geq 0} Y_0 \circ (X_0 \circ Y_0)^{\circ k}(x,z)\big[(X_0 \circ Y_0)^{\circ k}(x,z) - (X_0 \circ Y_0)^{\circ (k+1)}(x,z)\big].$$

$Q(0,y,z)$ *is obtained from the equality above by replacing* X_0 *(resp.* Y_0*) by* Y_0 *(resp.* X_0*). Moreover* $Q(0,0,z) = 0$ *and the function* $Q(x,y,z)$ *is obtained with (2).*

Theorem 2 is shown in [MR09] for both steps sets $\mathcal{S} = \{(-1,1),(1,1),(1,-1)\}$ and $\mathcal{S} = \{(-1,1),(0,1),(1,-1)\}$; its proof for the three other singular walks is obtained in the same way and we omit it.

Let us now turn to the 74 non-singular walks. We are going to state that both functions $c(x,z)Q(x,0,z)$ and $\tilde{c}(y,z)Q(0,y,z)$ verify a boundary value problem of Riemann-Carleman type, with boundary conditions on $X([y_1(z),y_2(z)],z)$ and $Y([x_1(z),x_2(z)],z)$, and thus we begin by recalling from Part 5.3 of [FIM99] some properties of these curves.

Lemma 3. *Consider* $X([y_1(z),y_2(z)],z)$ *and* $Y([x_1(z),x_2(z)],z)$*.* (i) *These two curves are symmetrical w.r.t. the real axis.* (ii) *They are connected and closed in* $\mathbb{C}\cup\{\infty\}$*.* (iii) *They split the complex plane into two connected components ; we call* $\mathscr{G}X([y_1(z),y_2(z)],z)$ *and* $\mathscr{G}Y([x_1(z),x_2(z)],z)$ *the connected components of* $x_1(z)$ *and* $y_1(z)$ *respectively. They verify* $\mathscr{G}X([y_1(z),y_2(z)],z) \subset \mathbb{C}\setminus[x_3(z),x_4(z)]$ *and* $\mathscr{G}Y([x_1(z),x_2(z)],z) \subset \mathbb{C}\setminus[y_3(z),y_4(z)]$*.*

Moreover, as illustrated by the example of Gessel's walk, see [KR09a], these curves are possibly not included in the unit disc and thus the functions $c(x,z)Q(x,0,z)$ and $\tilde{c}(y,z)Q(0,y,z)$ may *a priori* be not defined on them. For this reason we first need to continue the generating functions up to these curves : this is exactly the object of the following result, the proof of which being the subject of Section 2.

Theorem 4. *The functions* $c(x,z)Q(x,0,z)$ *and* $\tilde{c}(y,z)Q(0,y,z)$ *can be holomorphically continued from the open unit disc* $\mathscr{D}$ *to* $\mathscr{G}X([y_1(z),y_2(z)],z)\cup\mathscr{D}$ *and* $\mathscr{G}Y([x_1(z),x_2(z)],z)\cup\mathscr{D}$ *respectively.*

Now, exactly as in Section 0 of [KR09a] we obtain that :

$$\begin{aligned}
&\forall t \in X([y_1(z),y_2(z)],z) \,:\, c(t,z)Q(t,0,z) - c(\bar{t},z)Q(\bar{t},0,z) = tY_0(t,z) - \bar{t}Y_0(\bar{t},z),\\
&\forall t \in Y([x_1(z),x_2(z)],z) \,:\, \tilde{c}(t,z)Q(0,t,z) - \tilde{c}(\bar{t},z)Q(0,\bar{t},z) = X_0(t,z)t - X_0(\bar{t},z)\bar{t}.
\end{aligned} \tag{3}$$

Therefore, by using Theorem 4, we obtain that $c(x,z)Q(x,0,z)$ *and* $\tilde{c}(y,z)Q(0,y,z)$ *can be found among the functions holomorphic in* $\mathscr{G}X([y_1(z),y_2(z)],z)$ *and* $\mathscr{G}Y([x_1(z),x_2(z)],z)$ *and verifying the conditions (3) on the boundary of the latter sets.*

Such problems are known as boundary value problems of Riemann-Carleman type, see *e.g.* [FIM99]. A standard way to solve them consists in converting them into boundary value problems of Riemann-Hilbert type by use of conformal gluing functions (CGF).

Definition 5. *Let $\mathscr{C} \subset \mathbb{C} \cup \{\infty\}$ be an open and simply connected set, symmetrical w.r.t. the real axis and different from $\emptyset$, $\mathbb{C}$ and $\mathbb{C} \cup \{\infty\}$. A function u is said to be a CGF for the set $\mathscr{C}$ if* (i) *u is meromorphic in $\mathscr{C}$* (ii) *u establishes a conformal mapping of $\mathscr{C}$ onto the complex plane cut along some arc* (iii) *for all t in the boundary of $\mathscr{C}$, $u(t) = u(\overline{t})$.*

It is worth noting that the existence (but *no* explicit expression) of a CGF for a generic set $\mathscr{C}$ is ensured by general results on conformal gluing, see *e.g.* Chapter 2 of [Lit00].

Transforming the boundary value problems of Riemann-Carleman type into boundary value problems of Riemann-Hilbert type thanks to a CGF, solving these new problems and working out the solutions, we get as in [KR09a] the following.

Theorem 6. *Suppose that the walk is one of the 74 non-singular walks.*

The function $c(x,z)Q(x,0,z) - c(0,z)Q(0,0,z)$ has the following explicit expression for $z \in]0, 1/n[$ and $x \in \mathscr{G}X([y_1(z), y_2(z)], z)$:

$$c(x,z)Q(x,0,z) - c(0,z)Q(0,0,z) =$$
$$xY_0(x,z) + \frac{1}{\pi}\int_{x_1(z)}^{x_2(z)} \frac{t\big[-d(t,z)\big]^{1/2}}{2a(t,z)} \left[\frac{\partial_t w(t,z)}{w(t,z) - w(x,z)} - \frac{\partial_t w(t,z)}{w(t,z) - w(0,z)}\right] \mathrm{d}t,$$

w being a CGF for $\mathscr{G}X([y_1(z), y_2(z)], z)$.

The function $\tilde{c}(y,z)Q(0,y,z) - \tilde{c}(0,z)Q(0,0,z)$ has the following explicit expression for $z \in]0, 1/n[$ and $y \in \mathscr{G}Y([x_1(z), x_2(z)], z)$:

$$\widetilde{c}(y,z)Q(0,y,z) - \widetilde{c}(0,z)Q(0,0,z) =$$
$$X_0(y,z)y + \frac{1}{\pi}\int_{y_1(z)}^{y_2(z)} \frac{t\big[-\widetilde{d}(t,z)\big]^{1/2}}{2\widetilde{a}(t,z)} \left[\frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z) - \widetilde{w}(y,z)} - \frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z) - \widetilde{w}(0,z)}\right] \mathrm{d}t,$$

$\tilde{w}$ being a CGF for $\mathscr{G}Y([x_1(z), x_2(z)], z)$.

The explicit expression of $zQ(0,0,z)$ depends on the value of $c(0,z) = \tilde{c}(0,z) \in \{0, z\}$.

Suppose first that $c(0,z) = \tilde{c}(0,z) = z$. Then $\delta = 1$ in (2) and with (2) we obtain that for any x, y, z verifying $\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z = 0$ with $|x| \leq 1$, $|y| \leq 1$ and $z \in]0, 1/n[$:

$$zQ(0,0,z) = xy - \big[c(x,z)Q(x,0,z) - c(0,z)Q(0,0,z)\big] - \big[\widetilde{c}(y,z)Q(0,y,z) - \widetilde{c}(0,z)Q(0,0,z)\big].$$

Suppose now that $c(0,z) = \tilde{c}(0,z) = 0$, or equivalently that $\delta = 0$ in (2). Then the function $Q(0,0,z)$ equals the limit at x goes to 0 of

$$\frac{1}{c(x,z)}\left(xY_0(x,z) + \frac{1}{\pi}\int_{x_1(z)}^{x_2(z)} \frac{t\big[-d(t,z)\big]^{1/2}}{2a(t,z)} \left[\frac{\partial_t w(t,z)}{w(t,z) - w(x,z)} - \frac{\partial_t w(t,z)}{w(t,z) - w(0,z)}\right] \mathrm{d}t\right).$$

The function $Q(x,y,z)$ has the explicit expression obtained by putting in (2) the integral representations of $Q(x,0,z)$, $Q(0,y,z)$ and $Q(0,0,z)$ obtained just above.

As a consequence, for the generating functions $c(x,z)Q(x,0,z) - c(0,z)Q(0,0,z)$ and $\tilde{c}(y,z)Q(0,y,z) - \tilde{c}(0,z)Q(0,0,z)$ to be completely explicit, it remains to find explicitly suitable CGF w and $\tilde{w}$ for the sets $\mathscr{G}X([y_1(z), y_2(z)], z)$ and $\mathscr{G}Y([x_1(z), x_2(z)], z)$.

In this perspective let us first state the following fact – that unfortunately can be properly and completely written only in Section 4.

Theorem 7. *The functions w and $\tilde{w}$ explicitly defined in (10) of Section 4 are adequate CGF for the sets $\mathscr{G}X([y_1(z), y_2(z)], z)$ and $\mathscr{G}Y([x_1(z), x_2(z)], z)$ respectively.*

Having a CGF w for some set $\mathscr{C}$, it is clear that any of its fractional linear transformations $[\alpha w + \beta]/[\gamma w + \delta]$ with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha\gamma \neq 0$ is also a CGF for $\mathscr{C}$, see Definition 5. They are actually the only ones, see page 39 in Chapter 2 of [Lit00].

In particular, once Theorem 7 shown, we will have the explicit expression of *all* possible CGF, but the way in which they depend on the steps set $\mathcal{S}$ is yet unclear. In fact we have the following link – intrinsic, because independent of the choice of the CGF.

Theorem 8. *If the group of the walk is finite (resp. infinite), then any CGF for the sets $\mathscr{G}X([y_1(z), y_2(z)], z)$ or $\mathscr{G}Y([x_1(z), x_2(z)], z)$ is algebraic (resp. non-holonomic, and then, of course, non-algebraic). Moreover, in the case of a finite group, if in addition the covariance (i.e. the quantity $\sum_{(i,j)\in\mathcal{S}} ij$) is negative or zero (resp. positive), then any CGF is rational (resp. algebraic non-rational).*

Furthermore, in the case of a finite group, we will in Section 4 considerably simplify the explicit expression of the CGF : in Propositions 28, 29 and 31 we will consider the walks having a negative or zero covariance and we will then give the rational expression of the CGF, next in Proposition 30 we will concentrate on the walks having a positive covariance and we will then give minimal polynomials for the algebraic non-rational CGF.

Let us now make two remarks around Theorem 6.

Remark 9. *Theorem 6 can be extended without difficulty to the case of the generating function counting the number of paths confined in the quadrant, starting at (i_0, j_0) and ending at (i,j) in time k, for any initial state (i_0, j_0).*

Remark 10. *Both $c(x,z)Q(x,0,z)-c(0,z)Q(0,0,z)$ and $\tilde{c}(y,z)Q(0,y,z)-\tilde{c}(0,z)Q(0,0,z)$ are written in Theorem 6 as the sum of two functions not holomorphic near $[x_1(z), x_2(z)]$ and $[y_1(z), y_2(z)]$ respectively. However, these segments being included in the unit disc, the sum of these two functions is of course holomorphic in their neighborhood. In fact, by an application of the residue theorem exactly as in Section 4 of [KR09b], we could write both generating functions as functions clearly holomorphic near these segments and having actually their singularities near respectively $[x_3(z), x_4(z)]$ and $[y_3(z), y_4(z)]$.*

The rest of the article is organized as follows.

First in Section 2 we prove Theorem 4 and we comment on other continuation methods.

Next Section 3 is a technical and introductory part to Section 4. We study closely the surface $\left\{(x,y) \in \mathbb{C}^2 : \sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z = 0\right\}$ and particularly we uniformize it.

Finally in Section 4 we are interested in conformal gluing. There we prove Theorems 7-8 and Propositions 28-31.

2 Holomorphic continuation of the generating functions

Notation. For the sake of shortness we will drop, from now on, the dependence of the different quantities on $z \in]0, 1/n[$; it is implied that any statement in the sequel begins with "for any $z \in]0, 1/n[$".

This part aims at proving Theorem 4, in other words at showing that the generating functions $c(x)Q(x, 0)$ and $\tilde{c}(y)Q(0, y)$, already holomorphic in their open unit disc $\mathscr{D}$, can be holomorphically continued up to $\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}$ and $\mathscr{G}Y([x_1, x_2]) \cup \mathscr{D}$ respectively.

First of all, note that the location of the sets $\mathscr{G}X([y_1, y_2])$ and $\mathscr{G}Y([x_1, x_2])$ depends strongly on the steps set $\mathcal{S}$. In particular, it may happen that they are included in the unit disc – it is *e.g.* the case for both walks represented on Picture 6, as it will be illustrated on Picture 3 –, in that case Theorem 4 is obvious. On the other hand, there exist actually walks for which these sets don't belong to the unit disc – it is *e.g.* the case of Gessel's walk, see [KR09a].

Before starting the proof of Theorem 4, let us remark some facts around this important result. In [KR09a] we have also introduced a procedure of continuation for $c(x)Q(x, 0)$ and $\tilde{c}(y)Q(0, y)$; we choose here to present an other way in order to make this continuation, weaker but more elementary : weaker because Theorem 4 yields a continuation of the generating functions up to $\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}$ and $\mathscr{G}Y([x_1, x_2]) \cup \mathscr{D}$ and not up to $\mathbb{C} \setminus [x_3, x_4]$ and $\mathbb{C} \setminus [y_3, y_4]$ as in [KR09a], but more elementary because this continuation is done directly on the complex plane, rather than on the Riemann surface given by a uniformization of $\left\{(x, y) \in \mathbb{C}^2 : \sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z = 0\right\}$ as in [KR09a].

In order to be more accurate, let us first summarize the three-steps procedure of continuation we used in [KR09a] for Gessel's walk : firstly we had lifted the functions to the uniformization space defined in Section 3, secondly we had continued the lifted generating functions up to the whole uniformization space by use of the functional equation (2) and of the automorphisms (9), thirdly we had projected these continued lifted functions.

The advantages of this procedure are multiple : let us emphasize, among others, that it leads to a continuation up to the whole cut complex planes $\mathbb{C} \setminus [x_3, x_4]$ and $\mathbb{C} \setminus [y_3, y_4]$, and that it seems us to be quite natural by making symmetrical the roles of x and y.

On the other hand, if this procedure could be presented in a synthetic way for Gessel's walk, it could appear somewhat tedious in the general case, as the algorithmic procedure used in the second step of the continuation would depend strongly on the steps set $\mathcal{S}$ considered. This is why we have chosen to prove rather Theorem 4.

Proof of Theorem 4. We are going to explain the procedure of continuation only for $c(x)Q(x, 0)$, we would continue $\tilde{c}(y)Q(0, y)$ similarly.

To begin with, let us note that evaluating (2) at any x, y in the unit disc $\mathscr{D}$ such that $\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z = 0$ leads to $c(x)Q(x, 0) + \tilde{c}(y)Q(0, y) - z\delta Q(0, 0) - xy = 0$. As a consequence, if $x \in \{x \in \mathbb{C} : |Y_0(x)| < 1\} \cap \mathscr{D}$, we obtain

$$c(x)Q(x, 0) + \tilde{c}(Y_0(x))Q(0, Y_0(x)) - z\delta Q(0, 0) - xY_0(x) = 0. \tag{4}$$

Since $x \in \{x \in \mathbb{C} : |Y_0(x)| < 1\} \cap \mathscr{D}$ is non-empty, see (i) of Lemma 11 below, both functions $c(x)Q(x, 0)$ and $\tilde{c}(Y_0(x))Q(0, Y_0(x))$ as well as the identity (4) can be extended up to the connected component of $\{x \in \mathbb{C} : |Y_0(x)| < 1\} \cup \mathscr{D}$ containing $\{x \in \mathbb{C} : |Y_0(x)| < 1\} \cap \mathscr{D}$, by analytic continuation.

Therefore, in order to prove Theorem 4, it is enough to show that $\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}$ is connected and included in $\{x \in \mathbb{C} : |Y_0(x)| < 1\} \cup \mathscr{D}$, since $\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}$ has obviously a non-empty intersection with $\{x \in \mathbb{C} : |Y_0(x)| < 1\} \cap \mathscr{D}$. These facts are exactly the objects of (ii) and (iii) in Lemma 11 below.

It remains thus only to prove that this continuation of $c(x)Q(x, 0)$ is holomorphic on $\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}$.

On $\mathscr{D}$ this is immediate, $c(x)Q(x, 0)$ being there defined by its power series.

On $(\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}) \setminus \mathscr{D}$, it follows from (4) that the function $c(x)Q(x, 0)$ may have eventually the same singularities as Y_0 – namely the branch cuts $[x_1, x_2], [x_3, x_4]$ – and is holomorphic elsewhere. But these segments don't belong to $(\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}) \setminus \mathscr{D}$: with Section 1 we have that $[x_1, x_2]$ is included in $\mathscr{D}$ and by Lemma 3 we get that $[x_3, x_4]$ is exterior to $\mathscr{G}X([y_1, y_2])$.

The continuation of $c(x)Q(x, 0)$ is thus holomorphic on $\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}$ and Theorem 4 is proved. □

Lemma 11. (i) $\{x \in \mathbb{C} : |Y_0(x)| < 1\} \cap \mathscr{D}$ *is non-empty* (ii) $\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}$ *is connected* (iii) $\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}$ *is included in* $\{x \in \mathbb{C} : |Y_0(x)| < 1\} \cup \mathscr{D}$.

Proof. Note first that (i) is a straightforward consequence of Lemma 12. (ii) is also clear : both sets $\mathscr{G}X([y_1, y_2])$ and $\mathscr{D}$ are connected and the intersection $\mathscr{G}X([y_1, y_2]) \cap \mathscr{D}$ is non-empty, *e.g.* because x_1 belongs to both sets, so that the union $\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}$ is connected. And now we show (iii).

Clearly, it is enough to prove that $(\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}) \setminus \mathscr{D}$ is included in $\{x \in \mathbb{C} : |Y_0(x)| < 1\}$. This will follow from an application of the maximum modulus principle (see *e.g* [JS87]) to the function Y_0 on $(\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}) \setminus \mathscr{D}$.

First of all let us note that Y_0 is analytic on the latter domain, since thanks to Section 1 it is included in $\mathbb{C} \setminus ([x_1, x_2] \cup [x_3, x_4])$.

Next we prove that $|Y_0| < 1$ on the boundary of the set $(\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}) \setminus \mathscr{D}$, and for this it is enough to show that $|Y_0| < 1$ on $\{|x| = 1\} \cup X([y_1, y_2])$.

But by Lemma 12 it is immediate that $|Y_0| < 1$ on $\{|x| = 1\}$, and by Lemma 13 we get that $Y_0(X([y_1, y_2])) = [y_1, y_2]$, segment which is known to belong to the unit disc, thanks to Section 1.

In this way, the maximum modulus principle directly entails that $|Y_0| < 1$ on the domain $(\mathscr{G}X([y_1, y_2]) \cup \mathscr{D}) \setminus \mathscr{D}$. □

Lemma 12. $Y_0(\{|x| = 1\}) \subset \{|y| < 1\}$.

Proof. To avoid any confusion – in this proof we are going to use different values of z –, we reintroduce here the variable z. Let us first recall that Y_0 is one of the two solutions in y of $\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z = 0$, and that if Y_1 denotes the other one then we have $|Y_0| \leq |Y_1|$, see Lemma 1.

We are going to prove Lemma 12 first for $z = 1/n$ and we will deduce from this particular case the remaining cases $z \in]0, 1/n[$.

For $z=1/n$ the equality $\sum_{(i,j)\in\mathcal{S}} x^iy^j-1/z=0$ can be written $\sum_{(i,j)\in\mathcal{S}}(1/n)x^iy^j=1$; since $\sum_{(i,j)\in\mathcal{S}}(1/n)=1$ we can apply Lemma 2.3.4 of [FIM99] and in this way we directly obtain that $Y_0(\{|x|=1\},1/n)\subset\{|y|\leq 1\}$.

Suppose now that $z\in]0,1/n[$. In this case it is *not* possible to have $\sum_{(i,j)\in\mathcal{S}} x^iy^j-1/z=0$ with $|x|=|y|=1$: indeed, for $|x|=|y|=1$ we have $|\sum_{(i,j)\in\mathcal{S}} x^iy^j|\leq\sum_{(i,j)\in\mathcal{S}} 1=n<1/z$. As a consequence $Y_0(\{|x|=1\},z)\cap\{|y|=1\}=\emptyset$. By connectedness this implies that either $Y_0(\{|x|=1\},z)\subset\{|y|<1\}$ or $Y_0(\{|x|=1\},z)\subset\{|y|>1\}$.

But once again with Lemma 2.3.4 of [FIM99] we get $Y_0(\{|x|=1\},1/n)\cap\{|y|<1\}\neq\emptyset$, so that by continuity we obtain for $z\in]0,1/n[$ that $Y_0(\{|x|=1\},z)\subset\{|y|<1\}$. ☐

In the proof of Theorem 4, we have also used the following result, proved in Part 5.3 of [FIM99] for $z=1/n$; the proof for other values of z is completely similar and we omit it.

Lemma 13. *Suppose that the walk is non-singular – see Section 1. Then the two functions* $X_0:\mathscr{G}Y([x_1,x_2])\setminus[y_1,y_2]\to\mathscr{G}X([y_1,y_2])\setminus[x_1,x_2]$ *and* $Y_0:\mathscr{G}X([y_1,y_2])\setminus[x_1,x_2]\to\mathscr{G}Y([x_1,x_2])\setminus[y_1,y_2]$ *are conformal and reciprocal the one from the other.*

3 Uniformization

This part is introductory to Section 4 and consists in studying closely the set of zeros of the kernel, namely $\mathscr{K}=\big\{(x,y)\in\mathbb{C}^2:\sum_{(i,j)\in\mathcal{S}} x^iy^j-1/z=0\big\}$.

Proposition 14. *For any non-singular walk,* $\mathscr{K}$ *is a Riemann surface of genus one.*

Proof. From Section 1 we know that $\sum_{(i,j)\in\mathcal{S}} x^iy^j-1/z=0$ if and only if $(b(x)+2a(x)y)^2=d(x)$. But the Riemann surface of the square root of a third or fourth degree polynomial with distinct roots has genus one, see *e.g.* [JS87], therefore the genus of $\mathscr{K}$ is also one. ☐

Remark 15. *Note first that Proposition 14 can't be extended to the singular walks. Indeed, it follows from Section 1 that for the singular walks the polynomial d has a double root at* 0 *and two simple roots at* $x_3\neq x_4$, *and it is well-known, see e.g. [JS87], that the Riemann surface of the square root of such a polynomial has genus zero.*

Note also that Proposition 14, implicitly stated for $z\in]0,1/n[$, *can't be extended to* $z=0$ *or* $z=1/n$ *in the general case.*

Indeed, for $z=0$ *we have* $d(x)=x^2$ *and the Riemann surface of the square root of this polynomial is a disjoint union of two spheres, see e.g. [JS87].*

As for $z=1/n$, *it may happen that the genus of* $\mathscr{K}$ *is still one as it may happen that it becomes zero. In fact, Parts 2.3 and 6.1 of [FIM99] entail that it equals one (resp. zero) if and only if not the two (resp. the two) equalities* $\sum_{(i,j)\in\mathcal{S}} i=0$ *and* $\sum_{(i,j)\in\mathcal{S}} j=0$ *hold.*

With Proposition 14 it is immediate that $\mathscr{K}$ is isomorphic to some torus $\mathbb{C}/\Omega$. A suitable lattice Ω (in fact the *only possible* lattice, up to a homothetic transformation) is made explicit in Parts 3.1 and 3.3 of [FIM99], namely $\Omega=\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$, where

$$\omega_1=\imath\int_{x_1}^{x_2}\frac{\mathrm{d}x}{[-d(x)]^{1/2}},\qquad\qquad \omega_2=\int_{x_2}^{x_3}\frac{\mathrm{d}x}{[d(x)]^{1/2}}. \tag{5}$$

By using the same arguments as in Part 3.3 of [FIM99], we immediately obtain that we have in addition the *uniformization* $\mathscr{K}=\{(x(\omega),y(\omega)),\omega\in\mathbb{C}/\Omega\}$, where

$$x(\omega)=F(\wp(\omega),\wp'(\omega)),\qquad y(\omega)=G(\wp(\omega),\wp'(\omega)),\tag{6}$$

with $F(p,p')=x_4+d'(x_4)/(p-d''(x_4)/6)$, $G(p,p')=[-b(F(p,p'))+d'(x_4)p'/(2(\wp(\omega)-d''(x_4)/6)^2)]/[2a(F(p,p'))]$ if $x_4\neq\infty$ and $F(p,p')=(6p-d''(0))/d'''(0)$, $G(p,p')=[-b(F(p,p'))-3p'/d'''(0)]/[2a(F(p,p'))]$ if $x_4=\infty$, $\wp$ being the Weierstrass elliptic function with periods ω_1,ω_2.

It is well-known, see *e.g.* [JS87], that $\wp$ satisfies the differential equation

$$\wp'(\omega)^2=4[\wp(\omega)-\wp(\omega_1/2)][\wp(\omega)-\wp([\omega_1+\omega_2]/2)][\wp(\omega)-\wp(\omega_2/2)].\tag{7}$$

Moreover, it is proved in Part 3.3 of [FIM99] that $\wp(\omega_1/2)=f(x_3)$, $\wp([\omega_1+\omega_2]/2)=f(x_2)$ and $\wp(\omega_1/2)=f(x_1)$.

We are now going to be interested in the location on the parallelogram $[0,\omega_2[\times[0,\omega_1/\imath[$ of the reciprocal images through the uniformization of the important cycles that are the branch cuts $[x_1,x_2]$, $[x_3,x_4]$, $[y_1,y_2]$ and $[y_3,y_4]$. For this we need to define a new period, namely

$$\omega_3=\int_{X(y_1)}^{x_1}\frac{\mathrm{d}x}{[d(x)]^{1/2}}.\tag{8}$$

In Part 3.3 of [FIM99] is shown the following.

Lemma 16. $\omega_3\in]0,\omega_2[$.

By using then exactly the same analysis as in Part 5.5 of [FIM99], we get the following very nice result – one of the essential purposes of having introduced the uniformization (6).

Proposition 17. $x^{-1}([x_1,x_2])=[0,\omega_1[+\omega_2/2$, $x^{-1}([x_3,x_4])=[0,\omega_1[$, $y^{-1}([y_1,y_2])=[0,\omega_1[+(\omega_2+\omega_3)/2$ *and* $y^{-1}([y_3,y_4])=[0,\omega_1[+\omega_3/2$.

Now we would like to study more accurately ω_3. First, by use of the same arguments as the ones given page 14 of [KR09b], we obtain the location of ω_3 w.r.t. $\omega_2/2$.

Proposition 18. $\omega_3<\omega_2/2$ *(resp.* $\omega_3=\omega_2/2$, $\omega_3>\omega_2/2$*) if and only if the covariance (i.e. the quantity* $\sum_{(i,j)\in\mathcal{S}}ij$*) of the walk is negative (resp. zero, positive).*

It could appear surprising to introduce here the covariance, in fact it will turn out that on its sign depends interestingly a lot of quantities – in this perspective see Subsection 4.4.

If the group $W=\langle\Psi,\Phi\rangle$ defined in Section 1 is finite, we can precise Proposition 18.

Proposition 19. *For any* $k\geq 2$, *the group* $W=\langle\Psi,\Phi\rangle$ *has order* $2k$ *if and only if there exists an integer* $q\in\{1,\dots,k-1\}$, *independent of* z *and having no common divisor with* k, *such that for all* $z\in]0,1/n[$, $\omega_3=(q/k)\omega_2$.

In particular, since it is proved in [BMM09] that the group W takes only the orders 4, 6, 8 and ∞, Propositions 18-19 lead immediately to the following.

Corollary 20. *The group* $W = \langle \Psi, \Phi \rangle$ *has order* 4 *if and only if for all* $z \in]0, 1/n[$, $\omega_3 = \omega_2/2$. *For* $k \in \{3, 4\}$, *it has order* $2k$ *and a negative (resp. positive) covariance if and only if for all* $z \in]0, 1/n[$, $\omega_3 = \omega_2/k$ *(resp.* $\omega_3 = \omega_2 - \omega_2/k$*)).*

Before beginning the proof of Proposition 19, we emphasize that the birational transformations Ψ and Φ can obviously be understood as automorphisms of $\mathscr{K}$, and we recall from Part 3.1 of [FIM99] that thanks to the uniformization (6) these automorphisms of $\mathscr{K}$ become on $\mathbb{C}/\Omega$ the automorphisms ψ and ϕ with the following expressions :

$$\psi(\omega) = -\omega, \qquad \phi(\omega) = -\omega + \omega_3. \tag{9}$$

They are such that $\psi \circ \psi = \phi \circ \phi = \text{id}$, $x \circ \psi = x$, $y \circ \psi = [c(x)/a(x)]/y$, $x \circ \phi = [\tilde{c}(y)/\tilde{a}(y)]/x$ and $y \circ \phi = y$.

Proof of Proposition 19. Let us begin by proving the converse sense of Proposition 19 and for this let us, here again, reintroduce the variable z.

Suppose first that for *some* value $z \in]0, 1/n[$, $\omega_3/\omega_2 = q/k$, the integers q and k having no common divisor. This means that $\langle \psi, \phi \rangle$ is a finite group of order $2k$, see (9), which is equivalent to the fact $\langle \Psi, \Phi \rangle$ has order $2k$ on $\{(x, y) \in \mathbb{C}^2 : \sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z = 0\}$ – but *a priori* not on $\mathbb{C}^2$. This implies that for any $x \in \mathbb{C}$, the equalities $(\Phi \circ \Psi)^{\circ k}(x, Y_0(x, z)) = (x, Y_0(x, z))$ and $(\Phi \circ \Psi)^{\circ k}(x, Y_1(x, z)) = (x, Y_1(x, z))$ hold.

Suppose now that for *any* value $z \in]0, 1/n[$, ω_3/ω_2 is this same rational number q/k. In particular for any fixed $x \in \mathbb{C}$ and all $z \in]0, 1/n[$, $(\Phi \circ \Psi)^{\circ k}(x, Y_0(x, z)) = (x, Y_0(x, z))$. In other words, for any fixed $x \in \mathbb{C}$ and any $y \in \{Y_0(x, z) : z \in]0, 1/n[\}$, $(\Phi \circ \Psi)^{\circ k}(x, y) = (x, y)$. Since the set $\{Y_0(x, z) : z \in]0, 1/n[\}$ is not isolated, by analytic continuation we obtain that for any $x \in \mathbb{C}$ and any $y \in \mathbb{C}$, $(\Phi \circ \Psi)^{\circ k}(x, y) = (x, y)$, in such a way that $\langle \Psi, \Phi \rangle$ is finite as a group of birational transformations, of order less than or equal to $2k$.

$\langle \Psi, \Phi \rangle$ is of order exactly $2k$ because $\langle \psi, \phi \rangle$ has order $2k$ and thus we can find some elements (x, y) of order exactly $2k$. This entails the converse sense of Proposition 19.

Suppose now that $W = \langle \Psi, \Phi \rangle$ has order $2k$. The group $\langle \psi, \phi \rangle$ generated by ψ and ϕ is *a fortiori* finite, of order $2r(z) \leq 2k$, which means that $\inf\{p > 0 : (\phi \circ \psi)^{\circ p} = \text{id}\} = r(z)$. With (9) this immediately implies that $r(z)\omega_3$ is some point of the lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, contrary to $p\omega_3$ for $p \in \{1, \ldots, r(z) - 1\}$. But with Lemma 16 we have $\omega_3 \in]0, \omega_2[$, so that we get $r(z)\omega_3 = q(z)\omega_2$, where $q(z) \in \{1, \ldots, r(z) - 1\}$ has no common divisor with $r(z)$.

Moreover, from (5) and (8) we know that $\omega_3/\omega_2 = q(z)/r(z)$ is a holomorphic function of z ; taking rational values it has to be constant, say $\omega_3/\omega_2 = q/r$. Finally, if r was strictly smaller than k, then with the first part of the proof we would obtain that $W = \langle \Psi, \Phi \rangle$ has also order $2r$, and not $2k$ as assumed, so that $r = k$ and Proposition 19 is proved. □

The location of the reciprocal images through (6) of $[x_1, x_2]$, $[x_3, x_4]$, $[y_1, y_2]$ and $[y_3, y_4]$ being known, let us give, in Picture 3, some examples of the parallelogram $[0, \omega_2[\times[0, \omega_1/\imath[$ – in addition of the one corresponding to Gessel's walk that can be found in [KR09a].

Indeed, for the first walk of Picture 5, the group has order four, see [BMM09], and with Corollary 20 we get $\omega_3 = \omega_2/2$. In addition it is quite easy to show that in this case $X([y_1, y_2]) = X([y_3, y_4]) = Y([x_1, x_2]) = Y([x_3, x_4])$ is equal to the unit circle.

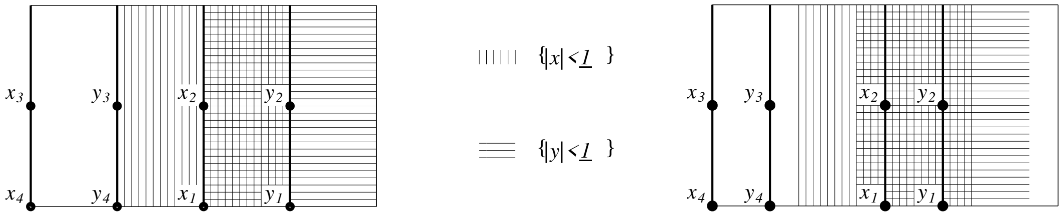

Figure 3: Examples of the parallelogram $[0, \omega_2[\times[0, \omega_1/\imath[$ with its important cycles, at the left for the first walk of Picture 5 below and at the right for the second walk of Picture 6

In particular, we obtain that $x^{-1}(\{|x| = 1\}) = ([0, \omega_1[+\omega_2/4) \cup ([0, \omega_1[+3\omega_2/4)$ and $y^{-1}(\{|y| = 1\}) = ([0, \omega_1[) \cup ([0, \omega_1[+\omega_2/2)$.

Also, for the second walk of Picture 6, the group has order six and the covariance is negative, hence with Corollary 20 we get $\omega_3 = \omega_2/3$. Then by using the same arguments as in the proof of Proposition 23 of [KR09a] we obtain the location of the cycles $x^{-1}(\{|x| = 1\})$ and $y^{-1}(\{|y| = 1\}$ for this walk, namely $x^{-1}(\{|x| = 1\}) = ([0, \omega_1[+\omega_2/4) \cup ([0, \omega_1[+3\omega_2/4)$ and $y^{-1}(\{|y| = 1\} = ([0, \omega_1[+5\omega_2/12) \cup ([0, \omega_1[+11\omega_2/12)$.

4 Conformal gluing functions

The main subject of Section 4 is to introduce and to study suitable CGF (see Definition 5) for the sets $\mathscr{G}X([y_1, y_2])$ and $\mathscr{G}Y([x_1, x_2])$ for the 74 non-degenerate walks.

For this we will first, in Subsection 4.1, find explicitly all appropriate CGF for both sets above (essentially thanks to the uniformization studied in Section 3) and in particular we will prove Theorem 7. Then we will remark that the behavior of these CGF is quite different according to the finiteness of the group W. This is why we will focus separately, in Subsections 4.2 and 4.3, on the walks having an infinite and next a finite group, and we will prove Theorem 8 and Propositions 28-31. Finally, in Subsection 4.4, we will compare this classification of the 79 walks according to the nature of the CGF with the one according to the nature of the series (1), obtained from [BMM09] and [BK09].

4.1 Finding all suitable conformal gluing functions

In [FIM99] is found explicitly *one* CGF and *only* for $z = 1/n$, see (10) below. Let us begin here by generalizing this result and by finding the expressions of *all* possible CGF for the sets $X([y_1, y_2])$ and $Y([x_1, x_2])$ for *any* value of z, and for this let us quote page 126 of [FIM99] : if we note $\hat{w} = w \circ x$ or $\hat{w} = w \circ y$, with x, y defined in (6), then the problem of finding a CGF w is equivalent *to find a function $\hat{w}$ meromorphic in $[\omega_2/2, (\omega_2+\omega_3)/2] \times \mathbb{R}$, ω_1 periodic, with only one simple pole in the domain $[\omega_2/2, (\omega_2+\omega_3)/2] \times [0, \omega_1/\imath]$ hatched at the left of Picture 4 below, and satisfying to the next two conditions :*

(i) *for all* $\omega \in [-\omega_1/2, \omega_1/2]$, $\hat{w}([\omega_1 + \omega_2]/2 + \omega) = \hat{w}([\omega_1 + \omega_2]/2 - \omega)$,

(ii) *for all* $\omega \in [-\omega_1/2, \omega_1/2]$, $\hat{w}([\omega_1 + \omega_2 + \omega_3]/2 + \omega) = \hat{w}([\omega_1 + \omega_2 + \omega_3]/2 - \omega)$.

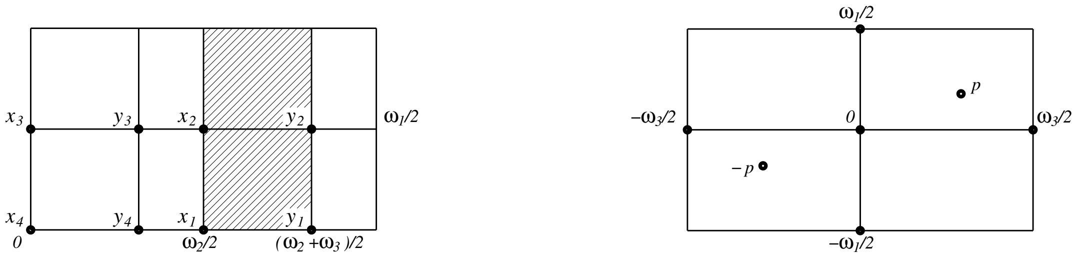

Figure 4: Domains of definition of $\hat{w}$ (at the left) and $\check{w}$ (at the right)

Setting $\check{w}(\omega) = \hat{w}([\omega_1 + \omega_2]/2 + \omega)$, the problem mentioned above obviously becomes *to find a function* $\check{w}$ *meromorphic in* $[0, \omega_3/2] \times \mathbb{R}$, ω_1 *periodic, with only one simple pole in the domain* $[0, \omega_3/2] \times [-\omega_1/(2\imath), \omega_1/(2\imath)]$, *and such that :*

(i') *for all* $\omega \in [-\omega_1/2, \omega_1/2]$, $\check{w}(\omega) = \check{w}(-\omega)$,

(ii') *for all* $\omega \in [-\omega_1/2, \omega_1/2]$, $\check{w}(\omega_3/2 + \omega) = \check{w}(\omega_3/2 - \omega)$.

Now we remark that by analytic continuation, (i') allows us to continue $\check{w}$ from $[0, \omega_3/2] \times \mathbb{R}$ up to $[-\omega_3/2, \omega_3/2] \times \mathbb{R}$, and next, also by analytic continuation, (ii') enables us to continue $\check{w}$ as a ω_3 periodic function – since evaluating (ii') at $\omega_3/2 + \omega$ and using (i') lead to $\check{w}(\omega_3 + \omega) = \check{w}(\omega)$. In particular, the problem of finding the CGF finally becomes : *to find* $\check{w}$ *an even elliptic function with periods* ω_1, ω_3 *with only two simple poles at* $\pm p$ *(or one double pole at* p *if* p *and* $-p$ *are congruent modulo the lattice) in the parallelogram* $[-\omega_3/2, \omega_3/2] \times [-\omega_1/(2\imath), \omega_1/(2\imath)]$ *drawn at the right of Picture 4 above.*

A crucial fact for us is then the following – below, $\wp_{1,3}$ denotes the Weierstrass elliptic function with periods ω_1, ω_3.

Lemma 21. *Let* $p \in [-\omega_3/2, \omega_3/2] \times [-\omega_1/(2\imath), \omega_1/(2\imath)]$.
If $p = 0$, *the only solutions of the problem above are the* $\alpha + \beta\wp_{1,3}(\omega)$ *for* $\alpha, \beta \in \mathbb{C}$.
If $p \neq 0$, *the only solutions are the* $\alpha + \beta/[\wp_{1,3}(\omega) - \wp_{1,3}(p)]$ *for* $\alpha, \beta \in \mathbb{C}$.

Proof. It is well-known, see *e.g.* Theorem 3.11.1 and Theorem 3.13.1 of [JS87], that an even elliptic function with periods ω_1, ω_3 and having with $2q$ poles in a period parallelogram is necessarily a rational transformation of order q of $\wp_{1,3}$. In our case, $\check{w}$, which has exactly two poles of order one or one pole of order two in $[-\omega_3/2, \omega_3/2] \times [-\omega_1/(2\imath), \omega_1/(2\imath)]$, is thus a fractional linear transformation of $\wp_{1,3}$.

In particular, it is immediate that $p = 0$ yields $\check{w}(\omega) = \alpha + \beta\wp_{1,3}(\omega)$.

If $p \neq 0$ then $\wp_{1,3}(p) \neq \infty$ and we obtain $\check{w}(\omega) = [\alpha\wp_{1,3}(\omega) + \gamma]/[\wp_{1,3}(\omega) - \wp_{1,3}(p)]$. ☐

Applying Lemma 21 for $p = 0$ we get that

$$w(t) = \wp_{1,3}\big(x^{-1}(t) - [\omega_1 + \omega_2]/2\big), \qquad \widetilde{w}(t) = \wp_{1,3}\big(y^{-1}(t) - [\omega_1 + \omega_2]/2\big) \tag{10}$$

are suitable CGF for the sets $X([y_1, y_2])$ and $Y([x_1, x_2])$, x^{-1} and y^{-1} denoting in (10) the reciprocal functions of the coordinates of the uniformization (6). Theorem 7 is thus proved.

Applying now Lemma 21 for *any* value of p yields that the fractional linear transformations of (10) are the only possible CGF.

Remark 22. *As said in Section 1, it can be deduced from [Lit00] – by using general results on Fredholm integral equations – that two CGF for the same set are necessarily fractional linear transformations the one from the other.*

It is worth noting that in our case, we have recovered this fact by using only the properties of the Weierstrass elliptic functions written in Lemma 21.

Remark 23. *The functions w and $\tilde{w}$ of (10) are related together through $\tilde{w} = w(X_0)$: indeed, (6) entails that $x(\omega) = X_0(y(\omega))$.*

Let us now concentrate on w and $\tilde{w}$ defined by (10), and note first that in Section 4 of [KR09b] we have proved their following global properties.

Proposition 24. *The function w (resp. $\tilde{w}$) defined by (10) is meromorphic on $\mathbb{C}\setminus[x_3, x_4]$ (resp. $\mathbb{C}\setminus[y_3, y_4]$) and has there one simple pole, at x_2 (resp. $Y(x_2)$), and the lower integer part of $\omega_2/(2\omega_3)$ double poles at some points of the segment $]x_2, x_3[\cap(\mathbb{C}\setminus\mathscr{G}X([y_1, y_2]))$ (resp. $]y_2, y_3[\cap(\mathbb{C}\setminus\mathscr{G}Y([x_1, x_2])))$.*

Now we remark that with (6) it is immediate that $w(t) = \wp_{1,3}\big(\wp^{-1}(f(t)) - [\omega_1+\omega_2]/2\big)$, with $f(t) = d''(x_4)/6 + d'(x_4)/(t - x_4)$ if $x_4 \neq \infty$ and $f(t) = d''(0)/6 + d'''(0)t/6$ if $x_4 = \infty$.

The CGF w is therefore equal to the Weierstrass elliptic function with periods ω_1, ω_3 evaluated at a translation of the reciprocal of the Weierstrass elliptic function with periods ω_1, ω_2. It turns out that the theory of transformation of elliptic functions – the basic result of which being here recalled in Lemma 25 below – entails that this expression admits a very nice simplification if ω_3/ω_2 is rational.

But Proposition 19 shows that the latter condition is related to the group $W = \langle\Psi, \Phi\rangle$ defined in Section 1, since it states that ω_3/ω_2 is rational for all $z \in]0, 1/n[$ if and only if the group W is finite, so that we will consider separately the study of w and $\tilde{w}$ according to the finiteness of this group.

First, in Subsection 4.2, we will concentrate on the case of an *infinite* group of the walk, and we will show that w and $\tilde{w}$ are then non-holonomic, which will prove the part of Theorem 8 concerning the infinite group.

Then, in Subsection 4.3, we will consider the case of a *finite* group. We will see that w and $\tilde{w}$ are then algebraic, and we will complete the proof of Theorem 8. Moreover, we will considerably simplify the explicit expressions of w and $\tilde{w}$ given in (10) for all the 23 walks having a finite group : see Proposition 28 (resp. 29-30, 31) for the walks associated with a group of order four (resp. six, eight).

It is worth noting that, even for $z = 1/n$, the forthcoming results of Subsections 4.2-4.3 on an in-depth study of the CGF are new – up to the best of our knowledge.

In order to close this part we state the following result, mentioned above and proved *e.g.* at `http://functions.wolfram.com/EllipticFunctions/WeierstrassP/16/06/03/` ; it will be of the highest importance for the proofs of Propositions 28-31.

Lemma 25. *Let $\breve{\wp}$ be the Weierstrass elliptic function with some periods $\acute{\omega}, \grave{\omega}$ and let p be some positive integer. Then the Weierstrass elliptic function with periods $\acute{\omega}, \grave{\omega}/p$ can be written in terms of $\breve{\wp}$ as $\breve{\wp}(\omega) + \sum_{k=1}^{n-1}\left[\breve{\wp}(\omega + k\grave{\omega}/p) - \breve{\wp}(k\grave{\omega}/p)\right]$.*

4.2 Case of an infinite group

Remark 26. *If the walk admits an infinite group, it may happen that ω_3/ω_2 is rational for some values of z, but ω_3/ω_2 has also to take non-rational values.*

Indeed, if for all $z \in]0, 1/n[$ ω_3/ω_2 was rational, then ω_3/ω_2 would be a rational constant, since with (5) and (8) ω_3/ω_2 is a holomorphic function of z, and Proposition 19 would then entail that the group is finite.

Let us now prove the part of Theorem 8 concerning the group infinite, and more precisely that for any $z \in]0, 1/n[$ such that ω_3/ω_2 is non-rational, the CGF w and $\tilde{w}$ defined in (10) are non-holonomic.

Proof. The class of holonomic functions being closed under algebraic substitution, see *e.g.* [FS09], it is enough to prove that $v(t) = w(f^{-1}(t)) = \wp_{1,3}(\wp^{-1}(t) - [\omega_1 + \omega_2]/2)$ is non-holonomic, since on the one hand f is rational and on the other hand $\tilde{w} = w(X_0)$ with X_0 algebraic. We are going first to show that v is non-algebraic, and then we will prove that if v is holonomic then it has to be algebraic, in such a way that v will be non-holonomic.

Suppose thus that v is algebraic, in other words assume that there exist polynomials $a_0, \ldots, a_q$ with $a_q \neq 0$ such that $\sum_{k=0}^{q} a_k(t)v(t)^k = 0$. By definition of v and since the Weierstrass elliptic function is non-algebraic, at least one of $a_0, \ldots, a_q$ is non-constant.

Evaluating now the last equality at $t = \wp(\omega + [\omega_1 + \omega_2]/2)$ and using the definition of v, we get $\sum_{k=0}^{q} a_k(\wp(\omega + [\omega_1 + \omega_2]/2))\wp_{1,3}(\omega)^k = 0$. Since $\wp(\omega + [\omega_1 + \omega_2]/2)$ is a rational transformation of $\wp(\omega)$, see *e.g.* the addition theorem (14) below, the previous equality reads $P(\wp(\omega), \wp_{1,3}(\omega)) = 0$, where P is a polynomial non-constant w.r.t. the two variables.

Now we recall from [JS87] that $\wp$ (resp. $\wp_{1,3}$) has poles at every point of the lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ (resp. $\omega_1\mathbb{Z} + \omega_3\mathbb{Z}$). Moreover, it is well-known that if ω_3/ω_2 is non-rational then $\omega_2\mathbb{Z} + \omega_3\mathbb{Z}$ is dense in $\mathbb{R}$. In particular, if ω_3/ω_2 is non-rational then the poles of $P(\wp(\omega), \wp_{1,3}(\omega))$ are non-isolated, which is a contradiction between the principle of analytic continuation and the fact that $P(\wp(\omega), \wp_{1,3}(\omega))$ is non-zero and meromorphic.

Suppose now that v is holonomic, *i.e.* that there are polynomials $a_0, \ldots, a_q$ with $a_q \neq 0$ such that $\sum_{k=0}^{q} a_k(t)v^{(k)}(t) = 0$, $v^{(k)}$ denoting the kth derivative of v, and let us show that in this case

$$\sum_{k=0}^{q} a_k(t)v^{(k)}(t) = U(t, v(t)) + V(t, v(t))v'(t), \tag{11}$$

where the dependence of U and V w.r.t. the first (resp. second) variable is rational (resp. polynomial), and where at least U or V is non-zero. Before showing these facts, let us explain how they entail the algebricity of v.

If V is identically zero then U has to be non-zero. Moreover, since U is rational w.r.t. the first variable and since with (11) we have $U(t, v(t)) = 0$, U has to be non-constant w.r.t. the second variable and (11) immediately yields that v is then algebraic.

Suppose now that V is not identically zero. From (11) and the supposed holonomy of v it follows that $U(t, v(t))^2 - V(t, v(t))^2 v'(t)^2 = 0$. Also, noting g the derivative of $\wp^{-1}$ – if g_2 and g_3 are the invariants of $\wp$ we have $g(t) = [4t^3 - g_2 t - g_3]^{-1/2}$ – we get $v'(t)^2 = g(t)^2 \wp_{1,3}'(\wp^{-1}(t) - [\omega_1 + \omega_2]/2)^2$. But if $g_{2,1,3}$ and $g_{3,1,3}$ are the invariants of $\wp_{1,3}$ we have $\wp_{1,3}'^2=4\wp_{1,3}^3-g_{2,1,3}\wp_{1,3}-g_{3,1,3}$, and finally we get $U(t,v(t))^2-V(t,v(t))^2g(t)^2[4v(t)^3-g_{2,1,3}v(t)-g_{3,1,3}]=0$. The latter quantity is rational in the first variable, since $g(t)^2$ is rational, and an odd degree polynomial in the second, in such a way that v is algebraic.

So it is enough to prove (11). For this we are going to show that for any k,

$$v^{(k)}(t)=U_k(t,v(t))+V_k(t,v(t))v'(t), \tag{12}$$

where the dependence of U_k (resp. V_k) is rational w.r.t. the first variable and polynomial of degree exactly $\lfloor k/2+1\rfloor$ (resp. $\lfloor(k-1)/2\rfloor$) w.r.t. the second variable, $\lfloor\cdot\rfloor$ denoting the lower integer part. Equality (11) will be then an immediate consequence of (12). Indeed, if q is even then by using (12) in (11) we obtain that the degree of U in v is exactly $\lfloor q/2+1\rfloor$ and U is thus obviously non-zero. Likewise, if q is odd we get that the degree of V is exactly $\lfloor(q-1)/2\rfloor$ and is thus clearly non-zero if $q\geq 3$. If $q=1$ then we easily make explicit $V_0=0$ and V_1, and we immediately deduce that V is also non-zero.

Let us now show (12). For $k=0$ this is obvious. As for $k\geq 1$, a straightforward calculation leads to $v^{(k)}(t)=\sum_{p=1}^k b_p(t)\wp_{1,3}^{(p)}(\wp^{-1}(t)-[\omega_1+\omega_2]/2)$, with $b_k=g(t)^k$, $b_{k-1}(t)=[k(k-1)/2]g'(t)g(t)^{k-2}$ and for $p\in\{2,\ldots,k-1\}$, b_{k-p} is a polynomial in the variables $g(t),g'(t),\ldots,g^{(p)}(t)$. Moreover, if p is even then b_p clearly is rational whereas if p is odd then b_p/g is rational. Next by a repeated use of Lemma 27 below and by using that $\wp_{1,3}'(\wp^{-1}(t)-[\omega_1+\omega_2]/2)=v'(t)/g(t)$ we obtain

$$v^{(k)}(t)=\sum_{p\neq 0\text{ even}} b_p(t)p_{p/2}(v(t))+v'(t)\sum_{p\text{ odd}}[b_p(t)/g(t)]p'_{(p-1)/2}(v(t)).$$

With the values of b_k and b_{k-1} given above this immediately yields (12), and therefore also (11), and finally the fact that v, w and $\tilde{w}$ are non-holonomic. ☐

Let us recall from [JS87] the following classical fact, that we have used in the proof above.

Lemma 27. *For any integer $r\geq 0$ we have $\wp^{(2r)}=p_r(\wp)$, where $p_r(x)$ is a polynomial with dominant coefficient equal to $(2r+1)!\,x^{r+1}$.*

4.3 Case of a finite group

Thanks to Proposition 19, we obtain that for the 23 walks having a finite group, ω_2/ω_3 is rational. Moreover, with [BMM09] and Corollary 20 it is enough to consider the cases ω_2/ω_3 equal to $2,3,3/2,4,4/3$. According to the classification of [BMM09] there are respectively $16,2,3,1,1$ such walks.

Proposition 28. *If the walk has a group of order four (i.e. if $\mathcal{S}$ has a vertical symmetry, see [BMM09]), or equivalently if $\omega_2/\omega_3=2$, then w defined in (10) is an affine combination of $[(t/x_4-1)(t-x_1)]/[(t-x_2)(t-x_3)]$. Moreover, suitable and explicit CGF for the set $\mathscr{G}Y([x_1,x_2])$ are $w(X_0(t))$, or, more symmetrically, $[(t/y_4-1)(t-y_1)]/[(t-y_2)(t-y_3)]$.*

Figure 5: Two examples among the sixteen walks with a group of order four

Proposition 29. *For both walks represented on Picture 6 – the only ones with a group of order six and a negative covariance, or equivalently verifying $\omega_2/\omega_3 = 3$ –, w and $\tilde{w}$ defined in (10) are affine combinations of respectively*

$$\frac{u(t)}{(t-x_2)(t-1/{x_2}^{1/2})^2}, \qquad \frac{\widetilde{u}(t)}{(t-y_2)(t-1/{y_2}^{1/2})^2},$$

with $u(t) = t^2$ and $\tilde{u}(t) = t$ (resp. $u(t) = \tilde{u}(t) = t(t+1)$) for the walk at the left (resp. at the right) of Picture 6.

Figure 6: The two walks with a group of order six and a negative covariance

Proposition 30. *For the three walks represented on Picture 7 – the only ones with a group of order six and a positive covariance, or equivalently verifying $\omega_2/\omega_3 = 3/2$ –, there exist $\alpha, \beta, \delta, \gamma$ which are algebraic w.r.t. z and made explicit in the proof, such that $w = \tilde{w}$ defined in (10) is the only solution with a pole at x_2 of*

$$w^2 + \left[\alpha + \beta\frac{u(t)}{(t-x_2)(t-1/{x_2}^{1/2})^2}\right] w + \left[\delta + \gamma\frac{u(t)}{(t-x_2)(t-1/{x_2}^{1/2})^2}\right] = 0,$$

with $u(t) = t^2$ (resp. $u(t) = t$, $u(t) = t(t+1)$) for the walk at the left (resp. in the middle, at the right) of Picture 7.

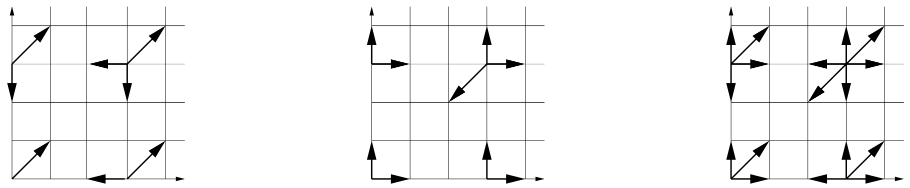

Figure 7: The three walks with a group of order six and a positive covariance

Proposition 31. *For the walk represented at the left of Picture 8 – the only one with a group of order eight and a negative covariance, or equivalently verifying $\omega_2/\omega_3 = 4$ –, w and $\tilde{w}$ defined in (10) are affine combinations of respectively*

$$\frac{t^2}{(t-x_2)(t-1)^2(t-x_3)}, \qquad \frac{t(t+1)^2}{(t-x_2)^2(t-x_3)^2}.$$

Figure 8: The two walks with a group of order eight

As for the walk at the right of Picture 8 above, known as Gessel's walk, the functions w and $\tilde{w}$ have been found explicitly in [KR09a].

Finite group of the walk and negative or zero covariance As a first enlightening example, let us consider the case of a covariance equal to zero – or equivalently $\omega_2/\omega_3 = 2$.

Proof of Proposition 28. It will follow from the equality (peculiar to the case $\omega_2/\omega_3 = 2$) :

$$\wp_{1,3}(\omega - [\omega_1 + \omega_2]/2) = -2f(x_1) + (f(x_2) - f(x_3))^2 \frac{\wp(\omega) - f(x_1)}{(\wp(\omega) - f(x_2))(\wp(\omega) - f(x_3))}. \quad (13)$$

Indeed, if (13) holds, let us evaluate it at $\omega = \wp^{-1}(f(t))$: with (10) we obtain that there exist two constants K_1 and K_2 such that $w(t) = K_1 + K_2[f(t) - f(x_1)]/[(f(t) - f(x_2))(f(t) - f(x_3))]$. But by using the explicit expression of f, it is immediate that if $x_4 \neq \infty$ then $w(t) = K_1 + K_3[(t-x_1)(t-x_4)]/[(t-x_2)(t-x_3)]$ and if $x_4 = \infty$ then $w(t) = K_1 + K_4[t - x_1]/[(t-x_2)(t-x_3)]$, K_3 and K_4 being some non-zero constants. Therefore for proving Proposition 28 it is enough to show (13).

For this, start by applying Lemma 25 for $p = 2$: we get $\wp_{1,3}(\omega - [\omega_1 + \omega_2]/2) = \wp(\omega - [\omega_1 + \omega_2]/2) + \wp(\omega - \omega_1/2) - \wp(\omega_2/2)$. Now, taking the usual notations $e_1 = \wp(\omega_1/2)$, $e_{1+2} = \wp([\omega_1 + \omega_2]/2)$ and $e_2 = \wp(\omega_2/2)$, we can state the two following particular cases of the addition formula (14) : $\wp(\omega - [\omega_1 + \omega_2]/2) = e_{1+2} + [(e_{1+2} - e_1)(e_{1+2} - e_2)]/[\wp(\omega) - e_{1+2}]$ and $\wp(\omega - \omega_1/2) = e_1 + [(e_1 - e_2)(e_1 - e_{1+2})]/[\wp(\omega) - e_1]$. In this way and after simplification we obtain $\wp_{1,3}(\omega - [\omega_1 + \omega_2]/2) = e_1 - e_2 + e_{1+2} + (e_1 - e_{1+2})^2[\wp(\omega) - e_2]/[(\wp(\omega) - e_1)(\wp(\omega) - e_{1+2})]$. Finally, by using the equalities $e_1 = f(x_3)$, $e_{1+2} = f(x_2)$ and $e_2 = f(x_1)$, see below (7), as well as $e_1 + e_{1+2} + e_2 = 0$, we immediately obtain (13). $\square$

Let us now consider the case of a negative covariance – or equivalently ω_2/ω_3 equal to 3 or 4 –, and begin with the situation $\omega_2/\omega_3 = 3$ which concerns the two walks of Picture 6.

Proof of Proposition 29. First of all, let us sketch the proof. We will first show that there exist and we will make explicit two third degree polynomials A and B such that $\wp_{1,3}(\omega - [\omega_1 + \omega_2]/2) = A(\wp(\omega))/B(\wp(\omega))$. In particular with (10) we will then obtain that $w(t) = A(f(t))/B(f(t))$. Next we will consider separately the two walks of Picture 6.

(i) For the walk at the left of Picture 6 we have $x_4 = \infty$ and then $A(f(t))$ and $B(f(t))$ are also third degree polynomials, see the expression of f below (10). We will show that $B(f(t)) = (t-x_2)(t-1/x_2{}^{1/2})^2$. In addition, if $r(t)$ denotes the rest of the euclidean division of $A(f(t))$ by $B(f(t))$, we will prove that $r(t) = (r''(0)/2)t^2$ with $r''(0) \neq 0$, in such a way that $w(t) = B(f(\infty))/A(f(\infty)) + (r''(0)/2)t^2/[(t-x_2)(t-1/x_2{}^{1/2})^2]$.

(ii) For the walk at the right of Picture 6 we have $x_4 \neq \infty$ and then $(t-x_4)^3 A(f(t))$ and $(t-x_4)^3 B(f(t))$ are third degree polynomials. We will show that $(t-x_4)^3 B(f(t)) = (t-x_2)(t-1/x_2{}^{1/2})^2$. If $r(t)$ denotes the rest of the euclidean division of $(t-x_4)^3 A(f(t))$ by $(t-x_4)^3 B(f(t))$, then we will prove that $r(t) = (r''(0)/2)t(t+1)$ with $r''(0) \neq 0$ so that $w(t) = B(f(\infty))/A(f(\infty)) + (r''(0)/2)t(t+1)/[(t-x_2)(t-1/x_2{}^{1/2})^2]$.

Finally, starting from the so obtained explicit formulation of w and by using that $\tilde{w} = w(X_0)$, an elementary calculation will lead, for each walk, to the expression of $\tilde{w}$ stated in Proposition 29, this will conclude the proof.

So we begin by finding explicitly, for both walks of Picture 6, two polynomials A and B of degree three such that $\wp_{1,3}(\omega - [\omega_1+\omega_2]/2) = A(\wp(\omega))/B(\wp(\omega))$.

Applying Lemma 25 with $p = 3$ we get $\wp_{1,3}(\omega - [\omega_1+\omega_2]/2) = \wp(\omega - [\omega_1+\omega_2]/2) + \wp(\omega - \omega_1/2 - \omega_2/6) - \wp(\omega_2/3) + \wp(\omega - \omega_1/2 + \omega_2/6) - \wp(2\omega_2/3)$. Then, using the addition formula for $\wp$, namely the following equality, see *e.g.* [JS87], valid for all $\acute{\omega}, \grave{\omega}$,

$$\wp(\acute{\omega} + \grave{\omega}) = -\wp(\acute{\omega}) - \wp(\grave{\omega}) + (1/4)[\wp'(\acute{\omega}) - \wp'(\grave{\omega})]^2/[\wp(\acute{\omega}) - \wp(\grave{\omega})]^2, \tag{14}$$

and noting $K = e_{1+2} - 2\wp(\omega_2/3) - 2\wp(\omega_1/2+\omega_2/6)$ we have that $\wp_{1,3}(\omega - [\omega_1+\omega_2]/2)$ equals

$$\frac{(e_{1+2}-e_1)(e_{1+2}-e_2)}{\wp(\omega)-e_{1+2}} - 2\wp(\omega) + \frac{1}{2}\frac{\wp'(\omega)^2 + \wp'(\omega_1/2+\omega_2/6)^2}{[\wp(\omega) - \wp(\omega_1/2+\omega_2/6)]^2} + K. \tag{15}$$

To obtain (15), we have used that $\wp(\omega_1/2+\omega_2/6) = \wp(\omega_1/2-\omega_2/6)$ and $\wp'(\omega_1/2+\omega_2/6) = -\wp'(\omega_1/2-\omega_2/6)$, got from the fact that $\wp(\omega_1/2+\omega)$ is even and $\wp'(\omega_1/2+\omega)$ is odd.

With (7) and (15), it is now clear that $\wp_{1,3}(\omega - [\omega_1+\omega_2]/2)$ can be written as $A(\wp(\omega))/B(\wp(\omega))$, moreover we can take $B(\wp(\omega)) = (\wp(\omega) - e_{1+2})(\wp(\omega) - \wp(\omega_1/2+\omega_2/6))^2$.

Now we show that $\wp(\omega_1/2+\omega_2/6) = f(1/x_2{}^{1/2})$. For this we will express $\wp(\omega_1/2+\omega_2/6)$ w.r.t. z. Since $\wp(\omega_1/2+\omega_2/6) = e_1 + [(e_1-e_2)(e_1-e_{1+2})]/[\wp(\omega_2/6) - e_1]$ it is enough to express $\wp(\omega_2/6)$ w.r.t. z. For this we will first find explicitly $\wp(\omega_2/3)$ and we will then use, for $\omega = \omega_2/3$, the fact $\wp(\omega/2)$ equals (see *e.g.* [JS87])

$$\wp(\omega) + [(\wp(\omega)-e_1)(\wp(\omega)-e_2)]^{1/2} + [(\wp(\omega)-e_1)(\wp(\omega)-e_{1+2})]^{1/2} + [(\wp(\omega)-e_2)(\wp(\omega)-e_{1+2})]^{1/2}.$$

In other words, for all coefficients in (15) to be explicit w.r.t. the variable z, it is enough to find only $\wp(\omega_2/3)$ in terms of z.

And now we show that for both walks of Picture 6, $\wp(\omega_2/3) = 1/3$. For this we use the following fact, already recalled in [KR09a] : the quantity $x = \wp(\omega_2/3)$ is the only positive root of $x^4 - g_2x^2/2 - g_3x - g_2^2/48$, $g_2 = -4[e_1e_2 + e_1e_{1+2} + e_2e_{1+2}]$ and $g_3 = 4e_1e_2e_{1+2}$ being the invariants of $\wp$. By using the explicit expressions of g_2 and g_3 (we recall that e_1,

e_2 and e_{1+2} are known explicitly w.r.t. z, see the proof of Proposition 28), we easily show that $1/3$ is a root of the polynomial above, being positive we get $\wp(\omega_2/3) = 1/3$.

Then an elementary calculation leads to $\wp(\omega_1/2 + \omega_2/6) = f(1/{x_2}^{1/2})$. Next with (7) we also obtain $\wp'(\omega_1/2 + \omega_2/6)$ and thus all coefficients in (15) are known in terms of z. In particular, this is also the case for the polynomials A and B. After a tedious but easy calculation we obtain the facts claimed in (i) and (ii) above, and, thus, Proposition 29. $\square$

Proposition 31 – concerning $\omega_2/\omega_3 = 4$ – could be obtained by applying Lemma 25 for $p = 4$, the details would be essentially the same as above, so that we omit to write them.

Finite group of the walk and positive covariance The only such walks are the three ones of Picture 7 as well as the one at the right of Picture 8.

The latter, known as Gessel's walk, has already been considered in [KR09a] : there we have shown that the CGF w and $\tilde{w}$ defined by (10) are algebraic (of degree three in t) and we have made explicit their minimal polynomials.

By using the same key idea as in [KR09a] – namely, a double application of Lemma 25 – , we are going now to prove Proposition 30, *i.e.* to show, for the three walks of Picture 7, that w and $\tilde{w}$ are algebraic (of degree two in t) and to find their minimal polynomials.

Proof of Proposition 30. Let us first of all recall that for the three walks considered here $\omega_2/\omega_3 = 3/2$, and define the auxiliary period $\omega_4 = \omega_2/3$.

First, with $\omega_4 = \omega_2/3$, thanks to Lemma 25 we will be able to express $\wp_{1,4}$ as a rational function of $\wp$. Moreover, since $\omega_4 = \omega_3/2$, once again with Lemma 25 we will write $\wp_{1,4}$ as a rational function of $\wp_{1,3}$. As an immediate consequence, $\wp_{1,3}$ will be an algebraic function of $\wp$. Then, with (10) and the addition formula (14) we will obtain that the CGF w defines an algebraic function of t.

Rational expression of $\wp_{1,4}$ in terms of $\wp$. By using exactly the same arguments as in the proof of Proposition 29, we obtain the three following facts : firstly $\wp_{1,4}(\omega - [\omega_1+\omega_2]/2)$ is equal to (15), secondly $\wp(\omega_2/3) = 1/3$ and thirdly the expression of all coefficients in (15) w.r.t. z is explicit. In this way we obtain that there exist K_1 and K_2 which depend only on z and could be made explicit, such that

$$\wp_{1,4}\big(x^{-1}(t) - [\omega_1 + \omega_2]/2\big) = K_1 + \frac{K_2 u(t)}{(t - x_2)(t - 1/{x_2}^{1/2})^2} \tag{16}$$

with $u(t)$ as described in the statement of Proposition 30.

Rational expression of $\wp_{1,4}$ in terms of $\wp_{1,3}$. Applying, as in the proof of Proposition 28, Lemma 25 for $p = 2$, we obtain that $\wp_{1,4}(\omega) = \wp_{1,3}(\omega) + \wp_{1,3}(\omega + \omega_3/2) - \wp_{1,3}(\omega_3/2)$. Denoting by $e_{1,1,3} = \wp_{1,3}(\omega_1/2)$, $e_{1+3,1,3} = \wp_{1,3}([\omega_1 + \omega_3]/2)$ and $e_{3,1,3} = \wp_{1,3}(\omega_3/2)$ we have $\wp_{1,3}(\omega+\omega_3/2) = e_{3,1,3}+[(e_{3,1,3}-e_{1,1,3})(e_{3,1,3}-e_{1+3,1,3})]/[\wp_{1,3}(\omega)-e_{3,1,3}]$. In particular we immediately obtain that

$$\wp_{1,3}(\omega)^2 - \big[e_{3,1,3} + \wp_{1,4}(\omega)\big]\wp_{1,3}(\omega) + \big[(e_{3,1,3} - e_{1,1,3})(e_{3,1,3} - e_{1+3,1,3}) + e_{3,1,3}\wp_{1,4}(\omega)\big] = 0. \tag{17}$$

Therefore, once the expressions of $e_{1,1,3}$, $e_{1+3,1,3}$ and $e_{3,1,3}$ will be known explicitly, Equations (10), (16) and (17) will immediately entail Proposition 30.

It remains thus to find explicitly $e_{1,1,3}$, $e_{1+3,1,3}$ and $e_{3,1,3}$. This will be a consequence of the possibility of expanding $\wp_{1,4}$ in two different ways.

First, we have seen just above that $\wp_{1,4}(\omega) = \wp_{1,3}(\omega) + [(e_{3,1,3} - e_{1,1,3})(e_{3,1,3} - e_{1+3,1,3})]/[\wp_{1,3}(\omega) - e_{3,1,3}]$, so that by using the usual expansion of $\wp_{1,3}$ at 0, namely $\wp_{1,3}(\omega) = \omega^{-2} + g_{2,1,3}\omega^2/20 + g_{3,1,3}\omega^4/28 + O(\omega^6)$, $g_{2,1,3}$ and $g_{3,1,3}$ being the invariants of $\wp_{1,3}$, as well as the following straightforward equality $(e_{3,1,3} - e_{1,1,3})(e_{3,1,3} - e_{1+3,1,3}) = 3{e_{3,1,3}}^2 - g_{2,1,3}/4$, we get

$$\wp_{1,4}(\omega) = \omega^{-2} + \big[3{e_{3,1,3}}^2 - g_{2,1,3}/5\big]\omega^2 + \big[g_{3,1,3} + 3{e_{3,1,3}}^3 - g_{2,1,3}e_{3,1,3}/4\big]\omega^4 + O(\omega^6). \quad (18)$$

Second, by applying Lemma 25 for $p = 3$ we obtain $\wp_{1,4}(\omega) = -\wp(\omega) + [\wp'(\omega)^2 + \wp'(\omega_2/3)^2]/[2(\wp(\omega) - \wp(\omega_2/3))] - 4\wp(\omega_2/3)$. Using that $\wp(\omega_2/3) = 1/3$ as well as (7) yield

$$\wp_{1,4}(\omega) = \omega^{-2} + \big[2/3 - 9g_2/20\big]\omega^2 + \big[10/27 - g_2/2 - 27g_3/28\big]\omega^4 + O(\omega^6). \quad (19)$$

By identifying the expansions (18) and (19) we obtain the expression of $g_{2,1,3}$ and $g_{3,1,3}$ in terms of g_2, g_3 and $e_{3,1,3}$.

In addition, $e_{3,1,3}$ is obviously a solution of $4{e_{3,1,3}}^3 - g_{2,1,3}e_{3,1,3} - g_{3,1,3} = 0$. If we use the expressions of $g_{2,1,3}$ and $g_{3,1,3}$ obtained just above, we obtain that ${e_{3,1,3}}^3 + [9g_2/16 - 5/6]e_{3,1,3} + [35/108 - 27g_3/32 - 7g_2/16] = 0$. We can solve this equation (we recall that g_2 and g_3 are known explicitly w.r.t. z) and in this way we get $e_{3,1,3}$. Next we obtain $g_{2,1,3}$ and $g_{3,1,3}$, or, equivalently, $e_{1,1,3}$ and $e_{1+3,1,3}$. In particular, the expansion (17) is now completely known and Proposition 30 is proved. $\square$

4.4 Concluding remarks

Some of the 23 walks associated with a finite group have been and are still the object of numerous studies, as illustrated by the recent and vivid interest in the famous Gessel's walk, see [KKZ09], [BK09] and [KR09a]. But other walks have also repeatedly caught the attention of the mathematical community, as Kreweras' walk (represented here at the left of Picture 7), see [BMM09] and the references therein, as well as Gouyou-Beauchamps' walk (drawn at the left of Picture 8), which, among others, has strong connections with the enumeration of non-crossing matchings, see [CDD+07].

However, certain questions concerning these walks – as the one of explaining some closed form expressions for quantities associated with them – are still open, see [BMM09]. We hope that Propositions 28-31 and the results of [KR09a], which give simple explicit expressions of the CGF w and $\tilde{w}$ for these 23 walks, and therefore thanks to Theorem 6 which yield an explicit expression of the trivariate generating function (1), with a different formulation from the one obtained in [BMM09] and [BK09], will lead to make progress in this perspective.

Another thing is that, although it is yet unclear how the nature of the CGF yields the nature of the generating function (1), we can make the following observation *a posteriori*. Subsections 4.2-4.3 imply that we can split the 23 walks having a finite group into two families, according to the nature of the CGF : for the 19 walks with negative or zero covariance the CGF is rational, see Propositions 28, 29 and 31, whereas for the 4 walks with positive covariance the CGF is algebraic non-rational : see Proposition 30 and [KR09a].

It is interesting to remark that this classification – that clearly implies that the sign of the covariance is a notable and natural quantity, thanks to Proposition 18 – is the same as the classification according to the nature of the series (1) : if the walk has negative or zero covariance then the generating function is holonomic non-algebraic, see [BMM09], whereas if the walk has positive covariance then (1) is algebraic, see [BMM09] and [BK09].

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