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# POINCARÉ INEQUALITY AND THE $\mathbb{L}^p$ CONVERGENCE OF SEMI-GROUPS.

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ABSTRACT. We prove that for symmetric Markov processes of diffusion type admitting a “carré du champ”, the Poincaré inequality is equivalent to the exponential convergence of the associated semi-group in one (resp. all)  $\mathbb{L}^p(\mu)$  spaces for  $1 < p < +\infty$ . Part of this result extends to the stationary non necessarily symmetric situation.

*Key words* : Poincaré inequality, rate of convergence.

*MSC 2010* : 26D10, 39B62, 47D07, 60G10, 60J60.

## 1. INTRODUCTION AND MAIN RESULTS.

Let  $X_t$  be a general Markov processes with infinitesimal generator  $L$  and with state space some Polish space  $E$ . We assume that the extended domain of the generator contains a nice core  $\mathcal{D}$  of uniformly continuous functions, containing the constant functions, which is an algebra, for which we may define the “carré du champ” operator

$$\Gamma(f, g) = \frac{1}{2} (L(fg) - fLg - gLf) .$$

Functions in  $\mathcal{D}$  will be called “smooth”. The associated Dirichlet form can thus be calculated for smooth  $f$ 's as

$$\mathcal{E}(f, f) := - \int f Lf d\mu = \int \Gamma(f, f) d\mu .$$

In addition we assume that  $L$  is  $\mu$ -symmetric for some probability measure defined on  $E$ . Thus  $L$  generates a  $\mu$ -symmetric (hence stationary) semi-group  $P_t$ , which is a contraction semi-group on all  $\mathbb{L}^p(\mu)$  for  $1 \leq p \leq +\infty$ , and the  $\mathbb{L}^2$  ergodic theorem (in the symmetric case) tells us that for all  $f \in \mathbb{L}^2(\mu)$ ,

$$\lim_{t \rightarrow +\infty} \| P_t f - \int f d\mu \|_{\mathbb{L}^2(\mu)} = 0 .$$

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For all this one can give a look at [3]. Here and in the sequel, for any  $p \in [1, \infty)$ ,  $\|f\|_{\mathbb{L}^p(\mu)}$ , or in a shorter way  $\|f\|_p$ , stands for the  $\mathbb{L}^p(\mu)$ -norm of  $f$  with respect to  $\mu$ :  $\|f\|_p^p := \int |f|^p d\mu$ .

It is then well known that the following two statements are equivalent

**(H-Poinc)**.  $\mu$  satisfies a Poincaré inequality, i.e. there exists a constant  $C_P$  such that for all smooth  $f$ ,

$$\text{Var}_\mu(f) := \int f^2 d\mu - \left( \int f d\mu \right)^2 \leq C_P \int \Gamma(f, f) d\mu.$$

**(H-2)**. There exists a constant  $\lambda_2$  such that

$$\text{Var}_\mu(P_t f) \leq e^{-2\lambda_2 t} \text{Var}_\mu(f).$$

If one of these assumptions is satisfied we have  $\lambda_2 = 1/C_P$ .

In the sequel we shall assume in addition that  $\Gamma$  comes from a derivation, i.e.

$$\Gamma(fg, h) = f \Gamma(g, h) + g \Gamma(f, h),$$

i.e. (in the terminology of [1]) that  $X$  is a diffusion. We also recall the chain rule: if  $\varphi$  is a  $C^2$  function,

$$L(\varphi(f)) = \varphi'(f) Lf + \varphi''(f) \Gamma(f, f).$$

In this note we shall establish the following theorem

**Theorem 1.1.** *For  $f \in \mathbb{L}^p(\mu)$  define  $N_p(f) := \|f - \int f d\mu\|_p$ . The following statements are equivalent*

- (1) **(H-Poinc)** is satisfied,
- (2) there exist some  $1 < p < +\infty$  and constants  $\lambda_p$  and  $K_p$  such that for all  $f \in \mathbb{L}^p(\mu)$ ,

$$N_p(P_t f) \leq K_p e^{-\lambda_p t} N_p(f),$$

- (3) for all  $1 < p < +\infty$ , there exist some constants  $\lambda_p$  and  $K_p$  such that for all  $f \in \mathbb{L}^p(\mu)$ ,

$$N_p(P_t f) \leq K_p e^{-\lambda_p t} N_p(f).$$

We shall denote by **(H-p)** the property (2) for a given  $p$ .

Of course (3) implies (2). The fact that (2) implies (1) is a consequence of the following Lemma which seems to be well known by the specialists in Statistical Physics (we learned this result from P. Caputo and P. Dai Pra) and is used in [10] (see lemma 2.6 therein). We shall give however a very elementary proof in the next section.

**Lemma 1.2.** *If it exists  $\beta > 0$  such that for all  $f \in \mathcal{C}$ ,  $\mathcal{C}$  being an everywhere dense subset of  $\mathbb{L}^2(\mu)$ , the following holds  $\text{Var}_\mu(P_t f) \leq c_f e^{-2\beta t}$ , then  $\text{Var}_\mu(P_t f) \leq e^{-2\beta t} \text{Var}_\mu(f)$  for all  $f \in \mathbb{L}^2(\mu)$ , i.e. the Poincaré inequality holds with  $C_P \leq 1/\beta$ .*

An immediate consequence of Lemma 1.2 is that (2) implies (1) in the statement of Theorem 1.1.

Indeed if **(H-p)** holds for  $p \geq 2$ ,

$$N_2(P_t f) \leq N_p(P_t f) \leq K_p e^{-\lambda_p t} N_p(f)$$

and applying Lemma 1.2 with  $\mathcal{C} = \mathbb{L}^p(\mu)$  we deduce that  $C_P \leq 1/\lambda_p$ .

If **(H-p)** holds for  $1 < p \leq 2$  we may similarly write

$$N_2(P_t f) \leq N_\infty^{(2-p)/2}(f) N_p^{p/2}(f) \leq (K_p)^{p/2} e^{-p\lambda_p t/2} N_\infty^{(2-p)/2}(f) N_p^{p/2}(f)$$

and applying Lemma 1.2 with  $\mathcal{C} = \mathbb{L}^\infty(\mu)$  we deduce that  $C_P \leq 2/(p\lambda_p)$ .

Of course if  $p$  and  $q$  are conjugate exponents, **(H-p)** and **(H-q)** are equivalent. More precisely we may write, with  $\mu(f) := \int f d\mu$

$$\begin{aligned} \left| \int P_t(f - \mu(f)) g d\mu \right| &= \left| \int P_t(f - \mu(f)) (g - \mu(g)) d\mu \right| = \left| \int (f - \mu(f)) P_t(g - \mu(g)) d\mu \right| \\ &\leq N_p(f) N_q(P_t g) \leq K_q e^{-\lambda_q t} N_p(f) N_q(g) \\ &\leq 2 K_q e^{-\lambda_q t} N_p(f) \|g\|_{\mathbb{L}^q(\mu)} \end{aligned}$$

if **(H-q)** holds. Hence

**Lemma 1.3.** *If  $\frac{1}{p} + \frac{1}{q} = 1$ , **(H-q)** implies **(H-p)** with  $K_p \leq 2K_q$  and  $\lambda_p \geq \lambda_q$ . Accordingly if **(H-p)** holds, **(H-Poinc)** holds with  $C_P \leq 1/\lambda_p$ .*

If  $p \leq 2$  we obtain a better bound than the one we obtained directly.

Lemma 1.3 also shows that, in order to complete the proof of Theorem 1.1, it is enough to show that (1) implies (3) for all  $p \geq 2$ .

Actually there are two interests in such a Theorem. The first one is obviously the rate of convergence at infinity for which what is important is to get the largest possible  $\lambda_p$ , despite the (reasonable) value of  $K_p$ . The second one is the opposite: get the result with  $K_p = 1$  so that the inequality becomes an equality at time  $t = 0$  in order to possibly use the result for isoperimetric controls for instance. The ideal situation is when we can reach these goals simultaneously (as for  $p = 2$ ). As we shall see however, for  $p > 2$  we will obtain two results described below.

**Theorem 1.4.** *If **(H-Poinc)** is satisfied, then for all  $p > 2$  **(H-p)** holds with  $K_p = 1$  and*

$$\lambda_p \geq \frac{2^{k+6}}{(27 \times 2^{k+1} C_P)} \quad \text{if} \quad 2^{k+1} \geq p > 2^k \quad \text{for some } k > 1.$$

*Consequently for  $1 < p < 2$ , **(H-p)** holds with  $K_p = 2$  and  $\lambda_p = \lambda_{p/(p-1)}$ .*

Note that for  $p = 2$  we recover a worse constant than the known  $\lambda_2 = 1/C_P$ .

We shall also prove

**Theorem 1.5.** *If **(H-Poinc)** is satisfied, then for all  $p > 2$  **(H-p)** holds with  $\lambda_p = 1/(p C_P)$  and  $K_p = 4^{1-\frac{1}{p}}$ .*

*If  $p = 2^k$  for  $k \geq 1$  one can improve these bounds in  $\lambda_{2^k} = 2/(2^k C_P)$  and  $K_p = 4^{1-\frac{2}{p}}$ .*

*Consequently for  $1 < p < 2$ , **(H-p)** holds with  $K_p = 2$  and  $\lambda_p = \lambda_{p/(p-1)}$ .*

Again we are losing some factor (but here only 2) for  $p > 2$  but close to 2. Of course the statements of both Theorem 1.4 and Theorem 1.5 indicate that the scheme of proof will be to get the result for the successive powers of 2 and then to interpolate between them.

The case  $p = 1$  is extensively studied in [4] and the Poincaré inequality is no more sufficient in general to obtain an exponential decay in  $\mathbb{L}^1(\mu)$ . Replacing  $\mathbb{L}^p$  norms by Orlicz norms (weaker than any  $N_p$  for  $p > 1$ ) is possible provided one reinforces the Poincaré inequality into a  $F$ -Sobolev inequality (see [4] Theorem 3.1) as it is well known in the case  $F = \log$  for the Orlicz space  $\mathbb{L} \log \mathbb{L}$ .

The question of exponential convergence in  $\mathbb{L}^p$  ( $p \neq 2$ ) was asked to us by M. Ledoux after a conversation with A. Naor. We did not find the statement of such a result in the literature. However recall that in [9], F.Y. Wang used the equivalent Beckner type formulation of Poincaré inequality to give a partial answer to the problem i.e., a Poincaré inequality with constant  $C_P$  is equivalent to the following: for any  $1 < p \leq 2$  and for any non-negative  $f$ ,

$$\int (P_t f)^p d\mu - \left( \int f d\mu \right)^p \leq e^{-\frac{4(p-1)t}{pC_P}} \left( \int (f)^p d\mu - \left( \int f d\mu \right)^p \right).$$

(One has to take care with the constants since some 2 may or may not appear in the definition of  $\Gamma$ , depending on authors and of papers by the same authors.) This result cannot be used to study the decay to the mean in  $\mathbb{L}^p$  norm, but it is of particular interest when studying densities of probability.

Note that the decay rate we obtain in Theorem 1.5 is not comparable with the one in Wang's result. Nevertheless, we recover here the  $\mathbb{L}^1$  decay obtained in [4] Example 2.3. so that, at least for powers of 2, the rate obtained in Theorem 1.5 seems to be almost optimal.

**Acknowledgement.** We warmly thank M. Ledoux for asking us about the exponential convergence in  $\mathbb{L}^p$  ( $p \neq 2$ ), but also especially because he preciously saved a copy of one of our main arguments that we loosed in our perfectly disordered office. The third author also would like to warmly thank Fabio Martinelli and the University of Rome 3 where part of this work was done.

## 2. POINCARÉ INEQUALITIES AND $\mathbb{L}^p$ SPACES.

We start with the **Proof of Lemma 1.2.**

*Proof.* The proof lies on the following lemma proven in [7] using the spectral resolution

**Lemma 2.1.**  $t \mapsto \log \| P_t f \|_{\mathbb{L}^2(\mu)}$  is convex.

Here is a direct proof that does not use the spectral resolution. If  $n(t) = \| P_t f \|_{\mathbb{L}^2(\mu)}^2$ , the sign of the second derivative of  $\log n$  is the one of  $n''n - (n')^2$ . But

$$n'(t) = 2 \int P_t f L P_t f d\mu$$

and

$$n''(t) = 2 \int (L P_t f)^2 d\mu + 2 \int P_t f L P_t L f d\mu = 4 \int (L P_t f)^2 d\mu,$$

so that lemma 2.1 is just a consequence of Cauchy-Schwarz inequality.

In order to prove lemma 1.2, assuming that  $\int f d\mu = 0$  which is not a restriction, it is enough to look at

$$t \mapsto \log \| P_t f \|_{\mathbb{L}^2(\mu)} + \beta t,$$

which is convex, according to lemma 2.1, and bounded since  $\text{Var}_\mu(P_t f) \leq c_f e^{-2\beta t}$ . But a bounded convex function on  $\mathbb{R}^+$  is necessarily non-increasing. Hence

$$\|P_t f\|_{\mathbb{L}^2(\mu)} \leq e^{-\beta t} \|P_0 f\|_{\mathbb{L}^2(\mu)}$$

for all  $f \in \mathcal{C}$ , the result follows using the density of  $\mathcal{C}$ .  $\square$

We come now to the proofs of our main theorems.

**Proof of Theorem 1.4.**

*Proof.* The natural idea to study the time derivative of  $N_p(P_t f)$ , namely

$$\frac{d}{dt} N_p^p(P_t f) = p \int \text{sign}(P_t f - \mu(f)) |P_t f - \mu(f)|^{p-1} L P_t f d\mu.$$

Hence we get an equivalence between

There exists a constant  $C(p)$  such that for all  $f$ ,

$$(2.2) \quad N_p^p(P_t f) \leq e^{-\frac{pt}{C(p)}} N_p^p(f).$$

There exists a constant  $C(p)$  such that for all  $f \in \mathcal{D}$  with  $\mu(f) = 0$ ,

$$(2.3) \quad N_p^p(f) \leq -C(p) \int \text{sign}(f) |f|^{p-1} L f d\mu.$$

In order to compare all the inequalities (2.3) to the Poincaré inequality (i.e.  $p = 2$ ) one is tempted to make the change of function  $f \mapsto \text{sign}(f) |f|^{2/p}$  (or  $f \mapsto \text{sign}(f) |f|^{p/2}$ ) and to use the chain rule. Unfortunately, first  $\varphi(u) = u^{2/p}$  is not  $C^2$ , second  $\mu(\text{sign}(f) |f|^{2/p}) \neq 0$  (the same for  $p/2$  for the second argument).

However, for  $p \geq 2$ , one can integrate by parts in (2.3) which thus becomes

$$(2.4) \quad N_p^p(f) \leq C(p) (p-1) \int |f|^{p-2} \Gamma(f, f) d\mu = C(p) \frac{4(p-1)}{p^2} \int \Gamma(|f|^{p/2}, |f|^{p/2}) d\mu.$$

It thus remains to show that the Poincaré inequality implies (2.4) for all  $p \geq 2$ . This will be done in two steps. First we will show the result for  $p = 4$ . Hence (2.2) hold for  $p = 2$  and  $p = 4$ . According to the Riesz-Thorin interpolation theorem, (2.2) (hence (2.4)) thus hold for all  $2 \leq p \leq 4$ . Next we shall show that if (2.4) holds for  $p$  it holds for  $2p$ . This will complete the proof by an induction argument. Of course the final step is the only necessary one (starting with  $p = 2$ ) but we think that the details for  $2p = 4$  will help to follow the scheme of proof for the general  $2p$  case.

We proceed with the proof for  $p = 4$ .

Assume that  $\mu(f) = 0$ . First, applying the Poincaré inequality to  $f^2$  we get

$$\int f^4 d\mu \leq \left( \int f^2 d\mu \right)^2 + 4C_P \int f^2 \Gamma(f, f) d\mu,$$

so that it remains to prove that

$$\left( \int f^2 d\mu \right)^2 \leq C \int f^2 \Gamma(f, f) d\mu,$$

for some constant  $C$ .

Let now, for every  $u > 0$ ,  $\varphi = \varphi_u : \mathbb{R} \mapsto \mathbb{R}$  be the 2-Lipschitz function defined by  $\varphi(s) = 0$  if  $|s| \leq u$ ,  $\varphi(s) = s$  if  $|s| \geq 2u$  and linear in between. Applying Poincaré inequality to  $\varphi(f)$  yields

$$\int (\varphi(f))^2 d\mu \leq \left( \int \varphi(f) d\mu \right)^2 + 4C_P \int_{\{|f| \geq u\}} \Gamma(f, f) d\mu.$$

But

$$\int (\varphi(f))^2 d\mu \geq \int_{\{|f| \geq 2u\}} f^2 d\mu \geq \int f^2 d\mu - 4u^2,$$

and since  $\mu(f) = 0$ ,

$$\left| \int \varphi(f) d\mu \right| \leq 4u.$$

Summarizing, it follows that

$$\begin{aligned} \int f^2 d\mu &\leq 20u^2 + 4C_P \int_{\{|f| \geq u\}} \Gamma(f, f) d\mu \\ &\leq 20u^2 + \frac{4}{u^2} C_P \int f^2 \Gamma(f, f) d\mu. \end{aligned}$$

Optimizing in  $u^2$  finally yields

$$\left( \int f^2 d\mu \right)^2 \leq 320 C_P \int f^2 \Gamma(f, f) d\mu,$$

i.e.

$$N_4^4(f) \leq 324 C_P \int f^2 \Gamma(f, f) d\mu.$$

The constant 324 is of course not optimal, but replacing the 2 by  $2a$  in the definition of  $\varphi$  yields of course the same constant.

Now assume that (2.4) holds for some  $p \geq 2$  and of course the Poincaré inequality holds with constant  $C_P$ . First we apply Poincaré inequality to the function  $|f|^p$ ,

$$\int |f|^{2p} d\mu \leq \left( \int |f|^p d\mu \right)^2 + C_P p^2 \int |f|^{2p-2} \Gamma(f, f) d\mu.$$

Now as in the previous step we introduce  $\varphi$  and remark that

$$\int |f|^p d\mu \leq \int |\varphi(f)|^p d\mu + 2^p u^p.$$

We write (2.4) for the function  $\varphi(f) - \mu(\varphi(f))$  and then apply  $|a + b|^q \leq 2^{q-1} (|a|^q + |b|^q)$  for  $q \geq 1$  and  $|a + b|^q \leq 2^q (|a|^q + |b|^q)$  if  $q \geq 0$ , and recalling that  $|\mu(\varphi(f))| \leq 4u$  in order to

obtain

$$\begin{aligned}
\int |\varphi(f)|^p d\mu &\leq 2^{p-1} \left( (p-1) C(p) \int |\varphi(f) - \mu(\varphi(f))|^{p-2} \Gamma(\varphi(f), \varphi(f)) d\mu + |\mu(\varphi(f))|^p \right) \\
&\leq 2^{p-1} (p-1) C(p) 4 \int_{\{|f| \geq u\}} 2^{p-2} (|f|^{p-2} + |\mu(\varphi(f))|^{p-2}) \Gamma(f, f) d\mu \\
&\quad + 2^{p-1} |\mu(\varphi(f))|^p \\
&\leq 2^{2p-1} (p-1) C(p) \int_{\{|f| \geq u\}} |f|^{p-2} \frac{|f|^p}{u^p} \Gamma(f, f) d\mu \\
&\quad + 2^{4p-5} (p-1) C(p) u^{p-2} \int_{\{|f| \geq u\}} \frac{|f|^{2p-2}}{u^{2p-2}} \Gamma(f, f) d\mu \\
&\quad + 2^{3p-1} u^p \\
&\leq 2^{3p-1} u^p + (2^{2p-1} + 2^{4p-5}) \frac{C(p)(p-1)}{u^p} \left( \int |f|^{2p-2} \Gamma(f, f) d\mu \right).
\end{aligned}$$

Again we optimize in  $u^p$  and obtain

$$\left( \int |f|^p d\mu \right)^2 \leq 4(2^{2p-1} + 2^{4p-5})(2^p + 2^{3p-1})(p-1) C(p) \left( \int |f|^{2p-2} \Gamma(f, f) d\mu \right),$$

and finally

$$\int |f|^{2p} d\mu \leq (4(2^{2p-1} + 2^{4p-5})(2^p + 2^{3p-1})(p-1) C(p) + p^2 C_P) \int |f|^{2p-2} \Gamma(f, f) d\mu,$$

and the proof is completed.  $\square$

Of course the final step is available for  $p = 2$  and  $C(2) = C_P$  but it furnishes a still worse constant than  $324C_P$ . The value of  $\lambda_p$  for  $p = 2^k$  can be obtained by induction.

### Proof of Theorem 1.5

*Proof.* We shall prove by induction that, provided **(H-Poinc)** holds, the following holds true for all  $k \geq 1$ : if  $p = 2^k$ , for all  $t \geq 0$

$$(2.5) \quad N_p^p(P_t f) \leq 4^{p-2} e^{-2t/C_P} N_p^p(f).$$

For  $k = 1$  (i.e  $p = 2$ ) (2.5) is equivalent to **(H-Poinc)**.

Now we proceed by induction. Without loss of generality we assume that  $\int f d\mu = 0$  and denote by  $U_k(t) := N_p^p(P_t f)$  for  $p = 2^k$ . Recall that

$$\begin{aligned}
U_k'(t) &= 2^k \int \text{sign}(P_t f) |P_t f|^{p-1} L P_t f d\mu \\
&= -2^k (2^k - 1) \int (P_t f)^{2^k-2} \Gamma(P_t f, P_t f) d\mu \\
&= -4(2^k - 1) 2^{-k} \int \Gamma((P_t f)^{2^{k-1}}, (P_t f)^{2^{k-1}}) d\mu \\
&\leq -3 \int \Gamma((P_t f)^{2^{k-1}}, (P_t f)^{2^{k-1}}) d\mu,
\end{aligned}$$

since for  $k \geq 1$ ,  $3 \leq 4(2^k - 1)2^{-k}$ . In addition the Poincaré inequality applied to  $(P_t f)^{2^{k-1}}$  yields

$$U_k(t) \leq U_{k-1}^2(t) + C_P \int \Gamma((P_t f)^{2^{k-1}}, (P_t f)^{2^{k-1}}) d\mu.$$

Putting these inequalities together we thus have

$$(2.6) \quad U_k'(t) \leq -\frac{3}{C_P} U_k(t) + \frac{3}{C_P} U_{k-1}^2(t).$$

We may thus apply Gronwall's lemma and obtain

$$U_k(t) \leq e^{-3t/C_P} \left( U_k(0) + \frac{3}{C_P} \int_0^t e^{3s/C_P} U_{k-1}^2(s) ds \right).$$

If (2.5) holds for  $p = 2^{k-1}$  with  $k-1 \geq 1$ , we thus obtain

$$\begin{aligned} U_k(t) &\leq e^{-3t/C_P} \left( U_k(0) + \frac{3}{C_P} \int_0^t e^{3s/C_P} \left( 4^{2^{k-1}-2} e^{-2s/C_P} U_{k-1}(0) \right)^2 ds \right) \\ &\leq e^{-3t/C_P} \left( U_k(0) + 3 \cdot 4^{2^k-4} U_{k-1}^2(0) \int_0^t \frac{1}{C_P} e^{-s/C_P} ds \right) \\ &\leq e^{-3t/C_P} \left( U_k(0) + 3 \cdot 4^{2^k-4} U_{k-1}^2(0) \right) \\ &\leq e^{-2t/C_P} U_k(0) \left( 1 + 3 \cdot 4^{2^k-4} \right), \end{aligned}$$

since  $U_{k-1}^2(0) \leq U_k(0)$  thanks to Cauchy-Schwarz inequality.

Finally remark that  $4^{2^k-2} \geq 1 + 3 \times 4^{2^k-4}$  for  $k \geq 2$ , so that the induction is completed.

Hence (2.5) is true for all  $p = 2^k$ . In order to apply again the Riesz-Thorin interpolation theorem for  $2^k < p \leq 2^{k+1}$  and complete the proof of the theorem it remains to note that

$$4^{1-\frac{1}{p}} e^{-t/p C_P} \geq \max \left( 4^{\frac{2^k-2}{2^k}} e^{-2t/2^k C_P}; 4^{\frac{2^{k+1}-2}{2^{k+1}}} e^{-2t/2^{k+1} C_P} \right).$$

□

### 3. ANOTHER PROOF OF THEOREM 1.1.

Let us start with a remark

**Remark 3.1.** Using Hölder inequality we see that (2.4) implies that for

$$\kappa(p) = (C(p)(p-1))^{p/2},$$

$$(3.2) \quad N_p^p(f) \leq \kappa(p) \int \Gamma^{p/2}(f, f) d\mu.$$

The latter is a  $\mathbb{L}^p$  Poincaré inequality which was used in [5] and particularly studied in [6].

As recalled by E. Milman, we can replace the mean  $\mu(f) = \int f d\mu$  by a median  $m_\mu(f)$  in (3.2). Indeed according to Lemma 2.1 in [6], for all  $1 \leq p < +\infty$

$$(3.3) \quad \frac{1}{2} N_p(f) \leq \|f - m_\mu(f)\|_p \leq 3 N_p(f).$$

Hence up to the constants we may replace  $\mu(f) = 0$  by  $m_\mu(f) = 0$  in (3.2). Now the transformations  $f \mapsto \text{sign}(f) |f|^h$  with  $h = 2/p$  or  $h = p/2$  is preserving the fact that 0 is a median so that we easily obtain (see [6] Proposition 2.5)

**Proposition 3.4.** *If  $\mu$  satisfies (3.2) for some  $p_0 \geq 1$  with a constant  $\kappa(p_0)$ , then it satisfies (3.2) for all  $p \geq p_0$ , with a constant*

$$\kappa(p) \leq \left( \frac{6p}{p_0} \right)^p \kappa^{p/p_0}(p_0).$$

◇

Unfortunately the same reasoning fails with (2.4) since there is no obvious comparison between  $\int |f - \mu(f)|^{p-2} \Gamma(f, f) d\mu$  and  $\int |f - m_\mu(f)|^{p-2} \Gamma(f, f) d\mu$ .

However we shall see that one can nevertheless use the median in order to prove Theorem 1.1, but that doing so furnishes disastrous constants.

Introduce some new notation. If  $f \in \mathbb{L}^p$ , denote by  $M_p^p(f) = \int |f - m_\mu(f)|^p d\mu$  and the new inequality

$$(3.5) \quad M_p^p(f) \leq B(p) \int |f - m_\mu(f)|^{p-2} \Gamma(f, f) d\mu.$$

we then have

**Theorem 3.6.** *All the inequalities (3.5) are equivalent (for  $+\infty > p \geq 2$  of course). Furthermore the best constants  $B(p)$  satisfy  $B(p) = \frac{p^2}{4} B(2)$ .*

*Proof.* Let  $f$  with  $m_\mu(f) = 0$ . If (3.5) holds for  $p = 2$  (i.e. the Poincaré inequality holds thanks to (3.3)), we apply it with  $g = \text{sign}(f) |f|^{p/2}$  and get

$$\int |f|^p d\mu = \int g^2 d\mu \leq B(2) \frac{p^2}{4} \int |f|^{p-2} \Gamma(f, f) d\mu,$$

i.e. (3.5) holds for  $p$  with  $B(p) \leq \frac{p^2}{4} B(2)$ .

Conversely if (3.5) holds for some  $p \geq 2$ , we apply it with the function

$$g = \text{sign}(f) |f|^{2/p} \mathbb{1}_{|f| \geq s} + s^{\frac{2-p}{p}} f \mathbb{1}_{|f| < s},$$

defined for  $s > 0$ . We thus obtain

$$\int_{|f| \geq s} |f|^2 d\mu + s^{2-p} \int_{|f| < s} |f|^p d\mu = \int |g|^p d\mu$$

and

$$\begin{aligned} \int |g|^p d\mu &\leq B(p) \left( \int_{|f| \geq s} |f|^{\frac{2}{p}(p-2)} \frac{4}{p^2} |f|^{\frac{2(2-p)}{p}} \Gamma(f, f) d\mu \right) + \\ &\quad + B(p) \left( s^{2-p} \int_{|f| < s} |f|^{p-2} \Gamma(f, f) d\mu \right) \\ &\leq B(p) \left( \frac{4}{p^2} \int_{|f| \geq s} \Gamma(f, f) d\mu + \int_{|f| < s} \Gamma(f, f) d\mu \right), \end{aligned}$$

so that by letting  $s$  go to 0 we obtain  $B(2) \leq \frac{4}{p^2} B(p)$ , hence the result.  $\square$

We shall now see how to use this theorem in order to study  $N_p(P_t f)$ .

First recall that, according to (3.3), (3.5) for  $p = 2$  is equivalent to the Poincaré inequality, and

$$\frac{1}{9} B(2) \leq C_P \leq 4 B(2).$$

It follows, using again (3.3) and Theorem 3.6, that if **(H-Poinc)** holds, for any  $p \geq 2$ ,

$$\begin{aligned} (3.7) \quad 2^{-p} N_p^p(f) &\leq M_p^p(f) \leq B(p) \int |f - m_\mu(f)|^{p-2} \Gamma(f, f) d\mu \\ &\leq B(p) \delta(p-2) \int |f - \mu(f)|^{p-2} \Gamma(f, f) d\mu + \\ &\quad + B(p) \delta(p-2) |\mu(f) - m_\mu(f)|^{p-2} \int \Gamma(f, f) d\mu \\ &\leq B(p) \delta(p-2) \int |f - \mu(f)|^{p-2} \Gamma(f, f) d\mu + \\ &\quad + B(p) \delta(p-2) 2^{(p-2)/2} (\text{Var}_\mu(f))^{(p-2)/2} \int \Gamma(f, f) d\mu, \end{aligned}$$

where we have used

$$|\mu(f) - m_\mu(f)| \leq \sqrt{2} (\text{Var}_\mu(f))^{1/2}$$

(see the proof of Lemma 2.1 in [6]) and

$$(u + v)^p \leq \delta(p)(u^p + v^p)$$

for any non-negative  $u$  and  $v$  and any  $p \geq 0$ , with  $\delta(p) = 2^{p-1}$  if  $p \geq 1$  and  $\delta(p) = 1$  if  $0 \leq p \leq 1$ ; hence finally  $\delta(p) = 1 \vee 2^{p-1}$ .

Now consider, for  $p \geq 2$ , the following “entropy functional” (in the terminology of P.D.E. specialists)

$$(3.8) \quad E_p(f) = a_p N_p^p(f) + b_p (\text{Var}_\mu(f))^{p/2} \leq (a_p + b_p) N_p^p(f),$$

where  $a_p$  and  $b_p$  are positive constants to be chosen later. Remark first that using Poincaré inequality and (3.7), we have (remember  $B(p) \leq 9C_P p^2/4$ )

$$(3.9) \quad E_p(f) \leq A(p) + D(p),$$

where

$$\begin{aligned} A(p) &= a_p 2^p \frac{9C_P p^2}{4} \delta(p-2) \int |f - \mu(f)|^{p-2} \Gamma(f, f) d\mu, \\ D(p) &= \left( a_p 2^p \frac{9C_P p^2}{4} 2^{(p-2)/2} \delta(p-2) + C_P b_p \right) (\text{Var}_\mu(f))^{(p-2)/2} \int \Gamma(f, f) d\mu. \end{aligned}$$

We have

$$\begin{aligned}
\frac{d}{dt}E_p(P_t f) &= -a_p p(p-1) \int |P_t f - \mu(P_t f)|^{p-2} \Gamma(P_t f, P_t f) d\mu \\
&\quad - b_p \frac{p}{2} (\text{Var}_\mu(P_t f))^{(p-2)/2} \int \Gamma(P_t f, P_t f) d\mu, \\
&= -\frac{p(p-1)}{\frac{9C_P p^2}{4} \delta(p-2) 2^p} a_p 2^p \frac{9C_P p^2}{4} \delta(p-2) \int |P_t f - \mu(f)|^{p-2} \Gamma(P_t f, P_t f) d\mu \\
&\quad - \frac{b_p \frac{p}{2}}{a_p 2^p \frac{9C_P p^2}{4} 2^{(p-2)/2} \delta(p-2) + C_P b_p} \left( a_p 2^p \frac{9C_P p^2}{4} 2^{(p-2)/2} \delta(p-2) + C_P b_p \right) \\
&\quad \times (\text{Var}_\mu(P_t f))^{(p-2)/2} \int \Gamma(P_t f, P_t f) d\mu.
\end{aligned}$$

Using (3.9), and choosing  $a_p, b_p$  such that

$$\frac{p(p-1)}{\frac{9C_P p^2}{4} \delta(p-2) 2^p} = \frac{b_p \frac{p}{2}}{a_p 2^p \frac{9C_P p^2}{4} 2^{(p-2)/2} \delta(p-2) + C_P b_p}$$

which is possible as  $p \geq 2$ , we thus get

$$\frac{d}{dt}E_p(P_t f) \leq -\frac{p(p-1)}{\frac{9C_P p^2}{4} \delta(p-2) 2^p} E_p(P_t f) = -\gamma_p E_p(P_t f)$$

so that applying Gronwall's lemma we deduce

$$N_p^p(P_t f) \leq \frac{1}{a_p} E_p(P_t f) \leq e^{-\gamma_p t} E_p(f) \leq \frac{a_p + b_p}{a_p} e^{-\gamma_p t} N_p^p(f).$$

Putting all our results together, we have thus shown:

**Theorem 3.10.** *If (H-Poinc) holds with constant  $C_P$ , then for all  $p \geq 2$ ,*

$$N_p(P_t f) \leq K_p e^{-\lambda_p t} N_p(f),$$

with

$$\lambda_p = \frac{4(p-1)}{9p^2 (1 \vee 2^{p-3}) 2^p C_P},$$

and

$$K_p^p = 1 + \frac{\frac{9p^2}{4} (p-1) 2^{(3p-2)/2} (1 \vee 2^{p-3})}{\frac{9p^2}{4} (1 \vee 2^{p-3}) 2^{p-1} - p + 1}.$$

In this result, the constant  $K_p$  is, at least for large  $p$ , smaller than the one obtained in Theorem 1.5 but of course the constant  $\lambda_p$  is quite bad, but however better than the one in Theorem 1.4.

## 4. SOME FINAL REMARKS.

We did not succeed in proving the analogue of Lemma 1.2 for  $p > 2$  (and actually we believe that such a statement is false). Hence both Theorems 1.4, 1.5 and 3.10 have their own interest.

Of course under stronger assumptions than the sole Poincaré inequality (logarithmic Sobolev inequality for instance), one can improve the bounds obtained in Theorem 1.1.

**Extension to the non-symmetric case.**

Notice that the only point where we used symmetry is the proof of Lemma 2.1, hence of Lemma 1.2. In particular if  $\mu$  is invariant but not necessarily symmetric, **(H-Poinc)** implies exponential decay in all the  $\mathbb{L}^p(\mu)$ ,  $p \geq 2$ , and our bounds are available, in particular we may choose  $K_p = 1$ .

But if **(H-p)** holds for some  $p > 2$  and with  $K_p = 1$  (which is crucial) then (2.4) is satisfied, which in return implies the same decay for the dual semi-group  $P_t^*$ . Hence the duality argument shows that **(H-q)** is satisfied for both  $P_t$  and  $P_t^*$ , where  $q$  is the conjugate exponent of  $p$ . Hence **(H-Poinc)** implies exponential decay in all the  $\mathbb{L}^p(\mu)$ ,  $1 < p < +\infty$ .

Conversely, assume that **(H-p)** holds for some  $p > 2$  and with  $K_p = 1$  (which is still crucial). The previous argument shows that **(H-q)** is satisfied. The Riesz-Thorin interpolation theorem then shows that **(H-s)** is satisfied for all  $q \leq s \leq p$ , hence for  $s = 2$ . But since we do not know that  $K_2 = 1$ , we cannot conclude that the Poincaré inequality is satisfied. Also note that the induction argument we used in the proofs calls explicitly upon the Poincaré inequality, so that we cannot deduce that **(H-s)** holds for  $s > p$ .

Finally recall that in the non-symmetric situation, exponential decay in  $\mathbb{L}^2$  can occur while the Poincaré inequality is not satisfied. Of course in this situation,  $K_2 > 1$ . This is the generic situation in many hypocoercive kinetic models like the kinetic Ornstein-Uhlenbeck process studied in [8] (also see [2] section 6).

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