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# Asymptotic behavior of stochastic PDEs with random coefficients

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## Abstract

We study the long time behavior of the solution of a stochastic PDEs with random coefficients assuming that randomness arises in a different independent scale. We apply the obtained results to 2D-Navier–Stokes equations.

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**Key words:** Random coefficients, Stochastic PDEs, Invariant measures, Ergodicity.

## 1 Introduction and setting of the problem

In this work, we consider partial differential equations driven by two different sources of randomness. One is of white noise type and the other one is smoother. This kind of problem is very natural. For instance, if a physical system is submitted to external random forces and if these have different time scales the fast ones can often be approximated by a white noise.

We thus study the following stochastic differential equation,

$$\begin{cases} dX = (AX + b(X) + g(X, \bar{Y}))dt + \sigma(X, \bar{Y})dW(t), \\ X(s) = x \in H, \quad s \leq t, \end{cases} \quad (1.1)$$

where  $A : D(A) \subset H \rightarrow H$  is the infinitesimal generator of a strongly continuous semigroup  $e^{tA}$ ,  $b : D(b) \subset H \rightarrow H$ ,  $g : D(g) \subset H \times K \rightarrow H$ ,  $\sigma : H \times K \rightarrow L_2(H)$  are suitable nonlinear mappings.

The two sources of randomness are the Wiener process  $W$  and the process  $\bar{Y}$ . We assume that they are independent. More precisely, we are given two separable Hilbert spaces  $H$  and  $K$  and two filtered probability spaces  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$  and  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\mathcal{F}}_t)_{t \geq 0})$ . We shall set  $\mathbb{E} = \mathbb{E}^{\mathbb{P}}$ ,  $\tilde{\mathbb{E}} = \tilde{\mathbb{E}}^{\tilde{\mathbb{P}}}$ . Then  $W(t)$  is a cylindrical Wiener process on  $H$  associated to the stochastic basis  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$  and  $\bar{Y}(t)$  is a  $K$ -valued Markov stationary process associated to the stochastic basis  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\mathcal{F}}_t)_{t \geq 0})$  independent of  $W$ .

Several problems can be written as equation (1.1). For instance, it may describe the evolution of a fluid and equation (1.1) is then an abstract form of the Navier-Stokes equation. Other models are reaction-diffusion equations, Ginzburg-Landau equations and so on.

In [3], [4], the case when  $\bar{Y}$  is deterministic and periodic in time has been studied. Since  $X$  is not an homogeneous Markov process, the notion of invariant measure does not make sense anymore. Instead, the longtime behaviour is described by an evolutionary system of measures  $(\mu_t)_{t \in \mathbb{R}}$ . It is periodic and is such that if the law of  $X$  at time  $s$  is  $\mu_s$  then it is  $\mu_t$  at time  $t$ . In fact, this evolutionary system can be constructed by disintegration of an invariant measure of an enlarged system which is Markov. Moreover, uniqueness, ergodicity and mixing properties have been generalized to this context.

Our aim in this work is to generalize the results obtained in [3] to more general driving forces. We also define systems of measures which generalize the concept of invariant measures and describe the longtime behaviour. It is also obtained by disintegration of an invariant measure but in a more complicated way.

We assume that (1.1) has a unique continuous solution. This is the case in the examples described above if the function  $g(X, \bar{Y})$  is Lipschitz in  $X$ , has polynomial growth in  $\bar{Y}$  and  $\bar{Y}$  has moments for instance. We denote the solution by  $X(t, s, x)$ .

We also assume that there is a continuous Markov process  $Y(t, s, y)$ ,  $t \geq s$ ,  $y \in K$  on  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\mathcal{F}}_t)_{t \geq 0})$  such that  $Y(s, s, y) = y$  and

$$Y(t, s, \bar{Y}(s)) = \bar{Y}(t), \quad t \geq s.$$

A typical example is when  $Y$  is the solution of a stochastic partial differential equation driven by another Wiener process and  $\bar{Y}$  is a stationary solution of this equation. The simplest case is given by an Ornstein Uhlenbeck process:

$$Y(t, s, y) = e^{(t-s)B}y + \int_s^t e^{(t-r)B}dV(r), \quad t > s, \quad (1.2)$$

where  $B : D(B) \subset K \rightarrow K$  is self-adjoint, strictly negative and such that  $B^{-1}$  is of trace class and  $V$  is a cylindrical Wiener process in  $\mathbb{R}$  with values in  $K$  independent of  $W$ .

We set

$$P_{s,t}\varphi(x) = P_{s,t}^{\tilde{\omega}}\varphi(x) = \mathbb{E}[\varphi(X(t, s, x))], \quad \varphi \in B_b(H).$$

Obviously,  $P_{s,t}\varphi(x)$  is a random variable in  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  and for each  $\tilde{\omega} \in \tilde{\Omega}$  it holds

$$P_{s,r}P_{r,t} = P_{s,t}, \quad s \leq r \leq t.$$

One should not confuse  $P_{s,t}\varphi(x)$  with  $\tilde{\mathbb{E}}[P_{s,t}\varphi(x)]$ . The latter does not fulfil the cocycle law since  $X(t, s, x)$  is not a Markov process in general.

In this paper we are interested in the asymptotic behavior of  $P_{s,t}\varphi(x)$ . We shall proceed as follows. First we construct an enlarged homogeneous Markov process  $Z(t, s, x, h)$  with state space  $H \times \mathcal{K}$ , where  $\mathcal{K} := C((-\infty, 0]; K)$ .

Then assuming that  $Z$  possesses an invariant measure  $\nu(dx, dh)$  we show that a disintegration of  $\nu$  produces a family  $(\mu_t)_{t \in \mathbb{R}}$  of random probability measures on  $H$  such that  $\tilde{\omega}$  a.s.

$$\int_H P_{s,t}^{\tilde{\omega}}\varphi(x)\mu_s(dx) = \int_H \varphi(x)\mu_t(dx), \quad \varphi \in B_b(H), \quad t > s. \quad (1.3)$$

Such a family of measures is called an evolutionary system of measures.

Then, we give sufficient conditions for uniqueness of an evolutionary system of measures. We generalize the classical criterion based on irreducibility and strong Feller property. We also show that the recent method developed in [10] generalizes to our context. Finally, we illustrate our results on the two dimensional Navier-Stokes.

## 2 Evolutionary systems of measures

### 2.1 Construction of a Markov process

Let us first define a new Markov process on  $\mathcal{K} := C((-\infty, 0]; K)$ , setting

$$H(t, s, h)(\theta) = \begin{cases} Y(t + \theta, s, h(0)), & \text{if } t + \theta \geq s, \\ h(\theta - s + t), & \text{if } t + \theta < s. \end{cases}, \quad h \in \mathcal{K}. \quad (2.1)$$

It is not difficult to check that

$$H(t, s, h) = H(t, r, H(r, s, h)), \quad t \geq r \geq s. \quad (2.2)$$

It follows easily that  $H$  is a Markov process which is clearly homogeneous.

We assume that for any  $h \in \mathcal{K}$  the following equation has a unique continuous solution

$$\begin{cases} dX^h &= (AX^h + b(X^h) + g(X^h, H(t, s, h)(0)))dt \\ &+ \sigma(X^h, H(t, s, h)(0))dW(t), \\ X^h(s) &= x \in H, \quad s \leq t. \end{cases} \quad (2.3)$$

Then we define the spaces

$$(\Omega_1, \mathcal{F}_1, \mathbb{P}_1, (\mathcal{F}_{1,t})_{t \geq 0}) = (\Omega \times \tilde{\Omega}, \mathcal{F} \times \tilde{\mathcal{F}}, \mathbb{P} \times \tilde{\mathbb{P}}, (\mathcal{F}_t \times \tilde{\mathcal{F}}_t)_{t \geq 0}),$$

(we shall denote by  $\mathbb{E}_1$  the expectation in this space) and

$$\mathcal{H} = H \times \mathcal{K},$$

and we consider the homogeneous Markov process

$$Z(t, s, x, h) = (X^h(t, s, x), H(t, s, h)).$$

We denote by  $Q_t$ ,  $t \geq 0$ , and  $R_t$ ,  $t \geq 0$  the transition semigroups associated to  $Z$  and  $H$  respectively.

We also denote by  $P_{t,s}^h$  the transition operators associated to  $X^h$ .

## 2.2 Stationary processes

If

$$h(\theta) = \bar{Y}(s + \theta), \quad \theta \leq 0,$$

then

$$H(t, s, h) = Y(t, s, h(0)) = \bar{Y}(t + \cdot), \quad t \geq 0.$$

Therefore  $H(t, s, h)(0) = \bar{Y}(t)$ , so that

$$X^h(t, s, x) = X(t, s, x),$$

where  $X(t, s, x)$  is the solution to (1.1). Moreover  $H(t, s, h)(\theta) = \bar{Y}(t + \theta)$  so that

$$H(t, s, h) = \tau_t \bar{Y}|_{(-\infty, t]}, \quad (2.4)$$

where, for a function  $f$  defined on  $(-\infty, t]$ ,  $\tau_t f$  is the function defined on  $(-\infty, 0]$  by  $\tau_t f(\theta) = f(t + \theta)$ . It follows that

$$Z(t, s, x, h) = (X(t, s, x), \tau_t \bar{Y}|_{(-\infty, t]}).$$

Note also that the stochastic process  $H(t, s, h)$  is stationary since the laws of  $\tau_t \bar{Y}|_{(-\infty, t]}$  and  $\tau_s \bar{Y}|_{(-\infty, s]}$  coincide for any  $t, s$ . Moreover, it does not depend on  $s$ . We shall denote it by  $\bar{H}$ , and by  $\lambda$  its invariant law.

Conversely, let  $\bar{H}$  be a stationary process associated to  $H$ . Then

$$\bar{H}(t)(\theta) = Y(t + \theta, s, \bar{H}(s)(0))$$

for all  $t + \theta \geq s$ . Therefore, if we define

$$\bar{Y}(t) = \bar{H}(t)(0)$$

we have  $\bar{Y}(t) = H(\tilde{t})(\tilde{\theta})$  provided  $\tilde{t} + \tilde{\theta} = t$ . In particular,  $\bar{Y}(t) = \bar{H}(0)(t)$  for  $t \leq 0$ .

Note also that  $\bar{Y}(t) = \bar{H}(t)(0) = Y(t, s, \bar{H}(s)(0)) = Y(t, s, \bar{Y}(s))$ . Since  $\bar{Y}$  is clearly stationary, it follows that it is a stationary process associated to  $Y$ .

This shows that we have a one to one correspondance between stationary processes for  $H$  and  $Y$ .

Let  $\lambda$  be an invariant measure for  $H$ :

$$\int_{\mathcal{X}} \alpha(H(t, s, h)) \lambda(dh) = \int_{\mathcal{X}} \alpha(h) \lambda(dh), \quad \alpha \in C_b(\mathcal{X}).$$

Take  $\alpha(h) = \gamma(h(0))$ ,  $h \in \mathcal{X}$ , where  $\gamma \in C_b(K)$ . Since  $H(t, 0, h)(0) = Y(t, 0, h(0))$ , we deduce

$$\int_K \tilde{\mathbb{E}}[\gamma(Y(t, 0, h(0)))] \lambda(dh) = \int_K \gamma(h(0)) \lambda(dh), \quad \forall \gamma \in C_b(K). \quad (2.5)$$

If we denote by  $\lambda_0$  the image measure of  $\lambda$  by the mapping

$$\mathcal{X} \rightarrow K, \quad h \mapsto h(0),$$

we deduce from (2.5) that

$$\int_{\mathcal{X}} \tilde{\mathbb{E}}[\gamma(Y(t, 0, \xi))] \lambda_0(d\xi) = \int_K \gamma(\xi) \lambda_0(d\xi), \quad \forall \gamma \in C_b(K). \quad (2.6)$$

Therefore  $\lambda_0$  is an invariant measure for  $Y$ .

We now prove that ergodicity is transferred to  $\bar{H}$ . In the proof of this result, we also prove that the law of the process  $\bar{Y}$  determines the law of  $\bar{H}$ .

**Proposition 2.1** *Assume that  $Y$  has a unique invariant measure, then the process  $\overline{H}$  is stationary and ergodic.*

**Proof.** Let  $\lambda$  be an invariant law of  $\overline{H}$ . Assume that

$$\lambda = \eta\lambda^1 + (1 - \eta)\lambda^2,$$

where  $\eta \in [0, 1]$  and  $\lambda^1, \lambda^2$  are invariant for  $\overline{H}$ . Given  $\theta \in \mathbb{R}$  consider the mapping

$$\mathcal{X} \rightarrow K, \quad h \mapsto h(\theta)$$

and denote by  $\lambda_\theta, \lambda_\theta^1, \lambda_\theta^2$  the image measures of  $\lambda, \lambda^1, \lambda^2$  respectively by this mapping. By the discussion above,  $\lambda_0, \lambda_0^1, \lambda_0^2$  are invariant for  $Y$ . Therefore

$$\lambda_0 = \lambda_0^1 = \lambda_0^2.$$

We have in addition for  $\gamma_1, \gamma_2 \in C_b(K)$  and  $\theta_1 \in \mathbb{R}, \theta_2 \geq 0$

$$\begin{aligned} & \int_{\mathcal{X}} \gamma_1(h(\theta_1))\gamma_2(h(\theta_1 + \theta_2))\lambda(dh) \\ &= \int_{\mathcal{X}} \tilde{\mathbb{E}}[\gamma_1(Y(t + \theta_1, 0, h(0)))\gamma_2(Y(t + \theta_1 + \theta_2, 0, h(0)))]\lambda(dh) \\ &= \int_K \tilde{\mathbb{E}}[\gamma_1(Y(t + \theta_1, 0, y))\gamma_2(Y(t + \theta_1 + \theta_2, 0, y))]\lambda_0(dy) \\ &= \int_K \tilde{\mathbb{E}}[\gamma_1(y)\gamma_2(Y(\theta_2, 0, y))]\lambda_0(dy). \end{aligned}$$

We have used the fact that  $\lambda_0$  is the image law of  $\lambda$  and the invariance of  $\lambda_0$ .

Therefore the joint law of  $h(\theta_1), h(\theta_1 + \theta_2)$  is given. We argue similarly for  $\lambda^1$  and  $\lambda^2$  and for  $\theta_1, \theta_1 + \theta_2, \dots, \theta_1 + \dots + \theta_n$  and deduce that  $\lambda, \lambda^1, \lambda^2$  have the same image laws for applications

$$\mathcal{X} \rightarrow K^n, \quad h \mapsto (h(\theta_1), h(\theta_1 + \theta_2), \dots, h(\theta_1 + \dots + \theta_n)).$$

This yields  $\lambda = \lambda^1 = \lambda^2$  and so,  $\lambda$  is an extremal point and an ergodic invariant measure.  $\square$

## 2.3 Evolutionary systems of measures

We now define the main new object of our article. We expect that the long time behaviour of the solutions (1.1) is described by a random family of measures  $(\mu_t)_{t \in \mathbb{R}}$  satisfying  $\tilde{\mathbb{P}}$  a.s.

$$\int_H \mathbb{E}[\varphi(X(t, s, x))]\mu_s(dx) = \int_H \varphi(x)\mu_t(dx), \quad (2.7)$$

for all  $\varphi \in C_b(H)$ ,  $t > s$ . This is a natural generalisation of the notion of invariant measure.

This definition is too general and we need to make some restrictions on the system  $(\mu_t)_{t \in \mathbb{R}, \tilde{\omega} \in \tilde{\Omega}}$ . Without loss of generality, we assume that the forcing is associated to a shift  $(\tau_t)_{t \in \mathbb{R}}: \bar{Y}(t, \tilde{\omega}) = \bar{Y}(s, \tau_{t-s}\tilde{\omega})$ ,  $t \geq s$ ,  $\tilde{\omega} \in \tilde{\Omega}$ . We consider the stationary process  $(\bar{H}(t))_{t \in \mathbb{R}}$  in  $\mathcal{H}$  constructed above and defined by  $\bar{H}(t) = \tau_t \bar{Y}|_{(-\infty, t]}$  and denote by  $\lambda$  its invariant measure. Then clearly  $\bar{H}(t, \tilde{\omega}) = \bar{H}(s, \tau_{t-s}\tilde{\omega})$ ,  $t \geq s$ ,  $\tilde{\omega} \in \tilde{\Omega}$ .

We assume that  $(\mu_t)_{t \in \mathbb{R}, \tilde{\omega} \in \tilde{\Omega}}$  satisfies the following consistency assumption:

$$\mu_t(\tilde{\omega}) = \mu_s(\tau_{t-s}\tilde{\omega}), \text{ a.s. } t \geq s, \text{ and } \tilde{\mathbb{P}} \text{ a.s. } \tilde{\omega} \in \tilde{\Omega}. \quad (2.8)$$

This says that two solutions of (1.1) with the same past define the same evolution.

It is also natural to assume that, for each  $t \in \mathbb{R}$ ,  $\mu_t$  is measurable with respect to the  $\sigma$ -field generated by  $\bar{Y}(s)$ ,  $s \leq t$ .

A random family of measures  $(\mu_t)_{t \in \mathbb{R}}$  satisfying these properties is called an *evolutionary system of measures*.

### 3 Invariant measures for $Z$ are associated to evolutionary systems of measures

Let us assume that  $Z$  possesses an invariant measure  $\nu$ :

$$\int_{\mathcal{H}} \mathbb{E}_1[\Phi(X^h(t, s, x), H(t, s, h))] \nu(dx, dh) = \int_{\mathcal{H}} \Phi(x, h) \nu(dx, dh), \quad \Phi \in C_b(\mathcal{H}). \quad (3.1)$$

Let us consider a disintegration of  $\nu$

$$\nu(dx, dh) = \mu_h(dx) \lambda(dh).$$

**Lemma 3.1**  *$\lambda$  is an invariant measure for  $H$ , that is*

$$\int_{\mathcal{H}} \tilde{\mathbb{E}}[\alpha(H(t, 0, h))] \lambda(dh) = \int_{\mathcal{H}} \alpha(h) \lambda(dh), \quad \forall t \geq 0, \alpha \in C_b(\mathcal{H}). \quad (3.2)$$



**Proof.** Let  $\alpha \in C_b(\mathcal{H})$ . Then by (3.1) with  $s = 0$  we have

$$\begin{aligned} \int_{\mathcal{H}} \mathbb{E}_1[\alpha(H(t, 0, h))] \nu(dx, dh) &= \int_{\mathcal{H}} \alpha(h) \nu(dx, dh) \\ &= \int_{\mathcal{H}} \alpha(h) \lambda(dh). \end{aligned}$$

But

$$\int_{\mathcal{H}} \mathbb{E}_1[\alpha(H(t, 0, h))] \nu(dx, dh) = \int_{\mathcal{H}} \tilde{\mathbb{E}}[\alpha(H(t, 0, h))] \lambda(dh).$$

Therefore  $\lambda$  is invariant for  $H$ .  $\square$

Let  $\bar{H}$  be a stationary process associated to  $H$  with invariant law  $\lambda$ . We set

$$\mu_t = \mu_{\bar{H}(t)}, \quad t \in \mathbb{R}.$$

**Theorem 3.2** *The family  $(\mu_t)_{t \in \mathbb{R}}$  is an evolutionary system of measures.*

**Proof.** We prove that (2.7) holds. The other properties of evolutionary systems are clearly satisfied.

Choosing for  $\varphi \in C_b(H)$ ,  $\alpha \in C_b(\mathcal{H})$ :

$$\Phi(x, h) = \varphi(x) \alpha(h),$$

we have by (3.1)

$$\int_{\mathcal{H}} \mathbb{E}_1[\varphi(X^h(t, s, x)) \alpha(H(t, s, h))] \mu_h(dx) \lambda(dh) = \int_{\mathcal{H}} \varphi(x) \alpha(h) \mu_h(dx) \lambda(dh).$$

Moreover, by the invariance of  $\lambda$  for the process  $H$  we get

$$\int_{\mathcal{H}} \varphi(x) \alpha(h) \mu_h(dx) \lambda(dh) = \int_{\mathcal{H}} \tilde{\mathbb{E}}[\varphi(x) \alpha(H(t, s, h)) \mu_{H(t, s, h)}](dx) \lambda(dh).$$

This yields

$$\begin{aligned} &\int_{\mathcal{H}} \tilde{\mathbb{E}} \left\{ \mathbb{E}[\varphi(X^h(t, s, x)) \alpha(H(t, s, h))] \right\} \mu_h(dx) \lambda(dh) \\ &= \int_{\mathcal{H}} \tilde{\mathbb{E}}[\varphi(x) \alpha(H(t, s, h)) \mu_{H(t, s, h)}(dx)] \lambda(dh). \end{aligned} \tag{3.3}$$

Since  $\overline{H}(s)$  has law  $\lambda$ ,  $H(t, s, \overline{H}(s)) = \overline{H}(t)$  and  $X^{\overline{H}(s)}(t, s, x) = X(t, s, x)$ , we can rewrite (3.3) as

$$\int_H \tilde{\mathbb{E}} \{ \mathbb{E}[\varphi(X(t, s, x))\alpha(\overline{H}(t))] \} \mu_{\overline{H}(s)}(dx) = \int_H \tilde{\mathbb{E}}[\varphi(x)\alpha(\overline{H}(t))\mu_{\overline{H}(t)}(dx)].$$

Note that  $\int_H \mathbb{E}[\varphi(X(t, s, x))]\mu_{\overline{H}(s)}(dx)$  and  $\int_H \varphi(x)\mu_{\overline{H}(t)}(dx)$  are  $\overline{H}(t)$ -measurable (their are functions of  $\overline{H}(t)$ ) and we deduce by the arbitrariness of  $\alpha$  that

$$\int_H \mathbb{E}[\varphi(X(t, s, x))]\mu_{\overline{H}(s)}(dx) = \int_H \varphi(x)\mu_{\overline{H}(t)}(dx), \quad \tilde{\mathbb{P}}\text{-a.s.},$$

which yields (2.7) since  $\mu_t := \mu_{\overline{H}(t)}$ ,  $t \in \mathbb{R}$ .

However, the set of all  $\tilde{\omega} \in \tilde{\Omega}$  for which identity (2.7) holds depends of  $t, s, \varphi$ . Modifying the disintegration of  $\nu$ , we can easily get rid of this dependence. We proceed as in [3]. First, taking  $\varphi = \mathbb{1}_C$  for  $C$  in a countable set generating Borel sets of  $H$ , we can show that (2.7) holds in fact for every  $\varphi \in B_b(H)$  for almost every  $\tilde{\omega}$  depending only on  $t, s$ . From Fubini's theorem we find that  $\tilde{\mathbb{P}}$ -a.s.

$$\int_H \mathbb{E}[\varphi(X(t, s, x))]\mu_s(dx) = \int_H \varphi(x)\mu_t(dx), \quad \text{for almost all } s \leq t. \quad (3.4)$$

Choose  $s_n \downarrow -\infty$  such that

$$P_{s_n, t}^* \mu_{s_n} = \mu_t, \quad \text{for almost all } t \geq s_n$$

and set

$$\tilde{\mu}_t^n = P_{s_n, t}^* \mu_{s_n}, \quad t \geq s_n.$$

From the continuity of  $t \mapsto X(t, s, x)$  for almost every  $\tilde{\omega}$ , we deduce that  $t \mapsto \tilde{\mu}_t^n$  is continuous. Moreover  $\tilde{\mu}_t^n = \tilde{\mu}_t^{n+1}$ , a.s.  $t \geq s_n$ , so that by continuity

$$\tilde{\mu}_t^n = \tilde{\mu}_t^{n+1}, \quad \forall t \geq s_n.$$

Now we define

$$\tilde{\mu}_t = \tilde{\mu}_t^n, \quad \forall t \geq s_n.$$

Obviously,  $\tilde{\mu}_t = \mu_t$  for almost all  $t \in \mathbb{R}$ , so that

$$P_{s_n, t}^* \tilde{\mu}_{s_n} = \tilde{\mu}_t, \quad \text{a.s. } t \geq s_n.$$

By the continuity in  $t$  we deduce for all  $t \in \mathbb{R}$

$$P_{s_n, t}^* \tilde{\mu}_{s_n} = \tilde{\mu}_t, \quad \forall t \geq s_n$$

and for  $t \geq s \geq s_n$  we have

$$P_{s,t}^* \tilde{\mu}_s = P_{s,t}^*(P_{s_n,s}^* \tilde{\mu}_{s_n}) = P_{s_n,t}^* \tilde{\mu}_{s_n} = \tilde{\mu}_t.$$

Therefore we can conclude that  $\tilde{\mathbb{P}}$ -a.s.

$$P_{s,t}^* \tilde{\mu}_s = \tilde{\mu}_t, \quad \forall t \geq s,$$

as claimed. Since  $\tilde{\mu}_t = \mu_t$  for almost all  $t \in \mathbb{R}$ , the consistency relation is still satisfied by  $\tilde{\mu}_t$ .  $\square$

Let us now see how an evolutionary system of measures  $(\tilde{\mu}_t)_{t \in \mathbb{R}, \tilde{\omega} \in \tilde{\Omega}}$  yields an invariant measure for  $Z$ . By definition,  $\tilde{\mu}_t$  is measurable with respect to the  $\sigma$ -field generated by  $\bar{Y}(s)$ ,  $s \leq t$ . This implies that there exists a measurable function  $h \mapsto \mu_h^t$  such that  $\tilde{\mu}_t = \mu_{\bar{H}(t)}^t$ . By the consistency assumption (2.8), we have  $\tilde{\mathbb{P}}$ -a.s. for  $t \geq s$  in a set  $I$  of full measure,

$$\mu_{\bar{H}(t,\tilde{\omega})}^t = \tilde{\mu}_t(\tilde{\omega}) = \tilde{\mu}_s(\tau_{t-s}\tilde{\omega}) = \mu_{\bar{H}(s,\tau_{t-s}\tilde{\omega})}^s = \mu_{\bar{H}(t,\tilde{\omega})}^s$$

so that  $\mu^t$  does not depend on  $t$  for  $t \in I$ . We choose  $s_0 \in I$ , set  $\mu_h = \mu_h^{s_0}$  and define

$$\nu(dx, dh) = \mu_h(dx) \lambda(dh).$$

Then, since  $\mathcal{L}(\bar{Y}(\cdot, s_0)) = \lambda$  and  $X^{\bar{H}(s_0)}(t, s_0, x) = X(t, s_0, x)$ , we have for  $\psi \in B_b(\mathcal{H})$ ,  $t \in I$ ,

$$\begin{aligned} \int_{\mathcal{H}} Q_{t-s_0} \psi(x, h) \nu(dx, dh) &= \int_{\mathcal{H}} \mathbb{E}_1[\psi(X^h(t, s_0, x), H^h(t, s_0, h))] \mu_h(dx) \lambda(dh) \\ &= \int_H \mathbb{E}_1[\psi(X(t, s_0, x), \bar{H}(t))] \mu_{\bar{H}(s_0)}(dx). \end{aligned}$$

Therefore, for  $\psi$  of the form  $\psi(x, h) = \varphi(x)\alpha(h)$ ,

$$\begin{aligned}
\int_{\mathcal{H}} Q_{t-s_0} \psi(x, h) \nu(dx, dh) &= \int_H \mathbb{E}_1[\varphi(X(t, s_0, x))\alpha(\overline{H}(t))]\tilde{\mu}_{s_0}(dx) \\
&= \tilde{\mathbb{E}} \left[ \int_H \mathbb{E}(\varphi(X(t, s_0, x)))\tilde{\mu}_{s_0}(dx)\alpha(\overline{H}(t)) \right] \\
&= \tilde{\mathbb{E}} \int_H \varphi(x)\tilde{\mu}_t(dx)\alpha(\overline{H}(t)) \\
&= \tilde{\mathbb{E}} \int_H \varphi(x)\mu_{\overline{H}(t)}(dx)\alpha(\overline{H}(t)) \\
&= \int_{\mathcal{H}} \varphi(x)\alpha(h)\mu_h(dx)\lambda(dh). \\
&= \int_{\mathcal{H}} \psi(x, h)\nu(dx, dh).
\end{aligned}$$

Since the left hand side is a continuous function of  $t$ , we deduce that the equality holds for all  $t \in \mathbb{R}$  and that  $\nu$  is an invariant measure for  $Z$ .

It follows from our discussion that the correspondance  $\nu \mapsto ((\mu_t)_{t \in \mathbb{R}}, \lambda)$  is a bijection. In particular, if there exists a unique invariant measure with marginal  $\lambda$  for  $Z$ , there exists a unique evolutionary system of measure.

## 4 A simple example

We consider the following special form of (1.1) in  $\mathbb{R}$ .

$$\begin{cases} dX &= (-X + \overline{Y})dt + dW, \quad t \geq s \\ X(s) &= x \in H, \end{cases} \quad (4.1)$$

where  $\overline{Y}$  is the stationary process

$$\overline{Y}(t) = \int_{-\infty}^t e^{-(t-r)} dV(r), \quad t \in \mathbb{R} \quad (4.2)$$

and  $V, W$  are independent Wiener processes.

We consider the homogeneous Markov process introduced above

$$Z(t, s, x, h) = (X^h(t, s, x), H(t, s, h)), \quad t \geq s, \quad x \in H, \quad h \in \mathcal{H}.$$

Let us write explicitly  $Z(t, s, x, h)$ . We have

$$X^h(t, s, x) = e^{-(t-s)}x + \int_s^t e^{-(t-r)}Y(r, s, h(0))dr + \int_s^t e^{-(t-r)}dW(r). \quad (4.3)$$

Then we have

$$\begin{cases} \lim_{s \rightarrow -\infty} X^h(t, s, x) = \int_{-\infty}^t e^{(t-r)A}\bar{Y}(r)dr + \int_{-\infty}^t e^{(t-r)A}dW(r) := \zeta_t \\ \lim_{s \rightarrow -\infty} H(t, s, h)(\theta) = \bar{Y}(t + \theta). \end{cases} \quad (4.4)$$

This implies that the probability measure on  $H \times \mathcal{K}$ ,

$$\nu = \mathcal{L}(\zeta_0, \bar{H}(0)),$$

is the unique invariant measure for  $Z$ . In other words we have

$$\int_{H \times \mathcal{K}} \Phi(x, h)\nu(dx, dh) = \mathbb{E}_1[\Phi(\zeta_0, \bar{H}(0))].$$

Let  $\nu(dx, dh) = \mu_x(dx)\lambda(dh)$  be a disintegration of  $\nu$ . Then we have

$$\lambda = \mathcal{L}(\bar{H}(0))$$

and

$$\mu_t = \mathcal{N} \left( \int_{-\infty}^0 e^{-\theta} \bar{Y}(t + \theta) d\theta, \frac{1}{2} \right).$$

We see on this simple example that it is necessary to parametrize the evolutionary system of measure by the whole history of the driving process.

## 5 Ergodicity in the regular case

We assume for simplicity in this section that the Markov process  $Y$  has a unique invariant measure. It is then the invariant law of  $\bar{Y}$  and  $\bar{Y}$  is ergodic. We construct the ergodic process  $\bar{H}$  as above and denote by  $\lambda$  its invariant law.

We generalize the famous Doob criterion (see for instance [5]) of ergodicity to evolutionary system of measures.

Let us set

$$\pi_{s,t}(x, E) = \pi_{s,t}^{\tilde{\omega}}(x, E) = P_{s,t}\mathbb{1}_E(x), \quad \forall I \in \mathcal{B}(H).$$

We say that  $P_{s,t}$  is *regular* at  $\tilde{\omega}$  if for any  $s < t$  we have

$$\pi_{s,t}^{\tilde{\omega}}(x, \cdot) \sim \pi_{s,t}^{\tilde{\omega}}(y, \cdot) \quad \text{for all } x, y \in H.$$

We notice the following straightforward identity,

$$\pi_{s,t}(x, E) = \int_H \pi_{s,s+h}(x, dy) \pi_{s+h,t}(y, E), \quad s+h < t, \quad h > 0, \quad E \in \mathcal{B}(H). \quad (5.1)$$

We say that  $P_{s,t}$  is strong Feller at  $\tilde{\omega}$  if for each  $s < t$ ,  $P_{s,t}$  maps  $\mathcal{B}(H)$  to  $C_b(H)$ . It is irreducible at  $\tilde{\omega}$  if, for any  $s < t$ ,  $x \in H$  and  $O \subset H$  open,  $\pi_{s,t}(x_0, O) > 0$ . The proofs of the following Propositions 5.1 and 5.2 are completely similar to that of [3, Proposition 4.1, Proposition 4.2].

**Proposition 5.1** *If the transition semigroup  $P_{s,t}$  is strong Feller and irreducible at  $\tilde{\omega}$ , then it is regular at  $\tilde{\omega}$ .*

**Proposition 5.2** *Assume that  $P_{s,t}$  is regular at  $\tilde{\omega}$  and that it possesses an invariant set of probabilities  $\mu_t$ ,  $t \in \mathbb{R}$ . Then  $\mu_t(\tilde{\omega})$  is equivalent to  $\pi_{s,t}^{\tilde{\omega}}(x, \cdot)$  for all  $s < t$  and  $x \in H$ .*

Next we prove the following theorem which can be used in several applications provided the noise  $W$  is non degenerate.

**Theorem 5.3** *We assume that  $P_{s,t}$  is regular for almost all  $\tilde{\omega}$  and that the Markov process  $Y$  has a unique invariant measure. Then  $Q_t$  has at most one invariant measure which is in addition ergodic. It follows that there exists a unique evolutionary system of measures.*

**Proof.** It suffices to prove that all invariant measures of  $Q_t$  are ergodic. Let  $\nu(dx, dh) = \mu_h(dx)\lambda(dh)$  be an invariant measure for  $Q_t$  and  $\Gamma \in \mathcal{B}(\mathcal{H})$  be an invariant set for  $\nu$ :

$$Q_t \mathbb{1}_\Gamma(x, h) = \mathbb{1}_\Gamma(x, h), \quad \nu\text{-a.e.}$$

Then for any  $\Phi \in B_b(\mathcal{H})$  we have

$$\int_{\mathcal{H}} (Q_t \mathbb{1}_\Gamma)(x, h) \Phi(x, h) \nu(dx, dh) = \int_{\mathcal{H}} \mathbb{1}_\Gamma(x, h) \Phi(x, h) \nu(dx, dh).$$

Setting

$$\Gamma_h = \{x \in H : (x, h) \in \Gamma\},$$

we have  $\mathbb{1}_\Gamma(x, h) = \mathbb{1}_{\Gamma_h}(x)$  and

$$\int_{\mathcal{H}} \mathbb{E}_1 \left( \mathbb{1}_{\Gamma_{H(t,0,h)}}(X^h(t, 0, x)) \Phi(x, h) \nu(dx, dh) \right) = \int_{\mathcal{H}} \mathbb{1}_{\Gamma_h}(x) \Phi(x, h) \nu(dx, dh).$$

On the other hand,

$$\begin{aligned} & \int_{\mathcal{H}} \mathbb{E}_1 \left( \mathbb{1}_{\Gamma_{H(t,0,h)}}(X^h(t, 0, x)) \Phi(x, h) \nu(dx, dh) \right) \\ &= \int_{\mathcal{H}} \tilde{\mathbb{E}} \left( P_{0,t}^h \mathbb{1}_{\Gamma_{H(t,0,h)}} \right) (x) \Phi(x, h) \mu_h(dx) \lambda(dh) \end{aligned}$$

We deduce

$$\tilde{\mathbb{E}} \left( P_{0,t}^h \mathbb{1}_{\Gamma_{H(t,0,h)}} \right) = \mathbb{1}_{\Gamma_h}, \quad \nu\text{-a.s.}$$

so that

$$P_{0,t}^h \mathbb{1}_{\Gamma_{H(t,0,h)}} = \mathbb{1}_{\Gamma_h}, \quad \tilde{\mathbb{P}} \times \nu\text{-a.s.}$$

and

$$P_{0,t}^{\overline{H}(0)} \mathbb{1}_{\Gamma_{\overline{H}(t)}} = \mathbb{1}_{\Gamma_{\overline{H}(0)}}, \quad \mu_{\overline{H}(0)} \times \tilde{\mathbb{P}}\text{-a.s.}$$

Therefore,  $\tilde{\mathbb{P}}$ -a.s., if  $\mu_{\overline{H}(0)}(\Gamma_{\overline{H}(0)}) \neq 0$  then for  $\mu_{\overline{H}(0)}$  almost every  $x \in \Gamma_{\overline{H}(0)}$  we have  $P_{0,t}^{\overline{H}(0)} \mathbb{1}_{\Gamma_{\overline{H}(t)}}(x) = 1$ , equivalently  $\pi_{0,t}^{\overline{H}(0)}(x, \Gamma_{\overline{H}(t)}) = 1$ . Since, by assumption,  $P_{0,t}^{\overline{H}(0)}$  is regular, we have

$$\pi_{0,t}^{\overline{H}(0)}(y, \Gamma_{\overline{H}(t)}) = 1, \quad \forall y \in H,$$

and so,

$$\mathbb{1}_{\Gamma_{\overline{H}(0)}}(y) = 1, \quad \mu_{\overline{H}(0)}\text{-a.s.}$$

Therefore

$$\mu_{\overline{H}(0)}(\Gamma_{\overline{H}(0)}) = 1.$$

We have proved that,  $\tilde{\mathbb{P}}$ -a.s.,

$$\mu_{\overline{H}(0)}(\Gamma_{\overline{H}(0)}) = 0 \text{ or } 1.$$

Similarly we show that

$$\mu_{\overline{H}(t)}(\Gamma_{\overline{H}(t)}) = 0 \text{ or } 1, \quad \forall t \in \mathbb{R}.$$

Set

$$\tilde{\Omega}_1 := \{\tilde{\omega} \in \tilde{\Omega} : \mu_{\overline{H}(1)}(\Gamma_{\overline{H}(1)}) = 1\}.$$

Then

$$\tau_1 \tilde{\Omega}_1 := \{\tilde{\omega} \in \tilde{\Omega} : \mu_{\overline{H}(0)}(\Gamma_{\overline{H}(0)}) = 1\},$$

where  $\theta_t$  is the shift defined in section 2.3

Write now

$$\begin{aligned} \mu_{\overline{H}(0)}(\Gamma_{\overline{H}(0)}) &= \int_H \mathbb{1}_{\Gamma_{\overline{H}(0)}} d\mu_{\overline{H}(0)} = \int_H P_{0,1}^{\overline{H}(0)} \mathbb{1}_{\Gamma_{\overline{H}(1)}} d\mu_{\overline{H}(0)} \\ &= \int_H \mathbb{1}_{\Gamma_{\overline{H}(1)}} d\mu_{\overline{H}(1)} = \mu_{\overline{H}(1)}(\Gamma_{\overline{H}(1)}). \end{aligned}$$

Therefore  $\tau_1 \tilde{\Omega}_1 = \tilde{\Omega}_1$  and the ergodicity implies  $\tilde{\mathbb{P}}(\tilde{\Omega}_1) = 0$  or  $1$ .

If  $\tilde{\mathbb{P}}(\tilde{\Omega}_1) = 0$  we have  $\nu(\Gamma) = \tilde{\mathbb{E}}\mu_{\overline{H}(0)}(\Gamma_{\overline{H}(0)}) = 0$  and if  $\tilde{\mathbb{P}}(\tilde{\Omega}_1) = 1$  we have  $\nu(\Gamma) = \tilde{\mathbb{E}}\mu_{\overline{H}(t)}(\Gamma_{\overline{H}(t)}) = 1$ .  $\square$

**Remark 5.4** The proof of Theorem 4.3 in [3] is not complete. In fact in that theorem we have proved only that  $\nu_h(\Gamma_h) = 0$  or  $1$ . One has to use the same argument as before to arrive at the conclusion.  $\square$

## 6 Uniqueness by asymptotic strong Feller property

We assume again that  $\overline{Y}$  is ergodic and that the process  $Y(\cdot, s, y)$  has a unique invariant measure. The following definition is a natural generalization to  $P_{s,t}$  of a concept introduced in [10].

**Definition 6.1** *We say that  $P_{s,t}$  is asymptotic strong Feller (ASF) at  $x \in H$  if there is a sequence of pseudo-metrics  $\{d_n\}$  on  $H$  such that*

$$d_n(x_1, x_2) \uparrow 1, \quad \forall x_1 \neq x_2, \quad (6.1)$$

and a sequence  $\{t_n\}$  of positive numbers such that,

$$\lim_{\gamma \rightarrow 0} \limsup_{n \rightarrow \infty} \tilde{\mathbb{E}} \left( \sup_{y \in B(x, \gamma)} W_{d_n}(\pi_{s, s+t_n}(x, \cdot), \pi_{s, s+t_n}(y, \cdot)) \right) = 0. \quad (6.2)$$

$P_{s,t}$  is called asymptotically strong Feller (ASF), if it is asymptotically strong Feller at any  $x \in H$  and  $s_0 \in \mathbb{R}$ .

As in [3] section 5.1,  $W_d$  denotes the Wasserstein metric on the space of probability measures on  $H$  associated to a pseudo metric  $d$ . Recall that  $d_n(x_1, x_2) \uparrow 1, \quad \forall x_1 \neq x_2$ , implies that  $W_{d_n}(\mu_1, \mu_2) \rightarrow \|\mu_1 - \mu_2\|_{TV}$ , the total variation distance between  $\mu_1$  and  $\mu_2$  (see [10], Corollary 3.5).



**Remark 6.2** The previous definition is independent of  $s$  because by stationarity of  $\bar{Y}$

$$\tilde{\mathbb{E}} \left[ \sup_{y \in B(x, \gamma)} W_{d_n}(\pi_{s, s+t_n}(x, \cdot), \pi_{s, s+t_n}(y, \cdot)) \right],$$

is independent of  $s$ .

The following result can be proved as in [10].

**Proposition 6.3** *Assume that for some  $s > 0$  there exist  $t_n \uparrow +\infty$ ,  $\delta_n \rightarrow 0$  and  $C(s, |x|)$  locally bounded with respect to  $|x|$  and such that for  $\gamma < 1$ ,*

$$\tilde{\mathbb{E}} \left( \sup_{y \in B(x, \gamma)} |DP_{s, s+t_n} \varphi(x)| \right) \leq C(s, |x|) (\|\varphi\|_\infty + \delta_n \|D\varphi\|_\infty). \quad (6.3)$$

Then  $P_{s, t}$  is ASF.

**Lemma 6.4** *Let  $\nu^1(dx, dh) = \mu_h^1(dx)\lambda(dh)$ ,  $\nu^2(dx, dh) = \mu_h^2(dx)\lambda(dh)$  be two invariant measures for  $Q_t$  such that  $\nu_1$  and  $\nu_2$  are singular. Then  $\mu_h^1$  and  $\mu_h^2$  are singular for  $\lambda$  almost all  $h \in \mathcal{H}$ .*

**Proof.** Let  $A, B \in \mathcal{B}(\mathcal{H})$  such that  $\nu^1(A) = \nu^2(B) = 1$  and  $A \cap B = \emptyset$ . For each  $h \in \mathcal{H}$  we define

$$A_h = \{x \in H : (x, h) \in A\}, \quad B_h = \{x \in H : (x, h) \in B\}.$$

Then we have  $A_h \cap B_h = \emptyset$  and  $\mu_h^1(A_h) = \mu_h^2(B_h) = 1$  because

$$\nu^1(A) = \int_{\mathcal{H}} \mu_h^1(A_h) \lambda(dh) = 1, \quad \nu^2(B) = \int_{\mathcal{H}} \mu_h^2(B_h) \lambda(dh) = 1.$$

□

Using the same proof of for Lemma 5.6 in [3], we prove the following result.

**Lemma 6.5** *Let  $d \leq 1$  be a pseudo-metric on  $\mathcal{H}$  and let  $\nu^1$  and  $\nu^2$  be two invariant measures for  $Q_t$  with the same marginal  $\lambda$ . Let us denote by  $(\mu_t^1)_{t \in \mathbb{R}}$  and  $(\mu_t^2)_{t \in \mathbb{R}}$  the system of random measures constructed in Section 3. Then we have*

$$W_d(\mu_{t+s}^1, \mu_{t+s}^2) = 1 - \mu_s^1(A) \wedge \mu_s^2(A) \left( 1 - \sup_{y, z \in A} W_d(\pi_{s, t+s}(y, \cdot) - \pi_{s, t+s}(z, \cdot)) \right). \quad (6.4)$$

**Theorem 6.6** Assume that  $(P_{s,t})_{t \geq s}$  is ASF and that there is  $x_0 \in H$  such that all invariant measure  $\nu$  of  $Q_t$

$$x_0 \in \text{supp } \mu_h, \quad \lambda \text{-a.s.} \quad (6.5)$$

Then there exists at most one invariant measure for  $(Q_t)_{t \geq 0}$ .

**Proof.** It is enough to show that two ergodic invariant measures of  $(Q_t)_{t \geq 0}$ ,  $\nu^1, \nu^2$  are identical. If not then  $\nu^1$  and  $\nu^2$  are necessarily singular and so, by Lemma 6.4,  $\mu_h^1, \mu_h^2$  are singular  $\lambda$ -a.s. Since  $Q_t$  is ASF there exists  $\gamma_0 > 0$ ,  $n \in \mathbb{N}$  such that

$$\tilde{\mathbb{E}} \left[ \sup_{y, z \in B(x_0, \gamma_0)} (W_{d_n}(\pi_{0, t_n}(y, \cdot), \pi_{0, t_n}(z, \cdot))) < \frac{1}{4} \right]$$

We deduce

$$\tilde{\mathbb{P}} \left[ \sup_{y, z \in B(x_0, \gamma_0)} (W_{d_n}(\pi_{0, t_n}(y, \cdot), \pi_{0, t_n}(z, \cdot))) < \frac{1}{2} \right] > \frac{1}{2}.$$

By Lemma 6.5 and stationarity

$$\begin{aligned} & \tilde{\mathbb{P}} [W_{d_n}(\mu_0^1, \mu_0^2) < 1] \\ &= \tilde{\mathbb{P}} [W_{d_n}(\mu_{t_n}^1, \mu_{t_n}^2) < 1] \\ &\geq \tilde{\mathbb{P}} \left[ W_{d_n}(\mu_{t_n}^1, \mu_{t_n}^2) < 1 - \frac{1}{2} \mu_0^1(B(x_0, \gamma_0)) \wedge \mu_0^2(B(x_0, \gamma_0)) \right] \\ &> \frac{1}{2}. \end{aligned}$$

Letting  $n \rightarrow \infty$  yields

$$\tilde{\mathbb{P}} [\|\mu_0^1 - \mu_0^2\|_{TV} < 1] > \frac{1}{2}.$$

Equivalently

$$\lambda\{h; \|\mu_h^1 - \mu_h^2\|_{TV} < 1\} > \frac{1}{2}$$

But this is impossible because  $\mu_h^1$  and  $\mu_h^2$  are almost surely singular.  $\square$

## 7 Application to 2D Navier–Stokes equations

We illustrate the above theory on the two-dimensional Navier–Stokes equations on a bounded domain  $\mathcal{O} \subset \mathbb{R}^2$  with Dirichlet boundary conditions and a stationary forcing term. The unknowns are the velocity  $X(t, \xi)$  and the pressure  $p(t, \xi)$  defined for  $t > 0$  and  $\xi \in \overline{\mathcal{O}}$ :

$$\begin{cases} dX(t, \xi) = [\Delta X(t, \xi) - (X(t, \xi) \cdot \nabla)X(t, \xi) + g(X(t, \xi), \overline{Y}(t, \xi))]dt \\ \quad - \nabla p(t, \xi)dt + f(t, \xi)dt + \sigma(X(t, \xi), \overline{Y}(t, \xi))dW, \\ \operatorname{div} X(t, \xi) = 0, \end{cases} \quad (7.1)$$

with Dirichlet boundary conditions

$$X(t, \xi) = 0, \quad t > 0, \quad \xi \in \partial\mathcal{O},$$

and supplemented with the initial condition

$$X(s, \xi) = x(\xi), \quad \xi \in \mathcal{O}.$$

For simplicity, we assume that  $f = 0$ . Following the usual notations we rewrite the equations as,

$$\begin{cases} dX(t) = (AX(t) + b(X(t)) + g(X(t), \overline{Y}(t)))dt \\ \quad + \sigma(X(t), \overline{Y}(t))dW(t), \quad s \leq t, \\ X(s) = x. \end{cases} \quad (7.2)$$

Here  $A$  is the Stokes operator

$$A = P\Delta, \quad D(A) = (H^2(\mathcal{O}))^2 \cap (H_0^1(\mathcal{O}))^2 \cap H,$$

where

$$H = \{x \in (L^2(\mathcal{O}))^2 : \operatorname{div} x = 0 \text{ in } \mathcal{O}\},$$

$P$  is the orthogonal projection of  $(L^2(\mathcal{O}))^2$  on  $H$  and  $b$  the operator

$$(b(y), z) = b(y, y, z), \quad y, z \in V = \{y \in (H_0^1(D))^2 \cap H : \nabla \cdot y = 0\},$$

where

$$b(y, \theta, z) = - \sum_{i,j=1}^2 \int_D y_i D_i \theta_j z_j d\xi, \quad y, \theta, z \in V.$$

Moreover  $W$  is a cylindrical Wiener process on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  in  $H$ . Finally,  $f: \mathbb{R} \rightarrow H$  is continuous and  $2\pi$ -periodic with respect to  $t$ . We denote by  $|\cdot|$  the norm in  $H$ , by  $\|\cdot\|$  the norm in  $V$  and by  $(\cdot, \cdot)$  the scalar product in  $H$ .

We assume that there exist  $K_1 > 0$  and  $h: \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\begin{aligned} |g(x, y)|_H &\leq h(|y|_K), \quad \forall x \in H, y \in K, \\ |\sigma(x, y)|_{\mathcal{L}_2(H)} &\leq h(|y|_K), \quad \forall x \in H, y \in K, \\ \tilde{\mathbb{E}}(h(|\bar{Y}(t)|_K)^2) &= K_1 < \infty. \end{aligned} \tag{7.3}$$

**Proposition 7.1** *There exists an invariant measure for  $Z$ .*

**Proof.** By Itô's formula we deduce that

$$\mathbb{E}|X(t)|^2 + \mathbb{E} \int_0^t \|X(s)\|^2 ds \leq |x|^2 + 2\mathbb{E} \int_0^t h(|\bar{Y}(s)|)^2 ds \tag{7.4}$$

and, by stationarity of  $\bar{Y}$  and (7.3),

$$\mathbb{E}_1|X(t)|^2 + \mathbb{E}_1 \int_0^t \|X(s)\|^2 ds \leq |x|^2 + 2tK_1. \tag{7.5}$$

Therefore

$$\frac{1}{t} \mathbb{E}_1 \int_0^t \|X(s)\|^2 ds \leq |x|^2 + 2K_1. \tag{7.6}$$

We deduce from (7.6), for any  $M > 0$ ,

$$\frac{1}{t} \int_0^t \mathbb{P}_1(X(0, -s; 0) \in B_{H^1}(0, M)) ds \geq 1 - \frac{2K_1}{M^2}. \tag{7.7}$$

On the other hand, for any  $\epsilon > 0$  there exists a compact set  $A_\epsilon \subset \mathcal{H}$  such that

$$\mathbb{P}(\bar{Y}_{(-\infty, 0]} \in A_\epsilon) \geq 1 - \epsilon$$

and by the stationarity of  $\bar{Y}$ ,

$$\mathbb{P}(H(\theta, -t; \bar{Y}_{(-\infty, 0]} \in A_\epsilon) \geq 1 - \epsilon, \quad \forall t \in \mathbb{R}.$$

By the Krylov–Bogoliubov theorem there exists an invariant measure  $\nu$  for  $Z$ .  $\square$

By Proposition 7.1 we deduce the existence of the family  $(\tilde{\mu}_t)_{t \in \mathbb{R}, \tilde{\omega} \in \tilde{\Omega}}$ . Concerning uniqueness, we can use the criteria derived in sections 5 and 6. For instance, assume that the noise is additive :  $\sigma(x, y) = \sqrt{C}$  and non degenerate in the following sense

$$\text{Tr } C < \infty, \quad C^{-1/2}(-A)^{-1/2} \in L(H). \quad (7.8)$$

Then using the arguments in [1], [2], [6], [9] it is not difficult to prove that the strong Feller property holds and that the transition semigroup is almost surely irreducible. Then by Theorem 5.3 and section 3, there exists a unique evolutionary system of measure.

The nondegeneracy assumptions can be weakened using the ASF property. If we consider periodic boundary condition instead of Dirichlet boundary then the argument in [10] section 4.5 can be adapted to our setting and prove that the ASF property holds if

$$\mathbb{E} (\exp (th(\overline{Y}(t))^2)) \leq c_1 \exp (c_2 t + 1)$$

for some  $c_1, c_2 \geq 0$  and if the noise acts on a sufficiently large number of modes. Then, uniqueness of evolutionary system of measures holds if one can prove that there exists  $x_0$  such that (6.5) holds.

It is probably also possible to extend the more difficult truly elliptic case treated in [10] section 4.6 but this requires much more work and is beyond the scope of this work.

## 7.1 Uniqueness by coupling

In the same spirit as in [3], we show that coupling arguments extend to our situation. We do not consider the most general case which requires lengthy proofs. Instead, we consider a non degenerate noise so that the argument is not too long. The case of degenerate noise treated for instance in [7], [11], [12], [13] could be treated by mixing the arguments in these papers and the ideas below.

We are here concerned with the equation (7.2) with a non degenerate additive noise  $\sigma(x, y) = \sqrt{C}$  satisfying (7.8).

We need a further assumption on the process  $\overline{Y}$ . We assume that it possesses a Lyapunov structure. More precisely that there exists  $\kappa_1, \kappa_2 > 0$  such that for all  $t \geq s$ :

$$\tilde{\mathbb{E}} (|\overline{Y}(t)|^2 | \mathcal{F}_s) \leq e^{-\kappa_1(t-s)} |\overline{Y}(s)|^2 + \kappa_2. \quad (7.9)$$

Also, for simplicity, we consider the case when (7.3) holds with  $h(r) = \kappa_3(1+x)$ ,  $x \geq 0$ . A different  $h$  would require a different Lyapunov structure.

By Ito's formula and Poincaré inequality:

$$\mathbb{E}(|X(t, s, x)|^2) \leq e^{-\lambda_1(t-s)}|x|^2 + \frac{2\kappa_3}{\lambda_1} + 2\kappa_3 \int_s^t e^{-\lambda_1(t-\sigma)}|\bar{Y}(\sigma)|^2 d\sigma,$$

where  $\lambda_1$  is the first eigenvalue of  $A$ . Assumption (7.9) implies:

$$2\kappa_3 \tilde{\mathbb{E}} \left( \int_s^t e^{-\lambda_1(t-\sigma)}|\bar{Y}(\sigma)|^2 d\sigma | \mathcal{F}_s \right) \leq \frac{2\kappa_3}{\kappa_1 + \lambda_1} |\bar{Y}(s)|^2 + \frac{2\kappa_2\kappa_3}{\lambda_1}.$$

By the Markov property, we obtain for  $t \geq r \geq s$

$$\begin{aligned} & \mathbb{E}_1 (|X(t, s, x)|^2 + \delta |\bar{Y}(t)|^2 | \mathcal{F}_r^1) \\ & \leq e^{-\lambda_1(t-r)}|X(r, s, x)|^2 + \frac{2\kappa_3}{\lambda_1} + \frac{2\kappa_3}{\kappa_1 + \lambda_1} |\bar{Y}(s)|^2 + \frac{2\kappa_2\kappa_3}{\lambda_1} + \delta e^{-\kappa_1(t-r)}|\bar{Y}(r)|^2 + \delta\kappa_2. \end{aligned}$$

Recall that  $\mathcal{F}_r^1 = \tilde{\mathcal{F}}_r \times \mathcal{F}_r$ . Let  $T \geq 0$  and set

$$\kappa_5 = \min\{\lambda_1, \frac{1}{2}\kappa_1\}, \quad \alpha = e^{-\kappa_5 T} - e^{-\kappa_1 T}, \quad \delta = \frac{2\kappa_3}{\alpha(\kappa_1 + \lambda_1)}, \quad \kappa_4 = \frac{2\kappa_3}{\lambda_1} + \frac{2\kappa_2\kappa_3}{\lambda_1} + \delta\kappa_2$$

then, for  $k \geq 0$ , we obtain

$$\begin{aligned} & \mathbb{E}_1 (|X((k+1)T + s, s, x)|^2 + \delta |\bar{Y}((k+1)T + s)|^2 | \mathcal{F}_{kT+s}^1) \\ & \leq e^{-\kappa_5 T} (|X(kT + s, s, x)|^2 + \delta |\bar{Y}(kT + s)|^2) + \kappa_4. \end{aligned}$$

Thanks to this inequality, Lemma 7.2 below can now be proved as [3, Lemma 6.1].

**Lemma 7.2** *Fix  $T > 0, M \in \mathbb{N}$  and set*

$$\tau = \inf\{kT + s : k \in \mathbb{N}, |X(kT + s, s, x)|^2 + \delta |\bar{Y}(kT + s)|^2 \leq M\kappa_4\}.$$

*Then there exists  $C(T), M(T) \in \mathbb{N}$  such that if  $M \geq M(T)$ , we have*

$$\mathbb{P}_1(\tau \geq kT + s) \leq C(T)e^{-\frac{1}{2}k\kappa_5 T}(1 + |x|^2) \quad (7.10)$$

*and there exists  $C(\alpha, T)$  such that for  $\alpha < \frac{1}{2}\kappa_5$*

$$\mathbb{E}_1(e^{\alpha\tau}) \leq C(\alpha, T)e^{\alpha s}(1 + |x|^2). \quad (7.11)$$

The following Lemma is also proved as in [3, Lemma 6.2].

**Lemma 7.3** For any  $x, y \in H$ ,  $t \geq 0$ ,  $s \in \mathbb{R}$ ,  $K > 0$ ,  $g \in C_b^1(H)$  such that  $\|g\|_0 \leq 1$  there exist  $c_1, c_2 > 0$  such that

$$\begin{aligned} |P_{s,t+s}g(x) - P_{s,t+s}g(y)| &\leq \frac{1}{K} e^{-\lambda_1 t}(|x|^2 + |y|^2) + \frac{\text{Tr } C}{K} \\ &+ \frac{2}{\lambda_1 K} \int_s^{s+t} e^{-\lambda_1(t+s-\sigma)} |\bar{Y}(\sigma)|^2 d\sigma + \frac{c_2}{t} e^{2c_1 K} |x - y|. \end{aligned} \quad (7.12)$$

**Corollary 7.4** For any  $t > 0$  there exists  $\delta > 0$  such that for any  $s \in \mathbb{R}$ ,

$$\tilde{\mathbb{E}}(\|P_{s,t+s}^* \delta_x - P_{s,t+s}^* \delta_y\|_{TV}) \leq \frac{1}{2}, \quad (7.13)$$

provided  $|x|, |y| \leq \delta$ .

**Proof.** Take the supremum in  $\|g\|_0 \leq 1$  and then the expectation  $\tilde{\mathbb{E}}$  in (7.12). The result follows taking first  $K$  large and then choosing  $\delta$  small.  $\square$

**Lemma 7.5** For any  $T > 0$ ,  $\rho_1 > 0$ ,  $\delta_1 > 0$  there exists  $K_0(\rho_1, \delta_1) \in \mathbb{N}$  and  $\alpha(\rho_1, \delta_1) > 0$  such that for any  $|x| \leq \rho_1$

$$\mathbb{P}_1(|X(K_0(\rho_1, \delta_1)T + s, s, x)| \leq \delta_1) \geq \alpha(\rho_1, \delta_1). \quad (7.14)$$

**Proof.** The proof is similar to that of [3, Lemma 7.4]. The left hand side in (7.14) is independent on  $s$ . It suffices to consider  $s = 0$ . We consider the deterministic problem

$$\begin{cases} \frac{d\tilde{X}}{dt} = A\tilde{X} + b(\tilde{X}), \\ \tilde{X}(0) = x. \end{cases} \quad (7.15)$$

It is easy to see that for any  $\rho_1 > 0, \delta_1 > 0$  there exists  $K_0(\rho_1, \delta_1) \in \mathbb{N}$  such that

$$|\tilde{X}(t, 0, x)|^2 \leq \frac{1}{4} \delta_1^2 \quad \text{for } t \geq K_0(\rho_1, \delta_1)T, \quad (7.16)$$

provided  $|x| \leq \rho_1$ . Let

$$R(t) = \int_0^t e^{(t-\sigma)A} \bar{Y}(\sigma) d\sigma + \int_0^t e^{(t-\sigma)A} \sqrt{C} dW(\sigma). \quad (7.17)$$

Since the noise is non degenerate we can prove that

$$\mathbb{P} \left( \sup_{t \in K_0(\rho_1, K_1)T} |R(t)| \leq \frac{\eta}{2} \right) \geq \gamma_1(\eta, \tilde{\omega}) > 0, \tilde{\mathbb{P}}\text{-a.s.}$$

Taking the expectation  $\tilde{\mathbb{E}}$  we find

$$\mathbb{P}_1 \left( \sup_{t \in K_0(\rho_1, K_1)T} |R(t)| \leq \frac{\eta}{2} \right) \geq \tilde{\mathbb{E}}(\gamma_1(\eta, \tilde{\omega})) > 0.$$

Now we can conclude the proof as in [3] with minor modifications.  $\square$

We are now ready to construct a coupling by proceeding as in the proof of [3, Proposition 6.5].

**Proposition 7.6** *There exist  $c > 0$  and  $\gamma > 0$  such that for any  $s \in \mathbb{R}$ ,  $k \in \mathbb{N}$  and any  $\varphi \in C_b(H)$*

$$\tilde{\mathbb{E}}(|P_{s,t}\varphi(x) - P_{s,t}\varphi(y)|) \leq c\|\varphi\|_0 e^{-\tilde{\gamma}(t-s)}(1 + |x|^2 + |y|^2). \quad (7.18)$$

**Proof.** Fix  $T > 0$ . We take  $\delta > 0$  as in Corollary 7.4. For  $x, y \in B_\delta$  we fix  $\tilde{\omega} \in \tilde{\Omega}$  and we choose a maximal coupling,  $(Z_1^s(x, y), Z_2^s(x, y))$ . Then

$$\mathbb{P}(Z_1^s(x, y) \neq Z_2^s(x, y)) = \|P_{s,t}^*\delta_x - P_{s,t}^*\delta_y\|_{TV}.$$

By Corollary 7.4 it follows that

$$\mathbb{P}_1(Z_1^s(x, y) \neq Z_2^s(x, y)) \leq \frac{1}{2}.$$

We notice that  $(Z_1^s(x, y), Z_2^s(x, y))$  is still a coupling of  $(X(T+s, s, x), X(T+s, s, y))$  considered as random variables on  $\omega_1 = (\omega, \tilde{\omega})$ . Now we continue as in [3, Proposition 6.5], proving finally that

$$\mathbb{P}_1(X_1^{s,h} \neq X_2^{s,h}) \leq ce^{-\tilde{\gamma}k}(1 + |x|^2 + |y|^2),$$

for some  $\tilde{\gamma}$ . It follows:

$$\tilde{\mathbb{E}}(|P_{s,kT+s}\varphi(x) - P_{s,kT+s}\varphi(y)|) \leq c\|\varphi\|_0 e^{-\tilde{\gamma}k}(1 + |x|^2 + |y|^2).$$

Writing

$$P_{s,t}\varphi(x) - P_{s,t}\varphi(y) = P_{s,kT+s}P_{kT+s,t}\varphi(x) - P_{s,kT+s}P_{kT+s,t}\varphi(y)$$

and  $\|P_{kT+s,t}\varphi\|_0 \leq \|\varphi\|_0$ , (7.18) follows with a different constant  $c$ .  $\square$

By Borel-Cantelli Lemma, we deduce  $\tilde{\mathbb{P}}$  almost sure exponential convergence.



**Corollary 7.7** For  $\tilde{\mathbb{P}}$  almost every  $\tilde{\omega}$ , there exists  $T_0(\tilde{\omega})$  such that for any  $t \geq T_0$

$$|P_{s,t+s}\varphi(x) - P_{s,t+s}\varphi(y)| \leq c\|\varphi\|_0 e^{-\tilde{\gamma}t/2}(1 + |x|^2 + |y|^2).$$

**Theorem 7.8** Let

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t, s, x))], \quad \varphi \in B_b(H),$$

where  $X(t, s, x)$  is the solution of the Navier-Stokes (NS). Then, if (7.9) holds, we have

$$\tilde{\mathbb{E}} \left| P_{s,t}\varphi(x) - \int_H \varphi(y)\mu_t(dy) \right| \leq c(s)\|\varphi\|_0 e^{-\tilde{\gamma}(t-s)}(1 + |x|^2) \quad (7.19)$$

and

$$\left| P_{s,t}\varphi(x) - \int_H \varphi(x)\mu_t(dx) \right| \leq K(\tilde{\omega}, s, x)\|\varphi\|_0 e^{-\tilde{\gamma}t/2} \quad \tilde{\mathbb{P}}\text{-a.s.} \quad (7.20)$$

**Proof.** Taking into account (2.7) we have

$$P_{s,t}\varphi(x) - \int_H \varphi(y)\mu_t(dy) = \int_H (P_{s,t}\varphi(x) - P_{s,t}\varphi(y))\mu_s(dy).$$

Therefore by (7.18) it follows that

$$\tilde{\mathbb{E}} \left| P_{s,t}\varphi(x) - \int_H \varphi(y)\mu_t(dy) \right| \leq c\|\varphi\|_0 e^{-\tilde{\gamma}(t-s)} \int_H (1 + |x|^2 + |y|^2)\mu_s(dy)$$

so that (7.19) follows. Again, (7.20) is obtained thanks to Borel Cantelli Lemma.  $\square$

Identity (7.20) can be interpreted by saying that as  $t \rightarrow +\infty$ , the observable  $P_{s,t}^{\tilde{\omega}}\varphi$  approaches exponentially fast a random limit curve

$$t \rightarrow \int_H \varphi(x)\mu_t(dx),$$

which forgets about  $s$ .

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