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On the vertical structure of wave forcing for the ocean circulation

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ABSTRACT

The conservation of momentum, when averaged over the phase of surface gravity waves can take two forms, whether or not the momentum variable contains the wave pseudo-momentum. The vertical profiles of the resulting wave-induced forces are discussed, with application to realistic condition. It was already proved that forces for the total momentum that use analytical functions of the local wave properties are necessarily inconsistent, and thus inaccurate at the lowest order. The consequences of these inaccuracies are explored here. In inviscid conditions, it is shown that large spurious currents of the order of 10 times the Stokes drift are generated on a sloping bottom, however small that slope is. These spurious velocities are reduced but are still significant when a strong vertical mixing is applied. In contrast, forces for the quasi-Eulerian mean momentum do not suffer from this inconsistency, and accurate numerical models can be developed. Choosing to solve for the quasi-Eulerian mean flow is also intrinsically more simple.

1. Introduction

The wave-averaged conservation of momentum can take two forms, whether or not the momentum variable contains the wave pseudo-momentum or not¹. With some approximations related to wave non-linearity, this question has been properly treated for depth-integrated mass and momentum balances (Longuet-Higgins and Stewart 1964; Garrett 1976; Smith 2006). The vertical profiles of the mass and momentum balances is much more complex and has lead to a number of investigations, and some misconceptions. Most of the difficulties arise from the vertical flux of wave momentum, which is often forgotten although it can be dominant. The purpose of the present paper is to review these different approaches, organize them logically, and expose a practical and correct way of extending three-dimensional primitive equations to include the phase-average effect of surface gravity waves. This takes the form of forcing terms in the momentum equations, and in the surface boundary condition for the mass conservation. We also wish to illustrate the errors induced by erroneous vertical profiles of the wave-induced forcing. A proper averaging of oceanic flows over the wave phases needs to address two essential difficulties, as summarized by figure 1.

¹The wave pseudo-momentum is defined as a quantity that only involves the zero-mean displacement of the water particles, and may differ from other definitions that could include the mean flow response, as explained by McIntyre (1981). For simplicity, we shall call 'momentum' the pseudo-momentum.

First, because of the large difference in density between air and water, the use of a standard Eulerian average, used by Rivero and Arcilla (1995) or Newberger and Allen (2007) is problematic in the region between crests and trough where both air and water are to be found. This Eulerian approach thus requires empirical ad hoc surface boundary conditions. Mathematical extension of the velocity field across the interface have been used by McWilliams et al. (2004), but it provides quantities that are difficult to interpret physically. Ardhuin et al. (2008b) showed that the resulting velocity actually corresponds to the quasi-Eulerian velocity ($\hat{u}, \hat{v}, \hat{w}$) first introduced by Jenkins (1989).

A surface-following coordinate system should be used to properly resolve the air-sea interface in the phase-averaged equations. Such averages naturally produce Lagrangian mean velocities, (U, V, W) , which can introduce some difficulties. First of all, the slow displacement of particles can be avoided by either following only the vertical motion (Mellor 2003; Ardhuin et al. 2008a), or by removing the horizontal mean drift by some other way, which is the case of the Generalized Lagrangian Mean (GLM) operator defined by Andrews and McIntyre (1978). The distorted coordinates that give divergent flows in GLM can be corrected by a vertical transformation, making the (U, V, W) flow non-divergent (Ardhuin et al. 2008b).

Second, and much more subtle, the evolution equation for the horizontal velocity components U and V , involves an acceleration term due to the vertical advection of wave

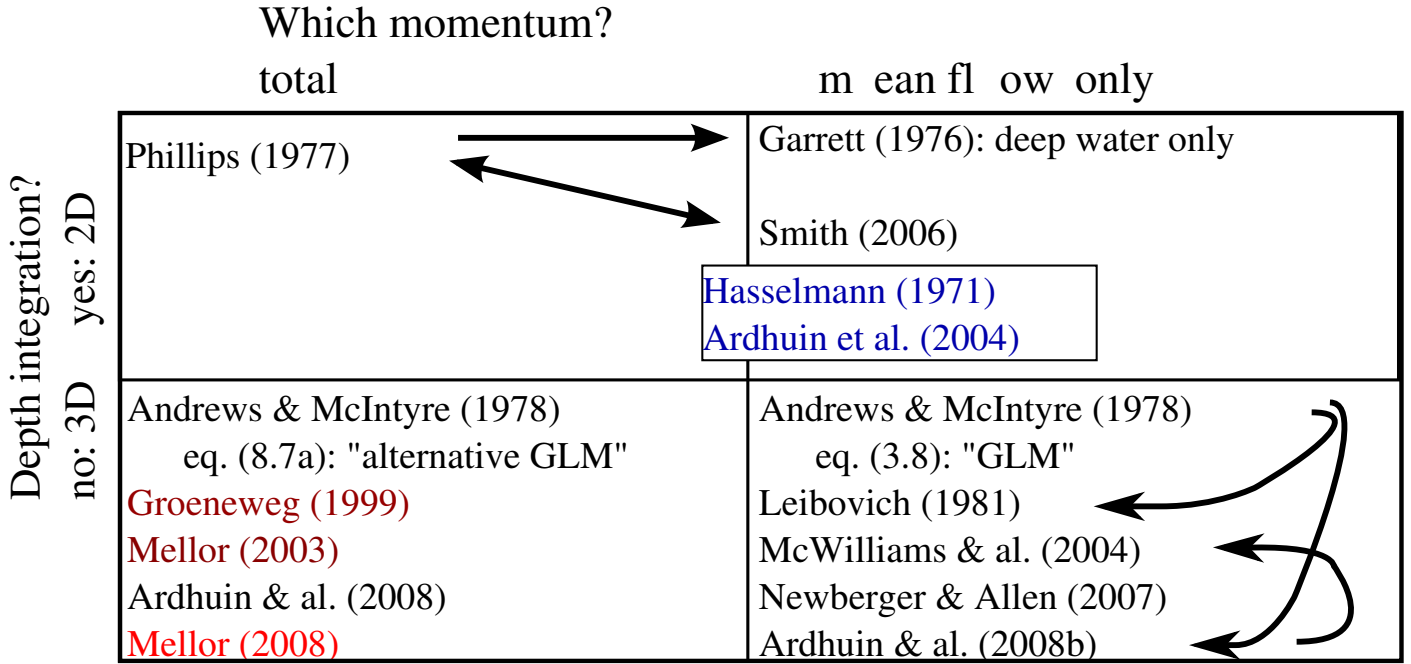


FIG. 1. General organization of wave-averaged theories according to their choice of momentum variable and depth integration. Names that appear in different shades of red correspond to theories that are not fully consistent with their originating hypotheses. In the case of 3D theories for the total momentum, the problem generally comes from the vertical flux of momentum and may only arise on a sloping bottom, not explicitly considered by Groeneweg (1999). In the case of Mellor (2008), the inconsistency arises from different averagings of different terms in the same equation, resulting in a strange mix of Eulerian and Lagrangian mean equations. Other theories not listed here are even less consistent and completely forgot about this vertical flux. The theory of Hasselmann (1971), extended by Arduin et al. (2004) does not quite fit the decomposition in mean flow and wave momentum as the momentum fluxes between the two (the interaction stresses) are not the same as the known fluxes (Ardhuin 2006). An arrow from a to b indicates a derivation link: b can be derived from a .

pseudo-momentum² $\rho_w U_s$ and $\rho_w V_s$. Because of the nature of the wave field, this vertical flux happens to be a non-local function of the water depth and thus requires a very complex wave model to be estimated properly (Ardhuin et al. 2008a). As a result, the evolution equation for U and V are not practical. Because the problematic vertical flux is a flux of wave momentum, this difficulty disappears in the case of evolution equations for the quasi-Eulerian momentum, defined by

$$(\hat{u}, \hat{v}, \hat{w}) = (U, V, W) - (U_s, V_s, W_s). \quad (1)$$

A simplified version of such an equation is given in section 2. In section 3, we illustrate the momentum balance and resulting inviscid flow for non-breaking shoaling waves, for which an exact solution exists. Because the erroneous equation set of Mellor (2003) for (U, V, W) has already been

²For the sake of simplicity we will ignore differences between the wave pseudo-momentum, as defined by Andrews and McIntyre (1978), and the more usual notion of Stokes drift, and thus chose the usual U_s notation. We will also omit below the ‘pseudo-’ prefix.

used in a numerical modelling publication by Haas and Warner (2009), we also illustrate on this shoaling example the strong spurious velocity that they produce, casting some doubt on the validity of similar numerical studies.

2. equations derived using the Generalized Lagrangian Mean

Starting from the general GLM equations of Andrews and McIntyre (1978), Arduin et al. (2008b) have given a ‘glm2- z ’ approximation: to second order in the wave non-linearity and with a change of the vertical coordinate. The resulting Jacobian associated with the averaging procedure is equal to 1, and both the resulting quasi-Eulerian flow field $(\hat{u}, \hat{v}, \hat{w})$ and Lagrangian-mean flow field (U, V, W) are non-divergent. We shall here consider the case in which the wave bottom boundary layer is not resolved. For a discussion of this, see Arduin et al. (2008b). To simplify the equations we generally give the wave forcing expressions for monochromatic waves as a function of the surface elevation

variance E . In the case of quasi-linear random waves the corresponding forcing is simply the sum of the monochromatic wave forcing with E replaced by the spectral density $E(f, \theta)$, as detailed in appendix B. The following glm2- z equations use the vertical z coordinate. Equations with the vertical ς coordinate are detailed in appendix A.

a. Momentum, mass, and tracer conservation

Adopting the notation defined here, and, for simplicity, neglecting the effect of the vertical current shear and partial standing waves in the wave orbital motion³, equation (42) in Arduin et al. (2008b) becomes

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial x} + \hat{v} \frac{\partial \hat{u}}{\partial y} + \hat{w} \frac{\partial \hat{u}}{\partial z} - f \hat{v} + \frac{1}{\rho} \frac{\partial p^H}{\partial x} - F_{m,x} \\ = \left[f + \left(\frac{\partial \hat{v}}{\partial x} - \frac{\partial \hat{u}}{\partial y} \right) \right] V_s - W_s \frac{\partial \hat{u}}{\partial z} \\ - \frac{\partial J}{\partial x} + F_{d,x}, \end{aligned} \quad (2)$$

and

$$\begin{aligned} \frac{\partial \hat{v}}{\partial t} + \hat{u} \frac{\partial \hat{v}}{\partial x} + \hat{v} \frac{\partial \hat{v}}{\partial y} + \hat{w} \frac{\partial \hat{v}}{\partial z} + f \hat{u} + \frac{1}{\rho} \frac{\partial p^H}{\partial y} - F_{m,y} \\ = - \left[f + \left(\frac{\partial \hat{v}}{\partial x} - \frac{\partial \hat{u}}{\partial y} \right) \right] U_s - W_s \frac{\partial \hat{v}}{\partial z} \\ - \frac{\partial J}{\partial y} + F_{d,y}, \end{aligned} \quad (3)$$

where the left hand side is the classical primitive equation model with p^H the hydrostatic pressure, $(F_{m,x}, F_{m,y})$ the mixing effects, and $(F_{d,x}, F_{d,y})$ the force induced by the wave dissipation. Mixing is also influenced by waves, but this aspect will not be discussed here (see Craig and Banner 1994; Groeneweg and Klopman 1998; Rascle and Arduin 2009). The second lines in eq. (2)–(3) are the vortex force introduced by Garrett (1976) in this context, and further discussed by Lane et al. (2007) and Smith (2006).

The mass conservation is

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{w}}{\partial z} = 0, \quad (4)$$

and the evolution of a conservative passive tracer concentration C is,

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x} [(\hat{u} + U_s)C] + \frac{\partial}{\partial y} [(\hat{v} + V_s)C] + \frac{\partial}{\partial z} [(\hat{w} + W_s)C] = 0. \quad (5)$$

All four conservation equations are valid from the bottom $z = -h$ to the local phase-averaged free surface $\hat{\eta}$.

³These simplified equations may be inconsistent due to the fact that the right hand side advection term $W_s \partial \hat{u} / \partial z$ is of the same order as the terms that arise from the vorticity of the wave motion induced by the current shear: these extra terms are neglected here but are given by McWilliams et al. (2004) and Arduin et al. (2008b). These terms are not dominant for mild current shears and bottom slopes.

b. Wave-induced forcing terms

The three-component Stokes drift (U_s, V_s, W_s) , wave-induced pressure term J , and momentum source due to wave dissipation $(F_{d,x}, F_{d,y})$, can all be computed from the local parameters. These include the wave-induced surface elevation variance E , the phase-averaged water depth $D = h + \hat{\eta}$, the wavenumber vector $\mathbf{k} = k(\cos \theta, \sin \theta)$, the intrinsic radian frequency $\sigma = \sqrt{gk \tanh(kD)}$, the bottom slope ... For random waves, these expression are easily extended by summing over the spectrum and replacing E by the spectral density $E(f, \theta)$ (see Appendix B).

The horizontal Stokes drift vector (U_s, V_s) is given by,

$$(U_s, V_s) = \sigma k (\cos \theta, \sin \theta) E \frac{\cosh(2kz + 2kh)}{\sinh^2(kD)}, \quad (6)$$

The less well-known vertical Stokes drift component happens to be given, at lowest order (e.g. Arduin et al. 2008a), by the horizontal divergence of (U_s, V_s) ,

$$W_s(z) = -U_s|_{z=-h} \frac{\partial h}{\partial x} - V_s|_{z=-h} \frac{\partial h}{\partial y} - \int_{-h}^z \frac{\partial U_s}{\partial x} + \frac{\partial V_s}{\partial y} dz. \quad (7)$$

In adiabatic conditions, the only other term is the wave-induced mean pressure J is,

$$J = g \frac{kE}{\sinh 2kD}. \quad (8)$$

No definite theory exists for the force induced by wave dissipation $(F_{d,x}, F_{d,y})$, as only the depth-integrated force is known (e.g. Smith 2006). An empirical parameterization for the vertical profile must be used. We may clearly distinguish between the force due to wave breaking and that due to bottom dissipation (Walstra et al. 2000). We know S_{oc} , the amount of energy given up by waves as they break, either in finite depth or deep water (e.g. Thornton and Guza 1983; Arduin et al. 2009b), and S_{bf} , the loss of energy due to bottom friction (e.g. Arduin et al. 2003). With a strong vertical mixing due to breaking waves, the vertical distribution of the momentum source is not very important (Rascle et al. 2006). One may parameterize the effect of wave dissipation as a surface stress, with a vertical profile given by the delta function $\delta_{z, \hat{\eta}}$,

$$\begin{aligned} (F_{d,x}, F_{d,y})(z) &= (\tau_{oc,x}, \tau_{oc,x}) \delta_{z, \hat{\eta}} \\ &= \int \frac{k}{\sigma} (\cos \theta, \sin \theta) S_{oc}(f, \theta) \delta_{z, \hat{\eta}} df d\theta, \end{aligned} \quad (9)$$

where $S_{oc}(f, \theta)$ is the spectral density of the waves-to-ocean energy flux (e.g. Arduin et al. 2009a), approximately equal to the dissipation source function in the spectral wave energy balance. Some authors have used a linear profile for $(F_{d,x}, F_{d,y})$ (Walstra et al. 2000). Because

the wave bottom boundary layer is not resolved, the effect of bottom friction is represented by the bottom boundary condition for the velocity (Longuet-Higgins 2005). If the wave bottom boundary layer were resolved, then one could introduce the source of momentum

$$(F_{d,x}, F_{d,y})(z) = \int \frac{k}{\sigma} (\cos \theta, \sin \theta) S_{bf}(f, \theta) G(z) df d\theta \quad (10)$$

near the bottom, where $G(z)$ is a function that integrates to 1 across the wave bottom boundary layer.

c. Boundary conditions

Starting from the bottom, at $z = -h$, for a non-resolved wave bottom boundary layer, the horizontal velocity should be prescribed as velocity at the bottom given by the streaming solution of (Longuet-Higgins 1953), still approximately valid for turbulent bottom boundary layers (e.g. Marin 2004),

$$(\hat{u}, \hat{v})|_{z=-h} = 1.5 (U_s, V_s)|_{z=-h} \quad (11)$$

and the vertical velocity is naturally

$$\hat{w} = -\hat{u} \frac{\partial h}{\partial x} - \hat{v} \frac{\partial h}{\partial y}. \quad (12)$$

The bottom stress can then be parameterized as

$$K_z \frac{\partial \hat{u}}{\partial z} \Big|_{z=-h} = \frac{\kappa^2}{\ln[(z+h)/z_0]} \Delta_u \sqrt{\Delta_u^2 + \Delta_v^2} \quad (13)$$

where $\Delta_u = \hat{u}(z) - \hat{u}(-h)$ and $\Delta_v = \hat{v}(z) - \hat{v}(-h)$, and K_z is the (varying) eddy viscosity. The bottom roughness for the current, z_0 , may be estimated with the model by Mathisen and Madsen (1996), which takes into account bottom streaming.

At the surface, the stresses are imposed, giving the upper boundary condition for the turbulent momentum flux,

$$K_z \frac{\partial \hat{u}}{\partial z} \Big|_{z=\hat{\eta}} = \tau_{a,x} - \tau_{aw,x} \quad (14)$$

where $\tau_{a,x}$ and $\tau_{aw,x}$ are, respectively, the x -component of the wind stress and of the wave-supported stress

$$(\tau_{aw,x}, \tau_{aw,y}) = \int \frac{k}{\sigma} (\cos \theta, \sin \theta) S_{atm}(f, \theta) df d\theta, \quad (15)$$

where $S_{atm}(f, \theta)$ is the spectral density of the wind to wave energy flux (e.g. Ardhuin et al. 2009a), approximately equal to the input source function in the spectral wave energy balance.

Finally, the surface kinematic boundary condition is given by

$$\frac{\partial \hat{\eta}}{\partial t} + (\hat{u} + U_s) \frac{\partial \hat{\eta}}{\partial x} + (\hat{v} + V_s) \frac{\partial \hat{\eta}}{\partial y} = \hat{w} + W_s. \quad (16)$$

TABLE 1. List of wave-forcing terms required to force an ocean circulation model solving for the quasi-Eulerian velocity. The J term is a 2D field when the effect of the vertical shear of the quasi-Eulerian current is neglected, as done here. In general J is a 3D forcing field (Ardhuin et al. 2008a).

term	type	see eq.
U_s	3D	6
V_s	3D	6
J	2D or 3D	8
$\tau_{aw,x}$	2D	15
$\tau_{aw,y}$	2D	15
$\tau_{oc,x}$	2D	9
$\tau_{oc,y}$	2D	9

For tracers (temperature, salinity, particulate or dissolved constituents ...), because the equations are unchanged (only for the explicit appearance of the Lagrangian mean velocity), the boundary conditions are unchanged from classical primitive equation models.

d. Summary of new terms introduced

As we have not discussed here the effects of waves on the turbulence closure, the forcing of the wave field on the ocean circulation requires the knowledge of all the fields listed in table 1.

Compared to equations for the Lagrangian mean velocity, such as those by Mellor (2003), the amount of data to be transferred is significantly reduced, since the latter form requires, the 3D fields S_{xx} , S_{yy} and S_{xy} , as well as the 3D fields U_s and V_s to correct the velocities before applying the turbulence closure (e.g. Walstra et al. 2000). This lower complexity of the quasi-Eulerian equations for the 3D case is contrary to the 2D case, in which seven 2D fields are needed, versus three to five for the depth-integrated Lagrangian equations. In both cases, for a full consistency of the ocean circulation and wave model, one should also use the wind stress of the wave model, and the surface flux of turbulent kinetic energy as discussed by Janssen et al. (2004), and a proxy of the breaking wave heights, possibly the wind sea wave height (Rasclé et al. 2008).

3. Example case of shoaling waves

a. Steady wave forcing

The first test of a 3D wave-forced model should be in conditions where the results are known, typically in the absence of dissipative effects. Such a test was proposed by Ardhuin et al. (2008b) with steady monochromatic waves shoaling on a slope without breaking nor bottom friction

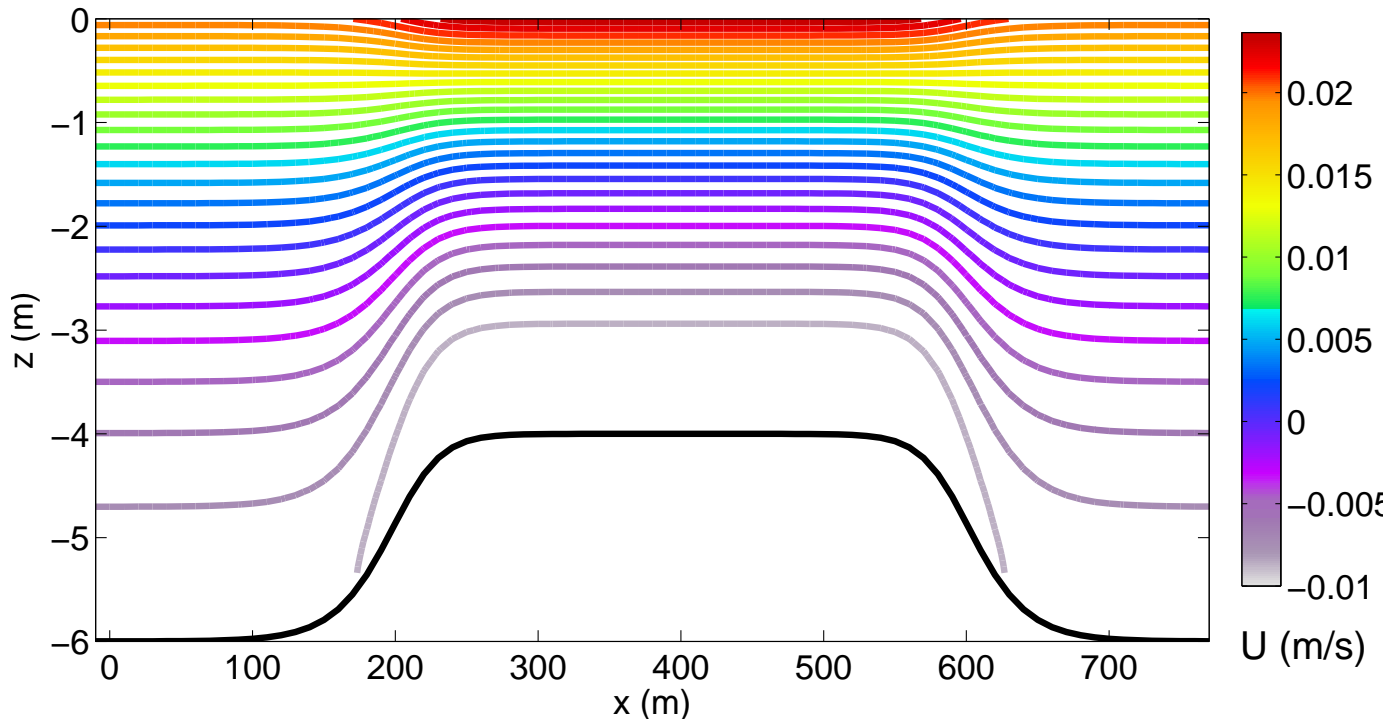


FIG. 2. Lagrangian velocity U for the inviscid sloping bottom case with $a = 0.36$ m and $T = 5.24$ s, obtained from the quasi-Eulerian analysis as $U = \hat{u} + U_s$. Contours are equally spaced from -0.001 to 0.025 m s $^{-1}$. The thick black line is the bottom elevation.

and for an inviscid fluid. The bottom slopes smoothly from a depth $D = 6$ to $D = 4$ m. Compared to Arduin et al. (2008) the bottom was extended by its symmetric, sloping back down to 6 m, in order to allow periodic boundary conditions if needed (figure 2). Taking the other parameters unchanged, we consider small incident wave amplitude of 0.12 m, and a period of 5.24 s. This was chosen to give a wave steepness $\epsilon_1 = 0.0266$, equal to the maximum bottom slope ϵ_2 . As a result the group velocity C_g varies little (5.4%) from 4.89 to 4.64 m s $^{-1}$, due to the fact that the non-dimensional depth kD is close to one. Because the current is much less than the group speed, the waves propagate with a nearly constant energy flux $C_g E$, resulting in a small increase of wave amplitude, by 2.7%, in the shallower part of the domain. The stationary wave elevation variance E is simply inferred from this constant energy flux.

The Eulerian analysis of the stationary situation was given by Longuet-Higgins (1967), who showed that the mean water level should be 0.32 mm lower in the shallow region. Both studies by Rivero and Arcilla (1995) and Lane et al. (2007) clearly show that there is no other dynamical effect: the Eulerian mean current is steady and simply compensates for the divergence of the wave-induced mass transport. The stationary numerical solution is given by Arduin et al. (2008b). Because the relative variation in

phase speed is more important, from 6.54 to 5.65 m s $^{-1}$, it produces a strong divergence of the Stokes drift, which accelerates in shallow water. The Eulerian velocity \hat{u} is irrotational and thus nearly depth-uniform. \hat{u} can be computed exactly by solving the Laplace equation to second order in the wave slope (e.g. Arduin et al. 2008b). For our practical purpose, \hat{u} is very nearly equal to the depth-uniform current with a convergence that compensates for the Stokes drift divergence. The Lagrangian velocity U is given by the sum of the two steady velocity fields, as shown in figure 2.

We now solve for the equations by Mellor (2003) who, if correct, should lead to the steady Lagrangian velocity shown in figure 2. In order to obtain the numerical solution we use a coupled modelling system that combines the WAVEWATCH III numerical wave model (Tolman 2008, 2009) and the MARS3D ocean circulation model (Lazure and Dumas 2008), coupled by the automatic coupler PALM (Buis et al. 2008). In order to simplify comparisons by others, we have de-activated the feedback from the flow to the waves.

The numerical simulation is non-stationary: starting from rest, a steady wave field propagates from left to right, quickly filling up the entire domain, and then becoming stationary. Here the flow boundary conditions are open.

The monochromatic wave amplitude $a = 0.12$ m translates, in the case of random waves with the same energy, in a significant wave height H_s of 0.34 m, which is small. We will thus also test the model with a higher amplitude $a = 0.36$ m, i.e. $H_s = 1.02$ m, still far from the breaking limit in 4 m depth.

The MARS3D model domain uses 100 sigma levels regularly spaced, 5 active points in the y direction, with 2 extra wall points, and 2 ghost points needed to define finite differences, and 78 active points in the x direction. The time step was set to 0.05 s for the $H_s = 1.02$ m tests (and 1 s for $H_s = 0.34$ m). For the sake of simplicity, the wave model time step is taken equal to the flow model time step and they are coupled at each time step. The first tests are done without any bottom friction nor internal mixing.

For this case, we use the equation in ζ coordinate, with ζ defined by $z = s(x, \zeta, t) = \hat{\eta} + \zeta D + \tilde{s}$. The fast time-varying part \tilde{s} is defined such that there is now wave-induced vertical velocity fluctuations in ζ coordinate: the coordinate follows the rapid up-and-down motion of material surfaces shaken by waves (Mellor 2003; Ardhuin et al. 2008b). The equation of motion, where we have neglected the Coriolis force, density stratification, and mixing, are given by Mellor (2003),

$$\frac{\partial(DU)}{\partial t} + \frac{\partial(DU^2)}{\partial x} + \frac{\partial(\Omega U)}{\partial \zeta} = F, \quad (17)$$

or, equivalently, using the mass conservation equation,

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + \frac{\Omega}{D} \frac{\partial U}{\partial \zeta} = \frac{F}{D}. \quad (18)$$

The forcing on the right hand side is

$$F = F_{eta} + F_{xx} + F_{x3} \quad (19)$$

with $F_{eta} = -gD\partial\hat{\eta}/\partial x$, the hydrostatic pressure gradient, in which $\hat{\eta}$ is the phase-averaged surface elevation. The other two terms in F are the divergences of the horizontal and vertical wave-induced fluxes of momentum, F_{xx} and F_{x3} , given in Appendix B. A correct estimation of F_{x3} , such as provided by Ardhuin et al. (2008a), produced a zero total force F/D for a steady wave field in the absence of dissipation (Ardhuin et al. 2008a). However, because Mellor (2003) used only the flat-bottom Airy theory to solve for the wave motion, he obtained an erroneous estimation of F_{x3} (see Appendix B). As a result the total force F/D according to Mellor (2003) integrates to zero over the vertical, but it has a vertical profile that exceeds 150% of the pressure gradient F_{eta} , instead of the correct value of zero.

In our case the wave field is steady after about 5 minutes, the time it takes for waves to propagate across the domain, and thus F is stationary after that time. On the upslope the steady forcing F is in the direction of wave

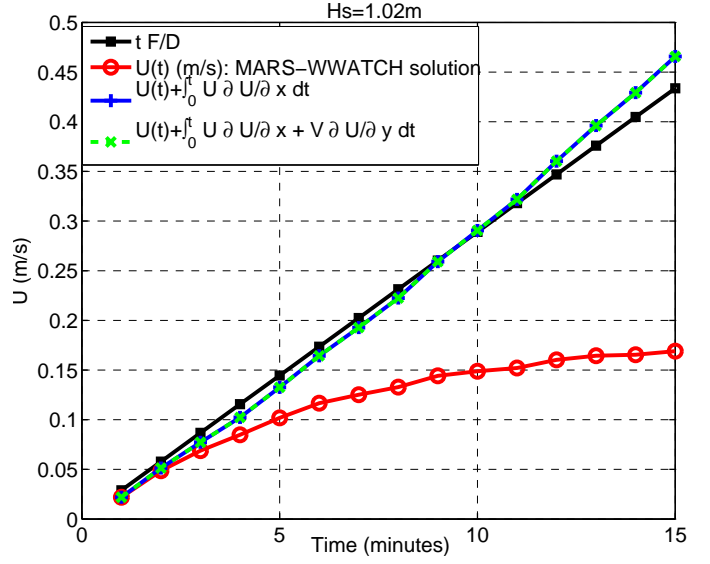


FIG. 3. Langrangian velocity $U(t)$ at the surface $z = \hat{\eta}$ and at the position $x = 610$ m, in the case $H_s = 1.02$ m and $T = 5.24$ s, without mixing. The MARS-WWATCH solution is compared to the linear trend given by a constant acceleration F/D , and the numerical integration of F/D .

propagation at the surface, and in the opposite direction at the bottom. Thus the Lagrangian velocity U increases linearly until the advection becomes significant. This is what the coupled model produces (figure 3). This surface velocity is associated to a countercurrent below (figure 4), producing a circulation pattern much stronger than the known correct solution (figure 2).

In spite of the small bottom slope and wave steepness, the resulting velocity reaches 17 cm s^{-1} in only 15 minutes, which is about 10 times the correct solution shown in figure 2. Further, if the model is integrated for a longer time, the region of positive velocity generated on the upslope meets the region of negative velocity generated on the downslope, resulting in large vertical velocities and further strange model adjustments. Figure 4 also shows that the solution at the surface is similar to a simple Lagrangian integration,

$$U(t) \simeq \int_0^t -U \frac{\partial U}{\partial x} + \frac{F}{D}, \quad (20)$$

where the only neglected terms in (18) are the advection along the y and z directions.

As shown by Ardhuin et al. (2008a), the forcing term F is proportional to $gD\varepsilon_1^2\varepsilon_2$, where ε_1 is the wave steepness, ε_2 is the maximum bottom slope. For the bottom shape and wave period chosen here, the maximum value of F is $0.29gD\varepsilon_1^2\varepsilon_2$. Obviously, the depth dependence of F plays an important role and F becomes depth-uniform for $kD \rightarrow$

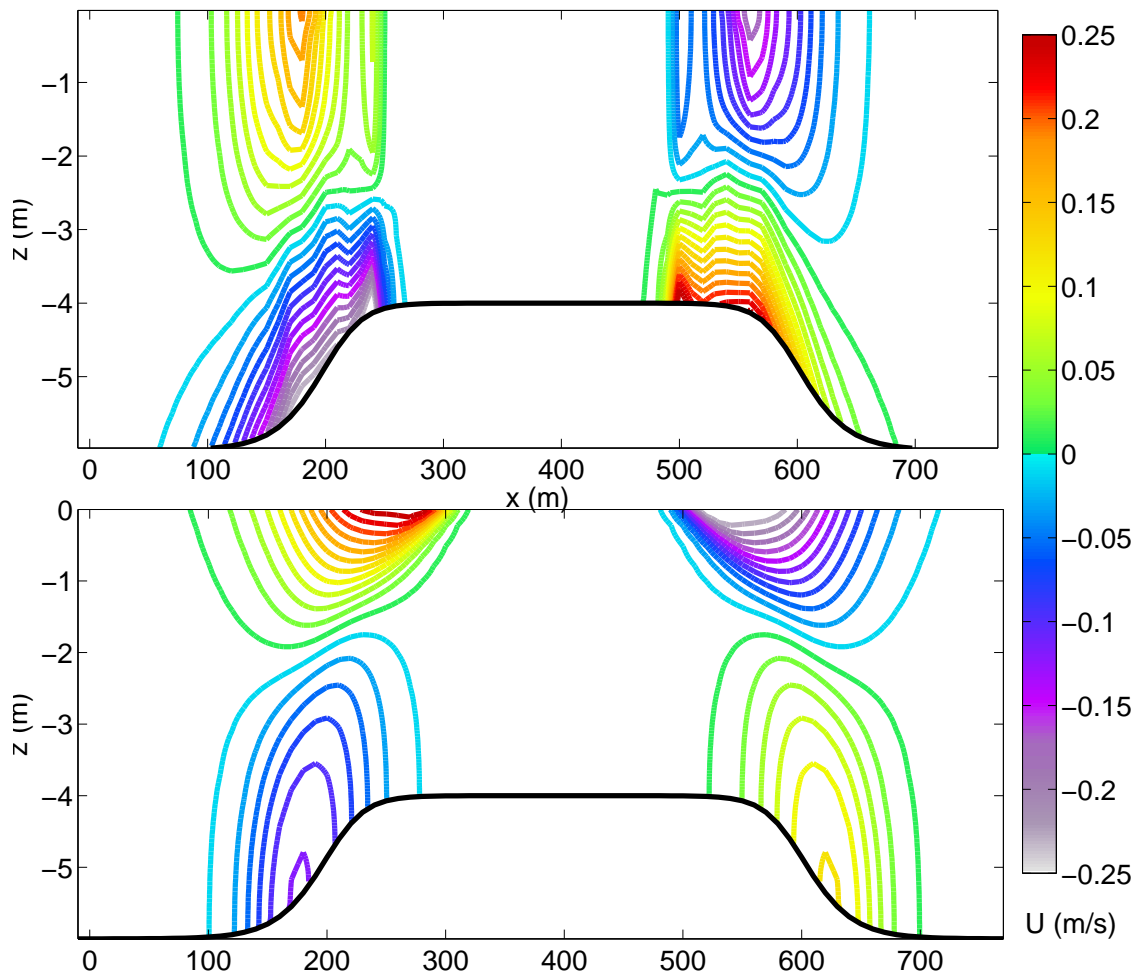


FIG. 4. Top panel: velocity U solution of the coupled model after 15 minutes of integration, bottom panel: simple integration in time of $F/D - U\partial U/\partial x$, using a first order Euler scheme coded in Matlab with the same time step as used in MARS3D.

0, so that one may expect that the problem could vanish in shallow water.

Unfortunately, in practice, the velocity at which the current first stabilizes (here after 15 minutes), is *independent* of ε_2 , provided that the change in water depth remains the same. If the bottom topography is stretched by a factor $1/\alpha$ in the x direction, the slope changes by a factor α and the change in advection makes up for the local modification of F by a factor α . Mathematically, equation (18) follows a Froude scaling: when x is replaced by $x' = \alpha x$ and t by $t' = \alpha^2 t$, the equation is unchanged if $F' = \alpha F$, and thus $U(x', t') = U(x, t)$.

As a result, for any wave field approaching the shore from deep water, even on a very gently sloping continental shelf, there will be a very large spurious onshore velocity at the surface. Based on the present case, this velocity should be at least of the order of 10 to 20 times the Stokes drift.

This surface momentum is generated where $kD \sim 1$, and self-adverts onshore. Obviously, some realistic mixing will reduce this effect. Using a large constant eddy viscosity of 2.8×10^{-3} only reduces the current by about a factor 2 to 3 (see table 2). This reduction factor depends on the bottom slope since the introduction of viscosity breaks the Froude scaling.

4. Conclusion

Any numerical model of the three-dimensional wave-forced circulation produces spurious velocities if it is formulated in terms of the Lagrangian mean velocity (total momentum) and uses analytical functions of the local wave field and topography: this is because the vertical flux of momentum is a non-local function of the water depth (Arduin et al. 2008b). We have shown here the very large magnitude of these spurious velocities when solving for the equa-

TABLE 2. Surface velocity at $x = 610$ m for different model settings. The settings corresponding to the test in Arduin et al. (2008b) are given in the second line $H_s = 0.34m$, $T = 5.6s$, $\nu_t = 0 \text{ m}^2 \text{ s}^{-1}$. The surface velocity values are written for $T = 900s$ where $Hs = 1.02m$ and for $T = 2700s$ where $Hs = 0.34m$.

$H_s(m)$	$T_p(s)$	$\nu_t(m^2.s^{-1})$	resulting $\bar{U}(m.s^{-1})$
1.02	5.6	0	0.1698
0.34	5.6	0	0.0537
0.34	13	0	0.0110
1.02	5.6	$2.8 \cdot 10^{-3}$	0.1094
0.34	5.6	$2.8 \cdot 10^{-3}$	0.0185
0.34	13	$2.8 \cdot 10^{-3}$	0.0026

tions proposed by Mellor (2003). This error likely plays a big part in the differences in vertical velocity profiles reported by Haas and Warner (2009), when comparing a version of ROMS solving the Mellor (2003) equations with SHORCIRC (see their figure 4). Another difference is due to the fact that SHORCIRC solves for the quasi-Eulerian mean velocity $U - U_s$ while the other model solves for the Lagrangian mean velocity U .

We can also mention that the equations proposed for U by Walstra et al. (2000) would give an incorrect set-down, and probably an opposite spurious acceleration because they have completely neglected the vertical flux term similar to S_{x3} that is present in the alternative GLM equation by Andrews and McIntyre (1978), and from which their equations are derived. There is thus no acceptable short-cut to a 3D equation for the Lagrangian velocity U : the only possibility would be to solve for the wave orbital motion to first order in the bottom slope. This requires a model of the kind developed by Athanassoulis and Belibassakis (1999) and Gerosthathis et al. (2005), with at least 10 vertical modes. Given the large effort required for a 4 by 4 km region with only 3 modes (Magne et al. 2007), this is hardly a practical solution. The only practical solution is thus the use of a momentum equation for the quasi-Eulerian velocity, such as proposed by McWilliams et al. (2004), Newberger and Allen (2007), or Arduin et al. (2008a).

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GLM equations in sigma coordinates : Momentum, mass, tracer conservation

When transforming to a vertical ζ coordinate (2) and (3) become

$$\begin{aligned}
\frac{\partial \hat{u}}{\partial t} &+ \hat{u} \frac{\partial \hat{u}}{\partial x} + \hat{v} \frac{\partial \hat{u}}{\partial y} + \widehat{W} \frac{\partial \hat{u}}{\partial \zeta} - f \hat{v} \\
&+ \frac{1}{\rho} \left(\frac{\partial p^H}{\partial x} + \frac{\partial p^H}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} \right) - F_{m,x} \\
&= \left[f + \left(\frac{\partial \hat{v}}{\partial x} + \frac{\partial \hat{v}}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} \right) \right] V_s \\
&- \left(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{u}}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial y} \right) V_s - \frac{W_s}{D} \cdot \frac{\partial \hat{u}}{\partial \zeta} \\
&- \frac{\partial J}{\partial x} - \frac{\partial J}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} + F_{d,x}, \tag{A1}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \hat{v}}{\partial t} &+ \hat{u} \frac{\partial \hat{v}}{\partial x} + \hat{v} \frac{\partial \hat{v}}{\partial y} + \widehat{W} \frac{\partial \hat{v}}{\partial \zeta} + f \hat{u} \\
&+ \frac{1}{\rho} \left(\frac{\partial p^H}{\partial y} + \frac{\partial p^H}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial y} \right) - F_{m,y} \\
&= - \left[f + \left(\frac{\partial \hat{v}}{\partial x} + \frac{\partial \hat{v}}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} \right) \right] U_s \\
&+ \left(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{u}}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial y} \right) U_s - \frac{W_s}{D} \cdot \frac{\partial \hat{v}}{\partial \zeta} \\
&- \frac{\partial J}{\partial y} - \frac{\partial J}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial y} + F_{d,y}, \tag{A2}
\end{aligned}$$

where

- $\zeta = \frac{z - \hat{\eta}}{D}$ is the sigma coordinate with $\hat{\eta}$ the mean elevation, h the bottom depth and $D = \hat{\eta} + h$ the mean water column depth,
- $\widehat{W} = \left(\frac{\partial \zeta}{\partial t} + \hat{u} \frac{\partial \zeta}{\partial x} + \hat{v} \frac{\partial \zeta}{\partial y} + \frac{\hat{w}}{D} \right)$,
- $\frac{\partial \zeta}{\partial x} = \frac{1}{D} \cdot \frac{\partial h}{\partial x} - \frac{\zeta}{D} \cdot \frac{\partial D}{\partial x}$,
- $\frac{\partial \zeta}{\partial y} = \frac{1}{D} \cdot \frac{\partial h}{\partial y} - \frac{\zeta}{D} \cdot \frac{\partial D}{\partial y}$,
- $\frac{\partial \zeta}{\partial t} = -\frac{\zeta}{D} \cdot \frac{\partial \hat{\eta}}{\partial t}$.

The mass conservation becomes

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{u}}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} + \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{v}}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial y} + \frac{1}{D} \cdot \frac{\partial \hat{w}}{\partial \zeta} = 0. \tag{A3}$$

The evolution of a conservative passive tracer concentration C is,

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{\partial C}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial t} \\ + \frac{\partial}{\partial x} [(\hat{u} + U_s)C] + \left(\frac{\partial}{\partial \zeta} [(\hat{u} + U_s)C] \right) \frac{\partial \zeta}{\partial x} \\ + \frac{\partial}{\partial y} [(\hat{v} + V_s)C] + \left(\frac{\partial}{\partial \zeta} [(\hat{v} + V_s)C] \right) \frac{\partial \zeta}{\partial y} \\ + \frac{1}{D} \cdot \frac{\partial}{\partial \zeta} [(\hat{w} + W_s)C] = 0. \end{aligned} \quad (\text{A4})$$

APPENDIX B

Explicit form of random wave forcing terms for the quasi-Eulerian velocity

For random waves, eq. (6) becomes

$$(U_s, V_s) = \int \sigma k (\cos \theta, \sin \theta) E(f, \theta) \frac{\cosh(2kz + 2kh)}{\sinh^2(kD)} df d\theta, \quad (\text{B1})$$

where $E(f, \theta)$ is the spectral density of the surface wave elevation variance, usually known as the wave spectrum, the state variable of most numerical wave models, and the wave-induced pressure term becomes,

$$J = \int g \frac{kE(f, \theta)}{\sinh 2kD} df d\theta. \quad (\text{B2})$$

APPENDIX C

Forcing terms for the Lagrangian mean velocity

The horizontal and vertical radiation stresses in ζ coordinate take the form,

$$F_{xx} = -\frac{\partial S_{xx}}{\partial x} = -\frac{\partial}{\partial x} \left(D\tilde{u}^2 + \tilde{p} \frac{\partial \tilde{s}}{\partial \zeta} \right). \quad (\text{C1})$$

Using Airy theory, S_{xx} is given by,

$$\begin{aligned} S_{xx} = \int kDE(f, \theta) [\cos^2 \theta F_{CS}F_{CC} \\ + (F_{CS}F_{CC} - F_{SS}F_{CS})] df d\theta, \end{aligned} \quad (\text{C2})$$

and the vertical profile function F_{CS} changes with f and is defined by

$$F_{CS} = \frac{\cosh[kD(1 + \zeta)]}{\sinh(kD)}, \quad (\text{C3})$$

with similar definitions for F_{SS} (respectively F_{CC}), replacing cosh in the numerator (respectively sinh in the denominator) by sinh (respectively cosh).

The horizontal force that is given by the vertical divergence of S_{x3} is

$$F_{x3} = -\frac{\partial S_{x3}}{\partial \zeta} = \frac{\partial}{\partial \zeta} \left(\overline{\tilde{p} \frac{\partial \tilde{s}}{\partial x}} \right). \quad (\text{C4})$$

In this case, Airy theory is insufficient for a consistent approximation. Yet Mellor (2003) still used Airy theory, thus producing the erroneous expression,

$$\begin{aligned} S_{x3} = - \int (F_{CC} - F_{SS}) \\ \times \left[E(f, \theta) \frac{\partial F_{SS}}{\partial x} + \frac{F_{SS}}{2} \frac{\partial E(f, \theta)}{\partial x} \right] df d\theta. \end{aligned} \quad (\text{C5})$$

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