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# POINCARÉ INEQUALITY AND THE $\mathbb{L}^p$ CONVERGENCE OF SEMI-GROUPS.

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ABSTRACT. We prove that for symmetric Markov processes of diffusion type admitting a “carré du champ”, the Poincaré inequality is equivalent to the exponential convergence of the associated semi-group in one (resp. all)  $\mathbb{L}^p(\mu)$  spaces for  $1 < p < +\infty$ . Part of this result extends to the stationary non necessarily symmetric situation.

*Key words* : Poincaré inequality, rate of convergence.

*MSC 2010* : . 26D10, 39B62, 47D07, 60G10, 60J60.

## 1. Introduction and main results.

Let  $X_t$  be a general Markov processes with infinitesimal generator  $L$  and with state space some Polish space  $E$ . We assume that the extended domain of the generator contains a nice core  $\mathcal{D}$  of uniformly continuous functions, containing the constant functions, which is an algebra, for which we may define the “carré du champ” operator

$$\Gamma(f, g) = \frac{1}{2} (L(fg) - fLg - gLf) .$$

Functions in  $\mathcal{D}$  will be called “smooth”. The associated Dirichlet form can thus be calculated for smooth  $f$ 's as

$$\mathcal{E}(f, f) := - \int f Lf d\mu = \int \Gamma(f, f) d\mu .$$

In addition we assume that  $L$  is  $\mu$ -symmetric for some probability measure defined on  $E$ . Thus  $L$  generates a  $\mu$ -symmetric (hence stationary) semi-group  $P_t$ , which is a contraction semi-group on all  $\mathbb{L}^p(\mu)$  for  $1 \leq p \leq +\infty$ , and the  $\mathbb{L}^2$  ergodic theorem (in the symmetric case) tells us that for all  $f \in \mathbb{L}^2(\mu)$ ,

$$\lim_{t \rightarrow +\infty} \| P_t f - \int f d\mu \|_{\mathbb{L}^2(\mu)} = 0 .$$

For all this one can give a look at [2].

It is then well known that the following two statements are equivalent

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**(H-Poinc).**  $\mu$  satisfies a Poincaré inequality, i.e. there exists a constant  $C_P$  such that for all smooth  $f$ ,

$$\mathrm{Var}_\mu(f) := \int f^2 d\mu - \left( \int f d\mu \right)^2 \leq C_P \int \Gamma(f, f) d\mu.$$

**(H-2).** There exists a constant  $\lambda_2$  such that

$$\mathrm{Var}_\mu(P_t f) \leq e^{-2\lambda_2 t} \mathrm{Var}_\mu(f).$$

If one of these assumptions is satisfied we have  $\lambda_2 = 1/C_P$ .

In the sequel we shall assume in addition that  $\Gamma$  comes from a derivation, i.e.

$$\Gamma(fg, h) = f \Gamma(g, h) + g \Gamma(f, h),$$

i.e. (in the terminology of [1]) that  $X$  is a diffusion. We also recall the chain rule: if  $\varphi$  is a  $C^2$  function,

$$L(\varphi(f)) = \varphi'(f) Lf + \varphi''(f) \Gamma(f, f).$$

In this note we shall establish the following theorem

**Theorem 1.1.** *For  $f \in \mathbb{L}^p(\mu)$  define  $N_p(f) := \|f - \int f d\mu\|_p$ . The following statements are equivalent*

- (1) **(H-Poinc)** is satisfied,
- (2) there exist some  $1 < p < +\infty$  and constants  $\lambda_p$  and  $K_p$  such that for all  $f \in \mathbb{L}^p(\mu)$ ,

$$N_p(P_t f) \leq K_p e^{-\lambda_p t} N_p(f),$$

- (3) for all  $1 < p < +\infty$ , there exist some constants  $\lambda_p$  and  $K_p$  such that for all  $f \in \mathbb{L}^p(\mu)$ ,

$$N_p(P_t f) \leq K_p e^{-\lambda_p t} N_p(f).$$

Of course we shall denote by **(H-p)** the property (2) for a given  $p$ .

We shall give two proofs of this theorem in the next two sections. The first one shows that if  $p \geq 2$  one can choose  $K_p = 1$  for some  $\lambda_p$  (see the remark at the end of the next section for an explicit bound). The second proof yields explicit values of  $K_p$  (but not 1) and  $\lambda_p$  for  $p \geq 2$ .

For  $p \leq 2$  a simple duality argument developed in the next section shows that, if  $\frac{1}{p} + \frac{1}{q} = 1$  one can choose  $K_p = 2K_q$  and  $\lambda_p = \lambda_q$ .

The case  $p = 1$  is extensively studied in [3] and the Poincaré inequality is no more sufficient in general to obtain an exponential decay in  $\mathbb{L}^1(\mu)$ . Replacing  $\mathbb{L}^p$  norms by Orlicz norms (weaker than any  $N_p$  for  $p > 1$ ) is possible provided one reinforces the Poincaré inequality into a  $F$ -Sobolev inequality (see [3] Theorem 3.1) as it is well known in the case  $F = \log$  for the Orlicz space  $\mathbb{L} \log \mathbb{L}$ .

The question of exponential convergence in  $\mathbb{L}^p$  ( $p \neq 2$ ) was asked to us by M. Ledoux after a conversation with A. Naor. We warmly thank M. Ledoux for preciously saving a copy of one of our main arguments that we loosed in our perfectly disordered office.

We did not find the statement of such a result in the literature. However recall that in [7], F.Y. Wang used the equivalent Beckner type formulation of Poincaré inequality to give a

partial answer to the problem i.e., a Poincaré inequality with constant  $C_P$  is equivalent to the following: for any  $1 \leq p \leq 2$  and for any non-negative  $f$ ,

$$\int (P_t f)^p d\mu - \left( \int f d\mu \right)^p \leq e^{-\frac{4(p-1)t}{pC_P}} \left( \int (f)^p d\mu - \left( \int f d\mu \right)^p \right).$$

(One has to take care with the constants since some 2 may or may not appear in the definition of  $\Gamma$ , depending on authors and of papers by the same authors.) This result cannot be used to study the decay to the mean in  $\mathbb{L}^p$  norm, but it is of particular interest when studying densities of probability.

## 2. POINCARÉ INEQUALITIES AND $\mathbb{L}^p$ SPACES.

We start with an improvement of the usual equivalence (**H-Poinc**) and (**H-2**) which seems to be well known by the specialists in Statistical Physics (we learned this result from P. Caputo and P. Dai Pra). We shall give however a very elementary proof

**Lemma 2.1.** *If it exists  $\beta > 0$  such that for all  $f \in \mathcal{C}$ ,  $\mathcal{C}$  being an everywhere dense subset of  $\mathbb{L}^2(\mu)$ , the following holds  $\text{Var}_\mu(P_t f) \leq c_f e^{-2\beta t}$ , then  $\text{Var}_\mu(P_t f) \leq e^{-2\beta t} \text{Var}_\mu(f)$  for all  $f \in \mathbb{L}^2(\mu)$ , i.e. the Poincaré inequality holds with  $C_P \leq 1/\beta$ .*

*Proof.* The proof lies on the following lemma proven in [6] using the spectral resolution

**Lemma 2.2.**  $t \mapsto \log \| P_t f \|_{\mathbb{L}^2(\mu)}$  is convex.

Here is a direct proof that does not use the spectral resolution. If  $n(t) = \| P_t f \|_{\mathbb{L}^2(\mu)}^2$ , the sign of the second derivative of  $\log n$  is the one of  $n''n - (n')^2$ . But

$$n'(t) = 2 \int P_t f L P_t f d\mu$$

and

$$n''(t) = 2 \int (L P_t f)^2 d\mu + 2 \int P_t f L P_t L f d\mu = 4 \int (L P_t f)^2 d\mu,$$

so that lemma 2.2 is just a consequence of Cauchy-Schwarz inequality.

In order to prove lemma 2.1, assuming that  $\int f d\mu = 0$  which is not a restriction, it is enough to look at

$$t \mapsto \log \| P_t f \|_{\mathbb{L}^2(\mu)} + \beta t,$$

which is convex, according to lemma 2.2, and bounded since  $\text{Var}_\mu(P_t f) \leq c_f e^{-2\beta t}$ . But a bounded convex function on  $\mathbb{R}^+$  is necessarily non-increasing. Hence

$$\| P_t f \|_{\mathbb{L}^2(\mu)} \leq e^{-\beta t} \| P_0 f \|_{\mathbb{L}^2(\mu)}$$

for all  $f \in \mathcal{C}$ , the result follows using the density of  $\mathcal{C}$ . □

An immediate consequence of Lemma 2.1 is that (2) implies (1) in the statement of Theorem 1.1.

Indeed if (**H-p**) holds for  $p \geq 2$ ,

$$N_2(P_t f) \leq N_p(P_t f) \leq K_p e^{-\lambda_p t} N_p(f)$$

and applying Lemma 2.1 with  $\mathcal{C} = \mathbb{L}^p(\mu)$  we deduce that  $C_P \leq 1/\lambda_p$ .

If **(H-p)** holds for  $1 < p \leq 2$  we may similarly write

$$N_2(P_t f) \leq N_\infty^{(2-p)/2}(f) N_p^{p/2}(f) \leq (K_p)^{p/2} e^{-p\lambda_p t/2} N_\infty^{(2-p)/2}(f) N_p^{p/2}(f)$$

and applying Lemma 2.1 with  $\mathcal{C} = \mathbb{L}^\infty(\mu)$  we deduce that  $C_P \leq 2/(p\lambda_p)$ .

Of course if  $p$  and  $q$  are conjugate exponents, **(H-p)** and **(H-q)** are equivalent. More precisely we may write, with  $\mu(f) := \int f d\mu$

$$\begin{aligned} \left| \int P_t(f - \mu(f)) g d\mu \right| &= \left| \int P_t(f - \mu(f)) (g - \mu(g)) d\mu \right| = \left| \int (f - \mu(f)) P_t(g - \mu(g)) d\mu \right| \\ &\leq N_p(f) N_q(P_t g) \leq K_q e^{-\lambda_q t} N_p(f) N_q(g) \\ &\leq 2 K_q e^{-\lambda_q t} N_p(f) \|g\|_{\mathbb{L}^q(\mu)} \end{aligned}$$

if **(H-q)** holds. Hence

**Lemma 2.3.** *If  $\frac{1}{p} + \frac{1}{q} = 1$ , **(H-q)** implies **(H-p)** with  $K_p \leq 2 K_q$  and  $\lambda_p \geq \lambda_q$ .*

*Accordingly if **(H-p)** holds, **(H-Poinc)** holds with  $C_P \leq 1/\lambda_p$ .*

If  $p \leq 2$  we obtain a better bound than the one we obtained directly.

Lemma 2.3 also shows that, in order to complete the proof of Theorem 1.1, it is enough to show that (1) implies (3) for all  $p \geq 2$ . We shall now proceed with this proof.

### A first proof of Theorem 1.1.

The natural idea to study the time derivative of  $N_p(P_t f)$ , namely

$$\frac{d}{dt} N_p^p(P_t f) = p \int \text{sign}(P_t f - \mu(f)) |P_t f - \mu(f)|^{p-1} L P_t f d\mu.$$

Hence we get an equivalence between

There exists a constant  $C(p)$  such that for all  $f$ ,

$$(2.4) \quad N_p^p(P_t f) \leq e^{-\frac{pt}{C(p)}} N_p^p(f).$$

There exists a constant  $C(p)$  such that for all  $f \in \mathcal{D}$  with  $\mu(f) = 0$ ,

$$(2.5) \quad N_p^p(f) \leq -C(p) \int \text{sign}(f) |f|^{p-1} L f d\mu.$$

In order to compare all the inequalities (2.5) to the Poincaré inequality (i.e.  $p = 2$ ) one is tempted to make the change of function  $f \mapsto \text{sign}(f) |f|^{2/p}$  (or  $f \mapsto \text{sign}(f) |f|^{p/2}$ ) and to use the chain rule. Unfortunately, first  $\varphi(u) = u^{2/p}$  is not  $C^2$ , second  $\mu(\text{sign}(f) |f|^{2/p}) \neq 0$  (the same for  $p/2$  for the second argument).

However, for  $p \geq 2$ , one can integrate by parts in (2.5) which thus becomes

$$(2.6) \quad N_p^p(f) \leq C(p) (p-1) \int |f|^{p-2} \Gamma(f, f) d\mu = C(p) \frac{4(p-1)}{p^2} \int \Gamma(|f|^{p/2}, |f|^{p/2}) d\mu.$$

It thus remains to show that the Poincaré inequality implies (2.6) for all  $p \geq 2$ . This will be done in two steps. First we will show the result for  $p = 4$ . Hence (2.4) hold for  $p = 2$  and  $p = 4$ . According to the Riesz-Thorin interpolation theorem, (2.4) (hence (2.6)) thus hold for all  $2 \leq p \leq 4$ . Next we shall show that if (2.6) holds for  $p$  it holds for  $2p$ . This will

complete the proof by an induction argument. Of course the final step is the only necessary one (starting with  $p = 2$ ) but we think that the details for  $2p = 4$  will help to follow the scheme of proof for the general  $2p$  case.

We proceed with the proof for  $p = 4$ .

Assume that  $\mu(f) = 0$ . First, applying the Poincaré inequality to  $f^2$  we get

$$\int f^4 d\mu \leq \left( \int f^2 d\mu \right)^2 + 4C_P \int f^2 \Gamma(f, f) d\mu,$$

so that it remains to prove that

$$\left( \int f^2 d\mu \right)^2 \leq C \int f^2 \Gamma(f, f) d\mu,$$

for some constant  $C$ .

Let now, for every  $u > 0$ ,  $\varphi = \varphi_u : \mathbb{R} \mapsto \mathbb{R}$  be the 2-Lipschitz function defined by  $\varphi(s) = 0$  if  $|s| \leq u$ ,  $\varphi(s) = s$  if  $|s| \geq 2u$  and linear in between. Applying Poincaré inequality to  $\varphi(f)$  yields

$$\int (\varphi(f))^2 d\mu \leq \left( \int \varphi(f) d\mu \right)^2 + 4C_P \int_{\{|f| \geq u\}} \Gamma(f, f) d\mu.$$

But

$$\int (\varphi(f))^2 d\mu \geq \int_{\{|f| \geq 2u\}} f^2 d\mu \geq \int f^2 d\mu - 4u^2,$$

and since  $\mu(f) = 0$ ,

$$\left| \int \varphi(f) d\mu \right| \leq 4u.$$

Summarizing, it follows that

$$\begin{aligned} \int f^2 d\mu &\leq 20u^2 + 4C_P \int_{\{|f| \geq u\}} \Gamma(f, f) d\mu \\ &\leq 20u^2 + \frac{4}{u^2} C_P \int f^2 \Gamma(f, f) d\mu. \end{aligned}$$

Optimizing in  $u^2$  finally yields

$$\left( \int f^2 d\mu \right)^2 \leq 320 C_P \int f^2 \Gamma(f, f) d\mu,$$

i.e.

$$N_4^4(f) \leq 324 C_P \int f^2 \Gamma(f, f) d\mu.$$

The constant 324 is of course not optimal, but replacing the 2 by  $2a$  in the definition of  $\varphi$  yields of course the same constant.

Now assume that (2.6) holds for some  $p \geq 2$  and of course the Poincaré inequality holds with constant  $C_P$ . First we apply Poincaré inequality to the function  $|f|^p$ ,

$$\int |f|^{2p} d\mu \leq \left( \int |f|^p d\mu \right)^2 + C_P p^2 \int |f|^{2p-2} \Gamma(f, f) d\mu.$$

Now as in the previous step we introduce  $\varphi$  and remark that

$$\int |f|^p d\mu \leq \int |\varphi(f)|^p d\mu + 2^p u^p.$$

We write (2.6) for the function  $\varphi(f) - \mu(\varphi(f))$  and then apply  $|a + b|^q \leq 2^{q-1}(|a|^q + |b|^q)$  for  $q \geq 1$  and  $|a + b|^q \leq 2^q(|a|^q + |b|^q)$  if  $q \geq 0$ , and recalling that  $|\mu(\varphi(f))| \leq 4u$  in order to obtain

$$\begin{aligned} \int |\varphi(f)|^p d\mu &\leq 2^{p-1} \left( (p-1) C(p) \int |\varphi(f) - \mu(\varphi(f))|^{p-2} \Gamma(\varphi(f), \varphi(f)) d\mu + |\mu(\varphi(f))|^p \right) \\ &\leq 2^{p-1} (p-1) C(p) 4 \int_{\{|f| \geq u\}} 2^{p-2} (|f|^{p-2} + |\mu(\varphi(f))|^{p-2}) \Gamma(f, f) d\mu \\ &\quad + 2^{2p-1} |\mu(\varphi(f))|^p \\ &\leq 2^{2p-1} (p-1) C(p) \int_{\{|f| \geq u\}} |f|^{p-2} \frac{|f|^p}{u^p} \Gamma(f, f) d\mu \\ &\quad + 2^{4p-5} (p-1) C(p) u^{p-2} \int_{\{|f| \geq u\}} \frac{|f|^{2p-2}}{u^{2p-2}} \Gamma(f, f) d\mu \\ &\quad + 2^{3p-1} u^p \\ &\leq 2^{3p-1} u^p + (2^{2p-1} + 2^{4p-5}) \frac{C(p)(p-1)}{u^p} \left( \int |f|^{2p-2} \Gamma(f, f) d\mu \right). \end{aligned}$$

Again we optimize in  $u^p$  and obtain

$$\left( \int |f|^p d\mu \right)^2 \leq 4(2^{2p-1} + 2^{4p-5})(2^p + 2^{3p-1})(p-1) C(p) \left( \int |f|^{2p-2} \Gamma(f, f) d\mu \right),$$

and finally

$$\int |f|^{2p} d\mu \leq (4(2^{2p-1} + 2^{4p-5})(2^p + 2^{3p-1})(p-1) C(p) + p^2 C_P) \int |f|^{2p-2} \Gamma(f, f) d\mu,$$

and the proof is completed.

Of course the final step is available for  $p = 2$  and  $C(2) = C_P$  but it furnishes a still worse constant than  $324C_P$ .

**Remark 2.7.** With the notation in Theorem 1.1 we thus have  $K_p = 1$  for  $p \geq 2$ . Now a rough estimate (asymptotically exact) yields

$$C(2^k) \leq 2^{7 \times 2^k} 2^{k(k-1)/2} 2^{-4} C_P,$$

and the method furnishes for  $2^{k+1} \geq p \geq 2^k$ ,  $C(p) = C(2^{k+1})$ .  $\diamond$

## 3. A SECOND PROOF OF THEOREM 1.1.

Let us start with a remark

**Remark 3.1.** Using Hölder inequality we see that (2.6) implies that for

$$(3.2) \quad \begin{aligned} \kappa(p) &= (C(p)(p-1))^{p/2}, \\ N_p^p(f) &\leq \kappa(p) \int \Gamma^{p/2}(f, f) d\mu. \end{aligned}$$

The latter is a  $\mathbb{L}^p$  Poincaré inequality which was used in [4] and particularly studied in [5]. As recalled by E. Milman, we can replace the mean  $\mu(f) = \int f d\mu$  by a median  $m_\mu(f)$  in (3.2). Indeed according to Lemma 2.1 in [5], for all  $1 \leq p < +\infty$

$$(3.3) \quad \frac{1}{2} N_p(f) \leq \|f - m_\mu(f)\|_p \leq 3 N_p(f).$$

Hence up to the constants we may replace  $\mu(f) = 0$  by  $m_\mu(f) = 0$  in (3.2). Now the transformations  $f \mapsto \text{sign}(f)|f|^h$  with  $h = 2/p$  or  $h = p/2$  is preserving the fact that 0 is a median so that we easily obtain (see [5] Proposition 2.5)

**Proposition 3.4.** *If  $\mu$  satisfies (3.2) for some  $p_0 \geq 1$  with a constant  $\kappa(p_0)$ , then it satisfies (3.2) for all  $p \geq p_0$ , with a constant*

$$\kappa(p) \leq \left(\frac{6p}{p_0}\right)^p \kappa^{p/p_0}(p_0).$$

◇

Unfortunately the same reasoning fails with (2.6) since there is no obvious comparison between  $\int |f - \mu(f)|^{p-2} \Gamma(f, f) d\mu$  and  $\int |f - m_\mu(f)|^{p-2} \Gamma(f, f) d\mu$ .

However we shall see that one can nevertheless use the median in order to prove Theorem 1.1.

Introduce some new notation. If  $f \in \mathbb{L}^p$ , denote by  $M_p^p(f) = \int |f - m_\mu(f)|^p d\mu$  and the new inequality

$$(3.5) \quad M_p^p(f) \leq B(p) \int |f - m_\mu(f)|^{p-2} \Gamma(f, f) d\mu.$$

we then have

**Theorem 3.6.** *All the inequalities (3.5) are equivalent (for  $+\infty > p \geq 2$  of course). Furthermore the best constants  $B(p)$  satisfy  $B(p) = \frac{p^2}{4} B(2)$ .*

*Proof.* Let  $f$  with  $m_\mu(f) = 0$ . If (3.5) holds for  $p = 2$  (i.e. the Poincaré inequality holds thanks to (3.3)), we apply it with  $g = \text{sign}(f)|f|^{p/2}$  and get

$$\int |f|^p d\mu = \int g^2 d\mu \leq B(2) \frac{p^2}{4} \int |f|^{p-2} \Gamma(f, f) d\mu,$$

i.e. (3.5) holds for  $p$  with  $B(p) \leq \frac{p^2}{4} B(2)$ .

Conversely if (3.5) holds for some  $p \geq 2$ , we apply it with the function

$$g = \text{sign}(f)|f|^{2/p} \mathbf{1}_{|f| \geq s} + s^{\frac{2-p}{p}} f \mathbf{1}_{|f| \leq s},$$



defined for  $s > 0$ . We thus obtain

$$\int_{|f| \geq s} |f|^2 d\mu + s^{2-p} \int_{|f| < s} |f|^p d\mu = \int |g|^p d\mu$$

and

$$\begin{aligned} \int |g|^p d\mu &\leq B(p) \left( \int_{|f| \geq s} |f|^{\frac{2}{p}(p-2)} \frac{4}{p^2} |f|^{\frac{2(2-p)}{p}} \Gamma(f, f) d\mu \right) + \\ &\quad + B(p) \left( s^{2-p} \int_{|f| < s} |f|^{p-2} \Gamma(f, f) d\mu \right) \\ &\leq B(p) \left( \frac{4}{p^2} \int_{|f| \geq s} \Gamma(f, f) d\mu + \int_{|f| < s} \Gamma(f, f) d\mu \right), \end{aligned}$$

so that by letting  $s$  go to 0 we obtain  $B(2) \leq \frac{4}{p^2} B(p)$ , hence the result.  $\square$

We shall now see how to use this theorem in order to study  $N_p(P_t f)$ .

First recall that, according to (3.3), (3.5) for  $p = 2$  is equivalent to the Poincaré inequality, and

$$\frac{1}{9} B(2) \leq C_P \leq 4 B(2).$$

It follows, using again (3.3) and Theorem 3.6, that if **(H-Poinc)** holds, for any  $p \geq 2$ ,

$$\begin{aligned} (3.7) \quad 2^{-p} N_p^p(f) &\leq M_p^p(f) \leq B(p) \int |f - m_\mu(f)|^{p-2} \Gamma(f, f) d\mu \\ &\leq B(p) \delta(p-2) \int |f - \mu(f)|^{p-2} \Gamma(f, f) d\mu + \\ &\quad + B(p) \delta(p-2) |\mu(f) - m_\mu(f)|^{p-2} \int \Gamma(f, f) d\mu \\ &\leq B(p) \delta(p-2) \int |f - \mu(f)|^{p-2} \Gamma(f, f) d\mu + \\ &\quad + B(p) \delta(p-2) 2^{(p-2)/2} (\text{Var}_\mu(f))^{(p-2)/2} \int \Gamma(f, f) d\mu, \end{aligned}$$

where we have used

$$|\mu(f) - m_\mu(f)| \leq \sqrt{2} (\text{Var}_\mu(f))^{1/2}$$

(see the proof of Lemma 2.1 in [5]) and

$$(u + v)^p \leq \delta(p)(u^p + v^p)$$

for any non-negative  $u$  and  $v$  and any  $p \geq 0$ , with  $\delta(p) = 2^{p-1}$  if  $p \geq 1$  and  $\delta(p) = 1$  if  $0 \leq p \leq 1$ ; hence finally  $\delta(p) = 1 \vee 2^{p-1}$ .

Now consider, for  $p \geq 2$ , the following ‘‘entropy functional’’ (in the terminology of P.D.E. specialists)

$$(3.8) \quad E_p(f) = a_p N_p^p(f) + b_p (\text{Var}_\mu(f))^{p/2} \leq (a_p + b_p) N_p^p(f),$$

where  $a_p$  and  $b_p$  are positive constants to be chosen later. Remark first that using Poincaré inequality and (3.7), we have (remember  $B(p) \leq 9C_P p^2/4$ )

$$(3.9) \quad E_p(f) \leq A(p) + D(p),$$

where

$$\begin{aligned} A(p) &= a_p 2^p \frac{9C_P p^2}{4} \delta(p-2) \int |f - \mu(f)|^{p-2} \Gamma(f, f) d\mu, \\ D(p) &= \left( a_p 2^p \frac{9C_P p^2}{4} 2^{(p-2)/2} \delta(p-2) + C_P b_p \right) (\text{Var}_\mu(f))^{(p-2)/2} \int \Gamma(f, f) d\mu. \end{aligned}$$

We have

$$\begin{aligned} \frac{d}{dt} E_p(P_t f) &= -a_p p(p-1) \int |P_t f - \mu(P_t f)|^{p-2} \Gamma(P_t f, P_t f) d\mu \\ &\quad - b_p \frac{p}{2} (\text{Var}_\mu(P_t f))^{(p-2)/2} \int \Gamma(P_t f, P_t f) d\mu, \\ &= -\frac{p(p-1)}{\frac{9C_P p^2}{4} \delta(p-2) 2^p} a_p 2^p \frac{9C_P p^2}{4} \delta(p-2) \int |P_t f - \mu(f)|^{p-2} \Gamma(P_t f, P_t f) d\mu \\ &\quad - \frac{b_p \frac{p}{2}}{a_p 2^p \frac{9C_P p^2}{4} 2^{(p-2)/2} \delta(p-2) + C_P b_p} \left( a_p 2^p \frac{9C_P p^2}{4} 2^{(p-2)/2} \delta(p-2) + C_P b_p \right) \\ &\quad \times (\text{Var}_\mu(P_t f))^{(p-2)/2} \int \Gamma(P_t f, P_t f) d\mu. \end{aligned}$$

Using (3.9), and choosing  $a_p, b_p$  such that

$$\frac{p(p-1)}{\frac{9C_P p^2}{4} \delta(p-2) 2^p} = \frac{b_p \frac{p}{2}}{a_p 2^p \frac{9C_P p^2}{4} 2^{(p-2)/2} \delta(p-2) + C_P b_p}$$

which is possible as  $p \geq 2$ , we thus get

$$\frac{d}{dt} E_p(P_t f) \leq -\frac{p(p-1)}{\frac{9C_P p^2}{4} \delta(p-2) 2^p} E_p(P_t f) = -\gamma_p E_p(P_t f)$$

so that applying Gronwall's lemma we deduce

$$N_p^p(P_t f) \leq \frac{1}{a_p} E_p(P_t f) \leq e^{-\gamma_p t} E_p(f) \leq \frac{a_p + b_p}{a_p} e^{-\gamma_p t} N_p^p(f).$$

Putting all our results together, we have thus shown

**Theorem 3.10.** *If (H-Poinc) holds with constant  $C_P$ , then for all  $p \geq 2$ ,*

$$N_p(P_t f) \leq K_p e^{-\lambda_p t} N_p(f),$$

with

$$\lambda_p = \frac{4(p-1)}{9p^2 (1 \vee 2^{p-3}) 2^p C_P},$$

and

$$K_p^p = 1 + \frac{\frac{9p^2}{4} (p-1) 2^{(3p-2)/2} (1 \vee 2^{p-3})}{\frac{9p^2}{4} (1 \vee 2^{p-3}) 2^{p-1} - p + 1}.$$

**Remark 3.11.** Due to several rough estimates, the previous proof furnishes  $K_2 = \sqrt{53/17}$  and  $\lambda_2 C_P = 1/36$  which is quite bad of course since we know that the optimal value is 1.

For  $p = 4$  we obtain  $K_4 = (2495/191)^{1/4} \leq 2$  and  $\lambda_4 C_P = 1/384$  which is worse than the 324 in the previous section. Actually, despite the constant  $K_p$ , the previous bound is better than the one we can expect from the first proof for  $p$ 's close to 2 (for which  $C_P \lambda_p$  is close to  $1/36$  instead of the  $1/324$  obtained via interpolation) and for large  $p$ 's since if  $p = 2^k$  we obtain  $C_P \lambda_p \sim 8/(9 \cdot 2^k \cdot 2^{2 \times 2^k})$ , so that we get a 2 instead of a 7 obtained by the first method.

Also note that we did not succeed in proving the analogue of Lemma 2.1 for  $p > 2$ . Hence the first proof furnishes a better result for small times (since we have the optimal  $K_p = 1$ ), while the second one is more interesting for large times.

Of course under stronger assumptions than the sole Poincaré inequality (logarithmic Sobolev inequality for instance), one can improve the bounds obtained in Theorem 1.1.  $\diamond$

**Remark 3.12. Extension to the non-symmetric case.**

Notice that the only point where we used symmetry is the proof of Lemma 2.2, hence of Lemma 2.1. In particular if  $\mu$  is invariant but not necessarily symmetric, **(H-Poinc)** implies exponential decay in all the  $\mathbb{L}^p(\mu)$ ,  $p \geq 2$ , and our bounds are available, in particular we may choose  $K_p = 1$ .

But if **(H-p)** holds for some  $p > 2$  and with  $K_p = 1$  (which is crucial) then (2.6) is satisfied, which in return implies the same decay for the dual semi-group  $P_t^*$ . Hence the duality argument shows that **(H-q)** is satisfied for both  $P_t$  and  $P_t^*$ , where  $q$  is the conjugate exponent of  $p$ . Hence **(H-Poinc)** implies exponential decay in all the  $\mathbb{L}^p(\mu)$ ,  $1 < p < +\infty$ .

Conversely, assume that **(H-p)** holds for some  $p > 2$  and with  $K_p = 1$  (which is still crucial). The previous argument shows that **(H-q)** is satisfied. The Riesz-Thorin interpolation theorem then shows that **(H-s)** is satisfied for all  $q \leq s \leq p$ , hence for  $s = 2$ . But since we do not know that  $K_2 = 1$ , we cannot conclude that the Poincaré inequality is satisfied. Also note that the induction argument we used in the first proof of the Theorem calls explicitly upon the Poincaré inequality, so that we cannot deduce that **(H-s)** holds for  $s > p$ .  $\diamond$

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