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Claire David

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Spurious lattice solitons for linear finite difference schemes

Claire David^{†*},

Université Pierre et Marie Curie-Paris 6

[†] Institut Jean Le Rond d'Alembert, UMR CNRS 7190,

Boîte courrier n^0162 , 4 place Jussieu, 75252 Paris, cedex 05, France

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Abstract

The goal of this work is to show that lattice traveling solitary wave are solution of the general linear finite-differenced version of the linear advection equation. The occurrence of such a spurious solitary waves, which exhibits a very long life time, results in a non-vanishing numerical error for arbitrary time in unbounded numerical domain. Such a behavior is referred here to has a structural instability of the scheme, since the space of solutions spanned by the numerical scheme encompasses types of solutions (lattice solitary waves in the present case) that are not solution of the original continuous equations.

Numerical schemes; spurious lattice solitary waves.

1 Introduction

The analysis and the control of numerical error in discretized propagation-type equations is of major importance for both theoretical analysis and practical applications. A huge amount of works has been devoted to the analysis of the numerical errors, its dynamics and its influence on the computed solution (the reader is referred to classical books, among which [1, ?, 4, 5]). It appears that existing works are mostly devoted to linear, one-dimensional numerical models, such as the linear advection equation

*Corresponding author: david@lmm.jussieu.fr; fax number: (+33) 1.44.27.52.59.

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad (1)$$

where c is a constant uniform advection velocity. A striking observation is that, despite the tremendous efforts devoted to the analysis of numerical schemes in this simple case, the full exact non-homogeneous error equation has been derived only very recently [?].

The two sources of numerical error are the dispersive and dissipative properties of the numerical scheme, which are very often investigated in unbounded or periodic domains thanks to a spectral analysis. In previous work [6], we analyzed of a linear dispersive mechanism which results in local error focusing, i.e. to a sudden local error burst in the L_∞ norm for polychromatic solutions, referred to as the spurious caustic phenomenon. We showed that, for some specific values of the Courant number, spurious caustics can exist for some popular finite-difference schemes.

In another work [7], [8] we have determined classes of traveling solitary wave solutions for a differential approximation of a finite difference scheme by means of a hyperbolic ansatz. We showed that spurious solitary waves can occur in finite-difference solutions of nonlinear wave equation. The occurrence of such a spurious solitary wave, which exhibits a very long life time, results in a non-vanishing numerical error for arbitrary time in unbounded numerical domain. Such a behavior is referred here to has a structural instability of the scheme, since the space of solutions spanned by the numerical scheme encompasses types of solutions (solitary waves in the present case) that are not solution of the original continuous equations.

We presently extend both works, in so far as we exhibit lattice solitary waves solution of the general linear finite-differenced version of the linear advection equation, rejoining the fact that there exists travelling solitary wave solutions for a differential approximation of a finite difference scheme.

The present analysis is restricted to interior stencil, and the influence of boundary conditions will not be considered.

The paper is organized as follows. The numerical schemes retained for the present analysis are briefly recalled in section 2. Solitary waves, and the related lattice ones are introduced and developped in section 3.

2 Test numerical schemes

For the sake of simplicity, the analysis will be restricted to schemes which involves at most three time levels and three grid points. The extension of the present analysis

Name	α	β	γ	δ	ϵ	ζ	η	θ	v
Leapfrog	$\frac{1}{2\tau}$	0	$\frac{-1}{2\tau}$	$\frac{c}{2h}$	$\frac{-c}{2h}$	0	0	0	0
Lax	$\frac{1}{\tau}$	0	0	$\frac{-1}{2\tau} + \frac{c}{2h}$	$\frac{-1}{2\tau} - \frac{c}{2h}$	0	0	0	0
Lax-Wendroff	$\frac{1}{\tau}$	$\frac{-1}{\tau} + \frac{c^2\tau}{h^2}$	0	$\frac{(1-\sigma)c}{2h}$	$\frac{-(1+\sigma)c}{2h}$	0	0	0	0
Crank-Nicolson	$\frac{2}{\tau}$	$\frac{2}{\tau}$	0	$\frac{-c}{2h}$	$\frac{c}{2h}$	$\frac{c}{2h}$	0	$\frac{-c}{2h}$	0

Table 1: Numerical scheme coefficients.

to other schemes is straightforward. For this class of schemes, the general finite-differenced version of the linear advection equation (1) can therefore be written as follows

$$\alpha u_j^{n+1} + \beta u_j^n + \gamma u_j^{n-1} + \zeta u_{j+1}^{n+1} + \delta u_{j+1}^n + \nu u_{j+1}^{n-1} + \theta u_{j-1}^{n+1} + \varepsilon u_{j-1}^n + \eta u_{j-1}^{n-1} = 0 \quad (2)$$

with

$$u_l^m = u(lh, m\tau) \quad (3)$$

where h and τ are the mesh size and time step respectively. For the sake of simplicity, these two quantities are assumed to be uniform. The CFL number is defined as $\sigma = c\tau/h$, while the non-dimensional wave number is defined as $\varphi = kh$, where k is the wave number of the signal under consideration.

A numerical scheme is specified by selecting appropriate values of the coefficients $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, v$ and θ in Eq. (2). Values corresponding to numerical schemes retained for the present works are given in Table 1.

3 Spurious lattice solitons

3.1 Analytical validation

Following [9], [10], [7], [8], we search solution of Eq. (1) under the form:

$$u(x, t, k) = A \operatorname{Sech}[k(x - vt)] + B \operatorname{Tanh}[k(x - vt)] \quad (4)$$

The discrete solution associated with a given numerical scheme will admit spurious solitary waves, and therefore spurious local energy pile-up and local sudden growth of the error, if the discrete relation is such that the condition (2) is satisfied by the solitary wave (4).

The related condition, which will be referred to as the solitary wave dispersion relation, is of the form

$$\mathcal{F}(i, n, \sigma, \tau, A, B, \text{Sech}[k(x - vt)], \text{Tanh}[k(x - vt)]) = 0 \quad (5)$$

where is a generic notation.

Depending on the existence of real numbers A, B, v, k , and of integers i, n satisfying this relation, spurious lattice solitary waves will or not appear.

4 Scheme study

4.1 The Lax scheme

Let us consider the Lax scheme. The solitary wave dispersion relation (5) yields:

$$\begin{aligned} () \quad & -(\sigma - 1)\text{sech}\left(\frac{k\tau(i - nv\sigma + 1)}{\sigma}\right) \left(A + B \sinh\left(\frac{k\tau(i - nv\sigma + 1)}{\sigma}\right)\right) \\ & + B(\sigma + 1)\text{Tanh}\left(\frac{k\tau(i - nv\sigma - 1)}{\sigma}\right) = 0 \end{aligned} \quad (6)$$

In the specific case where $\sigma = 1$, one obtains:

$$\text{sech}(k\tau(i - nv - 1))(A + B \sinh(k\tau(i - nv - 1))) = 0 \quad (7)$$

1. For $A = 0, v = 1$, the solitary wave dispersion relation is satisfied when

$$i = n - 1 \quad (8)$$

which occurs on the recursive calculation of the approximate solution.

It thus exhibits the existence of lattice solitons, related to the discrete numerical scheme, of the form

$$u_i = B \text{Tanh}[k(ih - (i + 1)\tau)] \quad (9)$$

2. More generally, numerical simulations usually dealing with values of the time step number $n \gg 1$, for $A = 0, v = \frac{1}{p}, p \in \mathbb{N}^*$, the solitary wave dispersion relation is satisfied when

$$n = p(i - 1) \quad (10)$$

It thus also exhibits the existence of lattice solitons, related to the discrete numerical scheme, of the form

$$u_i = B \operatorname{Tanh} [k (i h - p (i - 1) \tau)] \quad (11)$$

Also, when the time step number n goes towards infinity, $\operatorname{sech}(k \tau (i - n v - 1))$ tends towards zero, and the solitary wave dispersion relation (6) tends to be satisfied, accounting for the scheme to become numerically instable.

4.2 The Lax-Wendroff scheme

Let us consider the Lax-Wendroff scheme. The solitary wave dispersion relation (5) yields:

$$\begin{aligned} & \frac{\sigma(\sigma+1) \left(A \operatorname{sech} \left(k \left(\frac{\tau(i-1)}{\sigma} - \tau n v \right) \right) + B \operatorname{tanh} \left(k \left(\frac{\tau(i-1)}{\sigma} - \tau n v \right) \right) \right)}{2\tau} \\ & + \left(\frac{\sigma^2}{\tau} - \frac{1}{\tau} \right) \left(A \operatorname{sech} \left(k \left(\frac{\tau i}{\sigma} - \tau n v \right) \right) + B \operatorname{tanh} \left(k \left(\frac{\tau i}{\sigma} - \tau n v \right) \right) \right) \\ & + \frac{A \operatorname{sech} \left(k \left(\frac{\tau i}{\sigma} - \tau (n+1) v \right) \right) + B \operatorname{tanh} \left(k \left(\frac{\tau i}{\sigma} - \tau (n+1) v \right) \right)}{\tau} \\ & + \frac{(1-\sigma)\sigma \left(A \operatorname{sech} \left(k \left(\frac{\tau(i+1)}{\sigma} - \tau n v \right) \right) + B \operatorname{tanh} \left(k \left(\frac{\tau(i+1)}{\sigma} - \tau n v \right) \right) \right)}{2\tau} = 0 \end{aligned} \quad (12)$$

In the specific case where $\sigma = 1$, one obtains:

$$\begin{aligned} & A \left\{ -\operatorname{sech}(\tau k (i - n v - 1)) + \operatorname{sech}(\tau k (i - (n + 1) v)) \right\} \\ & + B \left\{ (\operatorname{tanh}(\tau k (-i + n v + 1)) + \operatorname{tanh}(\tau k (i - (n + 1) v))) \right\} = 0 \end{aligned} \quad (13)$$

1. For $A = 0$, $v = 1$, the solitary wave dispersion relation is satisfied when

$$n = i - 1 \quad (14)$$

which occurs on the recursive calculation of the approximate solution.

It thus exhibits the existence of lattice solitons, related to the discrete numerical scheme, of the form

$$u_i = B \operatorname{Tanh} [k (i h - (i - 1) \tau)] \quad (15)$$

2. As in the case of the Lax scheme, when the time step number n goes towards infinity, $\operatorname{sech}(k \tau (i - n v - 1))$ tends towards zero, and the solitary wave dispersion relation (12) tends to be satisfied, accounting for the scheme to become numerically instable.

4.3 The Leapfrog scheme

Let us consider the Leapfrog scheme. The solitary wave dispersion relation (5) yields:

$$A \operatorname{sech} \left(\frac{k \tau (i - (n-1) v \sigma)}{\sigma} \right) + A \sigma \operatorname{sech} \left(\frac{k \tau (i - n v \sigma - 1)}{\sigma} \right) - A \sigma \operatorname{sech} \left(\frac{k \tau (i - n v \sigma + 1)}{\sigma} \right) \\ + B \left(\tanh \left(\frac{k \tau (i - (n-1) v \sigma)}{\sigma} \right) + \sigma \left(\tanh \left(\frac{k \tau (i - n v \sigma - 1)}{\sigma} \right) - \tanh \left(\frac{k \tau (i - n v \sigma + 1)}{\sigma} \right) \right) \right) = 0 \quad (16)$$

For

$$v = v_{Leapfrog}^{sol} \frac{k(j+1)\tau + \sigma \cosh^{-1} \left(-\sqrt{\frac{A^2}{B^2} + 1} \right)}{k n \tau \sigma} \quad (17)$$

the solitary wave dispersion relation (16) is satisfied. It thus exhibits the existence of lattice solitons, related to the discrete numerical scheme, of the form

$$u_i^n = A \operatorname{Sech} [k(i h - n v_{Leapfrog}^{sol} \tau)] + B \operatorname{Tanh} [k(i h - n v_{Leapfrog}^{sol} \tau)] \quad (18)$$

4.4 The Crank-Nicolson scheme

Let us consider the Crank-Nicolson scheme. The solitary wave dispersion relation (5) yields:

$$\frac{\sigma^2 (A \operatorname{sech}(k(\frac{\tau(i-1)}{\sigma} - n v \tau)) + B \tanh(k(\frac{\tau(i-1)}{\sigma} - n v \tau)))}{2\tau^2} \\ - \frac{\sigma^2 (A \operatorname{sech}(k(\frac{\tau(i-1)}{\sigma} - \tau(n+1)v)) + B \tanh(k(\frac{\tau(i-1)}{\sigma} - \tau(n+1)v)))}{2\tau^2} \\ + \left(\frac{\sigma^2}{\tau^2} - \frac{1}{\tau} \right) (A \operatorname{sech}(k(\frac{\tau i}{\sigma} - \tau n v)) + B \tanh(k(\frac{\tau i}{\sigma} - \tau n v))) \\ + \left(\frac{\sigma^2}{\tau^2} + \frac{1}{\tau} \right) (A \operatorname{sech}(k(\frac{\tau i}{\sigma} - (n+1)v \tau)) + B \tanh(k(\frac{\tau i}{\sigma} - (n+1)v \tau))) \\ - \frac{\sigma^2 (A \operatorname{sech}(k(\frac{\tau(i+1)}{\sigma} - n v \tau)) + B \tanh(k(\frac{\tau(i+1)}{\sigma} - n v \tau)))}{2\tau^2} = 0 \quad (19)$$

5 Concluding remarks

The existence of spurious numerical lattice solitary waves in linear advection schemes has been proved. Such lattice solitary waves, which are not solutions of the exact continuous original equation, nevertheless satisfy the numerical scheme, appearing as parasitic solutions of the correct one. Such schemes will be referred to as structurally unstable ones. Such spurious solitary waves have constant energy, and therefore the numerical error norm does not vanish at arbitrary long integration times on unbounded numerical domains.

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