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SOME RESULTS OF KATZNELSON-TZAFRIRI TYPE.

M. ZARRABI

ABSTRACT. For semigroups $S = \mathbb{R}_+$, \mathbb{Z}_+ and \mathbb{Z}_+^k , we show that if T is a representation of S by contractions on a *Hilbert space* then $\inf_{t \in S} \|T(t)\widehat{f}(T)\| = \sup \{|f(\lambda)|, \lambda \in \text{Sp}_u(T, S)\}$, where $f \in L^1(S)$ and $\text{Sp}_u(T, S)$ is the unitary spectrum of T with respect to S . When T is a representation of a suitable semigroup S by contractions on a *Banach space*, we give sharp conditions on $\text{Sp}_u(T, S)$ which guarantee that the equality above holds. These conditions concern the thinness of $\text{Sp}_u(T, S)$ in the harmonic analysis sense. These results are related to theorems of Katznelson-Tzafriri type, which give conditions guaranteeing that $\inf_{t \in S} \|T(t)\widehat{f}(T)\|$ vanishes.

1. INTRODUCTION

Let G be a locally compact abelian group equipped with the Haar measure and S be a suitable subsemigroup of G . Let T be a representation of G by contractions on a Banach space X . For $f \in L^1(S)$, we set

$$\widehat{f}(T) : x \longmapsto \int_S f(t)T(t)xdt, \quad x \in X.$$

The subsemigroup S will be ordered in the following way: $s \preceq t \Leftrightarrow t - s \in S$. So we shall denote \lim_S , the limit as $t \rightarrow \infty$ through S . The purpose of this paper is to study the asymptotic behaviour of $\|T(t)\widehat{f}(T)\|$, as $t \rightarrow \infty$. This is related to Theorems of Katznelson-Tzafriri type ([14],[9],[10],[21],[3]). The unitary spectrum of T with respect to S is defined by

$$\text{Sp}_u(T, S) = \{\chi \in \Gamma; |\widehat{f}(\chi)| \leq \|\widehat{f}(T)\| \text{ for all } f \text{ in } L^1(S)\},$$

where Γ is the dual group of G . It is shown by Batty and Vũ in [3], that if $f \in L^1(S)$ satisfies spectral synthesis for $\text{Sp}_u(T, S)$, then $\lim_S \|T(t)\widehat{f}(T)\| = 0$. This result was first proved by Katznelson and Tzafriri ([14]) in the case $S = \mathbb{Z}_+$. More precisely, they showed that if T_1 is a contraction on a Banach

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space and if h is an analytic function on the unit disk \mathbb{D} , which has absolutely convergent Taylor series and satisfies spectral synthesis for $\sigma(T_1) \cap \mathbb{T}$, then

$$\lim_{n \rightarrow +\infty} \|T_1^n h(T_1)\| = 0. \quad (1.1)$$

Here $\sigma(T_1)$ denote the spectrum of T_1 and \mathbb{T} the unit circle. When T_1 is a contraction on a Hilbert space, Esterle, Strouse and Zouakia showed in [9] that (1.1) holds for every function h vanishing on $\sigma(T_1) \cap \mathbb{T}$. Recently, Léka has extended this result in [16] to a power bounded operators ($\sup_{n \in \mathbb{Z}_+} \|T_1^n\| < +\infty$) on a Hilbert space.

The case of C_0 -semigroups was also studied. It was shown independently in [10] and [21] that if $T = (T(t))_{t \geq 0}$ is a C_0 -semigroup, with generator A , then for every $f \in L^1(\mathbb{R}_+)$ which satisfies spectral synthesis for $i\sigma(A) \cap \mathbb{R}$, $\lim_{t \rightarrow +\infty} \|T(t)\hat{f}(T)\| = 0$.

If F is a locally compact Hausdorff space, we denote by $\mathcal{C}_0(F)$ the algebra of all continuous functions on F vanishing at infinity. If h a continuous bounded function on E and $E \subset F$, we set $\|h\|_{\mathcal{C}(E)} = \sup \{|f(\lambda)|, \lambda \in E\}$, with the understanding that $\|h\|_{\mathcal{C}(E)} = 0$ if E is empty.

Let T be a representation of S by contractions. Notice that we always have $\|\hat{f}\|_{\mathcal{C}(\text{Sp}_u(T,S))} \leq \lim_S \|T(t)\hat{f}(T)\|$. In this paper, we investigate conditions under which this inequality becomes an equality.

Let T be a representation by contractions on a *Hilbert space*. We show (Theorem 2.6) that if for every $f \in L^1(S)$, $\lim_S \|T(t)\hat{f}(T)\| \leq \|\hat{f}\|_{\mathcal{C}(\Gamma)}$ then

$$\lim_S \|T(t)\hat{f}(T)\| = \|\hat{f}\|_{\mathcal{C}(\text{Sp}_u(T,S))}. \quad (1.2)$$

As a consequence we obtain the following results:

(a) If $T = (T(t))_{t \geq 0}$ is a C_0 -semigroup of contractions on a Hilbert space with generator A , then for every $f \in L^1(\mathbb{R}^+)$,

$$\lim_{t \rightarrow +\infty} \|T(t)\hat{f}(T)\| = \|\hat{f}\|_{\mathcal{C}(i\sigma(A) \cap \mathbb{R})}.$$

(b) If T_1 is a contraction on a Hilbert space then for every function h in the disk algebra, we have $\lim_{n \rightarrow +\infty} \|T_1^n h(T_1)\| = \|h\|_{\mathcal{C}(\sigma(T_1) \cap \mathbb{T})}$.

(c) If T_1, T_2, \dots, T_k are finite many commuting contractions on a Hilbert space and $f \in \ell^1(\mathbb{Z}_+^k)$, then

$$\lim_{\min(n_1, n_2, \dots, n_k) \rightarrow \infty} \|T_1^{n_1} T_2^{n_2} \dots T_k^{n_k} \hat{f}(T_1, T_2, \dots, T_k)\| = \|\hat{f}\|_{\mathcal{C}(\sigma_{\mathcal{A}}(T_1, T_2, \dots, T_k) \cap \mathbb{T}^k)},$$

where \mathcal{A} is any unitary commutative Banach algebra that contains T_1, T_2, \dots, T_k and $\sigma_{\mathcal{A}}(T_1, T_2, \dots, T_k)$ is the joint spectrum of T_1, T_2, \dots, T_k relative to \mathcal{A} .

Notice that result (b) was obtained simultaneously in [27] and in [18] (the present paper is an improved version of [27]). For the proof of (a) and (b) we use in particular the von Neumann inequality, while a different method

is used for (c). Indeed the von Neumann inequality fails for three or more commuting contractions.

The above results are far from true for representations by contractions acting on a Banach space. We will see in section 5 that equality (1.2) depends on the thinness of $\text{Sp}_u(T, S)$.

Let E be a closed subset of Γ . We shall now denote by T a representation of S by contractions on a *Banach space*. We show (Corollary 5.5) that equality (1.2) holds for every T such that $\text{Sp}_u(T, S) = E$, if and only if E satisfies spectral synthesis and is a Helson set with $\alpha(E) = 1$, where $\alpha(E)$ is the Helson constant of E .

2. REPRESENTATIONS OF SUBSEMIGROUPS BY CONTRACTIONS ON A HILBERT SPACE

We shall adopt the terminology of [2] and [3]. Let G be a locally compact abelian group and S be a measurable subsemigroup of G with non empty interior in G and such that $G = S - S$. The group G is equipped with the Haar measure and S with the restriction of that measure. $L^1(S)$ will be identified with a subspace of $L^1(G)$. Denote by S^* the set of all the non-zero, continuous, bounded, homomorphisms of S into the multiplicative semigroup \mathbb{C} . Let

$$S_u^* = \{\chi \in S^*; |\chi(s)| = 1 \text{ for all } s \text{ in } S\}.$$

We shall identify S_u^* with the dual group Γ of G , in the natural way. For $f \in L^1(S)$ and $\chi \in S^*$, we set

$$\widehat{f}(\chi) = \int_S f(t)\chi(t)dt.$$

Finally we shall assume that $\{\widehat{f}, f \in L^1(S)\}$ separates the points of S^* from each other and from zero.

Examples. 1) Let $G = \mathbb{R}^n$, $S = \mathbb{R}_+^n$. The set S^* will be identified with $\overline{\mathbb{C}_-}^n$, where $\mathbb{C}_- = \{z \in \mathbb{C} : \text{Im}(z) < 0\}$. The identification map is given by $z \rightarrow \chi_z$, where $\chi_z(t) = e^{-it \cdot z}$, $z = (z_1, \dots, z_n) \in \overline{\mathbb{C}_-}^n$, $t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$ and $t \cdot z = t_1 z_1 + \dots + t_n z_n$. For $f \in L^1(\mathbb{R}_+^n)$ and $z \in \overline{\mathbb{C}_-}^n$, we have

$$\widehat{f}(z) = \widehat{f}(\chi_z) = \int_{\mathbb{R}_+^n} f(t)e^{-it \cdot z} dt.$$

Clearly $S_u^* = \mathbb{R}^n$.

2) Let $G = \mathbb{Z}^n$ and $S = \mathbb{Z}_+^n$, where \mathbb{Z}_+ is the set of all nonnegative integers; S^* will be identified with $\overline{\mathbb{D}}^n$ by the map $z = (z_1, \dots, z_n) \rightarrow \chi_z$, where $\chi_z(k) = z_1^{k_1} \dots z_n^{k_n}$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$. We have $S_u^* = \mathbb{T}^n$ and,

for $f \in L^1(\mathbb{Z}_+^n)$,

$$\widehat{f}(z) = \sum_{k \in \mathbb{Z}_+^n} f(k) z_1^{k_1} \dots z_n^{k_n}.$$

Let X be a Banach space, $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X . Let T be a representation of S by contractions on X , that is a strongly continuous homomorphism from S into $\mathcal{L}(X)$ such that $T(s)$ is a contraction whenever $s \in S$. For $f \in L^1(S)$, let $\widehat{f}(T) : X \rightarrow X$ be the bounded operator defined by

$$\widehat{f}(T) : x \mapsto \int_S f(t) T(t) x dt.$$

The spectrum of T with respect to S is defined by

$$\text{Sp}(T, S) = \{\chi \in S^*; |\widehat{f}(\chi)| \leq \|\widehat{f}(T)\| \text{ for all } f \text{ in } L^1(S)\}.$$

We shall mainly be interested in the unitary part of the spectrum of T , which is defined by $\text{Sp}_u(T, S) = \text{Sp}(T, S) \cap S_u^* = \text{Sp}(T, S) \cap \Gamma$.

We need the following version of the Douglas lemma ([7]) given in [2].

Lemma 2.1. *Let V be a representation of S by isometries on a Banach space Y . There exists a Banach space Y_d containing Y (by isometric isomorphism), and a representation V_d of G by isometries on Y_d such that $V_d(s)y = V(s)y$ ($y \in Y, s \in S$) and $\text{Sp}(V_d, G) = \text{Sp}_u(V, S)$.*

Lemma 2.2. *Let V be a representation of S by isometries on a Banach space Y and $y \in Y$. Assume that for every $f \in L^1(S)$,*

$$\|\widehat{f}(V)y\| \leq \|\widehat{f}\|_{C(\Gamma)}.$$

Then, for every $f \in L^1(S)$,

$$\|\widehat{f}(V)y\| \leq \|\widehat{f}\|_{C(\text{Sp}_u(V, S))}.$$

Proof. Let V_d be the representation of G by isometries on a Banach space Y_d , given by Lemma 2.1.

We assume that $Y \neq \{0\}$. By [2], Corollary 2.2, $\text{Sp}_u(T, S)$ is non-empty. Let $\epsilon > 0$ and $g \in L^1(G)$. Since S has a nonempty interior, there exists $s \in S$ such that

$$\int_{G \setminus S-s} |g(t)| dt < \epsilon.$$

Thus

$$\begin{aligned}
\|\widehat{g}(V_d)y\| &\leq \left\| \int_{S-s} g(t)V_d(t)y dt \right\| + \left\| \int_{G \setminus S-s} g(t)V_d(t)y dt \right\| \\
&\leq \left\| \int_S g(t-s)V_d(t-s)y dt \right\| + \epsilon\|y\| \\
&\leq \left\| V_d(-s) \left(\int_S g(t-s)V_d(t)y dt \right) \right\| + \epsilon\|y\| \\
&\leq \left\| \int_S g_s(t)V_d(t)y dt \right\| + \epsilon\|y\| = \|\widehat{g}_s(V_d)y\| + \epsilon\|y\|,
\end{aligned}$$

where

$$g_s(t) = \begin{cases} g(t-s) & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\|\widehat{g}_s(V_d)y\| = \|\widehat{g}_s(V)y\| \leq \|\widehat{g}_s\|_{\mathcal{C}(\Gamma)}.$$

On the other hand, for every $\chi \in \Gamma$, we have

$$\widehat{g}_s(\chi) = \widehat{g}(\chi) - \int_{G \setminus S-s} g(t)\chi(t) dt,$$

which implies that $\|\widehat{g}_s\|_{\mathcal{C}(\Gamma)} \leq \|\widehat{g}\|_{\mathcal{C}(\Gamma)} + \epsilon$. Therefore, we get

$$\|\widehat{g}(V_d)y\| \leq \|\widehat{g}\|_{\mathcal{C}(\Gamma)} + \epsilon(1 + \|y\|).$$

Letting $\epsilon \rightarrow 0$, we obtain

$$\|\widehat{g}(V_d)y\| \leq \|\widehat{g}\|_{\mathcal{C}(\Gamma)}.$$

Let $f \in L^1(S)$ and let $g \in L^1(G)$ be such that \widehat{g} vanishes on a neighborhood of $\text{Sp}_u(V, S)$. Since $\text{Sp}_u(V, S) = \text{Sp}(V_d, G)$, we have $\widehat{g}(V_d) = 0$ (see [15], p. 201). It follows then from the last inequality that

$$\|\widehat{f}(V)y\| \leq \|\widehat{f}(V_d)y - \widehat{g}(V_d)y\| \leq \|\widehat{f} - \widehat{g}\|_{\mathcal{C}(\Gamma)}.$$

Since $A(\Gamma) = \{\widehat{g}, g \in L^1(G)\}$ is dense in $\mathcal{C}_0(\Gamma)$ (see [22], Theorem 1.2.4, p. 9), it follows from the structure of closed ideals of $\mathcal{C}_0(\Gamma)$ that the set $\{\widehat{g}, g \in L^1(G) \text{ and } \widehat{g} = 0 \text{ on a neighborhood of } \text{Sp}_u(V, S)\}$ is dense in $\mathcal{I}(\text{Sp}_u(V, S)) := \{h \in \mathcal{C}_0(\Gamma), h = 0 \text{ on } \text{Sp}_u(V, S)\}$. Hence

$$\|\widehat{f}(V)y\| \leq \inf_{h \in \mathcal{I}(\text{Sp}_u(V, S))} \|\widehat{f} - h\|_{\mathcal{C}(\Gamma)} = \|\widehat{f}\|_{\mathcal{C}(\text{Sp}_u(V, S))}.$$

□

The following is Lemma 4.1 of [21].

Lemma 2.3. *Let T be a representation of S by contractions on a Banach space X . Then there exist a Banach space Y , a continuous homomorphism $\pi : X \mapsto Y$ with dense range, and a representation V of S by isometries on Y , such that*

- (i) $\text{Sp}(V, S) \subset \text{Sp}(T, S)$,
- (ii) $V(t) \circ \pi = \pi \circ T(t)$ for every t in S ,
- (iii) $\lim_S \|T(t)x\| = \|\pi x\|$ for every x in X .

Remark 2.4. In the above Lemma, when X is a Hilbert space then Y is a Hilbert space too. Indeed, let $\langle \cdot, \cdot \rangle$ be the inner product on X . By the polarization identity we see easily that the limit

$$\langle x, y \rangle_T := \lim_S \langle T(t)x, T(t)y \rangle,$$

exists for every $x, y \in X$. We set $\ell(x) = \lim_S \|T(t)x\| = \sqrt{\langle x, x \rangle_T}$ and we continue as in the proof of Lemma 4.1 of [21].

Lemma 2.5. *Let T be a representation of S by contractions on a Banach space X . Let $x \in X$. Assume that for every $f \in L^1(S)$,*

$$\lim_S \|T(t)\widehat{f}(T)x\| \leq \|\widehat{f}\|_{C(\Gamma)}.$$

Then, for every $f \in L^1(S)$,

$$\lim_S \|T(t)\widehat{f}(T)x\| \leq \|\widehat{f}\|_{C(\text{Sp}_u(T, S))}.$$

Proof. Let V and π be respectively, the isometry and the contractive homomorphism given by Lemma 2.3. Let $f \in L^1(S)$. Applying (ii) and then (iii) of Lemma 2.3, we obtain

$$\|\widehat{f}(V)\pi x\| = \|\pi(\widehat{f}(T)x)\| = \lim_S \|T(t)\widehat{f}(T)x\|.$$

Then we get $\|\widehat{f}(V)\pi x\| \leq \|\widehat{f}\|_{C(\Gamma)}$. It follows from Lemma 2.2 that

$$\lim_S \|T(t)\widehat{f}(T)x\| = \|\widehat{f}(V)\pi x\| \leq \|\widehat{f}\|_{C(\text{Sp}_u(V, S))} \leq \|\widehat{f}\|_{C(\text{Sp}_u(T, S))}.$$

The last inequality holds since $\text{Sp}_u(V, S) \subset \text{Sp}_u(T, S)$. □

Theorem 2.6. *Let T be a representation of S by contractions on a Banach space X . Assume that for every $f \in L^1(S)$,*

$$\lim_S \|T(t)\widehat{f}(T)\| \leq \|\widehat{f}\|_{C(\Gamma)}.$$

Then for every $f \in L^1(S)$,

$$\lim_S \|T(t)\widehat{f}(T)\| = \|\widehat{f}\|_{C(\text{Sp}_u(T, S))}.$$

Proof. Let $f \in L^1(S)$. For $t \in S$, we have

$$T(t)\widehat{f}(T) = \widehat{\tau_t f}(T)$$

where τ_t is the translation operator defined by: $\tau_t f = f(\cdot - t)$. Let $\chi \in \text{Sp}_u(T, S)$. We clearly have the following equality $\widehat{\tau_t f}(\chi) = \chi(t)\widehat{f}(\chi)$. Thus, using the inequality $|\widehat{\tau_t f}(\chi)| \leq \|\widehat{\tau_t f}(T)\|$, we get $|\widehat{f}(\chi)| \leq \|T(t)\widehat{f}(T)\|$. So $\|\widehat{f}\|_{\mathcal{C}(\text{Sp}_u(T, S))} \leq \lim_S \|T(t)\widehat{f}(T)\|$.

On the other hand, for $t \in S$, let $M(t)$ be the linear bounded operator defined on $\mathcal{L}(X)$ by

$$M(t) : A \mapsto T(t)A.$$

We shall now use a similar argument as in the proof of Theorem 4.3 of [3]. Let \mathcal{L}_0 be the set of all bounded operators A on X such that the map $t \rightarrow T(t)A$ is continuous; \mathcal{L}_0 is a closed linear subspace of $\mathcal{L}(X)$ that contains $\hat{u}(T)$, whenever $u \in L^1(S)$. Let M_0 be the restriction of M to \mathcal{L}_0 . Then M_0 is strongly continuous and $\text{Sp}_u(M_0, S) \subset \text{Sp}_u(T, S)$.

Take $s_0 \in \dot{S}$. There exists a sequence $(v_n)_n$ in $L^1(S)$ such that $|\widehat{v_n}| \leq 1$ and $f * v_n \rightarrow \tau_{s_0} f$, $n \rightarrow \infty$, in the L^1 -norm. For every $g \in L^1(S)$ we have

$$\lim_S \|M_0(t)\widehat{g}(M_0)\widehat{v_n}(T)\| = \lim_S \|T(t)(\widehat{g * v_n})(T)\| \leq \|\widehat{g * v_n}\|_{\mathcal{C}(\Gamma)} \leq \|\widehat{g}\|_{\mathcal{C}(\Gamma)}.$$

Applying now Lemma 2.5 to M_0 and f , we obtain

$$\begin{aligned} \lim_S \|T(t)(\widehat{f * v_n})(T)\| &= \lim_S \|M_0(t)\widehat{f}(M_0)\widehat{v_n}(T)\| \\ &\leq \|\widehat{f}\|_{\mathcal{C}(\text{Sp}_u(M_0, S))} \\ &\leq \|\widehat{f}\|_{\mathcal{C}(\text{Sp}_u(T, S))}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|f * v_n - \tau_{s_0} f\| = 0$ we have

$$\lim_{n \rightarrow \infty} \|(\widehat{f * v_n})(T) - T(s_0)\widehat{f}(T)\| = 0.$$

It follows that

$$\lim_S \|T(t + s_0)\widehat{f}(T)\| \leq \|\widehat{f}\|_{\mathcal{C}(\text{Sp}_u(T, S))},$$

which finishes the proof. □

3. C_0 -SEMIGROUPS AND CONTRACTIONS

Here we consider the case $G = \mathbb{R}$ and $S = \mathbb{R}_+$. We recall (see section 2) that $S^* = \overline{\mathbb{C}_-}$, $S^* = \Gamma = \mathbb{R}$, and for $z \in \overline{\mathbb{C}_-}$ and $f \in L^1(\mathbb{R}^+)$, we have $\widehat{f}(z) = \widehat{f}(\chi_z) = \int_0^{+\infty} f(t)e^{-itz} dt$.

Let $T = (T_t)_{t \geq 0}$ be a strongly continuous semigroup of contractions on a Hilbert space H and A be its generator. We have $\text{Sp}_u(T, \mathbb{R}_+) = i\sigma(A) \cap \mathbb{R}$ (see [3]).

Theorem 3.1. *Let $T = (T(t))_{t \geq 0}$ be a strongly continuous semigroup of contractions on a Hilbert space H and, let A be its generator. Then for every $f \in L^1(\mathbb{R}^+)$,*

$$\lim_{t \rightarrow +\infty} \|T(t)\widehat{f}(T)\| = \|\widehat{f}\|_{\mathcal{C}(i\sigma(A) \cap \mathbb{R})}.$$

Proof. Denote by \mathcal{D} the domain where A is defined. Let $v \in \mathcal{D}$, with $\|v\| = 1$. We have

$$\langle Av, v \rangle = \lim_{t \rightarrow +\infty} \left\langle \frac{1}{t}(T(t)v - v), v \right\rangle = \lim_{t \rightarrow +\infty} \frac{1}{t} (\langle T(t)v, v \rangle - 1),$$

where $\langle \cdot, \cdot \rangle$ is the scalar product of H . Since $|\langle T(t)v, v \rangle| \leq 1$, we have $\operatorname{Re}(\langle Av, v \rangle) \leq 0$. So the numerical range $\{\langle -Av, v \rangle, \|v\| = 1\}$, of $-A$ is contained in $-i\overline{\mathbb{C}_+}$. It follows from the von Neumann inequality (see, for instance, [6]) that for every rational function r bounded on $-i\overline{\mathbb{C}_+}$ and with poles in $i\mathbb{C}_+$, we have

$$\|r(-A)\| \leq \sup_{z \in -i\overline{\mathbb{C}_+}} |r(z)| = \sup_{z \in i\mathbb{R}} |r(z)|. \quad (3.1)$$

We denote by F the set of all functions of the form

$$f(t) = \sum_{k=1}^n a_k e^{\lambda_k t}, \quad (3.2)$$

where $(a_k)_{1 \leq k \leq n}$ and $(\lambda_k)_{1 \leq k \leq n}$ are a complex numbers such that for every k , λ_k has a negative real part. For f in F of the form (3.2), we have

$$\widehat{f}(z) = \int_0^{+\infty} f(t) e^{-izt} dt = \sum_{k=1}^n a_k (iz - \lambda_k)^{-1}, \quad z \in \overline{\mathbb{C}_-}.$$

and

$$\widehat{f}(T) = \sum_{k=1}^n a_k (-A - \lambda_k)^{-1} = r(-A),$$

where $r(z) = \widehat{f}(-iz) = \sum_{k=1}^n a_k (z - \lambda_k)^{-1}$, $z \in -i\overline{\mathbb{C}_+}$. It follows from (3.1) that

$$\|\widehat{f}(T)\| \leq \sup_{z \in i\mathbb{R}} |r(z)| = \sup_{x \in \mathbb{R}} |\widehat{f}(x)|$$

Since F is dense in $L^1(\mathbb{R}_+)$, the last inequality holds for every function f in $L^1(\mathbb{R}_+)$. Now we apply Theorem 2.6 to finish the proof. \square

Let $\lambda \in \mathbb{C}$, with positive real part. Under the hypothesis of the Theorem 3.1, we get $\lim_{t \rightarrow +\infty} \|T(t)(\lambda I - A)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(A) \cap i\mathbb{R})}$. More generally

if a_0, \dots, a_n are complex numbers, applying the above theorem to $f(t) = (\sum_{k=0}^n \frac{a_k}{k!} t^k) e^{-\lambda t}$, we obtain

$$\begin{aligned} \lim_{t \rightarrow +\infty} \left\| T(t) \sum_{k=0}^n a_k (\lambda I - A)^{-k-1} \right\| &= \sup_{x \in i\sigma(A) \cap \mathbb{R}} \left| \sum_{k=0}^n a_k (\lambda + ix)^{-k-1} \right| \\ &= \sup_{z \in \sigma(A) \cap i\mathbb{R}} \left| \sum_{k=0}^n a_k (\lambda - z)^{-k-1} \right|. \end{aligned}$$

Now we turn to the case $G = \mathbb{Z}$ and $S = \mathbb{Z}_+$. We have $S^* = \overline{\mathbb{D}}$ and $S_u^* = \Gamma = \mathbb{T}$. Every representation T of \mathbb{Z}_+ by contractions on H is given by $T = (T_1^n)_{n \in \mathbb{Z}_+}$, where T_1 is a contraction on H . We have $\text{Sp}_u(T, \mathbb{Z}_+) = \sigma(T_1) \cap \mathbb{T}$ (see [3]).

Theorem 3.2. *Let T_1 be a contraction on a Hilbert space H . Then for every h in the disk algebra $\mathcal{A}(\mathbb{D})$,*

$$\lim_{n \rightarrow +\infty} \|T_1^n h(T_1)\| = \|h\|_{\mathcal{C}(\sigma(T_1) \cap \mathbb{T})}.$$

Proof. Let $f = (f(n))_{n \geq 0} \in \ell^1(\mathbb{Z}_+)$. By the von Neumann inequality, we have $\lim_{n \rightarrow +\infty} \|T_1^n \widehat{f}(T)\| \leq \|\widehat{f}(T)\| \leq \|\widehat{f}\|_{\mathcal{C}(\mathbb{T})}$, where $T = (T_1^n)_{n \in \mathbb{Z}_+}$. Then we get from Theorem 2.6 that $\lim_{n \rightarrow +\infty} \|T_1^n \widehat{f}(T_1)\| = \|\widehat{f}\|_{\mathcal{C}(\text{Sp}_u(T, \mathbb{Z}_+)})$. The equality in the theorem follows since $\text{Sp}_u(T, \mathbb{Z}_+) = \sigma(T_1) \cap \mathbb{T}$ and since the set $\{\widehat{f}, f \in \ell^1(\mathbb{Z}_+)\}$ is dense in $\mathcal{A}(\mathbb{D})$. \square

4. SEVERAL COMMUTING CONTRACTIONS

Let $k \geq 2$ be an integer. We shall write $n = (n_1, \dots, n_k) \in \mathbb{Z}_+^k$. Let T be a representation of \mathbb{Z}_+^k by contractions on a Hilbert space H . We have $T = (T_1^{n_1} \dots T_k^{n_k})_{n \in \mathbb{Z}_+^k}$, where T_1, \dots, T_k are finite many commuting contractions on H .

There are several definitions of the joint spectrum. Let \mathcal{A} be a commutative Banach algebra that contains T_1, \dots, T_k and I be the identity map. We define the joint spectrum $\sigma_{\mathcal{A}}(T_1, \dots, T_k)$, relative to \mathcal{A} , to be the set of all k -tuples of complex numbers $\lambda = (\lambda_1, \dots, \lambda_k)$ such that the ideal (and/or the closed ideal) of \mathcal{A} generated by $\lambda_1 - T_1, \dots, \lambda_k - T_k$ is a proper subset of \mathcal{A} .

Let $f = (f(n))_{n \in \mathbb{Z}_+^k}$ in $\ell^1(\mathbb{Z}_+^k)$. We have

$$\widehat{f}(\lambda) = \sum_{n \in \mathbb{Z}_+^k} f(n) \lambda_1^{n_1} \dots \lambda_k^{n_k}, \quad \lambda = (\lambda_1, \dots, \lambda_k) \in \overline{\mathbb{D}}^k,$$

and

$$\widehat{f}(T) = \sum_{n \in \mathbb{Z}_+^k} f(n) T_1^{n_1} \dots T_k^{n_k}.$$

Let $\lambda \in \sigma_{\mathcal{A}}(T_1, \dots, T_k)$. It is easily seen that $\widehat{f}(T) - \widehat{f}(\lambda)$ is contained in the proper closed ideal generated by $\lambda_1 - T_1, \dots, \lambda_k - T_k$. So $\widehat{f}(\lambda) \in \sigma(\widehat{f}(T))$, which implies in particular that $|\widehat{f}(\lambda)| \leq \|\widehat{f}(T)\|$. Thus $\sigma_{\mathcal{A}}(T_1, \dots, T_k) \subset \text{Sp}(T, \mathbb{Z}_+^k)$.

Let now $\lambda \in \text{Sp}_u(T, \mathbb{Z}_+^k)$. Since $\text{Sp}(T, \mathbb{Z}_+^k)$ is σ -compact, λ is an ω -approximate eigenvalue of T , that is, there exists a sequence $(x_p)_p$ in H with $\|x_p\| = 1$ and such that $\|T_1^{n_1} \dots T_k^{n_k} x_p - \lambda_1^{n_1} \dots \lambda_k^{n_k} x_p\| \rightarrow 0$, as $p \rightarrow +\infty$, for each $n \in \mathbb{Z}_+^k$ (see [3], Proposition 2.2). In particular, for every $i = 0, \dots, k$, $\|(T_i - \lambda_i)x_p\| \rightarrow 0$ as $p \rightarrow +\infty$. It follows that the equation $(T_1 - \lambda_1)S_1 + \dots + (T_k - \lambda_k)S_k = I$ has no solution $S_1, \dots, S_k \in \mathcal{A}$. Thus $\lambda \in \sigma_{\mathcal{A}}(T_1, \dots, T_k)$.

We deduce from the above observations that $\sigma_{\mathcal{A}}(T_1, \dots, T_k) \cap \mathbb{T}^n = \text{Sp}_u(T, \mathbb{Z}_+^k)$. We see that $\sigma_{\mathcal{A}}(T_1, \dots, T_k) \cap \mathbb{T}^n$ is the same for any commutative Banach algebra that contains T_1, \dots, T_k and I .

We know by the Ando Theorem [1] that the von Neumann inequality holds for two commuting contractions on a Hilbert space while it fails for three (or more) commuting contractions (see [8]). So we get from Theorem 2.6, that if T_1 and T_2 are two commuting contractions on H and h is in the bidisc algebra, then

$$\lim_{\min(n_1, n_2) \rightarrow +\infty} \|T_1^{n_1} T_2^{n_2} h(T_1, T_2)\| = \|h\|_{\mathcal{C}(\sigma_{\mathcal{A}}(T_1, T_2) \cap \mathbb{T}^2)},$$

where \mathcal{A} is any commutative Banach algebra that contains T_1, T_2 and the identity map. A similar result remains true for many commuting contractions, but for the proof we need a different method. We shall use an argument based on the ultrapower spaces, suggested by the proof of Theorem 2.1 of Léka's paper ([16]).

Let \mathcal{U} be a free ultrafilter on \mathbb{Z}_+^k , $\ell^\infty(\mathbb{Z}_+^k, H)$ the Banach space of all bounded functions from \mathbb{Z}_+^k to H and $c_0(\mathbb{Z}_+^k, H, \mathcal{U})$ the set of all functions in $\ell^\infty(\mathbb{Z}_+^k, H)$ which converge to zero throughout the ultrafilter \mathcal{U} . The quotient space $H_{\mathcal{U}} := \ell^\infty(\mathbb{Z}_+^k, H) / c_0(\mathbb{Z}_+^k, H, \mathcal{U})$ is called an ultrapower space of H . For $x = (x_n)_{n \in \mathbb{Z}_+^k}$ we denote by \bar{x} the equivalence classe of x , that is $\bar{x} = (x_n)_{n \in \mathbb{Z}_+^k} + c_0(\mathbb{Z}_+^k, H, \mathcal{U})$. Notice that $H_{\mathcal{U}}$ is also a Hilbert space with the inner product

$$\langle \bar{x}, \bar{y} \rangle = \mathcal{U}\text{-}\lim_n \langle x_n, y_n \rangle,$$

For $S \in \mathcal{L}(H)$ we denote by $S_{\mathcal{U}}$ the operator defined on $H_{\mathcal{U}}$ by $S_{\mathcal{U}}\bar{x} = \overline{(Sx_n)_{n \in \mathbb{Z}_+^k}}$ (See [12]).

Let $T : \mathbb{Z}_+^k \rightarrow \mathcal{L}(H)$ be a representation of \mathbb{Z}_+^k by contractions on H . We set $T_{\mathcal{U}} = (T(n)_{\mathcal{U}})_{n \in \mathbb{Z}_+^k}$. Since the mapping $S \rightarrow S_{\mathcal{U}}$ is an isometric algebra homomorphism from $\mathcal{L}(H)$ into $\mathcal{L}(H_{\mathcal{U}})$, $T_{\mathcal{U}}$ is a representation of \mathbb{Z}_+^k by

contractions on $H_{\mathcal{U}}$. Moreover, we have $\text{Sp}(T_{\mathcal{U}}, \mathbb{Z}_+^k) = \text{Sp}(T, \mathbb{Z}_+^k)$ and if $T(n) = T_1^{n_1} \dots T_k^{n_k}$, $n \in \mathbb{Z}_+^k$, where T_1, \dots, T_k are finitely many commuting contractions on H , then $T_{\mathcal{U}}(n) = (T_1)_{\mathcal{U}}^{n_1} \dots (T_k)_{\mathcal{U}}^{n_k}$.

Theorem 4.1. *Let T_1, T_2, \dots, T_k be a commuting contractions on a Hilbert space H and $f \in \ell^1(\mathbb{Z}_+^k)$, then*

$$\lim_{\min(n_1, n_2, \dots, n_k) \rightarrow \infty} \|T_1^{n_1} T_2^{n_2} \dots T_k^{n_k} \widehat{f}(T_1, T_2, \dots, T_k)\| = \|\widehat{f}\|_{\mathcal{C}(\sigma_{\mathcal{A}}(T_1, T_2, \dots, T_k) \cap \mathbb{T}^k)},$$

where \mathcal{A} is any unital commutative Banach algebra that contains T_1, T_2, \dots, T_k and $\sigma_{\mathcal{A}}(T_1, T_2, \dots, T_k)$ is the joint spectrum of T_1, T_2, \dots, T_k relative to \mathcal{A} .

Proof. Let T be the representation of \mathbb{Z}_+^k defined by $T(n) = T_1^{n_1} \dots T_k^{n_k}$ for $n = (n_1, \dots, n_k) \in \mathbb{Z}_+^k$. Let V, π and Y be the elements associated to T and given by Lemma 2.3. Notice that by Remark 2.4, Y is a Hilbert space.

For every $f \in \ell^1(\mathbb{Z}_+^k)$ and every unit vector $x \in H$, we have

$$\lim_{\mathbb{Z}_+^k} \|T(n) \widehat{f}(T)x\| = \|\widehat{f}(V)\pi x\| \leq \|\widehat{f}(V)\|.$$

Notive that for every $n \in \mathbb{Z}_+^k$, $V(n) = V_1^{n_1} \dots V_k^{n_k}$, where V_1, \dots, V_k are commuting isometries on the Hilbert space Y . In this case the von Neumann inequality holds, that is $\|\widehat{f}(V)\| \leq \|\widehat{f}\|_{\mathcal{C}(\mathbb{T}^k)}$ (see [19], p. 28–29; [8]). So we get from Lemma 2.5 that

$$\lim_{\mathbb{Z}_+^k} \|T(n) \widehat{f}(T)x\| \leq \|\widehat{f}\|_{\mathcal{C}(\text{Sp}_u(T, \mathbb{Z}_+^k))}. \quad (4.1)$$

Take \mathcal{U} a free ultrafilter on \mathbb{Z}_+^k and let $H_{\mathcal{U}}$ be the ultrapower space of H defined previously.

Let $\epsilon > 0$. We set $\alpha := \lim_{\mathbb{Z}_+^k} \|T(n) \widehat{f}(T)\| = \inf_{n \in \mathbb{Z}_+^k} \|T(n) \widehat{f}(T)\|$. For every $m \in \mathbb{Z}_+^k$, there exists a unit vector x_m such that $(1-\epsilon)\alpha \leq \|T(m) \widehat{f}(T)x_m\|$. So for every $n \preccurlyeq m$, we have $(1-\epsilon)\alpha \leq \|T(n) \widehat{f}(T)x_m\|$.

Let $\bar{x} = (x_m)_{m \in \mathbb{Z}_+^k} + c_0(\mathbb{Z}_+^k, H, \mathcal{U})$. We have

$$\begin{aligned} \|T_{\mathcal{U}}(n) \widehat{f}(T_{\mathcal{U}})\bar{x}\| &= \left\| \left(T(n) \widehat{f}(T)x_m \right)_{m \in \mathbb{Z}_+^k} + c_0(\mathbb{Z}_+^k, H, \mathcal{U}) \right\| \\ &= \mathcal{U}\text{-}\lim_m \|T(n) \widehat{f}(T)x_m\| \\ &\geq \liminf_m \|T(n) \widehat{f}(T)x_m\| \geq (1-\epsilon)\alpha. \end{aligned}$$

Now applying (4.1) to the operator $T_{\mathcal{U}}$ and \bar{x} and letting $\epsilon \rightarrow 0$, we obtain

$$\alpha \leq \|\widehat{f}\|_{\mathcal{C}(\text{Sp}_u(T_{\mathcal{U}}, \mathbb{Z}_+^k))} = \|\widehat{f}\|_{\mathcal{C}(\text{Sp}_u(T, \mathbb{Z}_+^k))} = \|\widehat{f}\|_{\mathcal{C}(\sigma_{\mathcal{A}}(T_1, T_2, \dots, T_k) \cap \mathbb{T}^k)},$$

which finishes the proof. \square

5. REPRESENTATIONS OF SUBSEMIGROUPS BY CONTRACTIONS ON A BANACH SPACE

Let G , Γ and S be as in section 2. If I is a closed ideal of $L^1(G)$, we set

$$h(I) = \{\chi \in \Gamma : \widehat{f}(\chi) = 0, \text{ for all } f \in I\},$$

where $\widehat{f}(\chi) = \int_G f(t)\chi(t)dt$.

Let E be a closed subset of Γ . We set

$$I(E) = \{f \in L^1(G), \widehat{f} = 0 \text{ on } E\}$$

and

$$J(E) = \text{clos} \{f \in L^1(G), \widehat{f} = 0 \text{ on a neighborhood of } E\}.$$

Clearly $I(E)$ and $J(E)$ are closed ideals of $L^1(G)$, $J(E) \subset I(E)$ and we have $h(I(E)) = h(J(E)) = E$ (see [22], p. 161). The set E is said to satisfy spectral synthesis if $J(E) = I(E)$. Notice that E satisfies spectral synthesis if there exists a unique closed ideal I such that $h(I) = E$. A function $f \in L^1(G)$ (or \widehat{f}) is said to satisfy spectral synthesis for E if $f \in J(E)$.

Let $A(\Gamma)$ denote the Banach algebra of functions \widehat{f} , $f \in L^1(G)$ endowed with the norm $\|\widehat{f}\|_{A(\Gamma)} = \|f\|_{L^1(G)}$. Let $A(E) = \{\widehat{f}|_E, f \in L^1(G)\}$ be the Banach space equipped with the quotient norm of $L^1(G)/I(E)$, that is $\|\widehat{f}\|_{A(E)} = \inf_{g \in I(E)} \|f - g\|_{L^1(G)}$. It is well known that $A(E) \subset C_0(E)$. The set E is called a Helson set if $A(E) = C_0(E)$. Notice that E is a Helson set if and only if the canonical injection $i : A(E) \rightarrow C_0(E)$ is onto. This is equivalent to the fact that the quantity

$$\alpha(E) = \sup \left\{ \frac{\|\widehat{f}\|_{A(E)}}{\|\widehat{f}\|_{C(E)}}, f \in L^1(G) \text{ and } \widehat{f}|_E \neq 0 \right\},$$

is finite. By duality, we have

$$\alpha(E) = \sup \left\{ \frac{\|\mu\|}{\|\widehat{\mu}\|_\infty}, \mu \in M(E) \text{ and } \mu \neq 0 \right\},$$

where $M(E)$ is the set of Borel measures on Γ with support contained in E , $\|\mu\|$ is the total variation of μ and $\|\widehat{\mu}\|_\infty = \sup_{x \in G} |\widehat{\mu}(x)|$; $\alpha(E)$ is called the Helson constant of E (see [11] and [22]). At the end of this section we give various examples of Helson sets satisfying spectral synthesis.

Lemma 5.1. *Let V be a representation of S by isometries on a Banach space Y and $y \in Y$. Assume that $\text{Sp}_u(V, S) \subset E$, where $E \subset \Gamma$ is a closed set that satisfies spectral synthesis and which is a Helson set. Then for every $f \in L^1(S)$,*

$$\|\widehat{f}(V)y\| \leq \alpha(E) \|\widehat{f}\|_{C(E)} \|y\|.$$

Proof. Let V_d be as in Lemma 2.1, with $\text{Sp}(V_d, G) = \text{Sp}_u(V, S)$. Let $f \in L^1(S)$. Since E satisfies spectral synthesis, for every $g \in L^1(G)$ such that \widehat{g} vanishes on E , we have $\widehat{g}(V_d) = 0$ (see [15], p. 201). So

$$\begin{aligned} \|\widehat{f}(V)y\| &= \|\widehat{f}(V_d)y\| \leq \inf_{g \in I(E)} \|\widehat{f}(V_d)y - \widehat{g}(V_d)y\| \\ &\leq \inf_{g \in I(E)} \|\widehat{f} - \widehat{g}\| \|y\| = \|\widehat{f}\|_{A(E)} \|y\|. \end{aligned}$$

Now since E is a Helson set, we get $\|\widehat{f}(V)y\| \leq \alpha(E) \|\widehat{f}\|_{C(E)} \|y\|$. □

Lemma 5.2. *Let T be a representation of S by contractions on a Banach space X and $x \in X$. Assume that $E := \text{Sp}_u(T, S)$ satisfies spectral synthesis and is a Helson set. Then for every $f \in L^1(S)$,*

$$\lim_S \|T(t)\widehat{f}(T)x\| \leq \alpha(E) \|\widehat{f}\|_{C(E)} \|x\|.$$

Proof. Let V and π be respectively, the isometry and the contractive homomorphism given by Lemma 2.3. Let $f \in L^1(S)$. We have $\lim_S \|T(t)\widehat{f}(T)x\| = \|\widehat{f}(V)\pi x\|$ and by Lemma 5.1, we get

$$\lim_S \|T(t)\widehat{f}(T)x\| \leq \alpha(E) \|\widehat{f}\|_{C(E)} \|\pi x\| \leq \alpha(E) \|\widehat{f}\|_{C(E)} \|x\|.$$

□

We shall also need the following lemma.

Lemma 5.3. *Let T be a representation of G by isometries on a Banach space. Let $I = \{f \in L^1(G) : \widehat{f}(T) = 0\}$. Then I is a closed ideal of $L^1(G)$ and $\text{Sp}_u(T, S) = \text{Sp}(T, G) = h(I)$.*

Proof. It is easily seen that I is a closed ideal. Notice that $\text{Sp}(T, G)$ is the Arveson spectrum of T ([20]) or the finite L -spectrum of T ([15]). The equality $\text{Sp}(T, G) = h(I)$ follows from [20], Proposition 8.1.9.

The equality $\text{Sp}_u(T, S) = \text{Sp}(T, G)$ is stated without proof [4], p. 170. For the sake of completeness we include here the proof. The inclusion $\text{Sp}(T, G) \subset \text{Sp}_u(T, S)$ is straightforward. Let $\chi \in \text{Sp}_u(T, S)$, $f \in L^1(G)$ and $\epsilon > 0$. There exists $s \in S$ such that $\int_{G \setminus (S-s)} |f(t)| dt < \epsilon$. We set

$$f_s(t) = \begin{cases} f(t-s) & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases}$$

We have $f_s \in L^1(S)$ and

$$|\widehat{f}(\chi)| \leq |\widehat{f}_s(\chi)| + \epsilon \leq \|\widehat{f}_s(T)\| + \epsilon \leq \|\widehat{f}(T)\| + 2\epsilon.$$

Hence $|\widehat{f}(\chi)| \leq \|\widehat{f}(T)\|$, for every $f \in L^1(S)$ and thus $\chi \in \text{Sp}(T, G)$. Therefore $\text{Sp}(T, G) = \text{Sp}_u(T, S)$. □

Theorem 5.4. *Let $c \geq 1$ and E a closed subset of Γ . Then the following are equivalent.*

- (i) *E satisfies spectral synthesis and is a Helson set with $\alpha(E) \leq c$.*
- (ii) *For every representation T of S by contractions on a Banach space X such that $\text{Sp}_u(T, S) = E$, we have*

$$\lim_S \|T(t)\widehat{f}(T)\| \leq c\|\widehat{f}\|_{\mathcal{C}(E)}, \quad f \in L^1(S).$$

Proof. (i) \Rightarrow (ii) Let T be a representation of S by contractions on X with $\text{Sp}_u(T, S) = E$. Let $M(t) : A \mapsto T(t)A$. Then we apply Lemma 5.2 to the restriction of M to a closed subspace of $\mathcal{L}(X)$ as in the proof of Theorem 2.6.

(ii) \Rightarrow (i) Let I be a closed ideal of $L^1(G)$ with $h(I) = E$. Let $\pi : L^1(G) \rightarrow L^1(G)/I$ denote the canonical surjection and, for $t \in G$, τ_t the translation operator defined on $L^1(G)$ by $\tau_t f(x) = f(x-t)$, $x \in G$. We set $T(t)(\pi(f)) = \pi(\tau_t f)$, $f \in L^1(G)$. Since I is invariant under the translation group τ ([22], Theorem 7.1.2, p. 157), $T(t)$ is well defined and is an invertible isometry.

For $f \in L^1(G)$, $\widehat{f}(T)$ is the operator defined by: $\pi(g) \rightarrow \pi(f * g)$. Hence $\widehat{f}(T) = 0$ holds if and only if $f * g \in I$, whenever $g \in L^1(G)$. Since there exists a sequence $(g_n)_n$ in $L^1(G)$ such that $\|f - f * g_n\| \rightarrow 0$ (see [22], Theorem 1.1.8, p. 6), we see that $\widehat{f}(T) = 0$ holds if and only if $f \in I$. It follows from Lemma 5.3 that $\text{Sp}_u(T, S) = \text{Sp}(T, G) = E$.

Let $f \in L^1(G)$ and $\epsilon > 0$. There exists $s \in S$ such that $\int_{G \setminus (S-s)} |f(t)| dt < \epsilon$. We set

$$f_s(t) = \begin{cases} f(t-s) & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases}$$

We have $f_s \in L^1(S)$, $\|\widehat{f_s}\|_{\mathcal{C}(E)} \leq \|\widehat{f}\|_{\mathcal{C}(E)} + \epsilon$ and

$$\widehat{f_s}(T) = \int_S f(t-s)T(t)dt = T(s) \int_{S-s} f(t)T(t)dt.$$

Since for every t , $T(t)$ is an isometry with $\text{Sp}_u(T, S) = E$, we get

$$\left\| \int_{S-s} f(t)T(t)dt \right\| = \|\widehat{f_s}(T)\| = \lim_S \|T(t)\widehat{f_s}(T)\| \leq c\|\widehat{f_s}\|_{\mathcal{C}(E)} \leq c(\|\widehat{f}\|_{\mathcal{C}(E)} + \epsilon).$$

Hence

$$\begin{aligned} \|\widehat{f}(T)\| &\leq \left\| \int_{S-s} f(t)T(t)dt \right\| + \left\| \int_{G \setminus (S-s)} f(t)T(t)dt \right\| \\ &\leq c(\|\widehat{f}\|_{\mathcal{C}(E)} + \epsilon) + \int_{G \setminus (S-s)} |f(t)| dt \\ &\leq c\|\widehat{f}\|_{\mathcal{C}(E)} + (c+1)\epsilon. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we obtain

$$\|\widehat{f}(T)\| \leq c\|\widehat{f}\|_{\mathcal{C}(E)}. \quad (5.1)$$

Notice that $\widehat{f}(T)$ is the operator $\pi(g) \rightarrow \pi(f * g)$. Obviously, we have the inequality $\|\widehat{f}(T)\| \leq \|\pi(f)\|$. On the other hand side there exists a sequence $(u_n)_n$ of functions in $L^1(G)$ such that, for every n , $\|u_n\|_{L^1(G)} = 1$ and $f * u_n \rightarrow f$ (see ([22], Theorem 1.1.8, p. 6). It follows that $\|\widehat{f}(T)\| \geq \|\pi(f)\|$. Thus

$$\|\widehat{f}(T)\| = \|\pi(f)\|. \quad (5.2)$$

If $\pi(f) = 0$, that is $f \in I$ the last equality is obvious. It follows from (5.1) and (5.2) that

$$\|\pi(f)\| \leq c\|\widehat{f}\|_{\mathcal{C}(E)}. \quad (5.3)$$

We take $I = J(E)$. It follows from (5.3) that if $f \in L^1(G)$ with $\widehat{f} = 0$ on E , then $\pi(f) = 0$ and thus $f \in J(E)$. This shows that E satisfies spectral synthesis.

We take now $I = I(E)$. Let $i : A(E) \rightarrow \mathcal{C}_0(E)$ be the canonical injection. Inequality (5.3) shows that the range of i is closed. On the other hand, since $A(\Gamma)$ is dense in $\mathcal{C}_0(\Gamma)$ ([22], Theorem 1.2.4, p. 9), the range of i is also dense in $\mathcal{C}_0(E)$. So i is surjective, which implies that E is a Helson set with $\alpha(E) = \|i^{-1}\| \leq c$. \square

We get immediately the following result.

Corollary 5.5. *Let E be a closed subset of Γ . The following are equivalent.*

- (i) *E satisfies spectral synthesis and is a Helson set with $\alpha(E) = 1$.*
- (ii) *For every representation T of S by contractions on a Banach space such that $Sp_u(T, S) = E$, we have*

$$\lim_S \|T(t)\widehat{f}(T)\| = \|\widehat{f}\|_{\mathcal{C}(E)}, \quad f \in L^1(S).$$

In particular if T is a representation of S by contractions and if $Sp_u(T, S) = \{\chi\}$, then

$$\lim_S \|T(t)\widehat{f}(T)\| = |\widehat{f}(\chi)|, \quad f \in L^1(S).$$

Remark 5.6. In this remark we give some examples of Helson sets which satisfy spectral synthesis and we discuss the constant $\alpha(E)$. Let E be a closed subset of Γ ; E is called independent if for every χ_1, \dots, χ_k in E and every integers n_1, \dots, n_k ,

$$\chi_1^{n_1} \dots \chi_k^{n_k} = 1 \Rightarrow \chi_1^{n_1} = \dots = \chi_k^{n_k} = 1.$$

A compact subset E of Γ is called a Kronecker set if the following property holds: for every continuous function h on E with $|h| = 1$ and for every $\epsilon > 0$ there exists $t \in G$ such that $\sup_{\chi \in E} |h(\chi) - \chi(t)| < \epsilon$. Every Kronecker set E

satisfies spectral synthesis and is a Helson set with $\alpha(E) = 1$. Indeed, every measure μ with support contained in a Kronecker set satisfies $\|\mu\| = \|\hat{\mu}\|_\infty$ (see [22], Theorem 5.5.2, p. 113). It follows from the discussion made at the beginning of this section that a Kronecker set E is a Helson set with $\alpha(E) = 1$.

The fact that a Kronecker set is of spectral synthesis was proved by N. Varopoulos ([25]) for totally disconnected Kronecker sets and then extended to any Kronecker set by S. Saeki ([23]).

Notice that every Kronecker set is independent. If E is a finite independent set and if every element of E is of infinite order, then E is a Kronecker set (see [22], p. 98).

We recall also that if E is a finite set, then E satisfies spectral synthesis and is a Helson set with $\alpha(E) \leq \sqrt{\#E}$ where $\#E$ is the cardinal of E . On the other hand, for n a nonnegative integer, there exists E such that $\#E = 2^n$ and $\alpha(E) \geq \sqrt{\#E/2}$ (see [11], p. 33).

Finally we note that a countable, compact, independent set is a Helson set and satisfies spectral synthesis (see [22], Theorem 5.6.7, p. 117 and p.161).

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