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Almost sure asymptotics for the maximum local time in Brownian environment

Roland DIEL *

Abstract

We study the asymptotic behaviour of the maximum local time $\mathbf{L}^*(t)$ of the Brox's process, the diffusion in Brownian environment. Shi [15] proved that the maximum speed of $\mathbf{L}^*(t)$ is surprisingly, at least $t \log(\log(\log t))$ whereas in the discrete case it is t . We show here that $t \log(\log(\log t))$ is the proper rate and we prove that for the minimum speed the rate is the same as in the discrete case (Dembo, Gantert, Peres and Shi [5]) namely $t(\log(\log(\log t)))^{-1}$.

1 Introduction

Let $(W(x), x \in \mathbf{R})$ be a two-sided one-dimensional Brownian motion on \mathbf{R} with $W(0) = 0$. We call *diffusion process in the environment* W a process $(\mathbf{X}(t), t \in \mathbf{R}^+)$ whose infinitesimal generator given W is

$$\frac{1}{2}e^{W(x)} \frac{d}{dx} \left(e^{-W(x)} \frac{d}{dx} \right).$$

Notice that if we take W differentiable, $(\mathbf{X}(t), t \in \mathbf{R}^+)$ would be the solution of the following stochastic differential equation

$$\begin{cases} d\mathbf{X}(t) = d\beta(t) - \frac{1}{2}W'(\mathbf{X}(t))dt, \\ \mathbf{X}(0) = 0 \end{cases}$$

in which β is a standard one-dimensional Brownian motion independent of W . Of course as we choose for W a Brownian motion, the previous equation does not have any rigorous sense but it explains the denomination *environment* for W .

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This process was first introduced by S. Schumacher [14] and T. Brox [3]. The diffusion \mathbf{X} is recurrent and Brox shows in [3] that it is sub-diffusive with asymptotic behaviour in $(\log t)^2$. Moreover for a given instant t , \mathbf{X} has the property to be localized in the neighborhood of a random point $m_{\log t}$ only depending on t and W . The limit law of $m_{\log t}/(\log t)^2$ and therefore of $\mathbf{X}_t/(\log t)^2$ is independently made explicit by H. Kesten [11] and A. O. Golosov [8].

H. Kesten and A. O. Golosov's aim was to determine the limit law of the discrete time and space analogue of Brox's model introduced by F. Solomon [18] and then studied by Ya. G. Sinai [17]. This random walk in random environment, called Sinai's walk, $(S_n, n \in \mathbf{N})$ actually has the same limit distribution as Brox's.

Turning back to Brox's diffusion, one can note that H. Tanaka [21, 20] obtained a deeper localization and later Y. Hu and Z. Shi [9] got the almost sure rates of convergence of the \limsup and \liminf of X . It appears that these rates are the same as the ones for Sinai's walk. The question of an invariance principle between these two processes remains open (see Z. Shi [16] for a survey).

In the present paper, we are interested in the asymptotics of the maximum of the local time of \mathbf{X} . Indeed, to the diffusion \mathbf{X} corresponds a local time process $(\mathbf{L}(t, x), t \geq 0, x \in \mathbf{R})$ defined by the occupation time formula: \mathbf{L} is the unique a.s. jointly continuous process such that for any bounded Borel function f and for any $t \geq 0$,

$$\int_0^t f(\mathbf{X}_s) ds = \int_{\mathbf{R}} f(x) \mathbf{L}(t, x) dx.$$

We shall see later that such a process exists. The first results on the behavior of \mathbf{L} can be found in [15] and [10]. In particular it is proven in [10] that for any $x \in \mathbf{R}$,

$$\frac{\log(\mathbf{L}(t, x))}{\log t} \xrightarrow{\mathcal{L}} U \wedge \hat{U}, \quad t \rightarrow +\infty$$

where U and \hat{U} are independent variables uniformly distributed in $(0, 1)$ and $\xrightarrow{\mathcal{L}}$ is the convergence in law. Note that in the same paper Hu and Shi also prove that Sinai's walk local time has the same behaviour.

We now consider the maximum of the local time of \mathbf{X} ,

$$\mathbf{L}^*(t) := \sup_{x \in \mathbf{R}} \mathbf{L}(t, x).$$

Shi was the first one to be interested in this process; in [15] he gives a lower bound on the \limsup behaviour,

$$\limsup_{t \rightarrow \infty} \frac{\mathbf{L}^*(t)}{t \log_3(t)} \geq \frac{1}{32}$$

where for any $i \in \mathbf{N}$, $\log_{i+1} = \log \circ \log_i$ and $\log_1 = \log$. In the same paper, he computed the similar rate in the discrete case: there is constant $c \in (0, \infty)$ such that

$$\limsup_{n \rightarrow \infty} \frac{L_S^*(n)}{n} = c.$$

Thereby he highlighted a different behaviour for the discrete model and for the continuous one. The limit law of $\mathbf{L}^*(t)$ was then determined in [1]:

$$\frac{\mathbf{L}^*(t)}{t} \xrightarrow{\mathcal{L}} \frac{1}{\int_{-\infty}^{\infty} e^{-\tilde{W}(x)} dx}, \quad t \rightarrow +\infty$$

where \tilde{W} has the same law as the environment conditioned to stay positive. But unlike the discrete case (see [7]), this result does not allow to give an upper bound on the almost sure behaviour.

Here we prove that the quantity $\log_3(t)$ is the correct renormalization for the lim sup and an analog result for the lim inf:

Theorem 1.1 *Almost surely,*

$$\begin{aligned} \frac{1}{32} &\leq \limsup_{t \rightarrow \infty} \frac{\mathbf{L}^*(t)}{t \log_3(t)} \leq 2e \quad \text{and} \\ \frac{j_0^2}{64} &\leq \liminf_{t \rightarrow \infty} \frac{\log_3(t) \mathbf{L}^*(t)}{t} \leq e\pi^2. \end{aligned}$$

where j_0 is the smallest strictly positive root of Bessel function J_0 .

We can compare these results to the ones of the discrete case, see [15] (and [7] for the value of the constant) for the lim sup and [5] for the lim inf: there exists two constants $c, c' \in (0, \infty)$ such that almost surely,

$$\limsup_{n \rightarrow \infty} \frac{L_S^*(n)}{n} = c \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\log_3(n) L_S^*(n)}{n} = c'.$$

As Shi noted it, the lim sup in the continuous case and in the discrete case have a different normalization, but we see with Theorem 1.1 that the normalization is the same for the lim inf. There is the following heuristic interpretation: in the discrete case like in the continuous one, $\mathbf{L}^*(t)/t$ behaves approximately in the best case like the integral of the exponential of the "steepest" environment and in the worst case like the integral of the exponential of the "flattest" one. However if in discrete case there is a steepest environment, in the continuous case it is possible to be as steep as we want, and in the two cases, the environment can be as flat as we want. This explains the difference for the lim sup. To compute the exact rate, we need to study carefully both the behaviour of the "steepest" and "flattest" environment and the frequency of the change of the favourite point.

We now present a useful representation of \mathbf{X} we use throughout the article to study the process : the martingale representation theorem says

that, given the environment W , \mathbf{X} is a Brownian motion rescaled in time and space. Precisely, there is a Brownian motion B started at 0, independent of W such that if we define the scale function

$$\forall x \in \mathbb{R}, S_W(x) := \int_0^x e^{W(y)} dy \quad (1)$$

and the random time change

$$\forall t \geq 0, T_{W,B}(t) := \int_0^t e^{-2W(S_W^{-1}(B(s)))} ds, \quad (2)$$

then

$$\mathbf{X} = S_W^{-1} \circ B \circ T_{W,B}^{-1} \quad (3)$$

To simplify notations, we write when there is no possible confusion S and T for respectively S_W and $T_{W,B}$. We also denote by P the total probability and expectation and by P^W the probability and expectation given the environment W . Using (3), we easily obtain that the local time has to be defined by, for any $x \in \mathbb{R}$ and $t \geq 0$,

$$\mathbf{L}(t, x) = e^{-W(x)} L(T^{-1}(t), S(x)) \quad (4)$$

where L is the local time process of the Brownian motion B .

The rest of the paper is organized as follows: in Section 2, we present some technical estimates used in the article, in Section 3, we show that environment W has “good” properties with a large enough probability, Section 4 is centered on the study of the local time of the process \mathbf{X} , then in Section 5, we look at the behaviour of the maximum of the local time process \mathbf{L}^* and finally in Section 6, Theorem 1.1 is proved.

2 Three useful theorems and some technical estimates

Here are presented some results used throughout the article. We begin this section with some notations :

$$\begin{aligned} \forall x \geq 0, \quad \boldsymbol{\tau}(x) &:= \inf\{t \geq 0 / \mathbf{X}(t) = x\}, \\ \forall x \geq 0, \quad \tau(x) &:= \inf\{t \geq 0 / B(t) = x\}, \\ \forall x \geq 0, \forall r \geq 0, \quad \boldsymbol{\sigma}(r, x) &:= \inf\{t \geq 0 / \mathbf{L}(t, x) \geq r\} \text{ and} \\ \forall x \geq 0, \forall r \geq 0, \quad \sigma(r, x) &:= \inf\{t \geq 0 / L(t, x) \geq r\}. \end{aligned}$$

More generally, the quantities related to \mathbf{X} are written in bold font and the ones related to B in normal font. Furthermore, in the rest of the paper, letter K stands for a universal constant whose value can change from one line to another and for any process M , we denote by $\tau_M(x)$ the hitting time of height x by M .

As we saw in the previous section, the study of the process \mathbf{X} is reduced by scaling in time and space to the study of a Brownian motion. We therefore use in our proofs the following two theorems that describe the law of the local time of a Brownian motion stopped at some properly chosen random times:

Theorem 2.1 (Ray-Knight) *Let a be a positive real number.*

The process $\{L(\tau(a), a - y), y \in [0, a]\}$ is a square of a 2-dimensional Bessel process started at 0. And conditionally on $L(\tau(a), 0)$, $\{L(\tau(a), -y), y \geq 0\}$ is a square of a 0-dimensional Bessel process started at $L(\tau(a), 0)$ independent of $\{L(\tau(a), y), y \geq 0\}$.

Theorem 2.2 (Ray-Knight) *Let r be a positive real number.*

The processes $\{L(\sigma(r, 0), y), y \in \mathbf{R}^+\}$ and $\{L(\sigma(r, 0), -y), y \in \mathbf{R}^+\}$ are two independent squares of 0-dimensional Bessel processes started at r .

In the rest of the article, we denote by Z a square of a 0-dimensional Bessel process started at 1 and by Q a square of a 2-dimensional Bessel process started at 0. To use the previous theorems, we must estimate the behaviour of these two processes. This is done in the following lemma.

Lemma 2.3 *For all $v, \delta, M > 0$, we have*

1. $P\left(\sup_{0 \leq t \leq v} |Z(t) - 1| \geq \delta\right) \leq 4 \frac{\sqrt{(1 + \delta)v}}{\delta} \exp\left(-\frac{\delta^2}{8(1 + \delta)v}\right),$
2. $P\left(\sup_{t \geq 0} Z(t) \geq M\right) = \frac{1}{M},$
3. $P\left(\sup_{0 \leq t \leq v} Q(t) \geq M\right) \leq 4e^{-\frac{M}{2v}},$
4. *There is a constant $K > 0$ such that for any $0 < a < b$,*

$$P\left(\sup_{a \leq t \leq b} \frac{1}{t} Q(t) \geq M\right) \leq K e^{-\frac{M}{2 \log(8b/a)}}.$$

Proof:

1 and 2. These are proved in [19] Lemma 3.1.

3. Denote by B and \tilde{B} two independent Brownian motions. The processes Q and $B^2 + \tilde{B}^2$ have the same law, so

$$\begin{aligned} P\left(\sup_{0 \leq t \leq v} Q(t) \geq M\right) &\leq P\left(\left(\sup_{0 \leq t \leq v} |B|(t)\right)^2 + \left(\sup_{0 \leq t \leq v} |\tilde{B}|(t)\right)^2 \geq M\right) \\ &\leq 4P\left(\left(\sup_{0 \leq t \leq v} B(t)\right)^2 + \left(\sup_{0 \leq t \leq v} \tilde{B}(t)\right)^2 \geq M\right) \end{aligned}$$

where the second inequality comes from reflection principle. And thus,

$$P\left(\sup_{0 \leq t \leq v} Q(t) \geq M\right) \leq 4P(Q(v) \geq M) = 4e^{-\frac{M}{2v}}$$

as $Q(v)$ has exponential distribution of parameter $2v$.

4. Thanks to the scaling property of Q ,

$$\begin{aligned} P\left(\sup_{a \leq t \leq b} \frac{1}{t} Q(t) \geq M\right) &= P\left(\sup_{a/b \leq t \leq 1} \frac{1}{t} Q(t) \geq M\right) \\ &\leq P\left(\sup_{a/b \leq t \leq 1} \frac{Q(t)}{t \log(8/t)} \geq \frac{M}{\log(8b/a)}\right) \\ &\leq P\left(\sup_{0 \leq t \leq 1} \sqrt{\frac{Q(t)}{t \log(8/t)}} \geq \sqrt{\frac{M}{\log(8b/a)}}\right). \end{aligned}$$

We then conclude with Lemma 6.1 in [9] which says that there exist a constant $K > 0$ such that for all $x > 0$,

$$P\left(\sup_{0 \leq t \leq 1} \sqrt{\frac{Q(t)}{t \log(8/t)}} \geq x\right) \leq Ke^{-\frac{x^2}{2}}.$$

□

We also need to study the behaviour of the environment W . Let us start with a notation for the minimum of W on an interval

$$\underline{W}(x, y) := \begin{cases} \min_{z \in [x, y]} W(z) & \text{if } x \leq y, \\ +\infty & \text{if not} \end{cases}$$

and for the maximum

$$\overline{W}(x, y) := \begin{cases} \max_{z \in [x, y]} W(z) & \text{if } x \leq y, \\ -\infty & \text{if not.} \end{cases}$$

We can now define

$$H_v := \inf\{x \geq 0 ; W(x) - \underline{W}(0, x) \geq v\}$$

and

$$m_v := \inf\{x \geq 0 ; W(x) = \underline{W}(0, H_v)\}.$$

Brox showed in [3] that at time e^v , with high probability, the process \mathbf{X} has spent much of its time in the neighborhood of m_v or of the symmetric point on the negative real halfline. For our study, we need to know the law of the environment in the neighborhood of these points ; it is given by the following theorem due to Tanaka:

Theorem 2.4 (Tanaka [21]) *Let R be a Bessel process of dimension 3 started at 0 and*

$$\begin{aligned}\tau_R(v) &:= \inf\{x \leq 0/R(x) \geq v\}, \\ \zeta_R(v) &:= \inf\{x \leq 0/R(x) - \inf_{y \geq x} R(y) \geq v\} \text{ and} \\ \rho_R(v) &:= \sup\{x \leq \zeta_R(v)/R(x) - \inf_{y \geq x} R(y) = 0\}.\end{aligned}$$

Under P , the processes $(W(-x + m_v) - W(m_v))_{x \in [0, m_v]}$ and $(W(x + m_v) - W(m_v))_{x \in [0, H_v]}$ are independent and the following equality in law is true :

$$(W(-x + m_v) - W(m_v))_{x \in [0, m_v]} \stackrel{\mathcal{L}}{=} (R(x))_{x \in [0, \rho_R(v)]}$$

and

$$(W(x + m_v) - W(m_v))_{x \in [0, H_v]} \stackrel{\mathcal{L}}{=} (R(x))_{x \in [0, \tau_R(v)]}.$$

Therefore, to use this theorem it is necessary to have informations on the behaviour of Bessel process of dimension 3. It is done in the following lemma:

Lemma 2.5 *Let R be a 3-dimensional Bessel process started in 0. There is a positive real number K such that for every $a, t > 0$,*

$$\frac{a}{\sqrt{t}} e^{-a^2/2t} \leq P \left(\sup_{[0, t]} R > a \right) \leq K \left(\frac{a}{\sqrt{t}} + \frac{\sqrt{t}}{a} \right) e^{-a^2/2t}.$$

Proof: According to the reflection principle for Brownian motion, one can find a $K > 0$ such that

$$P \left(\sup_{[0, t]} R > a \right) \leq KP(R(t) > a).$$

Moreover, $P \left(\sup_{[0, t]} R > a \right) \geq P(R(t) > a)$. So the estimates of the lemma are a consequence of usual estimates for 3-dimensional Bessel process. \square

3 Estimates on the environment

We start with the definition of the environment reversed in time

$$\left(\widehat{W}(x); \quad x \in \mathbf{R} \right) := (W(-x); \quad x \in \mathbf{R})$$

which is still a Brownian motion. This allows us to restrict the study to W on \mathbf{R}^+ and to get similar results on \mathbf{R}^- by symmetry.

We study the environment on $[0, H_v]$ and particularly in the neighborhood of m_v as it is the place where the diffusion spends most of its time.

Unfortunately, the probability that, at time e^v , the process has reached the bottom of the valley m_v and, secondly it has not left the valley, tends to 1 but is not growing fast enough to derive almost sure results. So we rather study valleys of height $v - c_1 \log v$, where c_1 is a positive real number whose value will be determined later, so that, with high probability, at time e^v , the process has reached the bottom of the valley. We are also interested in the valley of height $v + c_3 \log v$ so that the process has not left this valley at time e^v . We therefore fix three constants $c_1, c_2, c_3 > 0$ and define recursively for any $v > 1$,

$$\begin{aligned} b_{v,0}^- &:= 0 \\ \forall i \geq 0, b_{v,i+1}^- &:= \inf\{x \geq b_{v,i}^-; W(x) - \underline{W}(b_{v,i}^-, x) \geq v - c_1 \log v\}, \\ m_{v,i+1}^- &:= \inf\{x \geq b_{v,i}^-; W(x) = \underline{W}(b_{v,i}^-, b_{v,i+1}^-)\} \text{ and} \end{aligned}$$

We then have to specify for any $i \in \mathbf{N}^*$,

$$\begin{aligned} a_{v,i}^- &:= \sup\{x \leq m_{v,i}^-; W(x) - W(m_{v,i}^-) \geq v - c_2 \log v\} \vee b_{v,i-1}^-, \\ c_v^+ &:= \inf\{x \geq 0; W(x) - \underline{W}(0, x) \geq v + c_3 \log v\}, \\ m_v^+ &:= \inf\{x \geq 0; W(x) = \underline{W}(0, c_v^+)\}, \\ b_v^+ &:= \inf\{x \geq m_v^+; W(x) - W(m_v^+) \geq v - c_1 \log v\} \text{ and} \\ a_v^+ &:= \sup\{x \leq m_v^+; W(x) - W(m_v^+) \geq v - c_2 \log v\} \vee 0. \end{aligned}$$

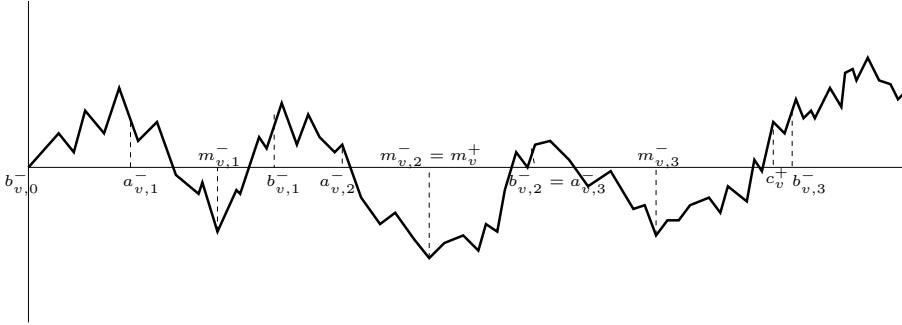


Figure 1: A sample path of W

There is a $i \in \mathbf{N}^*$ such that $m_v^+ = m_{v,i}^-$. We want to prove that, with a sufficiently high probability, $m_v^+ \in \{m_{v,1}^-; m_{v,2}^-\}$ and moreover that, after m_v^+ , there is no point of depth close to $W(m_v^+)$ in the valley $[0, c_v^+]$. This can be expressed formally as follows:

$$\Gamma_v^1 := \left\{ c_v^+ \leq b_{v,3}^-; \underline{W}(b_v^+, c_v^+) - W(m_v^+) \geq (c_1 + c_3) \log v \right\}.$$

We also need many more technical conditions to ensure that the environment does not stray too far from its average behaviour:

$$\begin{aligned} \Gamma_v^2 := & \left\{ b_{v,2}^- \leq v^6 ; W(m_{v,1}^-) \geq -v^2 ; W(m_{v,2}^-) - W(b_{v,1}^-) \geq -v^2 \right\} \\ & \cap \left\{ m_{v,1}^- - a_{v,1}^- \geq \frac{1}{v^2} ; m_{v,2}^- - a_{v,2}^- \geq \frac{1}{v^2} \right\} \\ & \cap \left\{ c_v^+ - m_v^+ \geq v ; W(c_v^+) - \underline{W}((c_v^+ - \log v) \vee m_v^+, c_v^+) \leq 2 \log v \right\} \\ & \cap \left\{ \overline{W}((m_{v,1}^- - \log v) \vee a_{v,1}^-, m_{v,1}^-) - W(m_{v,1}^-) \leq 2 \log v \right\} \\ & \cap \left\{ \overline{W}((m_{v,2}^- - \log v) \vee a_{v,2}^-, m_{v,2}^-) - W(m_{v,2}^-) \leq 2 \log v \right\}. \end{aligned}$$

We would also wish sometimes that $m_v^+ = m_{v,1}^-$ (which of course is obtained with a probability lower than the previous one). For that we use the event:

$$\Gamma_v^3 := \left\{ c_v^+ \leq b_{v,2}^- ; \underline{W}(b_v^+, c_v^+) - W(m_v^+) \geq (c_1 + c_3) \log v \right\}.$$

We define $\widehat{\Gamma}_v^1$, $\widehat{\Gamma}_v^2$ and $\widehat{\Gamma}_v^3$ similarly from \widehat{W} .

We therefore work on

$$\Gamma_v = \Gamma_v^1 \cap \Gamma_v^2 \cap \widehat{\Gamma}_v^1 \cap \widehat{\Gamma}_v^2 \quad (5)$$

or

$$\Gamma'_v = \Gamma_v^3 \cap \Gamma_v^2 \cap \widehat{\Gamma}_v^3 \cap \widehat{\Gamma}_v^2. \quad (6)$$

The first step as stated before is to show that these events occur with a high enough probability. We denote for every event A , $\overline{A} := \Omega \setminus A$.

Proposition 3.1 *There exists a constant $K > 0$ such that for any $v \geq 2$,*

$$P(\overline{\Gamma}_v) \leq K \left(\frac{\log v}{v - c_1 \log v} \right)^2 \quad \text{and} \quad P(\overline{\Gamma}'_v) \leq \frac{K \log v}{v - c_1 \log v}.$$

Proof: We start with an upper bound for $P(\overline{\Gamma}_v^1)$. Define

$$\begin{aligned} W_1 & := (W(b_{v,1}^- + x) - W(b_{v,1}^-))_{x \in [0, b_{v,2}^- - b_{v,1}^-]} \quad \text{and} \\ W_2 & := (W(b_{v,2}^- + x) - W(b_{v,2}^-))_{x \in [0, b_{v,3}^- - b_{v,2}^-]}. \end{aligned}$$

The event $\{c_v^+ > b_{v,3}^-\}$ can then be rewritten

$$\left\{ \sup_{[0, b_{v,2}^- - b_{v,1}^-]} W_1 \leq (c_1 + c_3) \log v ; \sup_{[0, b_{v,3}^- - b_{v,2}^-]} W_2 \leq (c_1 + c_3) \log v \right\}.$$

The processes W_1 et W_2 are independent and have the same law as $(W(x), x \in [0, b_{v,1}^-])$, then

$$\begin{aligned} P(c_v^+ > b_{v,3}^-) &= \left(P \left(\sup_{[0, b_{v,1}^-]} W \leq (c_1 + c_3) \log v \right) \right)^2 \\ &= (P(W \text{ hits } -v + (2c_1 + c_3) \log v \text{ before } (c_1 + c_3) \log v))^2 \\ &= \left(\frac{(c_1 + c_3) \log v}{v - c_1 \log v} \right)^2. \end{aligned}$$

For the second part of $\bar{\Gamma}_v^1$, according to Theorem 2.4, if we denote by R a Bessel process of dimension 3 strated at $v - c_1 \log v$,

$$\begin{aligned} P(\underline{W}(b_v^+, c_v^+) - W(m_v^+) < (c_1 + c_3) \log v) \\ &= P(\underline{R}(0, \tau_R(v + c_3 \log v)) < (c_1 + c_3) \log v) \\ &= \left(\frac{(c_1 + c_3) \log v}{v - c_1 \log v} \right)^2. \end{aligned}$$

(See for example Property 2.2.2 of part II, chap 5 in [2].) We obtain in the same way

$$P(\bar{\Gamma}_v^3) \leq \frac{(c_1 + c_3) \log v}{v - c_1 \log v}.$$

We continue with an upper bound for $P(\bar{\Gamma}_v^2)$. As $W - \underline{W}$ has the same law as $|W|$,

$$\begin{aligned} P(b_{v,2}^- > v^6) &\leq P(b_{v,2}^- - b_{v,1}^- > \frac{v^6}{2}) + P(b_{v,1}^- > \frac{v^6}{2}) \\ &= 2P(\tau_{|W|}(v - c_1 \log v) > \frac{v^6}{2}) \\ &\leq 2P(\tau_W(v) > \frac{v^6}{2}) \leq \frac{K}{v^2}. \end{aligned}$$

Moreover, $-W(m_{v,1}^-)$ and $W(b_{v,1}^-) - W(m_{v,2}^-)$ are exponentially distributed with mean $v - c_1 \log v$ (see for example the first lemma of [12]) and are independent. Thus

$$\begin{aligned} P(W(b_{v,1}^-) - W(m_{v,2}^-) < -v^2) &= P(W(m_{v,1}^-) < -v^2) \\ &\leq P(W(m_{v,1}^-) < -(v - c_1 \log v)^2) \\ &= v^{c_1} e^{-v}. \end{aligned}$$

According to Theorem 2.4, still denoting by R a Bessel process of dimension 3 but now started at 0,

$$\begin{aligned} P\left(m_{v,i}^- - a_{v,i}^- < \frac{1}{v^2}\right) &\leq P\left(m_{v,i}^- - a_{v,i}^- < \frac{1}{(v - c_1 \log v)^2}\right) \\ &\leq P\left(\tau_R(v - c_1 \log v) < \frac{1}{(v - c_1 \log v)^2}\right) \\ &\quad + P\left(m_{v,i}^- < \frac{1}{(v - c_1 \log v)^2}\right). \end{aligned}$$

Yet, according to the scaling property of Brownian motion and a lemma proved by Cheliotis in [4] (claim at the end of the proof of Lemma 13), there is a constant $K > 0$, such that

$$P\left(m_{v,i}^- < \frac{1}{(v - c_1 \log v)^2}\right) = P\left(m_1 < \frac{1}{(v - c_1 \log v)^4}\right) \leq \frac{K}{(v - c_1 \log v)^2}.$$

Moreover, according to Lemma 2.5,

$$P\left(\tau_R(v - c_1 \log v) < \frac{1}{(v - c_1 \log v)^2}\right) \leq K(v - c_1 \log v)^2 e^{-(v - c_1 \log v)^4/2}.$$

We also obtain the following upper bound:

$$\begin{aligned} P(c_v^+ - m_v^+ < v) &= P(\tau_R(v + c_3 \log v) < v) \\ &\leq P(\tau_R(v) < v) \\ &\leq K\sqrt{v}e^{-v/2}. \end{aligned}$$

It remains to control $P(W(c_v^+) - \underline{W}((c_v^+ - \log v) \vee m_v^+, c_v^+) > 2 \log v)$. Let $\beta = v + c_3 \log v$, using one more time Theorem 2.4, we see that

$$\begin{aligned} &W(c_v^+) - \underline{W}((c_v^+ - \log v) \vee m_v^+, c_v^+) \\ &= W(c_v^+) - W(m_v^+) - \left(\min_{t \in [0, \log v \wedge (c_v^+ - m_v^+)]} W((c_v^+ - t) - W(m_v^+)) \right) \end{aligned}$$

has the same law as

$$\beta - \min_{t \in [0, \log v \wedge \tau_R(\beta)]} R(\tau_R(\beta) - t) = \max_{t \in [0, \log v \wedge \tau_R(\beta)]} (\beta - R(\tau_R(\beta) - t)).$$

And according to Proposition 4.8, Chapter VII of [13], the processes

$$(\beta - R(\tau_R(\beta) - t) ; t \in [0, \tau_R(\beta)]) \text{ and } (R(t) ; t \in [0, \tau_R(\beta)])$$

have the same law. Therefore

$$\begin{aligned}
P\left(W(c_v^+) - \underline{W}((c_v^+ - \log v) \vee m_v^+, c_v^+) > 2 \log v\right) \\
&= P\left(\max_{t \in [0, \log v \wedge \tau_R(\beta)]} R(t) > 2 \log v\right) \\
&\leq P\left(\max_{t \in [0, \log v]} R(t) > 2 \log v\right) \\
&\leq K \frac{\sqrt{\log v}}{v^2}
\end{aligned}$$

where the last inequality comes from Lemma 2.5. Finally, we just have to obtain an upper bound for

$$P(\overline{W}((m_{v,1}^- - \log v) \vee a_{v,1}^-, m_{v,1}^-) - W(m_{v,1}^-) > 2 \log v)$$

to prove the lemma. This can be done in the same way as the previous one. \square

We can now proceed with the local time of the diffusion \mathbf{X} .

4 Asymptotic behaviour of \mathbf{L} at particular times

Let r be a positive real number. As for the c_i , its value will be fixed later. We define $\sigma_{v,1}^- := \sigma(re^v, m_{v,1}^-)$, and in the same way $\sigma_{v,2}^-$, $\widehat{\sigma}_{v,1}^-$, $\widehat{\sigma}_{v,2}^-$ and σ_v^+ . We show in this section that at these times, it is possible to estimate the local time of \mathbf{X} in the neighborhood of the corresponding point. We first give an estimate at fixed environment in Proposition 4.1, then in Proposition 4.4, we give an estimate independent of the environment provided that this one belongs to Γ_v .

Proposition 4.1 (Estimates depending on the environment)

We define for $i \in \{1, 2\}$,

$$\begin{aligned}
\mathcal{A}_v^i &:= \left\{ \forall x \in [a_{v,i}^-, b_{v,i}^-], \left| \frac{\mathbf{L}(\sigma_{v,i}^-, x)}{re^{v-W(x)+W(m_{v,i}^-)}} - 1 \right| \leq \delta \right\}, \\
\mathcal{B}_v^i &:= \left\{ \forall x \in [b_{v,i-1}^-, a_{v,i}^-], \mathbf{L}(\sigma_{v,i}^-, x) \leq \delta re^v \right\}, \\
\mathcal{C}_v &:= \left\{ \forall x \in [b_v^+, c_v^+], \mathbf{L}(\sigma_v^+, x) \leq \delta re^v \right\} \text{ and} \\
\mathcal{D}_v &:= \left\{ \forall x > c_v^+, \mathbf{L}(\sigma_v^+, x) = 0 \right\}.
\end{aligned}$$

We also define $\widehat{\mathcal{A}}_v^i$, $\widehat{\mathcal{B}}_v^i$, $\widehat{\mathcal{C}}_v$ and $\widehat{\mathcal{D}}_v$ from \widehat{W} in the same way. Then there is

a constant $K > 0$ such that for any $v \geq 2$, any $0 \leq \delta \leq 1$ and any $r > 0$,

$$\begin{aligned}
P^W(\overline{\mathcal{A}}_v^i) &\leq \frac{K}{\delta} \sqrt{\frac{m_{v,i}^-}{rv^{c_2}}} \exp\left(-\frac{\delta^2 rv^{c_2}}{Km_{v,i}^-}\right), \\
P^W(\overline{\mathcal{B}}_v^i) &\leq K \exp\left(-\frac{\delta rv^{c_1}}{2(m_{v,i}^- - b_{v,i-1}^-) \log\left(8 \frac{S(m_{v,i}^-) - S(b_{v,i-1}^-)}{(S(m_{v,i}^-) - S(a_{v,i}^-))}\right)}\right) + \frac{1}{\delta v^{c_1 - c_2}}, \\
P^W(\overline{\mathcal{C}}_v) &\leq \frac{1}{\delta} e^{-W(b_v^+, c_v^+) + W(m_v^+)} \text{ and} \\
P^W(\overline{\mathcal{D}}_v) &\leq \frac{re^{v+W(m_v^+)}}{2(S(c_v^+) - S(m_v^+))}.
\end{aligned}$$

We have similar estimates for $P^W(\widehat{\mathcal{A}}_v^i)$, $P^W(\widehat{\mathcal{B}}_v^i)$, $P^W(\widehat{\mathcal{C}}_v)$ and $P^W(\widehat{\mathcal{D}}_v)$.

Proof:

We estimate probabilities of the various events relative to W , we obtain results relative to \widehat{W} by symmetry. To simplify the notations, all along the proof, we shall not mark the index v in variables and events. Let us begin with events \mathcal{A}^1 and \mathcal{B}^1 (\mathcal{A}^2 and \mathcal{B}^2 can be studied in the same way).

We decompose the local time in two terms, the first one representing the contribution of the local time before $\tau(m_1^-)$ (the first time where \mathbf{X} reaches m_1^-) is negligible compared to the second one representing the contribution of local time between $\tau(m_1^-)$ and $\sigma_1^- = \sigma(re^v, m_1^-)$ (the inverse of local time in m_1^-):

$$\mathbf{L}(\sigma_1^-, x) = \mathbf{L}(\tau(m_1^-), x) + (\mathbf{L}(\sigma_1^-, x) - \mathbf{L}(\tau(m_1^-), x)).$$

Let us begin with a lemma describing the behaviour of the first process.

Lemma 4.2 For any $v, \delta, r > 0$,

$$\begin{aligned}
P_{\mathcal{A}^1,1}^W &:= P^W\left(\sup_{x \in [a_1^-, m_1^-]} \frac{\mathbf{L}(\tau(m_1^-), x)}{re^{v-(W(x)-W(m_1^-))}} > \delta\right) \\
&\leq K \exp\left(-\frac{\delta rv^{c_2}}{2m_1^-}\right), \\
P_{\mathcal{B}^1,1}^W &:= P^W\left(\sup_{x \in [b_0^-, a_1^-]} \mathbf{L}(\tau(m_1^-), x) > \delta re^v\right) \\
&\leq K \exp\left(-\frac{\delta rv^{c_1}}{2(m_i^- - b_{i-1}^-) \log\left(8 \frac{S(m_i^-) - S(b_{i-1}^-)}{(S(m_i^-) - S(a_i^-))}\right)}\right).
\end{aligned}$$

Proof: Thanks to (3), we easily verify that $\tau(m_1^-) = T(\tau(S(m_1^-)))$. So with (4), we obtain for any $x \geq 0$,

$$\mathbf{L}(\tau(m_1^-), x) = e^{-W(x)} L(\tau(S(m_1^-)), S(x)).$$

Thus,

$$P_{\mathcal{A}^1,1}^W = P^W \left(\sup_{x \in [a_1^-, m_1^-]} L(\tau(S(m_1^-)), S(x)) > \delta r e^{v+W(m_1^-)} \right).$$

According to the first Ray-Knight theorem, Theorem 2.1,

$$\{L(\tau(S(m_1^-)), S(m_1^-) - y), y \in [0, S(m_1^-)]\}$$

has the same law as a square of a Bessel process of dimension 2 started at 0. Therefore, with Item 3 of Lemma 2.3,

$$P_{\mathcal{A}^1,1}^W \leq K \exp \left(-\frac{\delta r e^{v+W(m_1^-)}}{2(S(m_1^-) - S(a_1^-))} \right).$$

and by definition of a_1^- ,

$$\begin{aligned} S(m_1^-) - S(a_1^-) &= \int_{a_1^-}^{m_1^-} e^{W(x)} dx \leq m_1^- e^{\overline{W}(a_1^-, m_1^-)} \\ &\leq m_1^- e^{v-c_2 \log v + W(m_1^-)}. \end{aligned}$$

Hence we obtain the initial upper bound of the lemma.

Let us continue with the second one, to highlight the fact that the computations are identical for \mathcal{B}^1 and for \mathcal{B}^2 we make the quantity b_0^- appears, although it is zero. Using a similar argument as above,

$$\begin{aligned} P_{\mathcal{B}^1,1}^W &= P^W \left(\sup_{x \in [b_0^-, a_1^-]} e^{-W(x)} Q(S(m_1^-) - S(x)) > \delta r e^v \right) \\ &= P^W \left(\sup_{x \in [b_0^-, a_1^-]} \frac{e^{-W(x)}(S(m_1^-) - S(x))}{S(m_1^-) - S(x)} Q(S(m_1^-) - S(x)) > \delta r e^v \right). \end{aligned}$$

By definition of m_1^- , P -a.s, for any $x \in [b_0^-, a_1^-]$,

$$\begin{aligned} e^{-W(x)}(S(m_1^-) - S(x)) &= e^{-W(x)} \int_x^{m_1^-} e^{W(y)} dy \\ &\leq (m_1^- - b_0^-) e^{-W(x) + \overline{W}(x, m_1^-)} \\ &\leq (m_1^- - b_0^-) e^{v-c_1 \log v} \end{aligned}$$

where the last inequality is due to m_1^- being the first positive real x such that $W(x) - \underline{W}(0, x) \geq v - c_1 \log v$. Thus, coming back to the probability $P_{\mathcal{B}^1, 1}^W$, we obtain

$$P_{\mathcal{B}^1, 1}^W \leq P^W \left(\sup_{u \in [S(m_1^-) - S(a_1^-), S(m_1^-) - S(b_0^-)]} \frac{1}{u} Q(u) > \frac{\delta r v^{c_1}}{m_1^- - b_0^-} \right).$$

According to Item 4 of Lemma 2.3, we finally have

$$P_{\mathcal{B}^1, 1}^W \leq K \exp \left(- \frac{\delta r v^{c_1}}{2(m_1^- - b_0^-) \log \left(8 \frac{S(m_1^-) - S(b_0^-)}{S(m_1^-) - S(a_1^-)} \right)} \right).$$

This concludes the proof of the lemma. \square

Lemma 4.3 For any $v, \delta, r > 0$,

$$\begin{aligned} P_{\mathcal{A}^1, 2}^W &:= P^W \left(\sup_{x \in [a_1^-, b_1^-]} \left| \frac{\mathbf{L}(\sigma_1^-, x) - \mathbf{L}(\tau(m_1^-), x)}{r e^{v - (W(x) - W(m_1^-))}} - 1 \right| > \delta \right) \\ &\leq \frac{8}{\delta} \sqrt{\frac{(1 + \delta) m_1^-}{r v^{c_2}}} \exp \left(- \frac{\delta^2 r v^{c_2}}{8(1 + \delta) m_1^-} \right), \\ P_{\mathcal{B}^1, 2}^W &:= P^W \left(\sup_{x \in [b_0^-, a_1^-]} \mathbf{L}(\sigma_1^-, x) - \mathbf{L}(\tau(m_1^-), x) > \delta r e^v \right) \\ &\leq \frac{1}{\delta v^{c_1 - c_2}}. \end{aligned}$$

Proof: It is easy to verify that the inverse of local time σ satisfies the following equality for every $r > 0$ and $y \in \mathbf{R}$,

$$\sigma(r, y) = T(\sigma(r e^{W(y)}, S(y))). \quad (7)$$

Thus, (4) implies that, for any $r > 0$ and $y \in \mathbf{R}$,

$$\mathbf{L}(\sigma_1^-, x) = e^{-W(x)} L(\sigma(r e^{W(m_1^-) + v}, S(m_1^-)), S(x)).$$

And we obtain the following expression for the local time,

$$\begin{aligned} &\mathbf{L}(\sigma_1^-, x) - \mathbf{L}(\tau(m_1^-), x) \\ &= e^{-W(x)} \left(L(\sigma(r e^{W(m_1^-) + v}, S(m_1^-)), S(x)) - L(\tau(S(m_1^-)), S(x)) \right) \\ &\stackrel{\mathcal{L}}{=} e^{-W(x)} r e^{W(m_1^-) + v} L(\sigma(1, 0), s(x)) \end{aligned}$$

where

$$s(x) := (S(x) - S(m_1^-)) \frac{e^{-W(m_1^-)-v}}{r}.$$

Denoting by Z a square of a 2-dimensional Bessel process started at 1, according to the second Ray-Knight theorem (Theorem 2.2), we have

$$\begin{aligned} P_{\mathcal{A}^1,2}^W &\leq P^W \left(\sup_{0 \leq y \leq |s(a_1^-)|} |Z(y) - 1| > \delta \right) \\ &\quad + P^W \left(\sup_{0 \leq y \leq s(b_1^-)} |Z(y) - 1| > \delta \right). \end{aligned}$$

Therefore, using Item 1 of Lemma 2.3 we conclude that

$$\begin{aligned} P_{\mathcal{A}^1,2}^W &\leq \frac{4}{\delta} \sqrt{(1+\delta)|s(a_1^-)|} \exp\left(-\frac{\delta^2}{8(1+\delta)|s(a_1^-)|}\right) \\ &\quad + \frac{4}{\delta} \sqrt{(1+\delta)s(b_1^-)} \exp\left(-\frac{\delta^2}{8(1+\delta)s(b_1^-)}\right). \end{aligned}$$

Moreover, the definition of a_1^- implies

$$\begin{aligned} |s(a_1^-)| &= \frac{e^{-W(m_1^-)-v}}{r} \int_{a_1^-}^{m_1^-} e^{W(x)} dx \\ &\leq m_1^- \frac{e^{W(a_1^-)-W(m_1^-)-v}}{r} \leq \frac{m_1^-}{rv^{c_2}}. \end{aligned}$$

This last upper bound and the similar one for $s(b_1^-)$ lead us to the first inequality of the lemma. Now we continue with the proof of the second inequality. Still reasoning in the same way, we can have

$$P_{\mathcal{B}^1,2}^W = P^W \left(\sup_{x \in [b_0^-, a_1^-]} e^{-W(x)+W(m_1^-)} L(\sigma(1,0), s(x)) > \delta \right).$$

One more time, thanks to the second Ray-Knight theorem and denoting by Z a Bessel process of dimension 0 started at 1, we obtain:

$$\begin{aligned} P_{\mathcal{B}^1,2}^W &\leq P^W \left(e^{-\underline{W}(b_0^-, a_1^-)+W(m_1^-)} \sup_{u \geq 0} Z(u) > \delta \right) \\ &= \frac{1}{\delta} e^{-\underline{W}(b_0^-, a_1^-)+W(m_1^-)}. \end{aligned}$$

The second line is a consequence of Item 2 of Lemma 2.3. Let us denote by n^1 the unique real number in $[b_0^-, a_1^-]$ such that $W(n^1) = \underline{W}(b_0^-, a_1^-)$ then

$$\begin{aligned} W(m_1^-) - W(n^1) &= \overline{W}(n^1, a_1^-) - W(n^1) - (\overline{W}(n^1, a_1^-) - W(m_1^-)) \\ &\leq v - c_1 \log v - (v - c_2 \log v) = (c_2 - c_1) \log v. \end{aligned}$$

Therefore,

$$P_{\mathcal{B}^1,2}^W \leq \frac{1}{\delta v^{c_1 - c_2}}$$

and the lemma is proved. \square

Combining the results of Lemmas 4.2 and 4.3 yields to the upper bounds of $P^W(\overline{\mathcal{A}^1})$ and $P^W(\overline{\mathcal{B}^1})$ of Proposition 4.1.

We now continue with the estimate of $P^W(\mathcal{C})$. Reducing once again process \mathbf{X} to a Brownian motion by a time and space change, we obtain

$$P^W(\overline{\mathcal{C}}) = P^W \left(\sup_{x \in [b^+, c^+]} e^{-W(x) + W(m^+)} L(\sigma(1, 0), s(x)) > \delta \right).$$

As before, if Z denotes a squared 0-dimensional Bessel process starting at 1, the second Ray-Knight theorem gives:

$$\begin{aligned} P^W(\overline{\mathcal{C}}) &\leq P^W \left(e^{-\underline{W}(b^+, c^+) + W(m^+)} \sup_{u \geq 0} Z(u) > \delta \right) \\ &\leq \frac{1}{\delta} e^{-\underline{W}(b^+, c^+) + W(m^+)} \end{aligned}$$

where for the second inequality, we used Item 2 of Lemma 2.3. Finally we show that with high probability at time σ_1^- , diffusion \mathbf{X} does not hit c^+ , that is to say, we find an upper bound for $P^W(\overline{\mathcal{D}})$. The scale change in time and space and the usual properties of Brownian motion give us (see eg [2] Formula 4.1.2 page 185)

$$\begin{aligned} P^W(\overline{\mathcal{D}}) &= P^W(\tau(c^+) < \sigma(re^v, m^+)) \\ &= P^W(\tau(S(c^+)) < \sigma(re^{v+W(m^+)}, S(m^+))) \\ &= 1 - \exp\left(-\frac{re^{v+W(m^+)}}{2(S(c^+) - S(m^+))}\right) \leq \frac{re^{v+W(m^+)}}{2(S(c^+) - S(m^+))}. \end{aligned}$$

This completes the proof of the proposition. \square

We now give upper bounds independent of the medium while it is in the event Γ_v defined in 5.

Proposition 4.4 *We use the same notations as in the previous proposition. There is a constant $K > 0$ such that for any $v \geq 2$, any $0 \leq \delta \leq 1$, and any*

$r > 0$, if $W \in \Gamma_v$,

$$\begin{aligned} P^W(\overline{\mathcal{A}}_v^i) &\leq \frac{K}{\delta \sqrt{rv^{c_2-6}}} \exp\left(-\frac{\delta^2 rv^{c_2-6}}{K}\right), \\ P^W(\overline{\mathcal{B}}_v^i) &\leq K \exp\left(-\frac{\delta rv^{c_1-8}}{K}\right) + \frac{1}{\delta v^{c_1-c_2}}, \\ P^W(\overline{\mathcal{C}}_v) &\leq \frac{1}{\delta v^{c_1+c_3}} \text{ and} \\ P^W(\overline{\mathcal{D}}_v) &\leq \frac{r}{v^{c_3-2} \log v}. \end{aligned}$$

We have again similar estimates for $P^W(\widehat{\mathcal{A}}_v^i)$, $P^W(\widehat{\mathcal{B}}_v^i)$, $P^W(\widehat{\mathcal{C}}_v)$, $P^W(\widehat{\mathcal{D}}_v)$.

Proof: We only have to control the values of the upper bounds of Proposition 4.1 when $W \in \Gamma_v$.

For $P^W(\overline{\mathcal{A}}_v^i)$, it is enough to notice that $m_{v,i}^- \leq b_{v,2}^-$ and that, on Γ_v , the variable $b_{v,2}^-$ is smaller than v^6 . This also allows us to obtain the upper bound $m_{v,i}^- - b_{v,i-1}^- \leq b_{v,2}^- \leq v^6$. To estimate $P^W(\overline{\mathcal{B}}_v^i)$, it remains to study

$$\frac{S(m_{v,i}^-) - S(b_{v,i-1}^-)}{S(m_{v,i}^-) - S(a_{v,i}^-)} \leq \frac{(m_{v,i}^- - b_{v,i-1}^-) e^{\overline{W}(b_{v,i-1}^-, m_{v,i}^-)}}{(m_{v,i}^- - a_{v,i}^-) e^{W(m_{v,i}^-)}}.$$

But, on one hand $\overline{W}(b_{v,i-1}^-, m_{v,i}^-) - W(b_{v,i-1}^-) \leq v - c_1 \log v \leq v$ and on the other hand, as we work on Γ_v , it is easy to find upper bound for the other quantities involved in the computation. And we finally obtain,

$$\frac{S(m_{v,i}^-) - S(b_{v,i-1}^-)}{S(m_{v,i}^-) - S(a_{v,i}^-)} \leq v^8 e^{v+v^2}.$$

which implies the result of the proposition.

As on Γ_v , we have $\underline{W}(b_v^+, c_v^+) - W(m_v^+) \geq (c_1 + c_3) \log v$, the third estimate is obtained immediately. It remains only the upper bound of $P^W(\overline{\mathcal{D}}_v)$. For this, it is enough to found a lower bound to

$$\begin{aligned} S(c_v^+) - S(m_v^+) &= \int_{m_v^+}^{c_v^+} e^{W(x)} dx \geq \int_{(c_v^+ - \log v) \vee m_v^+}^{c_v^+} e^{W(x)} dx \\ &\geq (\log v \wedge (c_v^+ - m_v^+)) e^{\underline{W}((c_v^+ - \log v) \vee m_v^+, c_v^+)}. \end{aligned}$$

Thus, as on Γ_v we have $c_v^+ - m_v^+ \geq v$ and $\underline{W}((c_v^+ - \log v) \vee m_v^+, c_v^+) \geq W(c_v^+) - 2 \log v$, the following is true:

$$P^W(\overline{\mathcal{D}}_v) \leq \frac{rv^2}{\log v} e^{v+W(m_v^+)-W(c_v^+)}.$$

Moreover $W(c_v^+) - W(m_v^+) = v + c_3 \log v$, we therefore obtain the last result of the proposition. \square

5 Asymptotics of the maximum local time

Thanks to Proposition 4.4, we can now study the process \mathbf{L} at the time $\sigma_v := \widehat{\sigma}_v \wedge \widehat{\sigma}_{v,1}^- \wedge \sigma_{v,1}^- \wedge \sigma_v^+$ the first instant the diffusion has "passed a time" re^v in one of the points $m_{v,1}^-, m_v^+, \widehat{m}_{v,1}^-$ or \widehat{m}_v^+ . Let us begin by fixing the values of constants c_i : we take a real number $c > 0$, then we choose c_1, c_2 and c_3 as follows $c_1 := 2c + 8$, $c_2 := c + 6$ and $c_3 := c + 2$. This yields to the following proposition:

Proposition 5.1 *Let $v \geq 2$. We define*

$$I_v^- := \int_{a_{v,1}^-}^{b_{v,1}^-} e^{-W(x)+W(m_{v,1}^-)} dx, \quad I_v^+ := \int_{a_v^+}^{b_v^+} e^{-W(x)+W(m_v^+)} dx.$$

We define similarly \widehat{I}_v^- and \widehat{I}_v^+ from \widehat{W} and

$$j_v := \widehat{I}_v^- \wedge \widehat{I}_v^+ \wedge I_v^- \wedge I_v^+ \text{ et } J_v := \widehat{I}_v^- + \widehat{I}_v^+ + I_v^- + I_v^+.$$

Then there is a constant $K > 0$ such that for any $0 < \delta < 1$ and any $r > 0$, if $W \in \Gamma_v$,

$$\begin{aligned} P^W \left(re^v \leq \mathbf{L}^*(\sigma_v) \leq re^v(1 + \delta) ; j_v(1 - \delta) \leq \frac{\sigma_v}{re^v} \leq J_v + 2v^6\delta \right) \\ \geq 1 - K \left(\frac{1}{\delta\sqrt{rv^c}} \exp\left(-\frac{\delta^2rv^c}{K}\right) + \exp\left(-\frac{\delta rv^c}{K}\right) + \frac{1}{\delta v^c} + \frac{r}{v^c} \right). \end{aligned}$$

Note that in the previous section we used the four points $m_{v,1}^-, m_{v,2}^-, \widehat{m}_{v,1}^-$ and $\widehat{m}_{v,2}^-$ whereas in the last proposition we use the points $m_{v,1}^-, m_v^+, \widehat{m}_{v,1}^-$ and \widehat{m}_v^+ . The former ones were interesting because they simplify the computations of Proposition 4.4, but the latter ones allow us to reduce the study of $I_v^-, I_v^+, \widehat{I}_v^-$ and \widehat{I}_v^+ to the study of the integral

$$I_v = \int_{a_v}^{b_v} e^{-W(x)+W(m_v)} dx$$

where

$$\begin{aligned} b_v &:= \inf\{x \geq m_v ; W(x) - W(m_v) \geq v - c_1 \log v\} \text{ and} \\ a_v &:= \sup\{x \leq m_v ; W(x) - W(m_v) \geq v - c_2 \log v\} \vee 0. \end{aligned}$$

Proof: We only have to prove that on the intersection of all the events of Proposition 4.4 the event

$$\left\{ re^v \leq \mathbf{L}^*(\sigma_v) \leq re^v(1 + \delta) ; j_v(1 - \delta) \leq \frac{\sigma_v}{re^v} \leq J_v + 2v^6\delta \right\}$$

is realized. As σ_v is the first time the diffusion has "passed a time" re^v in one of the points $m_{v,1}^-, m_v^+, \widehat{m}_{v,1}^-$ or \widehat{m}_v^+ we already have $re^v \leq \mathbf{L}^*(\sigma_v)$.

Moreover, the local time is non-decreasing, so for every $x \in \mathbf{R}$,

$$\mathbf{L}(\sigma_v, x) = \mathbf{L}(\widehat{\sigma}_v, x) \wedge \mathbf{L}(\widehat{\sigma}_{v,1}^-, x) \wedge \mathbf{L}(\sigma_{v,1}^-, x) \wedge \mathbf{L}(\sigma_v^+, x).$$

On Γ_v we have $\sigma_v^+ \in \{\sigma_{v,1}^-, \sigma_{v,2}^-\}$ and $\widehat{\sigma}_v \in \{\widehat{\sigma}_{v,1}^-, \widehat{\sigma}_{v,2}^-\}$. We then obtain the upper bound $\mathbf{L}^*(\sigma_v) \leq re^v(1 + \delta)$.

Let us continue with the estimate of σ_v : by definition of the local time,

$$\sigma_v = \int_{-\infty}^{+\infty} \mathbf{L}(\sigma_v, x) dx, \quad P\text{-a.s.}$$

If $m_v^+ = m_{v,1}^-$, then $b_v^+ = b_{v,1}^-$ and $\sigma_v^+ = \sigma_{v,1}^-$, therefore

$$\begin{aligned} \int_0^{+\infty} \mathbf{L}(\sigma_v, x) dx &\leq \int_0^{a_{v,1}^-} \mathbf{L}(\sigma_{v,1}^-, x) dx + \int_{a_{v,1}^-}^{b_{v,1}^-} \mathbf{L}(\sigma_{v,1}^-, x) dx + \int_{b_{v,1}^-}^{c_v^+} \mathbf{L}(\sigma_v^+, x) dx \\ &\leq (I_v^- + \delta c_v^+) re^v \end{aligned}$$

else if $m_v^+ = m_{v,2}^-$,

$$\begin{aligned} \int_0^{+\infty} \mathbf{L}(\sigma_v, x) dx &\leq \int_0^{a_{v,1}^-} \mathbf{L}(\sigma_{v,1}^-, x) dx + \int_{a_{v,1}^-}^{b_{v,1}^-} \mathbf{L}(\sigma_{v,1}^-, x) dx \\ &\quad + \int_{b_{v,1}^-}^{a_{v,2}^-} \mathbf{L}(\sigma_{v,2}^-, x) dx + \int_{a_{v,2}^-}^{b_{v,2}^-} \mathbf{L}(\sigma_{v,2}^-, x) dx + \int_{b_{v,2}^-}^{c_v^+} \mathbf{L}(\sigma_v^+, x) dx \\ &\leq (I_v^- + I_v^+ + \delta c_v^+) re^v. \end{aligned}$$

The integral $\int_{-\infty}^0 \mathbf{L}(\sigma_v, x) dx$ has a similar upper bound. As on Γ_v we have $c_v^+ + \widehat{c}_v^+ \leq 2v^6$, the upper bound of the proposition follows immediately.

For the lower bound, if $\sigma_v = \sigma_{v,1}^-$, we have

$$\begin{aligned} \frac{\sigma_{v,1}^-}{re^v} &\geq \int_{a_{v,1}^-}^{b_{v,1}^-} \mathbf{L}(\sigma_{v,1}^-, x) dx \\ &\geq I_v^- (1 - \delta). \end{aligned}$$

Doing the same computation if σ_v takes one of the three other possible values, the lower bound stated in the proposition is obtained immediately. \square

We are now interested in the behaviour of the local time in deterministic time:

Proposition 5.2 *Let $v \geq 2$. For any $0 < \delta \leq 1/2$, if $W \in \Gamma_v$,*

$$\begin{aligned} P^W \left(\frac{e^v}{J_v + 2v^6\delta} \leq \mathbf{L}^*(e^v) \leq \frac{e^v(1+\delta)}{j_v(1-\delta)} \right) \\ \geq 1 - K \left(\frac{1}{\delta\sqrt{v^{c-6}}} \exp\left(-\frac{\delta^2 v^{c-6}}{K}\right) + \exp\left(-\frac{\delta v^{c-6}}{K}\right) + \frac{1}{\delta v^c} + \frac{1}{v^{c-4}} \right). \end{aligned}$$

Proof: We now use the real number r which appears in all propositions since the beginning. Furthermore we use the result of the previous proposition. Let us define

$$\rho(v) := \frac{1}{J_v + 2v^6\delta} \text{ and } r(v) := \frac{1}{j_v(1-\delta)}.$$

We also write σ_v^ρ for the time σ_v associated with ρ and σ_v^r for the one associated with r . We now consider the events

$$\begin{aligned} \Omega^\rho &:= \left\{ \frac{\sigma_v^\rho}{\rho(v)e^v} \leq J_v + v^3\delta \right\} = \{ \sigma_v^\rho \leq e^v \}, \\ \Omega^r &:= \left\{ \mathbf{L}^*(\sigma_v^r) \leq r(v)e^v(1+\delta) ; j_v(1-\delta) \leq \frac{\sigma_v^r}{r(v)e^v} \right\} \\ &= \{ \mathbf{L}^*(\sigma_v^r) \leq r(v)e^v(1+\delta) ; e^v \leq \sigma_v^r \}. \end{aligned}$$

As the maximum of the local time is a non decreasing function,

$$\begin{aligned} \text{on } \Omega^r, \mathbf{L}^*(e^v) \leq \mathbf{L}^*(\sigma_v^r) \leq r(v)e^v(1+\delta) \\ \text{and on } \Omega^\rho, \rho(v)e^v \leq \mathbf{L}^*(\sigma_v^\rho) \leq \mathbf{L}^*(e^v). \end{aligned}$$

Therefore it is enough to find a lower bound of $P(\Omega^r \cap \Omega^\rho)$. According to the previous proposition, we only have to estimate r and ρ . On Γ_v it is true that

$$j_v \leq J_v \leq c_v^+ - \widehat{c}_v^+ \leq 2v^6.$$

Moreover we have $m_{v,1}^- - a_{v,1}^- \geq \frac{1}{v^2}$ and

$$\overline{W}(m_{v,1}^- - \log v \vee a_{v,1}^-, m_{v,1}^-) - W(m_{v,1}^-) \leq 2 \log v,$$

thus

$$\begin{aligned} I_v^- &= \int_{a_{v,1}^-}^{b_{v,1}^-} e^{-W(x)+W(m_{v,1}^-)} dx \\ &\geq \left((m_{v,1}^- - a_{v,1}^-) \wedge (\log v) \right) e^{-\overline{W}(m_{v,1}^- - \log v \vee a_{v,1}^-, m_{v,1}^-) + W(m_{v,1}^-)} \\ &\geq \frac{1}{v^4}. \end{aligned}$$

The lower bounds for I_v^+ , \widehat{I}_v^- and \widehat{I}_v^+ are found in the same way. Finally,

$$\frac{K}{v^6} \leq \rho(v) \leq r(v) \leq 2v^4$$

and it is now easy to obtain the estimate of the proposition. \square

Using the result of the previous proposition with $\delta = v^{-7}$ and $c > 20$ and the upper bound for $P(\Gamma_v)$ of Proposition 3.1, this yields for any $v \geq 2$ to

$$P\left(\frac{e^v}{J_v + 2v^{-1}} \leq \mathbf{L}^*(e^v) \leq \frac{e^v(1+v^{-1})}{j_v(1-v^{-1})}\right) \geq 1 - K\left(\frac{\log v}{v - c_1 \log v}\right)^2. \quad (8)$$

We also obtain, using this time the event Γ'_v ,

$$P\left(\frac{e^v}{I_v + \widehat{I}_v + 2v^{-1}} \leq \mathbf{L}^*(e^v) \leq \frac{e^v(1+v^{-1})}{I_v \wedge \widehat{I}_v(1-v^{-1})}\right) \geq 1 - \frac{K \log v}{v - c_1 \log v}. \quad (9)$$

6 Proof of Theorem 1.1

As shown by Proposition 5.2, the asymptotic behaviour of \mathbf{L}^* has a direct link with the ones of j_v , J_v and more generally with the behaviour of integral $I_v = \int_{a_v}^{b_v} e^{-W(x)+W(m_v)} dx$. So to prove the theorem we need to study this integral.

6.1 Maximum and minimum speed of process I_v

Let us begin with a lower bound for the maximum speed:

Lemma 6.1 *Let $v_n = e^n$. Then P -a.s.,*

$$\limsup_{n \rightarrow \infty} \frac{I_{v_n} \wedge \widehat{I}_{v_n}}{\log_2 v_n} \geq \frac{1}{e\pi^2}.$$

Proof: We first define the sequence of events

$$E_n := \left\{ m_{v_n} > m_{v_{n-1}} ; \int_{m_{v_n}}^{b_{v_n}} e^{-W(x)+W(m_{v_n})} dx \geq \frac{\log n}{e\pi^2} \right\}.$$

Denote by $(\mathcal{F}_n)_{n \in \mathbf{N}}$ the filtration generated by $(W(t) ; 0 \leq t \leq b_{v_n})$, we also put $W_n = W(\cdot + b_{v_n}) - W(b_{v_n})$. The events

$$\{m_{v_n} > m_{v_{n-1}}\} = \{W_{n-1} \text{ hits } -v_{n-1} \text{ before } v_n - v_{n-1}\}$$

and

$$\left\{ \int_{m_{v_n}}^{b_{v_n}} e^{-W(x)+W(m_{v_n})} dx \geq \frac{\log n}{e\pi^2} \right\}$$

are independent of one another and of \mathcal{F}_{n-1} . Moreover,

$$P(W_n \text{ hits } -v_{n-1} \text{ before } v_n - v_{n-1}) = \frac{v_n - v_{n-1}}{v_n} = (1 - e^{-1})$$

and according to Theorem 2.4,

$$\begin{aligned} P\left(\int_{m_{v_n}}^{b_{v_n}} e^{-W(x)+W(m_{v_n})} dx \geq \frac{\log n}{e\pi^2}\right) &\geq P\left(\int_0^{T_R(1)} e^{-R(x)} dx \geq \frac{\log n}{e\pi^2}\right) \\ &\geq P\left(T_R(1) \geq \frac{\log n}{\pi^2}\right). \end{aligned}$$

According to Theorem 2 of [22] (recall that $j_{1/2,0} = \pi$.) there is a real number $K > 0$ such that

$$P(T_R(1) \geq x) \sim \frac{1}{K} e^{-\pi^2 \frac{x}{2}}.$$

Then we have, (the value of K can change)

$$P(T_R(1) \geq x) \geq \frac{1}{K} e^{-\pi^2 \frac{x}{2}}$$

and so

$$P(E_n) \geq \frac{1 - e^{-1}}{K\sqrt{n}}.$$

We now define the similar event for \widehat{W} :

$$\widehat{E}_n := \left\{ \widehat{m}_{v_n} > \widehat{m}_{v_{n-1}} ; \int_{\widehat{m}_{v_n}}^{\widehat{b}_{v_n}} e^{-W(x)+W(\widehat{m}_{v_n})} dx \geq \frac{\log n}{e\pi^2} \right\}.$$

The events E_n, \widehat{E}_n are pairwise independent, thus

$$P(E_n \cap \widehat{E}_n) = P(E_n)P(\widehat{E}_n) \geq \frac{(1 - e^{-1})^2}{K^2 n}.$$

We can now conclude thanks to the second Borel-Cantelli lemma. \square

We are not interested in an upper bound of the minimum speed because this would lead, except for the value of the constant, to the result obtained by Shi in [15]. We now search almost sure bounds of the integral. To this end,

we start first by studying the successive values μ_n the process $\{m_v, v \geq 1\}$ can take. This is precisely defined as follows:

$$\begin{aligned}\beta_1 &:= \inf\{x \geq 0 ; W(x) - \underline{W}(0, x) = 1\}, \\ \mu_1 &:= \inf\{x \geq 0 ; W(x) = \underline{W}(0, \beta_1)\}, \\ \gamma_1 &:= \inf\{x \geq \beta_1 ; W(x) = W(\mu_1)\}, \\ M_1 &:= \inf\{x \geq \beta_1 ; W(x) = \overline{W}(\beta_1, \gamma_1)\}, \\ h_1 &:= W(M_1) - W(\mu_1)\end{aligned}$$

and recursively for any $n \in \mathbf{N}^*$,

$$\begin{aligned}\beta_{n+1} &:= \inf\{x \geq \gamma_n ; W(x) - \underline{W}(\gamma_n, x) = h_{n-1}\}, \\ \mu_{n+1} &:= \inf\{x \geq \gamma_n ; W(x) = \underline{W}(\gamma_n, \beta_{n+1})\}, \\ \gamma_{n+1} &:= \inf\{x \geq \beta_{n+1} ; W(x) = W(\mu_n)\}, \\ M_{n+1} &:= \inf\{x \geq \beta_{n+1} ; W(x) = \overline{W}(\beta_{n+1}, \gamma_{n+1})\}, \\ h_n &:= W(M_n) - W(\mu_n) \text{ and} \\ \mathcal{F}_n &:= \sigma(W(x), 0 \leq x \leq \gamma_n).\end{aligned}$$

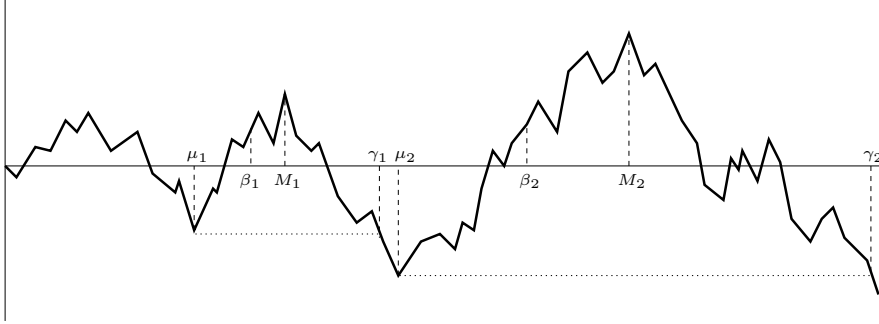


Figure 2: The variables for a sample path of W

Lemma 6.2 *There is a real $K > 0$ such that for any $n > 0$ and any $\lambda > 0$,*

$$\begin{aligned}P\left(\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)} dx \geq \lambda\right) &\leq K e^{-j_0^2 \frac{\lambda}{16}} \text{ and} \\ P\left(\int_{\mu_n}^{\beta_n} e^{-W(x)+W(\mu_n)} dx \leq \lambda\right) &\leq K \left(\frac{1}{\sqrt{e\lambda}} + \sqrt{e\lambda}\right) e^{-1/(2e\lambda)}\end{aligned}$$

where j_0 is the smallest strictly positive root of Bessel function J_0 .

Proof: The process $(W(\gamma_{n-1} + x) - W(\gamma_{n-1}))_{x \geq 0}$ is a Brownian motion independent of \mathcal{F}_{n-1} . Therefore given $h_{n-1} = h$, Theorem 2.4 gives us the law of the process

$$(W(\mu_n + x) - W(\mu_n), -\mu_n + \gamma_{n-1} \leq x \leq \beta_n - \mu_n).$$

So denoting by R and \tilde{R} two independent Bessel processes of dimension 3 started at 0, we have

$$P\left(\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)} dx \geq \lambda\right) \leq P\left(\int_0^\infty e^{-R(x)} dx + \int_0^\infty e^{-\tilde{R}(x)} dx \geq \lambda\right).$$

Moreover using Le Gall's Ray-Knight Theorem (Proposition 1.1 [6]), it can be proved that $\int_0^\infty e^{-R(x)} dx$ has the same law as $4T_Q(1)$ where $T_Q(1)$ is the hitting time of height 1 by a square of a 2-dimensional Bessel process started in 0. Thereby, we denote by $\tilde{T}_Q(1)$ a independent copy of $T_Q(1)$ and we have

$$\begin{aligned} P\left(\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)} dx \geq \lambda\right) &\leq P\left(T_Q(1) + \tilde{T}_Q(1) \geq \frac{\lambda}{4}\right) \\ &\leq 2P\left(T_Q(1) \geq \frac{\lambda}{8}\right). \end{aligned}$$

Moreover according to Theorem 2 of [22], as in the proof of the previous lemma,

$$\begin{aligned} P\left(\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)} dx \geq \lambda\right) &\leq P\left(T_Q(1) + \tilde{T}_Q(1) \geq \frac{\lambda}{4}\right) \\ &\leq Ke^{-j_0^2 \frac{\lambda}{16}}. \end{aligned}$$

We still have thanks to Theorem 2.4, denoting $T_R(1) := \inf\{x \geq 0, R(x) \geq 1\}$,

$$\begin{aligned} P\left(\int_{\mu_n}^{\beta_n} e^{-W(x)+W(\mu_n)} dx \leq \lambda\right) &\leq P\left(\int_0^{T_R(1)} e^{-R(x)} dx \leq \lambda\right) \\ &\leq P(T_R(1) \leq e\lambda) \\ &\leq K\left(\frac{1}{\sqrt{e\lambda}} + \sqrt{e\lambda}\right) e^{-1/(2e\lambda)}. \end{aligned}$$

This concludes the proof. \square

We also need the following lemma:

Lemma 6.3 *Let $a < e < b$. Then P -a.s., for n large enough,*

$$a^n < h_n < b^n, \quad a^n < W(\mu_n) - W(\mu_{n+1}) < b^n \quad \text{and} \quad a^{2n} < \gamma_n < b^{2n}.$$

Proof: We begin with the law of the sequence (h_n) . For any $h \geq 1$, any $n \in \mathbf{N}$ and any $x \geq 1$,

$$\begin{aligned} P\left(\frac{h_{n+1}}{h_n} \leq h \mid h_n = x\right) &= P\left(\frac{h_{n+1} - h_n}{h_n} \leq h - 1 \mid h_n = x\right) \\ &= P(\tau_W((h-1)x) \geq \tau_W(-x)) = 1 - \frac{1}{h}. \end{aligned}$$

Thus the variables $r_n := h_{n+1}/h_n$ are independent and so $\log r_n$ is exponentially distributed with mean 1. Therefore $\log h_n = \sum \log r_k$ has the law $\Gamma(n, 1)$ and for any $1 < a < e$, $n \in \mathbf{N}$,

$$\begin{aligned} P(h_n \leq a^n) &= P(\log h_n \leq n \log a) = \int_0^{n \log a} \frac{x^{n-1}}{(n-1)!} e^{-x} dx \\ &\leq \frac{(n \log a)^n}{a^n (n-1)!} \end{aligned}$$

because the function $x \rightarrow x^{n-1} e^{-x}$ is non decreasing on $[0, n \log a]$. Stirling Formula $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ gives us

$$\frac{(n \log a)^n}{a^n (n-1)!} \sim \sqrt{\frac{n}{2\pi}} \left(\frac{e \log a}{a}\right)^n.$$

As for any $a \in]1, e[$, $0 < \frac{e \log a}{a} < 1$, the serie $\sum P(h_n \leq a^n)$ converges. Then Borel-Cantelli lemma yields directly the first part of the lemma. We prove the upper bound in the same way.

For the second result, we begin by noting that, given $h_{n-1} = x$, the process $W(\mu_{n-1}) - W(\mu_n)$ has the same law as $W(m_x)$. Therefore,

$$P(W(\mu_{n-1}) - W(\mu_n) < h | h_{n-1} = x) = 1 - e^{-h/x} \leq \frac{h}{x}.$$

Taking $1 < c < a < e$, we obtain

$$\begin{aligned} P(W(\mu_{n-1}) - W(\mu_n) < c^{n-1}) \\ &\leq P(W(\mu_{n-1}) - W(\mu_n) < c^{n-1} ; h_{n-1} > a^{n-1}) + P(h_{n-1} \leq a^{n-1}) \\ &\leq \left(\frac{c}{a}\right)^{n-1} + P(h_{n-1} \leq a^{n-1}). \end{aligned}$$

The result of the previous proof implies that the sum of

$$P(W(\mu_{n-1}) - W(\mu_n) < c^{n-1})$$

converges and Borel-Cantelli Lemma shows that for large n ,

$$W(m_{n-1}) - W(m_n) \geq c^{n-1}.$$

The second inequality can be proved in the same way.

We reason similarly to find an upper bound of γ_n : first, we show that for $c > 0$,

$$P\left(\frac{\gamma_n - \gamma_{n-1}}{h_n^2} > c\right) \leq 2P(\tau_W(1) \geq c) \leq \frac{K}{\sqrt{c}}. \quad (10)$$

Indeed, as before, conditioning by $W(\mu_n)$,

$$P\left(\frac{\gamma_n - \beta_n}{h_n^2} > c\right) \leq P\left(\frac{\gamma_n - \beta_n}{W(\mu_n)^2} > c\right) = P(\tau_W(1) \geq c)$$

and conditioning by h_{n-1} ,

$$P\left(\frac{\beta_n - \gamma_{n-1}}{h_n^2} > c\right) \leq P\left(\frac{\beta_n - \gamma_{n-1}}{h_{n-1}^2} > c\right) \leq P(\tau_W(1) \geq c).$$

The bound of (10) follows immediately. Now, let $\epsilon > 0$ and $\gamma_0 = 0$, we have for any $n \geq 1$,

$$\begin{aligned} P\left(\frac{\gamma_n}{h_n^2} > (1 + \epsilon)^{2n}\right) &\leq P\left(\sum_{k=1}^n \frac{\gamma_k - \gamma_{k-1}}{h_k^2} > (1 + \epsilon)^{2n}\right) \\ &\leq \sum_{k=1}^n P\left(\frac{\gamma_k - \gamma_{k-1}}{h_k^2} > \frac{(1 + \epsilon)^{2n}}{n}\right) \leq \frac{Kn^{3/2}}{(1 + \epsilon)^n}. \end{aligned}$$

Therefore Borel-Cantelli Lemma implies that P -a.s., for n large enough,

$$\gamma_n \leq (1 + \epsilon)^{2n} h_n^2.$$

It is then easy to deduce the upper bound for γ_n . And the lower bound can be obtain easily using same techniques as before. \square

It is now possible to control the asymptotic behaviour of the integral:

Proposition 6.4 *P -almost surely,*

$$\begin{aligned} \liminf_{v \rightarrow \infty} \log_2 v \int_{a_v}^{b_v} e^{-W(x)+W(m_v)} dx &\geq \frac{1}{2e} \text{ and} \\ \limsup_{v \rightarrow \infty} \frac{1}{\log_2 v} \int_{a_v}^{b_v} e^{-W(x)+W(m_v)} dx &\leq \frac{16}{j_0^2}. \end{aligned}$$

Proof: Let us begin with the lower bound. Fix $c > 2e$, according to Lemma 6.2, for any $n \in \mathbf{N}^*$,

$$P\left(\int_{\mu_n}^{\beta_n} e^{-W(x)+W(\mu_n)} dx \leq \frac{1}{c \log(n-1)}\right) \leq K \frac{\sqrt{\log(n-1)}}{(n-1)^{c/2e}}.$$

According to the first Borel-Cantelli Lemma, P -a.s. for n large enough,

$$\int_{\mu_n}^{\beta_n} e^{-W(x)+W(\mu_n)} dx > \frac{1}{c \log(n-1)}.$$

We come back to the m_v . We fix $v > 1$. P -a.s. there is a unique $n \in \mathbf{N}^*$ such that $h_{n-1} < v \leq h_n$. Thus necessarily $m_v = \mu_n$ and $b_v \geq \beta_n$. Therefore

$$\int_{m_v}^{b_v} e^{-W(x)+W(m_v)} dx \geq \int_{\mu_n}^{\beta_n} e^{-W(x)+W(\mu_n)} dx > \frac{1}{c \log(n-1)}.$$

Let $a \in (1, e)$, according to Lemma 6.3, if n is large enough, $h_{n-1} > a^{n-1}$. Thereby P -a.s., for v large enough,

$$\int_{m_v}^{b_v} e^{-W(x)+W(m_v)} dx \geq \frac{1}{c(\log_2 v - \log_2 a)}.$$

When c goes to $2e$, we obtain the result of the proposition.

Now we continue with the upper bound. Fix $c > 16/j_0^2$. One more time thanks to Lemma 6.2 and Borel-Cantelli Lemma, P -a.s. for n large enough,

$$\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)} dx < c \log(n-1).$$

Fix $v > 1$, P -a.s., there is a unique $n \in \mathbf{N}^*$ such that $h_{n-1} < v \leq h_n$. Therefore $m_v = \mu_n$, $b_v \leq M_n$ and taking $1 < a < e < b$,

$$\begin{aligned} \int_{a_v}^{b_v} e^{-W(x)+W(m_v)} dx &\leq \int_0^{M_n} e^{-W(x)+W(\mu_n)} dx \\ &\leq c_n e^{-W(\mu_{n-1})+W(\mu_n)} + \int_{\gamma_n}^{M_n} e^{-W(x)+W(\mu_n)} dx \\ &\leq b^n e^{-a^n} + c \log(n-1) \\ &\leq b^n e^{-a^n} + c(\log_2 v - \log_2 a). \end{aligned}$$

Thus,

$$\limsup \frac{1}{\log_2 v} \int_{a_v}^{b_v} e^{-W(x)+W(m_v)} dx \leq c$$

and when c goes to $16/j_0^2$, this yields to the upper bound. \square

We can now come back to the process \mathbf{L}^* .

6.2 End of the proof of Theorem 1.1

The previous results allow us to know the asymptotic behaviour of \mathbf{L}^* . Using (8) with $v_n = n^{2/3}$ and Borel-Cantelli Lemma, we obtain that, P -almost surely for n large enough,

$$\frac{e^{v_n}}{J_{v_n} + 1/v_n} \leq \mathbf{L}^*(e^{v_n}) \leq \frac{e^{v_n}(1 + 1/v_n)}{j_{v_n}(1 - 1/v_n)}.$$

Thereby Proposition 6.4 gives us

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\mathbf{L}^*(e^{v_n})}{e^{v_n} \log_2 v_n} &\leq \frac{1}{\liminf j_{v_n} \log_2 v_n} \leq 2e \text{ and} \\ \liminf_{n \rightarrow \infty} \frac{\log_2 v_n \mathbf{L}^*(e^{v_n})}{e^{v_n}} &\geq \liminf_{n \rightarrow \infty} \frac{\log_2 v_n}{J_{v_n}} \geq \frac{j_0^2}{64}. \end{aligned}$$

Moreover as \mathbf{L}^* is non decreasing, denoting by $[x]$ the integer part of x ,

$$\begin{aligned} \frac{j_0^2}{64} &\leq \liminf_{v \rightarrow \infty} \frac{\mathbf{L}^*(e^{\lfloor v^{3/2} \rfloor^{2/3}})}{e^{\lfloor v^{3/2} \rfloor^{2/3}}} \leq \liminf_{v \rightarrow \infty} \frac{\mathbf{L}^*(e^v)}{e^v} \text{ et} \\ \limsup_{v \rightarrow \infty} \frac{\mathbf{L}^*(e^v)}{e^v} &\leq \limsup_{v \rightarrow \infty} \frac{\mathbf{L}^*(e^{(\lfloor v^{3/2} \rfloor + 1)^{2/3}})}{e^{(\lfloor v^{3/2} \rfloor + 1)^{2/3}}} \leq 2e \end{aligned}$$

and we have shown the first part of Theorem 1.1.

For the second part, we use this time (9) with $v_n = e^n$ and Borel-Cantelli Lemma. We then have, P -almost surely for n large enough,

$$\frac{e^{v_n}}{I_{v_n}^- + \widehat{I}_{v_n}^- + 1/n} \leq \mathbf{L}^*(e^{v_n}) \leq \frac{e^{v_n}(1 + 1/n)}{I_{v_n}^- \wedge \widehat{I}_{v_n}^- (1 - 1/n)}.$$

Lemma 6.1 leads directly to

$$\liminf_{v \rightarrow \infty} \frac{\mathbf{L}^*(e^v)}{e^v} \leq e\pi^2.$$

And the proof of the theorem is completed.

7 What can be added

The methods we used allow us to obtain an other result on the position of the favorite point of the diffusion. We denote by $m^*(t)$ the favorite point with the smaller modulus for the diffusion \mathbf{X} . Using a method similar to the one in the proof of Proposition 5.2, it can be shown that P -a.s., for v large enough, there exists a real $m_v^\# \in \{m_{v,1}^{R,-}, m_v^{R,+}, \widehat{m}_{v,1}^{R,-}, \widehat{m}_v^{R,+}\}$, such that with natural notations, $m_{e^v}^* \in [a_v^\#, b_v^\#]$ and

$$0 \leq W(m_{e^v}^*) - W(m_v^\#) \leq 2 \log \frac{r(v)}{R(v)} \leq 2 \log_3 v.$$

We can also prove that for any $\epsilon > 0$, P -a.s., for v large enough, for any $t \in [a_v, m_v - (\log v)^{2+\epsilon}] \cup [m_v + (\log v)^{2+\epsilon}, b_v]$,

$$W(t) - W(m_v) > 3 \log_3 v.$$

Hence we obtain the following:

Theorem 7.1 *Let c be a real number strictly larger than 2. P -almost surely,*

$$\limsup_{t \rightarrow \infty} \frac{|m^*(t) - m_{\log t}^\#|}{(\log_2 t)^c} = 0.$$

This theorem has to be compared with the corollary in the introduction of [4], note that accuracy gained in time is lost in space.

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