



HAL
open science

Numerical schemes for the rough heat equation

Aurélien Deya

► **To cite this version:**

| Aurélien Deya. Numerical schemes for the rough heat equation. 2010. hal-00460666v2

HAL Id: hal-00460666

<https://hal.science/hal-00460666v2>

Preprint submitted on 15 Dec 2010 (v2), last revised 27 Feb 2016 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NUMERICAL SCHEMES FOR THE ROUGH HEAT EQUATION

AURÉLIEN DEYA

ABSTRACT. This paper is devoted to the study of numerical approximation schemes for the heat equation on $(0, 1)$ perturbed by a non-linear rough signal. It is the continuation of [9, 8], where the existence and uniqueness of a solution has been established. The approach combines rough paths methods with standard considerations on discretizing stochastic PDEs. The results apply to a geometric 2-rough path, which covers the case of the multidimensional fractional Brownian motion with Hurst index $H > 1/3$.

1. INTRODUCTION

This paper is part of an ongoing project whose general objective is to extend the scope of applications of the rough paths method to infinite-dimensional equation, with as a target the possibility of a pathwise approach to stochastic PDEs (see [15, 9, 3, 4, 11]). The equation we mean to focus on here is the following:

$$y_0 = \psi \in L^2(0, 1) \quad , \quad dy_t = \Delta y_t dt + \sum_{i=1}^m f_i(y_t) dx_t^i \quad , \quad t \in [0, 1], \quad (1)$$

where:

- Δ is the Laplacian operator on $L^2(0, 1)$ with Dirichlet boundary conditions,
- $f_i(y_t)(\xi) := f_i(y_t(\xi))$ for some regular function $f_i : \mathbb{R} \rightarrow \mathbb{R}$,
- $x : [0, 1] \rightarrow \mathbb{R}^m$ is a geometric rough path of order 1 (see Assumption **(X1)** $_\gamma$) or 2 (see Assumption **(X2)** $_\gamma$).

Owing to the results of [5], we know that the latter hypothesis includes in particular the case where x is a fractional Brownian motion (fBm in the sequel) with Hurst index $H > 1/3$. Thus, Equation (1) provides in this situation a model that can deal with the long-range dependance property at the core of many applications in engineering, biophysics or mathematical finance (see for instance [7, 22, 26]). It also worth mentioning that in the fBm case, the equation can also be handled with Malliavin calculus tools (see [29, 24, 28, 18]), but for $H > 1/2$ or for very particular choices of f_i only ($f_i = 1$ or $f_i = \text{Id}$).

The theoretical treatment of (1) under its general form has been established in [9] and [8]. More precisely:

(i) When x is a geometric 1-rough path, it is proved in [9] that (1) admits a unique global solution for any regular enough initial condition ψ , and this is obtained by means of an abstract fixed-point argument in a well-chosen class of processes.

Date: December 15, 2010.

2000 Mathematics Subject Classification. 60H35, 60H15, 60G22.

Key words and phrases. Rough paths theory; Stochastic PDEs; Approximation schemes; Fractional Brownian motion.

(ii) When x is a geometric 2-rough path, the existence and uniqueness of a global solution has been shown in [8] via a time-discretization of the equation.

We will go back to the exact statement of those two results in Sections 3 and 4. Let us only point out here that in both situations, explicit solutions are rarely known and the arguments at the basis of these existence results are not sufficiently constructive to provide a representation of the solution. This paper is meant to remedy this problem by introducing easily-implementable numerical schemes for the two configurations (i) and (ii). The approximation procedure will stem from two successive discretizations, in accordance with the classical strategy displayed for Wiener SPDEs (see [16] or [17]): we first turn to a time-discretization of the problem and then perform a space-discretization of the algorithm, following the Galerkin projection method.

The schemes will actually be derived from the theoretical interpretations of (1) contained in [9, 8]. For this reason, let us remind the reader with a few key-points of the approach displayed in the latter references:

- The equation is in fact analyzed in its mild form, namely

$$y_t = S_t\psi + \int_0^t S_{t-u} dx_u^i f_i(y_u) \quad , \quad t \in [0, 1], \quad (2)$$

where S stands for the semigroup generated by Δ . This is a classical change of viewpoint in the study of (stochastic) PDEs (see [6]), which allows to resort to the numerous regularizing properties of S (summed up in Subsection 2.5).

- As with rough standard systems, the interpretation of the right-hand-side of (2) relies on the expansion of the convolutional integral $\int_s^t S_{t-u} dx_u^i f_i(y_u)$, which gives rise to a decomposition such as

$$\int_s^t S_{t-u} dx_u^i f_i(y_u) = P_{ts} + R_{ts}, \quad (3)$$

where P is a "main" term and R a "residual" term of high regularity w.r.t (s, t) , which is likely to disappear from an infinitesimal point of view. Once endowed with this decomposition, the time-discretization is naturally obtained by keeping the main term P only between two successive times of the partition:

$$y_0^M = \psi \quad , \quad y_{t_{k+1}}^M = S_{t_{k+1}-t_k} y_{t_k} + P_{t_{k+1}t_k}, \quad (4)$$

with for instance $t_k = t_k^M = k/M$. The reasoning can here be compared with the recent approach of Jentzen and Kloeden for the treatment of a Wiener noise ([19, 20, 21]): in order to deduce efficient approximation schemes, the two authors lean on a Taylor expansion of the solution, which indeed fits the pattern given by (3).

- Then, in comparison with the standard case, an additional step has to be performed so as to retrieve a practically-implementable algorithm: roughly speaking, it consists in projecting the (intermediate) scheme (4) onto (increasing) finite-dimensional subspaces of $L^2(0, 1)$. We will thus carefully examine how to combine this projection with the rough paths machinery.

To the best of our knowledge, this is the first occurrence of (explicit) approximation schemes for a PDE involving a fractional noise. The convergence of those schemes will hold for any geometric 2-rough path. We hope that the strategy as well as the technical arguments displayed in this paper will make possible the approximation of a larger class

of rough evolution equations, with for instance a more general operator or a fractional distribution-valued noise. For the time being, we cannot handle this task though, just because theoretical (global) solutions have not been obtained in those situations yet.

The article is organized as follows: in Section 2, we first elaborate on the assumptions underlying our study. We also introduce the two algorithms that will be brought into play and state the main convergence results. Section 3 is devoted to the treatment of the above case **(i)**. Only developments of order 1 will be involved in this section, so that the scheme can be seen as an adapted version of the usual Euler scheme. In Section 4, we will handle the scheme for the situation **(ii)**, which requires developments of order 2 and is thus closer to the well-known Milstein approximation for standard differential systems. Finally, Appendix A puts together some technical proofs that have been postponed for sake of clarity, while in Appendix B we give an insight into possible implementations of the algorithm in the fBm case.

2. SETTINGS AND MAIN RESULTS

2.1. Framework. We focus on the Laplacian operator Δ on to the Hilbert space $\mathcal{B} := L^2(0, 1)$ with Dirichlet boundary conditions. We fix from now on a basis of \mathcal{B} made of eigenvectors:

$$e_n(\xi) := \sqrt{2} \sin(\pi n \xi) \quad (n \in \mathbb{N}^*), \quad \text{with associated eigenvalues } \lambda_n := \pi^2 n^2.$$

For any $N \in \mathbb{N}^*$, P_N will stand for the projection operator onto the finite-dimensional subspace $V_N := \text{Vect}\{e_n, 1 \leq n \leq N\}$. It is a well-known fact that the fractional Sobolev spaces are likely to play a prominent role for the study of a stochastic PDE (see e.g. [23]):

Notation 2.1. For any $\kappa \geq 0$, we denote by \mathcal{B}_κ the fractional Sobolev space associated to $(-\Delta)^\kappa$ and characterized by

$$\mathcal{B}_\kappa = \left\{ y \in L^2(0, 1) : \sum_{n=1}^{\infty} \lambda_n^{2\kappa} (y^n)^2 < \infty \right\}, \quad (5)$$

where the (y^n) are the components of y in the basis (e_n) . This space is naturally provided with the norm

$$\|y\|_{\mathcal{B}_\kappa}^2 = \|(-\Delta)^\kappa y\|_{\mathcal{B}}^2 = \sum_{n=1}^{\infty} \lambda_n^{2\kappa} (y^n)^2, \quad (6)$$

and we extend the definition of \mathcal{B}_κ to any $\kappa < 0$ through the characterization formula (5).

2.2. Assumptions. As in [8, 9], we are interested in the mild formulation of the equation, namely

$$y_t = S_t \psi + \int_0^t S_{tu} dx_u^i f_i(y_u) \quad , \quad t \in [0, 1] \quad , \quad \psi \in \mathcal{B}, \quad (7)$$

where S is the semigroup generated by Δ and $S_{tu} := S_{t-u}$. A priori, the equation only makes sense for a regular (ie piecewise differentiable) process x . In this context, interpreting the rough version of (7) means extending the convolutional integral to a γ -Hölder process x , $\gamma \in (0, 1)$. For sake of simplicity, we will only consider in this paper the case γ is strictly greater than $1/3$, which covers in particular the Brownian motion

case. As in the classical rough paths theory, we will also be led to assume, depending on the regularity of x , that one of the two following assumptions is satisfied.

Assumption (X1) $_{\gamma}$: $x : [0, 1] \rightarrow \mathbb{R}^m$ is a geometric 1-rough path of order γ . In other words, x is a γ -Hölder process and there exists a sequence of piecewise differentiable process (x^M) such that

$$u_M := \mathcal{N}[x - x^M; \mathcal{C}_1^{\gamma}([0, 1]; \mathbb{R}^m)] \xrightarrow{M \rightarrow \infty} 0,$$

where $\mathcal{N}[\cdot; \mathcal{C}_1^{\gamma}([0, 1]; \mathbb{R}^m)]$ is just the usual Hölder norm, ie

$$\mathcal{N}[y; \mathcal{C}_1^{\gamma}([0, 1]; \mathbb{R}^m)] := \sup_{0 \leq s < t \leq 1} \frac{\|y_t - y_s\|_{\mathbb{R}^m}}{|t - s|^{\gamma}}.$$

Assumption (X2) $_{\gamma}$: $x : [0, 1] \rightarrow \mathbb{R}^m$ is a geometric 2-rough path of order γ . In other words, x is a γ -Hölder process and there exists a sequence of piecewise differentiable process (x^M) such that $\mathcal{N}[x - x^M; \mathcal{C}_1^{\gamma}([0, 1]; \mathbb{R}^m)] \xrightarrow{M \rightarrow \infty} 0$ and the sequence $(\mathbf{x}^{2,M})$ of Lévy areas associated to (x^M) , ie

$$\mathbf{x}_{ts}^{2,M,ij} := \int_s^t dx_u^{M,i} (x_u^{M,j} - x_s^{M,j}), \quad i, j = 1, \dots, m, \quad s < t \in [0, 1],$$

converges to an element \mathbf{x}^2 with respect to the norm

$$\mathcal{N}[y; \mathcal{C}_2^{2\gamma}([0, 1]; \mathbb{R}^{m,m})] := \sup_{0 \leq s < t \leq 1} \frac{\|y_{ts}\|_{\mathbb{R}^{m,m}}}{|t - s|^{2\gamma}}.$$

In brief,

$$v_M := \mathcal{N}[x - x^M; \mathcal{C}_1^{\gamma}([0, 1]; \mathbb{R}^m)] + \mathcal{N}[\mathbf{x}^{2,M} - \mathbf{x}^2; \mathcal{C}_2^{2\gamma}([0, 1]; \mathbb{R}^{m,m})] \xrightarrow{M \rightarrow \infty} 0.$$

Example: The main process we have in mind in this paper is the (m -dimensional) fractional Brownian motion $x = B^H$ with Hurst index $H > 1/3$. It has been indeed proved in [5] that this process satisfies Assumption **(X2) $_{\gamma}$** (and accordingly Assumption **(X1) $_{\gamma}$**) for any $1/3 < \gamma < H$, when taking for x^M the linear interpolation of x with uniform mesh $\frac{1}{M}$, ie

$$t_k = t_k^M := \frac{k}{M}, \quad x_t^M := x_{t_k} + M \cdot (t - t_k) \cdot (x_{t_{k+1}} - x_{t_k}) \quad \text{if } t \in [t_k, t_{k+1}). \quad (8)$$

Besides, the following sharp estimates have been established in [10]: for any $1/3 < \gamma < H$, there exists an almost surely finite random variable C_{γ} such that $v_M \leq C_{\gamma} \sqrt{\log M} \cdot M^{\gamma-H}$. This control can then be injected in our final estimates (14) and (15) so as to retrieve explicit (almost sure) rates in this fBm case. Note that Conditions **(X1) $_{\gamma}$** or **(X2) $_{\gamma}$** are actually fulfilled by a larger class of Gaussian processes, as reported in [13] or in [12].

As far as the regularity of the vector field f is concerned, it will be governed by the additional condition (k is a parameter in \mathbb{N}):

Assumption (F) $_k$: for every $i \in \{1, \dots, m\}$, f_i belongs to the space $\mathcal{C}^{k,\mathbf{b}}(\mathbb{R}; \mathbb{R})$ of k -time differentiable functions, bounded, with bounded derivatives.

2.3. Schemes. In order to introduce the two schemes we intend to study, let us define, for any piecewise differentiable process $\tilde{x} : [0, 1] \rightarrow \mathbb{R}^m$, the following operator-valued processes: for every $i, j = 1, \dots, m$, for any $s < t \in [0, 1]$,

$$X_{ts}^{\tilde{x},i} := \int_s^t S_{tu} d\tilde{x}_u^i, \quad X_{ts}^{\tilde{x},ij} := \int_s^t S_{tu} d\tilde{x}_u^i (x_u^j - x_s^j), \quad (9)$$

We suppose in addition that either Assumption $(\mathbf{X1})_\gamma$ or Assumption $(\mathbf{X2})_\gamma$ is satisfied, for some parameter $\gamma \in (0, 1)$ and some regularizing sequence (x^M) , and that Assumption $(\mathbf{F})_1$ holds true, so that f'_i is well-defined. With those conditions in mind, here is the two schemes that will come into play in the sequel:

Euler scheme: $y_0^{M,N} = P_N \psi$ and

$$y_{t_{k+1}}^{M,N} = S_{t_{k+1}t_k} y_{t_k}^{M,N} + X_{t_{k+1}t_k}^{x^M,i} P_N f_i(y_{t_k}^{M,N}), \quad (10)$$

where $t_k = t_k^M = \frac{k}{M}$.

Milstein scheme: $y_0^{M,N} = P_N \psi$ and

$$y_{t_{k+1}}^{M,N} = S_{t_{k+1}t_k} y_{t_k}^{M,N} + X_{t_{k+1}t_k}^{x^{2M},i} P_N f_i(y_{t_k}^{M,N}) + X_{t_{k+1}t_k}^{x^{2M},ij} P_N \left(f'_i(y_{t_k}^{M,N}) \cdot (P_N f_j(y_{t_k}^{M,N})) \right), \quad (11)$$

where $t_k = t_k^M = \frac{k}{2^M}$ and the notation $\phi \cdot \psi$ stands for the pointwise product of functions, ie $(\phi \cdot \psi)(\xi) := \phi(\xi)\psi(\xi)$.

Remark 2.2. The name we have given to the schemes is just a reference to the classical algorithms for standard stochastic differential equations. It indicates that (10) involves developments of order one only, while (11) appeal to second-order terms.

Remark 2.3. When x^M is the linear interpolation of x given by (8), the two sequences of operators $X_{t_{k+1}t_k}^{x^M,i}$, $X_{t_{k+1}t_k}^{x^M x^M,ij}$ that intervene in the schemes reduce to

$$X_{t_{k+1}t_k}^{x^M,i} = M \cdot (x_{t_{k+1}}^i - x_{t_k}^i) \cdot \int_{t_k}^{t_{k+1}} S_{t_{k+1}u} du \quad (12)$$

$$X_{t_{k+1}t_k}^{x^M x^M,ij} = M^2 \cdot (x_{t_{k+1}}^i - x_{t_k}^i) \cdot (x_{t_{k+1}}^j - x_{t_k}^j) \cdot \int_{t_k}^{t_{k+1}} S_{t_{k+1}u} du (u - t_k). \quad (13)$$

Consequently, in this case, the computations of Formulas (10) and (11) only require the a priori knowledge of the successive increments $x_{t_{k+1}} - x_{t_k}$, which makes the implementation of the algorithms very easy, as we shall see in Appendix B for the fBm case.

2.4. Main results. We are now in position to state our main convergence results. As in the standard rough paths theory results, we make a clear distinction between the case $\gamma > \frac{1}{2}$ and the case $\gamma \in (\frac{1}{3}, \frac{1}{2})$. Theorem 2.4 (resp. Theorem 2.5) will be proved in Section 3 (resp. Section 4).

Theorem 2.4. *Let $\gamma \in (\frac{1}{2}, 1)$ and suppose that Assumptions $(\mathbf{X1})_\gamma$ and $(\mathbf{F})_2$ are satisfied. Let also $\gamma' \in (\max(1 - \gamma, \frac{\gamma}{2}), \frac{1}{2})$ and $\psi \in \mathcal{B}_{\gamma'}$. Then there exists a function $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that if y is the solution of (7) with initial*

condition ψ (see Theorem 3.5) and $y^{M,M}$ is the process generated by the Euler scheme (10) with $M = N$,

$$\sup_{k \in \{0, \dots, M\}} \|y_{t_k^M} - y_{t_k^{M,M}}\|_{\mathcal{B}_{\gamma'}} \leq C \left(\|\psi\|_{\mathcal{B}_{\gamma'}}, \|x\|_{\gamma} \right) \left\{ \|\psi - P_M \psi\|_{\mathcal{B}_{\gamma'}} + u_M + \frac{1}{M^{\gamma + \gamma' - 1}} \right\}, \quad (14)$$

where we have used the shortcut $\|x\|_{\gamma} := \mathcal{N}[x; \mathcal{C}_1^{\gamma}([0, 1]; \mathbb{R}^m)]$.

Theorem 2.5. Let $\gamma \in (\frac{1}{3}, \frac{1}{2})$ and suppose that Assumptions $(\mathbf{X2})_{\gamma}$ and $(\mathbf{F})_3$ are satisfied. Let also $\gamma' \in (1 - \gamma, 2\gamma]$ and $\psi \in \mathcal{B}_{\gamma'}$. Then for every parameters

$$0 < \beta < \inf \left(\gamma + \gamma' - 1, \gamma - \gamma' + \frac{1}{2} \right) \quad , \quad 0 < \lambda < \gamma + \gamma' - 1,$$

there exists a function $C = C_{\beta, \lambda} : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that if y is the solution of (7) in $\mathcal{B}_{\gamma'}$ with initial condition ψ (see Theorem 4.5) and $y^{M,N}$ is the process generated by the Milstein scheme (11),

$$\sup_{k \in \{0, \dots, 2^M\}} \|y_{t_k^M} - y_{t_k^{M,N}}\|_{\mathcal{B}_{\gamma'}} \leq C \left(\|\psi\|_{\mathcal{B}_{\gamma'}}, \|\mathbf{x}\|_{\gamma} \right) \left\{ \|\psi - P_N \psi\|_{\mathcal{B}_{\gamma'}} + v_{2^M} + \frac{1}{(2^M)^{\beta}} + \frac{1}{N^{2\lambda}} \right\}, \quad (15)$$

where we have used the shortcut $\|\mathbf{x}\|_{\gamma} := \mathcal{N}[x; \mathcal{C}_1^{\gamma}([0, 1]; \mathbb{R}^m)] + \mathcal{N}[\mathbf{x}^2; \mathcal{C}_2^{2\gamma}([0, 1]; \mathbb{R}^{m,m})]$.

Remark 2.6. The particular choice $N = M$ in Theorem 2.4 has only been made so as to get a nice expression for the final estimate (14). Nevertheless, it is not hard to obtain a more general result with possibly different N, M , following the arguments of Section 3.

Remark 2.7. As we shall see in Section 4, the use of dyadic intervals in the Milstein scheme (11) is justified by the need of a decreasing sequence of partitions in the patching argument of Proposition 4.9. However, our convergence result can probably be extended to any sequence of partitions whose meshes tend to 0, at the price of more intricate local considerations in the proof of the latter proposition.

2.5. Tools of algebraic integration. Before going further, let us draw up a list of the properties at our disposal as far as the fractional Sobolev spaces \mathcal{B}_{κ} and the semigroup are concerned (the proof of those classical results can be found in [2], [25] or [27]):

- *Sobolev inclusions:* If $\kappa > 1/4$, \mathcal{B}_{κ} is a Banach algebra continuously included in the space $L^{\infty}([0, 1])$ of bounded functions on $[0, 1]$.
- *Projection:* For all $0 \leq \kappa < \alpha$ and for any $\varphi \in \mathcal{B}_{\alpha}$,

$$\|\varphi - P_N \varphi\|_{\mathcal{B}_{\kappa}} \leq \lambda_N^{-(\alpha - \kappa)} \|\varphi\|_{\mathcal{B}_{\alpha}}. \quad (16)$$

- *Contraction:* For any $\kappa \geq 0$, S is a contraction operator on \mathcal{B}_{κ} .
- *Regularization:* For any $t > 0$ and for all $-\infty < \kappa < \alpha < \infty$, S_t sends \mathcal{B}_{κ} into \mathcal{B}_{α} and

$$\|S_t \varphi\|_{\mathcal{B}_{\alpha}} \leq c_{\alpha, \kappa} t^{-(\alpha - \kappa)} \|\varphi\|_{\mathcal{B}_{\kappa}}. \quad (17)$$

- *Hölder regularity:* For all $t > 0, \alpha > 0$ and for any $\varphi \in \mathcal{B}_{\alpha}$,

$$\|S_t \varphi - \varphi\|_{\mathcal{B}} \leq c_{\alpha} t^{\alpha} \|\varphi\|_{\mathcal{B}_{\alpha}} \quad , \quad \|\Delta S_t \varphi\|_{\mathcal{B}} \leq c_{\alpha} t^{-1 + \alpha} \|\varphi\|_{\mathcal{B}_{\alpha}}. \quad (18)$$

- *Composition*: if $\kappa \in [0, 1/2)$ and $f \in \mathcal{C}^{1,\mathbf{b}}(\mathbb{R}; \mathbb{R})$ (see Assumption **(F)**_k for the latter notation),

$$\|f(\varphi)\|_{\mathcal{B}_\kappa} \leq c_{\kappa,f} \{1 + \|\varphi\|_{\mathcal{B}_\kappa}\}, \quad (19)$$

while if $\kappa \in [1/2, 1)$ and $f \in \mathcal{C}^{2,\mathbf{b}}(\mathbb{R}; \mathbb{R})$,

$$\|f(\varphi)\|_{\mathcal{B}_\kappa} \leq c_{\kappa,f} \{1 + \|\varphi\|_{\mathcal{B}_\kappa}^2\}, \quad (20)$$

where, in both cases, $f(\varphi)$ is understood in the sense of composition, ie $f(\varphi)(\xi) := f(\varphi(\xi))$.

- *Pointwise product*: if $\kappa \in [0, 1/2)$ and $\varphi, \psi \in \mathcal{B}_\kappa \cap L^\infty([0, 1])$,

$$\|\varphi \cdot \psi\|_{\mathcal{B}_\kappa} \leq c_\kappa \{\|\varphi\|_{L^\infty} \|\psi\|_{\mathcal{B}_\kappa} + \|\varphi\|_{\mathcal{B}_\kappa} \|\psi\|_{L^\infty}\}, \quad (21)$$

while if $\varphi \in \mathcal{B}_{-\kappa}$, $\psi \in \mathcal{B}_\alpha$, with $\kappa \geq 0$ and $\alpha > \max(\kappa, \frac{1}{4})$,

$$\|\varphi \cdot \psi\|_{\mathcal{B}_{-\kappa}} \leq c_{\kappa,\alpha} \|\varphi\|_{\mathcal{B}_{-\kappa}} \|\psi\|_{\mathcal{B}_\alpha}. \quad (22)$$

Remember that $\varphi \cdot \psi$ is understood as a pointwise product, ie $(\varphi \cdot \psi)(\xi) = \varphi(\xi)\psi(\xi)$.

With these properties in hand, the rough paths treatment of Equation (26) is based on the controlled expansion of the convolutional integral

$$\int_s^t S_{tu} dx_u^i f_i(y_u). \quad (23)$$

In order to express this expansion with the highest accuracy, we provide ourselves with a few tools and notations inspired by the algebraic integration theory for standard systems (see [14]).

Notations. For $k \in \{1, 2, 3\}$ and for any interval $I \subset [0, 1]$, denote

$$\mathcal{S}_k(I) := \{(t_1, \dots, t_k) \in I^k : t_1 \geq \dots \geq t_k\}.$$

Then for all processes $y : I \rightarrow \mathcal{B}$ and $z : \mathcal{S}_2(I) \rightarrow \mathcal{B}$, we set, for $s \leq u \leq t \in I$,

$$(\delta y)_{ts} := y_t - y_s \quad , \quad (\hat{\delta} y)_{ts} := (\delta y)_{ts} - a_{ts} y_s, \quad (24)$$

$$(\hat{\delta} z)_{tus} := z_{ts} - z_{tu} - S_{tu} z_{us}, \quad (25)$$

where $a_{ts} := S_{ts} - \text{Id}$.

To give an idea on how those operators arise from the handling of (7), let us observe for instance that the variations of the solution y are governed by the equation

$$(\delta y)_{ts} = \int_s^t S_{tu} dx_u^i f_i(y_u) + a_{ts} \int_0^s S_{su} dx_u^i f_i(y_u) = \int_s^t S_{tu} dx_u^i f_i(y_u) + a_{ts} y_s,$$

and (7) can thus be equivalently written as

$$y_0 = \psi \quad , \quad (\hat{\delta} y)_{ts} = \int_s^t S_{tu} dx_u^i f_i(y_u). \quad (26)$$

Let us also observe, in this convolutional context, the following elementary properties, that we label for further use:

Proposition 2.8. *Let $y : [0, 1] \rightarrow \mathcal{B}$, $z : \mathcal{S}_2 \rightarrow \mathcal{B}$, and let $x : [0, 1] \rightarrow \mathbb{R}$ a regular process. Then it holds:*

- *Telescopic sum*: $\hat{\delta}(\hat{\delta} y)_{tus} = 0$ and $(\hat{\delta} y)_{ts} = \sum_{i=0}^{n-1} S_{tt_{i+1}} (\hat{\delta} y)_{t_{i+1}t_i}$ for any partition $\{s = t_0 < t_1 < \dots < t_n = t\}$ of an interval $[s, t]$ of $[0, 1]$.

- *Chasles relation:* if $\mathcal{J}_{ts} := \int_s^t S_{tu} dx_u y_u$, then $\hat{\delta}\mathcal{J} = 0$.
- *Cohomology:* if $\hat{\delta}z = 0$, there exists $h : [0, 1] \rightarrow \mathcal{B}$ such that $\hat{\delta}h = z$.

On top of those algebraic considerations, if one wants to measure the regularity of the terms involved in the expansion of $\int_s^t S_{tu} dx_u^i f_i(y_u)$, one is led to introduce the following suitable semi-norms, that can be seen as generalizations of the classical Hölder norm: thus, if $y : [0, 1] \rightarrow V$, $z : \mathcal{S}_2 \rightarrow V$ and $h : \mathcal{S}_3 \rightarrow V$, where V is any Banach space, we will denote, for any $\lambda > 0$,

$$\mathcal{N}[y; \hat{\mathcal{C}}_1^\lambda([a, b]; V)] := \sup_{a \leq s < t \leq b} \frac{\|(\hat{\delta}y)_{ts}\|_V}{|t - s|^\lambda}, \quad \mathcal{N}[y; \mathcal{C}_1^0([a, b]; V)] := \sup_{t \in [a, b]} \|y_t\|_V, \quad (27)$$

$$\mathcal{N}[z; \mathcal{C}_2^\lambda([a, b]; V)] := \sup_{a \leq s < t \leq b} \frac{\|z_{ts}\|_V}{|t - s|^\lambda}, \quad \mathcal{N}[h; \mathcal{C}_3^\lambda([a, b]; V)] := \sup_{a \leq s < u < t \leq b} \frac{\|h_{tus}\|_V}{|t - s|^\lambda}. \quad (28)$$

Then $\hat{\mathcal{C}}_1^\lambda([a, b]; V)$ naturally stands for the set of processes $y : [0, 1] \rightarrow V$ such that $\mathcal{N}[y; \hat{\mathcal{C}}_1^\lambda([a, b]; V)] < \infty$, and we define $\mathcal{C}_1^0([a, b]; V)$, $\mathcal{C}_2^\lambda([a, b]; V)$ and $\mathcal{C}_3^\lambda([a, b]; V)$ along the same line. With these notations, observe for instance that if $y \in \mathcal{C}_2^\lambda([a, b]; \mathcal{L}(V, W))$ and $z \in \mathcal{C}_2^\beta([a, b]; V)$, the process h defined as $h_{tus} := y_{tu} z_{us}$ ($s \leq u \leq t$) belongs to $\mathcal{C}_3^{\lambda+\beta}([a, b]; W)$.

When $[a, b] = [0, 1]$, we will more simply write $\mathcal{C}_k^\lambda(V) := \mathcal{C}_k^\lambda([a, b]; V)$.

The following notational convention also turns out to be useful as far as products of processes are concerned:

Notation 2.9. *If $g : \mathcal{S}_n \rightarrow \mathcal{L}(V, W)$ and $h : \mathcal{S}_m \rightarrow W$, then the product $gh : \mathcal{S}_{n+m-1} \rightarrow W$ is defined by the formula*

$$(gh)_{t_1 \dots t_{n+m-1}} := g_{t_1 \dots t_n} h_{t_n \dots t_{n+m-1}}.$$

With this convention, it is readily checked that if $g : \mathcal{S}_2 \rightarrow \mathcal{L}(\mathcal{B}_\kappa, \mathcal{B}_\alpha)$ and $h : \mathcal{S}_n \rightarrow \mathcal{B}_\kappa$, then $\hat{\delta}(gh) : \mathcal{S}_{n+1} \rightarrow \mathcal{B}_\alpha$ is given by

$$\hat{\delta}(gh) = (\hat{\delta}g)h - g(\delta h). \quad (29)$$

To end up with this toolbox, let us report one of the cornerstone results of [15], which will allow us, in Section 4, to cope with the high-order terms popping out of the expansion of (23):

Theorem 2.10. *Fix an interval $I \subset [0, 1]$, a parameter $\kappa \geq 0$ and let $\mu > 1$. For any $h \in \mathcal{C}_3^\mu(I; \mathcal{B}_\kappa) \cap \text{Im } \hat{\delta}$, there exists a unique element*

$$\hat{\Lambda}h \in \bigcap_{\alpha \in [0, \mu]} \mathcal{C}_2^{\mu-\alpha}(I; \mathcal{B}_{\kappa+\alpha})$$

such that $\hat{\delta}(\hat{\Lambda}h) = h$. Moreover, $\hat{\Lambda}h$ satisfies the following contraction property: for all $\alpha \in [0, \mu)$,

$$\mathcal{N}[\hat{\Lambda}h; \mathcal{C}_2^{\mu-\alpha}(I; \mathcal{B}_{\kappa+\alpha})] \leq c_{\alpha, \mu} \mathcal{N}[h; \mathcal{C}_3^\mu(I; \mathcal{B}_\kappa)]. \quad (30)$$

3. YOUNG CASE

This section is devoted to the proof of Theorem 2.4. Consequently, we fix from now on the two parameters $\gamma \in (\frac{1}{2}, 1)$ and $\gamma' \in (\max(\frac{1}{4}, 1 - \gamma), \frac{1}{2})$, as well as the initial condition $\psi \in \mathcal{B}_{\gamma'}$. Under Assumptions $(\mathbf{X1})_\gamma$ and $(\mathbf{F})_2$, the convolution integral (23) can be extended to x via a first-order expansion. To do so, observe that if \tilde{x} is a piecewise differentiable process, one has, for any \mathcal{B} -valued differentiable process z ,

$$\int_s^t S_{tu} d\tilde{x}_u^i z_u = X_{ts}^{\tilde{x},i} z_s + \hat{\Lambda}_{ts}(X^{\tilde{x},i} \delta z), \quad (31)$$

where $X^{\tilde{x},i}$ is the operator-valued process defined by (9). Indeed, if we denote

$$J_{ts} := \int_s^t S_{tu} d\tilde{x}_u^i z_u - X_{ts}^{\tilde{x},i} z_s = \int_s^t S_{tu} d\tilde{x}_u^i (\delta z)_{us} \in \mathcal{C}_2^2(\mathcal{B}),$$

one has, with the help of Theorem 2.10, $\hat{\delta}(J - \hat{\Lambda}(X^{\tilde{x},i} \delta z)) = 0$, hence, owing to Proposition 2.8, $J - \hat{\Lambda}(X^{\tilde{x},i} \delta z) = \hat{\delta}h \in \mathcal{C}_2^2(\mathcal{B})$, which easily entails $\hat{\delta}h = 0$ (use the telescopic-sum property of Proposition 2.8). One can then rely on the following natural extension result:

Lemma 3.1 ([9]). *Under Assumption $(\mathbf{X1})_\gamma$, the sequence of operator-valued processes*

$$X_{ts}^{x^M,i} := \int_s^t S_{tu} dx_u^{M,i}$$

converges to an element $X^{x,i}$ with respect to the topology of the spaces $\mathcal{C}_2^{\gamma-\lambda}(\mathcal{L}(\mathcal{B}_\kappa, \mathcal{B}_{\kappa+\lambda}))$ ($\lambda \in [0, \gamma)$, $\kappa \in \mathbb{R}$) and $\mathcal{N}[X^{x,i}; \mathcal{C}_2^{\gamma-\lambda}(\mathcal{L}(\mathcal{B}_\kappa, \mathcal{B}_{\kappa+\lambda}))] \leq c_{\kappa,\lambda} \|x\|_\gamma$, as well as

$$\mathcal{N}[X^{x,i} - X^{x^M,i}; \mathcal{C}_2^{\gamma-\lambda}(\mathcal{L}(\mathcal{B}_\kappa, \mathcal{B}_{\kappa+\lambda}))] \leq c_{\kappa,\lambda} u_M. \quad (32)$$

Moreover, $X^{x,i}$ commutes with the projection P_N and satisfies the algebraic relation $\hat{\delta}X^{x,i} = 0$.

Remark 3.2. The underlying topology of this convergence result is of course closely related to the properties of the semigroup recalled in Subsection 2.5. In other words, the fact that $X^{x,i} \in \mathcal{C}_2^{\gamma-\lambda}(\mathcal{L}(\mathcal{B}_\kappa, \mathcal{B}_{\kappa+\lambda}))$ for $\kappa \in \mathbb{R}, \lambda \in [0, \gamma)$, is a consequence of the regularizing property (17).

Remark 3.3. Through the continuity result (32), one can see that the process X^x only depends on x and not on the particular approximating sequence x^M . This comment also holds for the forthcoming Lemma 4.1.

Once endowed with X^x , it is readily checked that the right-hand-side of (31) can also be extended to a less regular process z , which provides the expected interpretation:

Proposition 3.4 ([9]). *Under Assumption $(\mathbf{X1})_\gamma$, we define, for any process $z = (z^1, \dots, z^m)$ such that $z^i \in \mathcal{C}_1^0(\mathcal{B}_\kappa) \cap \mathcal{C}_1^\kappa(\mathcal{B})$ with $\gamma + \kappa > 1$, the integral*

$$\mathcal{J}_{ts}(\hat{dx} z) := X_{ts}^{x,i} z_s^i + \hat{\Lambda}_{ts}(X^{x,i} \delta z^i). \quad (33)$$

Then:

- $\mathcal{J}(\hat{dx} z)$ is well-defined via Theorem 2.10. It coincides with the Lebesgue integral $\int_s^t S_{tu} dx_u^i z_u^i$ when x is a piecewise differentiable process.
- The following estimate holds true:

$$\mathcal{N}[\mathcal{J}(\hat{dx} z); \mathcal{C}_2^\gamma(\mathcal{B}_\kappa)] \leq c \|x\|_\gamma \{ \mathcal{N}[z; \mathcal{C}_1^0(\mathcal{B}_\kappa)] + \mathcal{N}[z; \mathcal{C}_1^\kappa(\mathcal{B})] \}. \quad (34)$$

3.1. Previous results. The main result of [9] for the Young case is summed up by the following statement:

Theorem 3.5 ([9]). *Under Assumptions $(\mathbf{X1})_\gamma$ and $(\mathbf{F})_2$, Equation (7), interpreted thanks to the previous proposition, admits a unique solution y in $\hat{\mathcal{C}}_1^{\gamma'}(\mathcal{B}_{\gamma'})$, and the following estimates hold true:*

$$\mathcal{N}[y; \mathcal{C}_1^0(\mathcal{B}_{\gamma'})] + \mathcal{N}[y; \hat{\mathcal{C}}_1^{\gamma'}(\mathcal{B}_{\gamma'})] \leq C \left(\|\psi\|_{\mathcal{B}_{\gamma'}}, \|x\|_\gamma \right), \quad (35)$$

for some function $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets. Moreover, if y (resp. \tilde{y}) is the solution of (7) associated to a process x (resp. \tilde{x}) that satisfies Assumption $(\mathbf{X1})_\gamma$, with initial condition ψ (resp. $\tilde{\psi}$) in $\mathcal{B}_{\gamma'}$,

$$\mathcal{N}[y - \tilde{y}; \mathcal{C}_1^0(\mathcal{B}_{\gamma'})] \leq c_{x, \tilde{x}, \psi, \tilde{\psi}} \left\{ \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma'}} + \|x - \tilde{x}\|_\gamma \right\}, \quad (36)$$

with $c_{x, \tilde{x}, \psi, \tilde{\psi}} := C'(\|x\|_\gamma, \|\tilde{x}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}}, \|\tilde{\psi}\|_{\mathcal{B}_{\gamma'}})$, for some function $C' : (\mathbb{R}^+)^4 \rightarrow \mathbb{R}^+$ bounded on bounded sets.

Remark 3.6. It is worth noticing that (35) and (34) implies in particular

$$\mathcal{N}[y; \hat{\mathcal{C}}_1^\gamma(\mathcal{B}_{\gamma'})] \leq c_{\psi, x}.$$

Indeed, since y is solution to the system, one has

$$\begin{aligned} \|(\hat{\delta}y)_{ts}\|_{\mathcal{B}_{\gamma'}} &\leq \|\mathcal{J}_{ts}(\hat{d}x f(y))\|_{\mathcal{B}_{\gamma'}} \\ &\leq c_x |t - s|^\gamma \left\{ \mathcal{N}[f(y); \mathcal{C}_1^0(\mathcal{B}_{\gamma'})] + \mathcal{N}[f(y); \mathcal{C}_1^{\gamma'}(\mathcal{B})] \right\}. \end{aligned}$$

Then, thanks to (19) and (18), it holds $\mathcal{N}[f(y); \mathcal{C}_1^0(\mathcal{B}_{\gamma'})] \leq c \{1 + \mathcal{N}[y; \mathcal{C}_1^0(\mathcal{B}_{\gamma'})]\}$ and

$$\mathcal{N}[f(y); \mathcal{C}_1^{\gamma'}(\mathcal{B})] \leq c \mathcal{N}[y; \mathcal{C}_1^{\gamma'}(\mathcal{B})] \leq c \left\{ \mathcal{N}[y; \mathcal{C}_1^0(\mathcal{B}_{\gamma'})] + \mathcal{N}[y; \hat{\mathcal{C}}_1^{\gamma'}(\mathcal{B}_{\gamma'})] \right\}.$$

The continuity result (36) provides us with a control over the discretization of the driving signal x . This is the first step towards Theorem 2.4:

Notation 3.7. For any $M \in \mathbb{N}^*$, we denote by \bar{y}^M the Wong-Zakai approximation associated to x^M (with the same initial condition ψ), or otherwise stated the solution to Equation (7) when x is replaced with its interpolation x^M .

Corollary 3.8. With the above notations, there exists a function $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that, for any $M \in \mathbb{N}^*$,

$$\sup_{k \in \{0, \dots, M\}} \|y_{t_k} - \bar{y}_{t_k}^M\|_{\mathcal{B}_{\gamma'}} \leq C(\|x\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}}) u_M. \quad (37)$$

3.2. A uniform control. The second step of our reasoning consists in controlling the process $y^{M,N}$ generated by (10), uniformly in M and N . To do so, let us first extend $y^{M,N}$ on $[0, 1]$ through the formula: if $t \in [t_k, t_{k+1})$,

$$y_t^{M,N} := S_{tt_k} y_{t_k}^{M,N} + X_{tt_k}^{x^M, i} P_N f_i(y_{t_k}^{M,N}). \quad (38)$$

Now observe that by setting $r_{ts}^{M,N} := \hat{\Lambda}_{ts} \left(X^{x^M, i} \delta P_N f_i(y^{M,N}) \right)$, one can write, for any $k \in \{0, \dots, M-1\}$,

$$y_{t_{k+1}}^{M,N} = S_{t_{k+1}t_k} y_{t_k}^{M,N} + \int_{t_k}^{t_{k+1}} S_{t_{k+1}u} dx_u^{i, M} P_N f_i(y_u^{M,N}) - r_{t_{k+1}t_k}^{M,N}. \quad (39)$$

Extending the expression to all times $s < t$ gives rise to the two formulas:

Lemma 3.9. *If $t_p \leq s < t_{p+1} < \dots < t_q \leq t < t_{q+1}$, then*

$$(\hat{\delta}y^{M,N})_{ts} = \int_s^t S_{tu} dx_u^{i,M} P_N f_i(y_u^{M,N}) - y_{ts}^{M,N,\sharp}, \quad (40)$$

with

$$y_{ts}^{M,N,\sharp} := r_{tt_q}^{M,N} - S_{ts} r_{st_p}^{M,N} + \sum_{k=p}^{q-1} S_{tt_{k+1}} r_{t_{k+1}t_k}^{M,N}, \quad (41)$$

while if $t_p \leq s < t < t_{p+1}$,

$$(\hat{\delta}y^{M,N})_{ts} = X_{ts}^{x^M,i} P_N f_i(y_{t_p}^{M,N}). \quad (42)$$

Proof. Formula (42) is a straightforward consequence of the relation $\hat{\delta}X^{x^M,i} = 0$. As for (40), it follows from the association of (39) and the telescopic-sum property contained in Proposition 2.8, which gives here

$$\begin{aligned} (\hat{\delta}y^{M,N})_{ts} &= \sum_{k=p+1}^{q-1} S_{tt_{k+1}} (\hat{\delta}y^{M,N})_{t_{k+1}t_k} + (\hat{\delta}y^{M,N})_{tt_q} + S_{tt_{p+1}} (\hat{\delta}y^{M,N})_{t_{p+1}s} \\ &= \left[\int_{t_{p+1}}^{t_q} S_{tu} dx_u^{i,M} P_N f_i(y_u^{M,N}) - \sum_{k=p+1}^{q-1} S_{tt_{k+1}} r_{t_{k+1}t_k}^{M,N} \right] \\ &\quad + \left[\int_{t_q}^t S_{tu} dx_u^{i,M} P_N f_i(y_u^{M,N}) - r_{tt_q}^{M,N} \right] + S_{tt_{p+1}} (\hat{\delta}y^{M,N})_{t_{p+1}s} \end{aligned} \quad (43)$$

Since

$$\begin{aligned} (\hat{\delta}y^{M,N})_{t_{p+1}s} &= (\hat{\delta}y^{M,N})_{t_{p+1}t_p} - S_{t_{p+1}s} (\hat{\delta}y^{M,N})_{st_p} \\ &= \left[\int_{t_p}^{t_{p+1}} S_{t_{p+1}u} dx_u^{i,M} P_N f_i(y_u^{M,N}) + r_{t_{p+1}t_p}^{M,N} \right] \\ &\quad - S_{t_{p+1}s} \left[\int_{t_p}^s S_{su} dx_u^{i,M} P_N f_i(y_u^{M,N}) - r_{st_p}^{M,N} \right], \end{aligned} \quad (44)$$

it suffices to inject (44) in (43) to get (40). \square

We are going to lean on the two expressions (40) and (42) in order to establish the expected uniform estimate:

Proposition 3.10. *There exists a function $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that for every $M, N \in \mathbb{N}$,*

$$\mathcal{N}[y^{M,N}; \mathcal{C}_1^0([0, 1], \mathcal{B}_{\gamma'})] + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{\gamma'}([0, 1], \mathcal{B}_{\gamma'})] \leq C(\|x\|_{\gamma}, \|\psi\|_{\mathcal{B}_{\gamma'}}), \quad (45)$$

where $y^{M,N}$ is extended on $[0, 1]$ through Formula (38).

Proof. For sake of conciseness, denote here

$$\mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}(I)] := \mathcal{N}[y^{M,N}; \mathcal{C}_1^0(I, \mathcal{B}_{\gamma'})] + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{\gamma'}(I, \mathcal{B}_{\gamma'})].$$

With this notation in hand, we will actually prove the following assertion: there exists a time $T_0 = T_0(\|x\|_\gamma) > 0$ and a sequence of radii $R_l = R_l(\|x\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}})$ such that for any l ,

$$\mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, lT_0])] \leq R_l.$$

For $l = 0$, take $R_0 := \|\psi\|_{\mathcal{B}_{\gamma'}}$. Now assume that the property holds true for l , and let $s, t \in [0, (l+1)T_0]$.

1st case: $s, t \in [lT_0, (l+1)T_0]$.

1st subcase: $t_p \leq s < t_{p+1} < \dots < t_q \leq t < t_{q+1}$, with $|t - s| \geq \frac{1}{M}$. Then, from (40),

$$(\hat{\delta}y^{M,N})_{ts} = \int_s^t S_{tu} dx_u^{i,M} P_N f_i(y_u^{M,N}) - y_{ts}^{M,N,\#}.$$

Owing to the estimate (34) (applied to $x = x^M$), one easily deduces

$$\left\| \int_s^t S_{tu} dx_u^{i,M} P_N f_i(y_u^{M,N}) \right\|_{\mathcal{B}_{\gamma'}} \leq c_x |t - s|^{\gamma'} T_0^{\gamma - \gamma'} \left\{ 1 + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, (l+1)T_0]) \right\}.$$

Besides, thanks to the contraction property (30) of $\hat{\Lambda}$, one gets

$$\|r_{ts}^{M,N}\|_{\mathcal{B}} \leq c_x |t - s|^{\gamma + \gamma'} \left\{ 1 + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, (l+1)T_0]) \right\},$$

as well as

$$\|r_{ts}^{M,N}\|_{\mathcal{B}_{\gamma'}} \leq c_x |t - s|^\gamma \left\{ 1 + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, (l+1)T_0]) \right\}.$$

Thus,

$$\begin{aligned} \|y_{ts}^{M,N,\#}\|_{\mathcal{B}_{\gamma'}} &\leq \|r_{tq}^{M,N}\|_{\mathcal{B}_{\gamma'}} + \|r_{st_p}^{M,N}\|_{\mathcal{B}_{\gamma'}} + \|r_{t_q t_{q-1}}^{M,N}\|_{\mathcal{B}_{\gamma'}} + c_{\gamma'} \sum_{k=p}^{q-2} |t - t_{k+1}|^{-\gamma'} \|r_{t_{k+1} t_k}^{M,N}\|_{\mathcal{B}} \\ &\leq c_x \left\{ 1 + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, (l+1)T_0]) \right\} \cdot \\ &\quad \left\{ |t - s|^\gamma + \frac{1}{M^{\gamma + \gamma' - 1}} \left(\frac{1}{M} \sum_{k=p}^{q-2} |t - t_{k+1}|^{-\gamma'} \right) \right\} \\ &\leq c_x \left\{ 1 + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, (l+1)T_0]) \right\} \left\{ |t - s|^\gamma + \frac{|t - s|^{1-\gamma'}}{M^{\gamma + \gamma' - 1}} \right\} \\ &\leq c_x |t - s|^\gamma \left\{ 1 + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, (l+1)T_0]) \right\}. \end{aligned}$$

2nd subcase: $t_p \leq s < t < t_{p+1}$. Then $(\hat{\delta}y^{M,N})_{ts} = X_{ts}^{x^M, i} P_N f_i(y_{t_p}^{M,N})$, so that

$$\|(\hat{\delta}y^{M,N})_{ts}\|_{\mathcal{B}_{\gamma'}} \leq c_x |t - s|^\gamma \left\{ 1 + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, (l+1)T_0]) \right\}.$$

3rd subcase: $t_p \leq s < t_{p+1} \leq t < t_{p+2}$ with $|t - s| \leq 1/M$. Just notice that $\|(\hat{\delta}y^{M,N})_{ts}\|_{\mathcal{B}_{\gamma'}} \leq \|(\hat{\delta}y^{M,N})_{tt_{p+1}}\|_{\mathcal{B}_{\gamma'}} + \|(\hat{\delta}y^{M,N})_{t_{p+1}s}\|_{\mathcal{B}_{\gamma'}}$, so that we can go back to the second subcase.

Conclusion of the 1st case:

$$\mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{\gamma'}([lT_0, (l+1)T_0])] \leq c_x T_0^{\gamma-\gamma'} \left\{ 1 + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}^{0,\gamma'}([0, (l+1)T_0]) \right\}.$$

2nd case: $s < lT_0 \leq t \leq (l+1)T_0$. One has $\|(\hat{\delta}y^{M,N})_{ts}\|_{\mathcal{B}_{\gamma'}} \leq \|(\hat{\delta}y^{M,N})_{t,lT_0}\|_{\mathcal{B}_{\gamma'}} + \|(\hat{\delta}y^{M,N})_{lT_0,s}\|_{\mathcal{B}_{\gamma'}}$, and so, owing to the recurrence assumption,

$$\|(\hat{\delta}y^{M,N})_{ts}\|_{\mathcal{B}_{\gamma'}} \leq |t-s|^{\gamma'} \left\{ \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{\gamma'}([lT_0, (l+1)T_0])] + R_l \right\}.$$

The association of the two cases gives

$$\mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{\gamma'}([0, (l+1)T_0])] \leq c_x^1 T_0^{\gamma-\gamma'} \left\{ 1 + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, (l+1)T_0]) \right\} + R_l.$$

Since, for any $t \in [0, (l+1)T_0]$, $\|y_t^{M,N}\|_{\mathcal{B}_{\gamma'}} \leq \|\psi\|_{\mathcal{B}_{\gamma'}} + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{\gamma'}([0, (l+1)T_0])]$, one deduces

$$\mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, (l+1)T_0])] \leq \|\psi\|_{\mathcal{B}_{\gamma'}} + 2R_l + 2c_x^1 T_0^{\gamma-\gamma'} \left\{ 1 + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}^{0,\gamma'}([0, (l+1)T_0]) \right\}.$$

To complete the proof, it now suffices to pick T_0 such that $2c_x^1 T_0^{\gamma-\gamma'} = 1/2$ and to set

$$R_{l+1} = 2\|\psi\|_{\mathcal{B}_{\gamma'}} + 4R_l + 1.$$

□

3.3. Space discretization. This is the final step, that will lead us from \bar{y}^M to $y^{M,N}$. As in the previous subsection, we extend $y^{M,N}$ on $[0, 1]$ via (38) and use the notations $r^{M,N}, y^{M,N,\#}$ introduced in Lemma 3.9.

Lemma 3.11. *There exists a function $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that if $t_p \leq s < t_{p+1} < \dots < t_q \leq t < t_{q+1}$, with $|t-s| \geq 1/M$, then*

$$\|y_{ts}^{M,M,\#}\|_{\mathcal{B}_{\gamma'}} \leq \frac{C(\|x\|_{\gamma}, \|\psi\|_{\mathcal{B}_{\gamma'}})}{M^{\gamma+\gamma'-1}} |t-s|^{\gamma'}.$$

Proof. Thanks to the uniform control given by Proposition 3.10, one has

$$\begin{aligned} & \|y_{ts}^{M,M,\#}\|_{\mathcal{B}_{\gamma'}} \\ & \leq \|r_{tt_q}^{M,M}\|_{\mathcal{B}_{\gamma'}} + \|r_{st_p}^{M,M}\|_{\mathcal{B}_{\gamma'}} + \|r_{t_q t_{q-1}}^{M,M}\|_{\mathcal{B}_{\gamma'}} + c_{\gamma'} \sum_{k=p}^{q-2} |t-t_{k+1}|^{-\gamma'} \|r_{t_{k+1} t_k}^{M,M}\|_{\mathcal{B}} \\ & \leq c_{x,\psi} \left\{ \frac{1}{M^{\gamma}} + \frac{1}{M^{\gamma+\gamma'-1}} \left(\frac{1}{M} \sum_{k=p}^{q-1} |t-t_{k+1}|^{-\gamma'} \right) \right\} \\ & \leq c_{x,\psi} \left\{ \frac{|t-s|^{\gamma'}}{M^{\gamma-\gamma'}} + \frac{|t-s|^{1-\gamma'}}{M^{\gamma+\gamma'-1}} \right\} \leq c_{x,\psi} \frac{|t-s|^{\gamma'}}{M^{\gamma+\gamma'-1}} \end{aligned}$$

where, for the last inequality, we have used the fact that $1/4 < \gamma' < 1/2 < \gamma < 1$. □

Lemma 3.12. *There exists a function $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that if $t_p \leq s < t_{p+1} < \dots < t_q \leq t < t_{q+1}$, with $|t-s| \geq 1/M$, one has*

$$\left\| \int_s^t S_{tu} dx_u^{i,M} (P_M - Id) f_i(y_u^{M,M}) \right\|_{\mathcal{B}_{\gamma'}} \leq \frac{C(\|x\|_{\gamma}, \|\psi\|_{\mathcal{B}_{\gamma'}})}{M^{2(\gamma-\gamma')}} |t-s|^{\gamma'}.$$

Proof. As P_M commutes with the semigroup, one can of course write

$$\begin{aligned} \int_s^t S_{tu} dx_u^{i,M} (P_M - \text{Id}) f_i(y_u^{M,M}) \\ = X_{ts}^{x,i,M} (P_M - \text{Id}) f_i(y_s^{M,M}) + (P_M - \text{Id}) \hat{\Lambda}_{ts} (X^{x,i,M} \delta f_i(y^{M,M})). \end{aligned}$$

From this expression, the uniform control given by Proposition 3.10 easily yields

$$\begin{aligned} \|X_{ts}^{x,i,M} (P_M - \text{Id}) f_i(y_s^{M,M})\|_{\mathcal{B}_{\gamma'}} &\leq c_x |t-s|^{\gamma-\gamma'} \|(P_M - \text{Id}) f_i(y_s^{M,M})\|_{\mathcal{B}} \\ &\leq c_x \frac{|t-s|^{\gamma-\gamma'}}{M^{2\gamma'}} \|f_i(y_s^{M,M})\|_{\mathcal{B}_{\gamma'}} \\ &\leq c_{x,\psi} \frac{|t-s|^{\gamma'}}{M^\gamma}, \end{aligned}$$

while

$$\begin{aligned} \|(P_M - \text{Id}) \hat{\Lambda}_{ts} (X^{x,i,M} \delta f_i(y^{M,M}))\|_{\mathcal{B}_{\gamma'}} \\ \leq \frac{1}{M^{2(\gamma-\gamma')}} \|\hat{\Lambda}_{ts} (X^{x,i,M} \delta f_i(y^{M,M}))\|_{\mathcal{B}_{\gamma}} \leq c_{x,\psi} \frac{|t-s|^{\gamma'}}{M^{2(\gamma-\gamma')}}. \end{aligned}$$

□

We are now in position to prove the main result of this subsection, which, associated to Corollary 3.8, completes the proof of Theorem 2.4.

Proposition 3.13. *There exists a function $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that for any $M \in \mathbb{N}^*$,*

$$\sup_{k \in \{0, \dots, M\}} \|\bar{y}^M - y^{M,M}\|_{\mathcal{B}_{\gamma'}} \leq C(\|x\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}}) \left\{ \|\psi - P_M \psi\|_{\mathcal{B}_{\gamma'}} + \frac{1}{M^{\gamma+\gamma'-1}} \right\}. \quad (46)$$

Proof. As in the previous subsection, we use the shortcut

$$\mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}(I)] := \mathcal{N}[\bar{y}^M - y^{M,M}; \mathcal{C}_1^0(I, \mathcal{B}_{\gamma'})] + \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{\gamma'}(I, \mathcal{B}_{\gamma'})].$$

Local result. Consider first an interval $I_0 = [0, T_0]$, with T_0 a time to be precised at the end of this first step, and let $s, t \in [0, T_0]$.

1st case: if $t_p \leq s < t < t_{p+1}$, then

$$\hat{\delta}(\bar{y}^M - y^{M,M})_{ts} = (\hat{\delta} \bar{y}^M)_{ts} - X_{ts}^{x,i,M} P_M f_i(y_{t_p}^{M,M}),$$

hence

$$\|\hat{\delta}(\bar{y}^M - y^{M,M})_{ts}\|_{\mathcal{B}_{\gamma'}} \leq c_{\psi,x} |t-s|^\gamma \leq c_{\psi,x} \frac{|t-s|^{\gamma'}}{M^{\gamma-\gamma'}}.$$

2nd case: if $t_p \leq s < t_{p+1} \leq t < t_{p+2}$, we go back to the previous case by noticing that

$$\|\hat{\delta}(\bar{y}^M - y^{M,M})_{ts}\|_{\mathcal{B}_{\gamma'}} \leq \|\hat{\delta}(\bar{y}^M - y^{M,M})_{tt_{p+1}}\|_{\mathcal{B}_{\gamma'}} + \|\hat{\delta}(\bar{y}^M - y^{M,M})_{t_{p+1}s}\|_{\mathcal{B}_{\gamma'}}.$$

3rd case: $t_p \leq s < t_{p+1} < \dots < t_q \leq t < t_{q+1}$ with $|t - s| \geq 1/M$. Then

$$\begin{aligned} & \hat{\delta}(\bar{y}^M - y^{M,M})_{ts} \\ &= \int_s^t S_{tu} dx_u^{i,M} [f_i(\bar{y}_u^M) - P_M f_i(y_u^{M,M})] + y_{ts}^{M,M,\#} \\ &= \int_s^t S_{tu} dx_u^{i,M} [f_i(\bar{y}_u^M) - f_i(y_u^{M,M})] \\ & \quad + \int_s^t S_{tu} dx_u^{i,M} (\text{Id} - P_M) f_i(y_u^{M,M}) + y_{ts}^{M,M,\#}. \end{aligned}$$

According to the two previous lemmas, one can assert that

$$\left\| \int_s^t S_{tu} dx_u^{i,M} (\text{Id} - P_M) f_i(y_u^{M,M}) + y_{ts}^{M,M,\#} \right\|_{\mathcal{B}_{\gamma'}} \leq c_{\psi,x} \frac{|t - s|^{\gamma'}}{M^{\gamma + \gamma' - 1}}.$$

Besides, it is not hard to see that

$$\left\| \int_s^t S_{tu} dx_u^{i,M} [f_i(\bar{y}_u^M) - f_i(y_u^{M,M})] \right\|_{\mathcal{B}_{\gamma'}} \leq c_{\psi,x}^1 |t - s|^{\gamma'} T_0^{\gamma - \gamma'} \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, T_0])],$$

for some constant $c_{\psi,x}^1$ that we fix for the rest of the proof.

Summing up the three cases, we get

$$\mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{\gamma'}([0, T_0]; \mathcal{B}_{\gamma'})] \leq \frac{c_{\psi,x}^2}{M^{\gamma + \gamma' - 1}} + c_{\psi,x}^1 T_0^{\gamma - \gamma'} \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, T_0])].$$

In order to estimate $\mathcal{N}[\bar{y}^M - y^{M,M}; \mathcal{C}_1^0([0, T_0], \mathcal{B}_{\gamma'})]$, it now suffices to observe that $\bar{y}_s^M - y_s^{M,M} = \hat{\delta}(\bar{y}^M - y^{M,M})_{s0} + S_{s0}(\psi - P_M \psi)$, and so

$$\begin{aligned} & \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, T_0])] \\ & \leq \|\psi - P_M \psi\|_{\mathcal{B}_{\gamma'}} + \frac{2c_{\psi,x}^2}{M^{\gamma + \gamma' - 1}} + 2c_{\psi,x}^1 T_0^{\gamma - \gamma'} \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, T_0])]. \end{aligned}$$

Thus, pick T_0 such that $2c_{\psi,x}^1 T_0^{\gamma - \gamma'} = 1/2$ to obtain

$$\mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, T_0])] \leq 2\|\psi - P_M \psi\|_{\mathcal{B}_{\gamma'}} + \frac{4c_{\psi,x}^2}{M^{\gamma + \gamma' - 1}}. \quad (47)$$

Extending the result: By following the same steps as in the local reasoning, we clearly get, for any $\eta > 0$,

$$\begin{aligned} & \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{\gamma'}([T_0, T_0 + \eta]; \mathcal{B}_{\gamma'})] \\ & \leq \frac{c_{\psi,x}^2}{M^{\gamma + \gamma' - 1}} + c_{\psi,x}^1 \eta^{\gamma - \gamma'} \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, T_0 + \eta], \mathcal{B}_{\gamma'})], \end{aligned}$$

which, together with (47), leads to

$$\begin{aligned} & \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{\gamma'}([0, T_0 + \eta]; \mathcal{B}_{\gamma'})] \\ & \leq 2\|\psi - P_M \psi\|_{\mathcal{B}_{\gamma'}} + \frac{5c_{\psi,x}^2}{M^{\gamma + \gamma' - 1}} + c_{\psi,x}^1 \eta^{\gamma - \gamma'} \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, T_0 + \eta], \mathcal{B}_{\gamma'})], \end{aligned}$$

and then

$$\begin{aligned} & \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, T_0 + \eta])] \\ & \leq 5 \|\psi - P_M \psi\|_{\mathcal{B}_{\gamma'}} + \frac{10 c_{\psi,x}^2}{M^{\gamma+\gamma'-1}} + 2 c_{\psi,x}^1 \eta^{\gamma-\gamma'} \mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, T_0 + \eta])]. \end{aligned}$$

By taking $\eta = T_0$, we deduce

$$\mathcal{N}[\bar{y}^M - y^{M,M}; \hat{\mathcal{C}}_1^{0,\gamma'}([0, 2T_0])] \leq 10 \|\psi - P_M \psi\|_{\mathcal{B}_{\gamma'}} + \frac{20 c_{\psi,x}^2}{M^{\gamma+\gamma'-1}}.$$

We repeat the procedure until the whole interval $[0, 1]$ is covered. \square

4. ROUGH CASE

We now turn to the proof of Theorem 2.5. Thus, let us fix $\gamma \in (\frac{1}{3}, \frac{1}{2})$, $\gamma' \in (1 - \gamma, 2\gamma]$, $\psi \in \mathcal{B}_{\gamma'}$, and suppose that Assumptions $(\mathbf{X2})_\gamma$ and $(\mathbf{F1})_3$ are satisfied. We will follow (almost) the same steps as in the previous section: we first use pre-existing continuity results to reduce the problem to the study of the Wong-Zakai approximation \bar{y}^M , and then lean on a uniform bound for $y^{M,N}$ to control the transition from \bar{y}^M to $y^{M,N}$.

Before we trigger the procedure, let us remind the reader with a few considerations taken from [8] on how to give sense to the equation under Assumption $(\mathbf{X2})_\gamma$. As in the Young case, the interpretation is based on an expansion of the regular equation: observe that if \tilde{x} is a piecewise differentiable process, then

$$\int_s^t S_{tu} d\tilde{x}_u^i f_i(y_u) = X_{ts}^{\tilde{x},i} f_i(y_s) + X_{ts}^{\tilde{x}\tilde{x},ij} (f'_i(y_s) \cdot f_j(y_s)) + J_{ts}^y, \quad (48)$$

where the operator-valued processes $X^{\tilde{x},i}$, $X^{\tilde{x}\tilde{x},ij}$ have been defined by (9) and $J_{ts}^y := \int_s^t S_{tu} d\tilde{x}_u^i M_{us}^i$, with

$$\begin{aligned} M_{us}^i & := \int_0^1 dr [f'_i(y_s + r(\delta y)_{us}) - f'_i(y_s)] \cdot (\delta y)_{us} \\ & \quad + \left[a_{us} y_s + \int_s^u S_{uv} d\tilde{x}_v^i \delta(f_i(y))_{vs} + \int_s^u a_{uv} dx_v^j f_j(y_s) \right] \cdot f'_i(y_s). \end{aligned} \quad (49)$$

On top of the result of Lemma 3.1, one can here rely on the following extensions (we also anticipate on the sequel by introducing the additional process X^{ax}):

Lemma 4.1 ([9]). *The sequence of operator-valued processes*

$$X_{ts}^{ax^M,i} := \int_s^t a_{tu} dx_u^{M,i} \quad , \quad \text{resp.} \quad X_{ts}^{x^M x^M,ij} := \int_s^t S_{tu} dx_u^{M,i} (\delta x^{M,j})_{us},$$

converges to an element $X^{ax,i}$ (resp. $X^{xx,ij}$) with respect to the topology of

$$\mathcal{C}_2^{\gamma+\kappa}(\mathcal{L}(\mathcal{B}_{\alpha+\kappa}, \mathcal{B}_\alpha)) \quad (\alpha \geq 0, \kappa \in [0, 1)),$$

$$\text{resp.} \quad \mathcal{C}_2^{2\gamma-\kappa}(\mathcal{L}(\mathcal{B}_\alpha, \mathcal{B}_{\alpha+\kappa})) \quad (\alpha \in \mathbb{R}, \kappa \in [0, 2\gamma)).$$

Moreover,

$$\begin{aligned} & \mathcal{N}[X^{xx,ij}; \mathcal{C}_2^{2\gamma-\kappa}(\mathcal{L}(\mathcal{B}_\alpha, \mathcal{B}_{\alpha+\kappa}))] \leq c_{\alpha,\kappa} \|\mathbf{x}\|_\gamma, \\ & \mathcal{N}[X^{x^M x^M,ij} - X^{xx,ij}; \mathcal{C}_2^{2\gamma-\kappa}(\mathcal{L}(\mathcal{B}_\alpha, \mathcal{B}_{\alpha+\kappa}))] \leq c_{\alpha,\kappa} v_M, \end{aligned}$$

and the same controls hold for $X^{ax,i}$ in $\mathcal{C}_2^{\gamma+\kappa}(\mathcal{L}(\mathcal{B}_{\alpha+\kappa}, \mathcal{B}_\alpha))$. Finally, $X^{ax,i}$ and $X^{xx,ij}$ commute with the projection P_N and satisfy the following algebraic relations:

$$(\hat{\delta}X^{xx,ij})_{tus} = X_{tu}^{x,i}(\delta x^j)_{us} \quad , \quad X_{ts}^{ax,i} = X_{ts}^{x,i} - (\delta x^i)_{ts}, \quad (50)$$

where $X^{x,i}$ is the process given by Lemma 3.1.

Now, from a heuristic point of view, if we go back to the γ -Hölder process x in (48), the expression (49) allows to identify J^y as a \mathcal{B} -valued process of order $\mu := \inf(3\gamma, \gamma + \gamma') > 1$. This (partially) accounts for the definition:

Definition 4.2. Let $\kappa \in (0, 1)$ and $\psi \in \mathcal{B}_\kappa$. A process $y : [0, 1] \rightarrow \mathcal{B}_\kappa$ is said to be a rough solution of (7) in \mathcal{B}_κ if there exists two parameters $\mu > 1, \varepsilon > 0$ such that

$$y_0 = \psi \quad \text{and} \quad \hat{\delta}y - X^{x,i}f_i(y) - X^{xx,ij}(f'_i(y) \cdot f_j(y)) \in \mathcal{C}_2^\mu(\mathcal{B}) \cap \mathcal{C}_2^\varepsilon(\mathcal{B}_\kappa).$$

In accordance with the above considerations, one has in particular:

Proposition 4.3 ([8]). If x is a piecewise differentiable process (resp. a standard Brownian motion) and if the initial condition ψ belongs to \mathcal{B}_η with $\eta \in (0, 1)$ (resp. $\eta \in (\frac{1}{2}, 1)$), then the classical (resp. Stratonovich) solution of (7) is also a rough solution in \mathcal{B}_η .

Remark 4.4. Let us go back here to the Young setting, ie when $\gamma > 1/2$. In order to connect the above interpretation of (7) with the notion of solution derived from Proposition 3.4, observe the following equivalence: under the assumptions of Theorem 3.5, a process $y \in \hat{\mathcal{C}}_1^{\gamma'}(\mathcal{B}_{\gamma'})$ is solution of (7) (in the sense of Proposition 3.4) if and only if $y_0 = \psi$ and there exists $\mu > 1, \varepsilon > 0$ such that $\hat{\delta}y - X^{x,i}f_i(y) \in \mathcal{C}_2^\mu(\mathcal{B}) \cap \mathcal{C}_2^\varepsilon(\mathcal{B}_{\gamma'})$. Indeed, if y is the solution given by Theorem 3.5, then, owing to the contraction property (30), $\hat{\delta}y - X^{x,i}f_i(y) = \hat{\Lambda}(X^{x,i}\delta f_i(y)) \in \mathcal{C}_2^{\gamma+\gamma'}(\mathcal{B}) \cap \mathcal{C}_2^\gamma(\mathcal{B}_{\gamma'})$. On the other hand, if $\hat{\delta}y - X^{x,i}f_i(y) \in \mathcal{C}_2^\mu(\mathcal{B}) \cap \mathcal{C}_2^\varepsilon(\mathcal{B}_{\gamma'})$ and z is defined by $z_0 = \psi, \hat{\delta}z = X^{x,i}f_i(y) + \hat{\Lambda}(X^{x,i}\delta f_i(y))$, one has $\hat{\delta}(y - z) \in \mathcal{C}_2^\mu(\mathcal{B})$, with $\tilde{\mu} = \inf(\mu, \gamma + \gamma') > 1$. As $y_0 = z_0$, this easily entails $y = z$.

4.1. Previous results. With the above definition in mind, the main result of [8] can be summed up in the following way:

Theorem 4.5 ([8]). Under the assumptions of Theorem 2.5, Equation (7) admits a unique rough solution in $\mathcal{B}_{\gamma'}$ in the sense of Definition 4.2. Moreover, if y (resp. \tilde{y}) is the rough solution in $\mathcal{B}_{\gamma'}$ of (7) associated to a process x (resp. \tilde{x}) that satisfies $(\mathbf{X2})_\gamma$, with initial condition ψ (resp. $\tilde{\psi}$) in $\mathcal{B}_{\gamma'}$, then

$$\mathcal{N}[y - \tilde{y}; \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\gamma'})] \leq C \left(\|x\|_\gamma, \|\tilde{x}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}}, \|\tilde{\psi}\|_{\mathcal{B}_{\gamma'}} \right) \left\{ \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma'}} + \|x - \tilde{x}\|_\gamma \right\}, \quad (51)$$

for some function $C : (\mathbb{R}^+)^4 \rightarrow \mathbb{R}^+$ bounded on bounded sets.

As in the Young case, denote \bar{y}^M the Wong-Zakai solution of (7), which corresponds to the classical (or equivalently rough) solution of the equation when x is replaced with x^{2^M} . The continuity result (51) allows to control the transition from y to \bar{y}^M :

Corollary 4.6. Under the assumptions of Theorem 2.5, there exists a function $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that for any M ,

$$\sup_{k \in \{0, \dots, 2^M\}} \|y_{t_k} - \bar{y}_{t_k}^M\|_{\mathcal{B}_{\gamma'}} \leq C(\|x\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}}) v_{2^M}.$$

Now it is worth noticing that the time-discretization of the equation has been analyzed in [8], too. In other words, we already know how to control the difference between \bar{y}^M and the process y^M generated by the intermediate Milstein scheme: $y_0^M = \psi$ and

$$y_{t_{k+1}}^M = S_{t_{k+1}t_k} y_{t_k}^M + X_{t_{k+1}t_k}^{x^{2^M},i} f_i(y_{t_k}^M) + X_{t_{k+1}t_k}^{x^{2^M},ij} (f'_i(y_{t_k}^M) \cdot f_j(y_{t_k}^M)), \quad (52)$$

where $t_k = t_k^M = \frac{k}{2^M}$. To express this result, let us denote (Π^M) the sequence of dyadic partitions of $[0, 1]$, and introduce the two processes

$$K_{ts}^M := (\hat{\delta}y^M)_{ts} - X_{ts}^{x^{2^M},i} f_i(y_s^M) \quad , \quad J_{ts}^M := K_{ts}^M - X_{ts}^{x^{2^M},ij} (f'_i(y_s^M) \cdot f_j(y_s^M)),$$

for every $s < t \in \Pi^M$. For sake of clarity, we will also appeal, in the sequel, to the discrete versions of the generalized Hölder norms introduced in Subsection 2.5. Thus, for any $M \in \mathbb{N}$, we denote $\llbracket a, b \rrbracket_M := [a, b] \cap \Pi^M$ and

$$\mathcal{N}[h; \hat{\mathcal{C}}_1^\lambda(\llbracket t_p^n, t_q^n \rrbracket_M, \mathcal{B}_{\alpha,p})] := \sup_{\substack{t_p^n \leq s < t \leq t_q^n \\ s, t \in \Pi^M}} \frac{\|(\hat{\delta}h)_{ts}\|_{\mathcal{B}_{\alpha,p}}}{|t - s|^\lambda},$$

We define the quantities

$$\mathcal{N}[:, \mathcal{C}_1^0(\llbracket a, b \rrbracket_M; \mathcal{B}_{\alpha,p})] \quad , \quad \mathcal{N}[:, \mathcal{C}_2^\lambda(\llbracket a, b \rrbracket_M; \mathcal{B}_{\alpha,p})] \quad , \quad \mathcal{N}[:, \mathcal{C}_3^\lambda(\llbracket a, b \rrbracket_M; \mathcal{B}_{\alpha,p})],$$

along the same line.

Proposition 4.7 ([8]). *Under the assumptions of Theorem 4.5, for every*

$$0 < \beta < \inf \left(\gamma + \gamma' - 1, \gamma - \gamma' + \frac{1}{2} \right),$$

there exists a function $C = C_\beta : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that for any M ,

$$\sup_{k=0, \dots, 2^M} \|\bar{y}_{t_k}^M - y_{t_k}^M\|_{\mathcal{B}_{\gamma'}} \leq \frac{C(\|\mathbf{x}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}})}{(2^M)^\beta}, \quad (53)$$

where y^M is the process generated by the intermediate Milstein scheme (52). Moreover, there exists another function $C' : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that the following uniform control holds: For every M ,

$$\mathcal{N}[y^M; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_M; \mathcal{B}_{\gamma'})] + \mathcal{N}[y^M; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, 1 \rrbracket_M; \mathcal{B})] + \mathcal{N}[K^M; \mathcal{C}_2^{2\gamma}(\llbracket 0, 1 \rrbracket_M; \mathcal{B})] \leq c_{x,\psi}. \quad (54)$$

where $c_{x,\psi} := C'(\|\mathbf{x}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}})$.

It now remains to study the transition from y^M to $y^{M,N}$, which is the purpose of the two following subsections.

4.2. A uniform control. The aim here is to exhibit a uniform estimate for $y^{M,N}$, to which we will extensively appeal in the next subsection. As in the time-discretization procedure, the two following processes will play a prominent role in our reasoning: for every M, N and every $s < t \in \Pi^M$, define

$$K_{ts}^{M,N} := (\hat{\delta}y^{M,N})_{ts} - X_{ts}^{x^{2^M},i} P_N f_i(y_s^{M,N}),$$

$$J_{ts}^{M,N} := K_{ts}^{M,N} - X_{ts}^{x^{2^M},ij} P_N (f'_i(y_s^{M,N}) \cdot P_N f_j(y_s^{M,N})).$$

Proposition 4.8. *There exists a function $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that for every M, N ,*

$$\begin{aligned} \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, 1 \rrbracket_M; \mathcal{B})] + \mathcal{N}[y^{M,N}; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_M; \mathcal{B}_{\gamma'})] + \mathcal{N}[K^{M,N}; \mathcal{C}_2^{2\gamma}(\llbracket 0, 1 \rrbracket_M; \mathcal{B})] \\ \leq C(\|\mathbf{x}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}}). \end{aligned} \quad (55)$$

Proposition 4.8 is actually a spin-off of the following successive controls on $J^{M,N}$:

Proposition 4.9. *Fix ε, μ such that*

$$\gamma + \gamma' > \mu > 1 \quad , \quad \gamma - (\gamma' - \frac{1}{2}) > \varepsilon > 0.$$

There exists two integers $M_0 = M_0(\|\mathbf{x}\|_\gamma)$, $N_0 = N_0(\|\mathbf{x}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}})$ and a time $T_0 = T_0(\|\mathbf{x}\|_\gamma) > 0$, $T_0 \in \Pi^{M_0}$, such that for every $M \geq M_0$, $N \geq N_0$, for any k ,

$$\mathcal{N}[J^{M,N}; \mathcal{C}_2^\mu(\llbracket kT_0, (k+1)T_0 \wedge 1 \rrbracket_M; \mathcal{B})] \leq 1 + \|y_{kT_0}^{M,N}\|_{\mathcal{B}_{\gamma'}} \quad (56)$$

and

$$\mathcal{N}[J^{M,N}; \mathcal{C}_2^\varepsilon(\llbracket kT_0, (k+1)T_0 \wedge 1 \rrbracket_M; \mathcal{B}_{\gamma'})] \leq 1 + \|y_{kT_0}^{M,N}\|_{\mathcal{B}_{\gamma'}}. \quad (57)$$

The proof of Proposition 4.9 resorts to the following technical lemmas:

Lemma 4.10 ([8]). *Let $\varepsilon > 0$ and $\mu > 1$. There exists a constant $c = c_{\varepsilon, \mu}$ such that for any M and any process $A : \mathcal{S}_2 \rightarrow \mathcal{B}_{\gamma'}$ satisfying $A_{t_{k+1}^M, t_k^M} = 0$ for every $k \in \{0, \dots, M-1\}$,*

$$\|A_{ts}\|_{\mathcal{B}} \leq c |t - s|^\mu \mathcal{N}[\hat{\delta}A; \mathcal{C}_3^\mu(\llbracket s, t \rrbracket_M; \mathcal{B})] \quad (58)$$

and

$$\|A_{ts}\|_{\mathcal{B}_{\gamma'}} \leq c \left\{ |t - s|^\varepsilon + |t - s|^{\mu - \gamma'} \right\} \left\{ \mathcal{N}[\hat{\delta}A; \mathcal{C}_3^\mu(\llbracket s, t \rrbracket_M; \mathcal{B})] + \mathcal{N}[\hat{\delta}A; \mathcal{C}_3^\varepsilon(\llbracket s, t \rrbracket_M; \mathcal{B}_{\gamma'})] \right\}. \quad (59)$$

for any $s < t \in \Pi^M$.

Lemma 4.11. *There exists a function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ bounded on bounded sets such that for any M and every $s < t \in \Pi^M$,*

$$\begin{aligned} \mathcal{N}[\hat{\delta}J^{M,N}; \mathcal{C}_3^\mu(\llbracket s, t \rrbracket_M; \mathcal{B})] \leq c_x |t - s|^{\gamma + \gamma' - \mu} \\ \left\{ \mathcal{N}[y^{M,N}; \mathcal{C}_1^0(\llbracket s, t \rrbracket_M; \mathcal{B}_{\gamma'})] + \mathcal{N}[y^{M,N}; \hat{\mathcal{C}}_1^\gamma(\llbracket s, t \rrbracket_M; \mathcal{B})] + \mathcal{N}[K^{M,N}; \mathcal{C}_2^{2\gamma}(\llbracket s, t \rrbracket_M; \mathcal{B})] \right\} \\ \left\{ 1 + N^{1-2\gamma'} \mathcal{N}[y^{M,N}; \mathcal{C}_1^0(\llbracket s, t \rrbracket_M; \mathcal{B}_{\gamma'})]^2 \right\} \end{aligned} \quad (60)$$

and

$$\begin{aligned} \mathcal{N}[\hat{\delta}J^{M,N}; \mathcal{C}_3^\varepsilon(\llbracket s, t \rrbracket_M; \mathcal{B}_{\gamma'})] \leq c_x |t - s|^{\gamma - (\gamma' - \frac{1}{2}) - \varepsilon} \\ \left\{ 1 + \mathcal{N}[y^{M,N}; \mathcal{C}_1^0(\llbracket s, t \rrbracket_M; \mathcal{B}_{\gamma'})] \right\} \left\{ 1 + N^{1-2\gamma'} \mathcal{N}[y^{M,N}; \mathcal{C}_1^0(\llbracket s, t \rrbracket_M; \mathcal{B}_{\gamma'})]^2 \right\}, \end{aligned} \quad (61)$$

where $c_x := C(\|\mathbf{x}\|_\gamma)$.

Proof. See Appendix. □

Proof of Proposition 4.9. For sake of clarity, we write here x for x^{2^M} .

Step 1: $k = 0$. This is an iteration procedure over the points of the partition. Assume that both inequalities (56) and (57) hold true on $\llbracket 0, t_q^M \rrbracket_M$ and $t_{q+1}^M \leq T_0$ (T_0 will actually be precised in the course of the reasoning). Then, for every $t \in \llbracket 0, t_q^M \rrbracket_M$, one has

$$\begin{aligned} & \|y_t^{M,N}\|_{\mathcal{B}_{\gamma'}} \\ & \leq \|J_{t_0}^{M,N}\|_{\mathcal{B}_{\gamma'}} + \|S_{t_0}\psi\|_{\mathcal{B}_{\gamma'}} + \|X_{t_0}^{x,i}P_N f_i(\psi)\|_{\mathcal{B}_{\gamma'}} + \|X_{t_0}^{xx,ij}P_N(f'_i(\psi) \cdot P_N f_j(\psi))\|_{\mathcal{B}_{\gamma'}} \\ & \leq 1 + 2\|\psi\|_{\mathcal{B}_{\gamma'}} + c_x \left\{ t^{\gamma - (\gamma' - \frac{1}{2})} \|f_i(\psi)\|_{\mathcal{B}_{1/2}} + t^{2\gamma - (\gamma' - \eta)} \|f'_i(\psi) \cdot P_N f_j(\psi)\|_{\mathcal{B}_\eta} \right\}, \end{aligned}$$

where $\eta \in (0, \frac{1}{2})$ is picked such that $2\gamma > \gamma' - \eta$. Now $\|f_i(\psi)\|_{\mathcal{B}_{1/2}} \leq c \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\}$ and, as in (67) and (66),

$$\|f'_i(\psi) \cdot P_N f_j(\psi)\|_{\mathcal{B}_\eta} \leq c \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\} \left\{ 1 + \frac{\|\psi\|_{\mathcal{B}_{\gamma'}}^2}{N^{2\gamma'-1}} \right\},$$

hence

$$\mathcal{N}[y^{M,N}; \mathcal{C}_1^0(\llbracket 0, t_q^M \rrbracket_M; \mathcal{B}_{\gamma'})] \leq c_x^1 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\} \left\{ 1 + \frac{\|\psi\|_{\mathcal{B}_{\gamma'}}^2}{N^{2\gamma'-1}} \right\}.$$

At this point, let us introduce an integer $N_0^{1,1}$ such that $\frac{\|\psi\|_{\mathcal{B}_{\gamma'}}^2}{(N_0^{1,1})^{2\gamma'-1}} \leq 1$, so that for any $N \geq N_0^{1,1}$, $\mathcal{N}[y^{M,N}; \mathcal{C}_1^0(\llbracket 0, t_q^M \rrbracket_M; \mathcal{B}_{\gamma'})] \leq 2c_x^1 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\}$. Besides, if $s < t \in \llbracket 0, t_q^M \rrbracket_M$,

$$\begin{aligned} & \|(\hat{\delta}y^{M,N})_{ts}\|_{\mathcal{B}} \\ & \leq \|J_{ts}^{M,N}\|_{\mathcal{B}} + \|X_{ts}^{x,i}P_N f_i(y_s^{M,N})\|_{\mathcal{B}} + \|X_{ts}^{xx,ij}P_N(f'_i(y_s^{M,N}) \cdot P_N f_j(y_s^{M,N}))\|_{\mathcal{B}} \\ & \leq c_x |t - s|^\gamma \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\}, \end{aligned}$$

and

$$\begin{aligned} \|K_{ts}^{M,N}\|_{\mathcal{B}} & \leq \|J_{ts}^{M,N}\|_{\mathcal{B}} + \|X_{ts}^{xx,ij}P_N(f'_i(y_s^{M,N}) \cdot P_N f_j(y_s^{M,N}))\|_{\mathcal{B}} \\ & \leq c |t - s|^{2\gamma} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\}. \end{aligned}$$

By using (60), we deduce, for any $N \geq N_0^{1,1}$ (remember that $t_{q+1}^M \leq T_0$),

$$\begin{aligned} & \mathcal{N}[\hat{\delta}J^{M,N}; \mathcal{C}_3^\mu(\llbracket 0, t_{q+1}^M \rrbracket_M; \mathcal{B})] \\ & \leq c_x T_0^{\gamma + \gamma' - \mu} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\} \left\{ 1 + \frac{\mathcal{N}[y^{M,N}; \mathcal{C}_1^0(\llbracket 0, t_q^M \rrbracket_M; \mathcal{B}_{\gamma'})]^2}{N^{2\gamma'-1}} \right\} \\ & \leq c_x T_0^{\gamma + \gamma' - \mu} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\} \left\{ 1 + \frac{c_x^2 \|\psi\|_{\mathcal{B}_{\gamma'}}^2}{N^{2\gamma'-1}} \right\}, \end{aligned}$$

and in the same way, according to (61),

$$\mathcal{N}[\hat{\delta}J^{M,N}; \mathcal{C}_3^\varepsilon(\llbracket 0, t_{q+1}^M \rrbracket_M; \mathcal{B}_{\gamma'})] \leq c_x T_0^{\gamma - (\gamma' - \frac{1}{2}) - \varepsilon} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\} \left\{ 1 + \frac{c_x^2 \|\psi\|_{\mathcal{B}_{\gamma'}}^2}{N^{2\gamma'-1}} \right\}.$$

Let us fix an integer $N_0^{1,2} \geq N_0^{1,1}$ such that $\frac{c_x^2 \|\psi\|_{\mathcal{B}_{\gamma'}}^2}{N^{2\gamma'-1}} \leq 1$. Then, thanks to (58) and (59) (applied to $A = J^{M,N}$), we obtain, for any $N \geq N_0^{1,2}$,

$$\mathcal{N}[J^{M,N}; \mathcal{C}_2^\mu(\llbracket 0, t_{q+1}^M \rrbracket_M; \mathcal{B})] \leq c_x^3 \left\{ T_0^{\gamma+\gamma'-\mu} + T_0^{\gamma-(\gamma'-\frac{1}{2})-\varepsilon} \right\} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\},$$

and

$$\mathcal{N}[J^{M,N}; \mathcal{C}_2^\varepsilon(\llbracket 0, t_{q+1}^M \rrbracket_M; \mathcal{B}_{\gamma'})] \leq c_x^3 \left\{ T_0^{\gamma+\gamma'-\mu} + T_0^{\gamma-(\gamma'-\frac{1}{2})-\varepsilon} \right\} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\}.$$

Consider now a real $T'_0 > 0$ such that $c_x^3 \left\{ (T'_0)^{\gamma+\gamma'-\mu} + (T'_0)^{\gamma-(\gamma'-\frac{1}{2})-\varepsilon} \right\} \leq 1$ and let $M_0 = M_0(\|\mathbf{x}\|_\gamma)$ an integer such that $1/(2^{M_0}) \leq T'_0$. We fix T_0 in the non empty set $(0, T'_0) \cap \Pi^{M_0}$ so as to retrieve the expected controls, namely: for every $M \geq M_0, N \geq N_0^{1,2}$

$$\mathcal{N}[J^{M,N}; \mathcal{C}_2^\mu(\llbracket 0, t_{q+1}^M \rrbracket_M; \mathcal{B})] \leq 1 + \|\psi\|_{\gamma'}, \quad \mathcal{N}[J^{M,N}; \mathcal{C}_2^\varepsilon(\llbracket 0, t_{q+1}^M \rrbracket_M; \mathcal{B}_{\gamma'})] \leq 1 + \|\psi\|_{\gamma'},$$

which completes Step 1, that is to say the proof of (56) and (57) on $\llbracket T_0, 2T_0 \rrbracket_M$.

Step 2: $k = 1$. We henceforth fix $M \geq M_0$. With the same arguments as in Step 1, we first deduce, if both controls (56) and (57) are checked on $\llbracket T_0, t_q^M \rrbracket_M$ (with $t_{q+1}^M \leq 2T_0$),

$$\mathcal{N}[y^{M,N}; \mathcal{C}_1^0(\llbracket T_0, t_q^M \rrbracket_M; \mathcal{B}_{\gamma'})] \leq c_x^1 \left\{ 1 + \|y_{T_0}^{M,N}\|_{\mathcal{B}_{\gamma'}} \right\} \left\{ 1 + \frac{\|y_{T_0}^{M,N}\|_{\mathcal{B}_{\gamma'}}^2}{N^{2\gamma'-1}} \right\}. \quad (62)$$

Remember that for any $N \geq N_0^{1,1}$, $\|y_{T_0}^{M,N}\|_{\mathcal{B}_{\gamma'}} \leq 2c_x^1 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma'}} \right\}$. Consequently, we introduce an integer $N_0^{2,1} \geq N_0^{1,2}$ such that $\frac{(2c_x^1 \{1 + \|\psi\|_{\mathcal{B}_{\gamma'}}\})^2}{(N_0^{2,1})^{2\gamma'-1}} \leq 1$, and (62) entails, for any $N \geq N_0^{2,1}$,

$$\mathcal{N}[y^{M,N}; \mathcal{C}_1^0(\llbracket T_0, t_q^M \rrbracket_M; \mathcal{B}_{\gamma'})] \leq 2c_x^1 \left\{ 1 + \|y_{T_0}^{M,N}\|_{\mathcal{B}_{\gamma'}} \right\}.$$

Then, with the same estimates as in Step 1, we get, for any $N \geq N_0^{2,1}$,

$$\mathcal{N}[J^{M,N}; \mathcal{C}_2^\mu(\llbracket T_0, t_{q+1}^M \rrbracket_M; \mathcal{B})] \leq \frac{1}{2} \left\{ 1 + \frac{c_x^2 \|y_{T_0}^{M,N}\|_{\mathcal{B}_{\gamma'}}^2}{N^{2\gamma'-1}} \right\} \left\{ 1 + \|y_{T_0}^{M,N}\|_{\mathcal{B}_{\gamma'}} \right\}$$

and this control also holds for $\mathcal{N}[J^{M,N}; \mathcal{C}_2^\varepsilon(\llbracket T_0, t_{q+1}^M \rrbracket_M; \mathcal{B}_{\gamma'})]$. Introduce finally $N_0^{2,2} \geq N_0^{2,1}$ such that $\frac{c_x^2 (2c_x^1 \{1 + \|\psi\|_{\mathcal{B}_{\gamma'}}\})^2}{(N_0^{2,2})^{2\gamma'-1}} \leq 1$ to get (56) and (57) on $\llbracket T_0, t_{q+1}^M \rrbracket_M$ (for $N \geq N_0^{2,2}$), which completes the proof of (56) and (57) on $\llbracket T_0, 2T_0 \rrbracket_M$.

We repeat the procedure until Step L , where $L = L(\|\mathbf{x}\|_\gamma)$ is the smallest integer such that $LT_0 \geq 1$. □

Once endowed with the estimates of Proposition 4.9, the proof of Proposition 4.8 follows the same lines as the proof of Theorem 2.10 in [8]. For sake of conciseness, the reader is referred to the latter paper for further details on the procedure.

4.3. Space discretization. We are now in position to compare y^M with $y^{M,N}$. To this end, let us introduce the intermediate quantity: for every $s < t \in \Pi^M$,

$$\begin{aligned} \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket s, t \rrbracket_M)] &:= \mathcal{N}[y^M - y^{M,N}; \mathcal{C}_1^0(\llbracket s, t \rrbracket_M; \mathcal{B}_{\gamma'})] \\ &\quad + \mathcal{N}[y^M - y^{M,N}; \hat{\mathcal{C}}_1^\gamma(\llbracket s, t \rrbracket_M; \mathcal{B})] + \mathcal{N}[K^M - K^{M,N}; \mathcal{C}_2^{2\gamma}(\llbracket s, t \rrbracket_M; \mathcal{B})]. \end{aligned}$$

Lemma 4.12. *For any $\lambda \in (0, \gamma + \gamma' - 1)$, there exists a function $C = C_\lambda : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ bounded on bounded sets such that for any M, N , for every $s < t \in \Pi^M$,*

$$\mathcal{N}[\hat{\delta}(J^M - J^{M,N}); \mathcal{C}_3^{\gamma+\gamma'-\lambda}(\llbracket s, t \rrbracket_M; \mathcal{B})] \leq c_{x,\psi} \left\{ |t - s|^\lambda \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket s, t \rrbracket_M)] + \frac{1}{N^{2\lambda}} \right\},$$

and

$$\mathcal{N}[\hat{\delta}(J^M - J^{M,N}); \mathcal{C}_3^{\gamma-\lambda}(\llbracket s, t \rrbracket_M; \mathcal{B}_{\gamma'})] \leq c_{x,\psi} \left\{ |t - s|^\lambda \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket s, t \rrbracket_M)] + \frac{1}{N^{2\lambda}} \right\},$$

where $c_{x,\psi} := C(\|\mathbf{x}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma'}})$.

Proof. See Appendix. □

Proof of Theorem 2.5. For sake of clarity, we write here x for x^{2M} . Consider a time $T_1 \in \llbracket 0, 1 \rrbracket_M$. For any $t \in \llbracket 0, T_1 \rrbracket_M$, one has

$$\begin{aligned} y_t^M - y_t^{M,N} &= [\psi - P_N \psi] + [X_{t_0}^{x,i} f_i(\psi) - X_{t_0}^{x,i} P_N f_i(P_N \psi)] \\ &\quad + [X_{t_0}^{xx,ij} (f'_i(\psi) \cdot f_j(\psi)) - X_{t_0}^{xx,ij} P_N (f'_i(P_N \psi) \cdot P_N f_j(P_N \psi))] + [J_{t_0}^M - J_{t_0}^{M,N}]. \end{aligned}$$

Thanks to Lemma 4.10 (applied to $A = J^M - J^{M,N}$) and Lemma 4.12, we already know that

$$\|J_{t_0}^M - J_{t_0}^{M,N}\|_{\mathcal{B}_{\gamma'}} \leq c_{x,\psi} \{T_1^\gamma \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_M)] + N^{-2\lambda}\}.$$

Then

$$\begin{aligned} &\|X_{t_0}^{x,i} f_i(\psi) - X_{t_0}^{x,i} P_N f_i(P_N \psi)\|_{\mathcal{B}_{\gamma'}} \\ &\leq \|X_{t_0}^{x,i} [f_i(\psi) - f_i(P_N \psi)]\|_{\mathcal{B}_{\gamma'}} + \|X_{t_0}^{x,i} (\text{Id} - P_N) f_i(P_N \psi)\|_{\mathcal{B}_{\gamma'}} \\ &\leq c_{x,\psi} \left\{ \|\psi - P_N \psi\|_{\mathcal{B}_{\gamma'}} + N^{-2\lambda} \right\} \end{aligned}$$

and with similar arguments

$$\begin{aligned} &\|X_{t_0}^{xx,ij} (f'_i(\psi) \cdot f_j(\psi)) - X_{t_0}^{xx,ij} P_N (f'_i(P_N \psi) \cdot P_N f_j(P_N \psi))\|_{\mathcal{B}_{\gamma'}} \\ &\leq c_{x,\psi} \left\{ \|\psi - P_N \psi\|_{\mathcal{B}_{\gamma'}} + N^{-2\lambda} \right\}, \end{aligned}$$

so that

$$\begin{aligned} &\mathcal{N}[y^M - y^{M,N}; \mathcal{C}_1^0(\llbracket 0, T_1 \rrbracket_M; \mathcal{B}_{\gamma'})] \\ &\leq c_{x,\psi} \left\{ T_1^\gamma \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_M)] + \|\psi - P_N \psi\|_{\mathcal{B}_{\gamma'}} + N^{-2\lambda} \right\}. \end{aligned}$$

Let us now analyze (in \mathcal{B}) the decomposition: for every $s < t \in \llbracket 0, T_1 \rrbracket_M$,

$$\begin{aligned} \hat{\delta}(y^M - y^{M,N})_{ts} &= [X_{ts}^{x,i} f_i(y_s^M) - X_{ts}^{x,i} P_N f_i(y_s^{M,N})] \\ &\quad + [X_{ts}^{xx,ij} (f'_i(y_s^M) \cdot f_j(y_s^M)) - X_{ts}^{xx,ij} P_N (f'_i(y_s^{M,N}) \cdot P_N f_j(y_s^{M,N}))] + [J_{ts}^M - J_{ts}^{M,N}]. \end{aligned}$$

According to Lemmas 4.10 and 4.12,

$$\|J_{ts}^M - J_{ts}^{M,N}\|_{\mathcal{B}} \leq c_{x,\psi} |t - s|^{2\gamma} \left\{ T_1^{\gamma' - \gamma} \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_M)] + N^{-2\lambda} \right\}. \quad (63)$$

Moreover,

$$\begin{aligned} & \|X_{ts}^{x,i} f_i(y_s^M) - X_{ts}^{x,i} P_N f_i(y_s^{M,N})\|_{\beta} \\ & \leq c_x |t - s|^\gamma \|y_s^M - y_s^{M,N}\|_{\mathcal{B}} + \|X_{ts}^{x,i} (\text{Id} - P_N) f_i(y_s^{M,N})\|_{\mathcal{B}} \\ & \leq c_{x,\psi} |t - s|^\gamma \left\{ T_1^\gamma \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_M)] + \|\psi - P_N \psi\|_{\mathcal{B}_{\gamma'}} + N^{-2\lambda} \right\} \end{aligned}$$

and this kind of argument leads to

$$\begin{aligned} & \mathcal{N}[y^M - y^{M,N}; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, T_1 \rrbracket_M; \mathcal{B}_{\gamma'})] \\ & \leq c_{x,\psi} \left\{ T_1^\gamma \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_M)] + \|\psi - P_N \psi\|_{\mathcal{B}_{\gamma'}} + N^{-2\lambda} \right\}. \end{aligned}$$

Finally,

$$\begin{aligned} & K_{ts}^M - K_{ts}^{M,N} = \\ & \left[X_{ts}^{xx,ij} (f'_i(y_s^M) \cdot f_j(y_s^M)) - X_{ts}^{xx,ij} P_N (f'_i(y_s^{M,N}) \cdot P_N f_j(y_s^{M,N})) \right] + \left[J_{ts}^M - J_{ts}^{M,N} \right], \end{aligned}$$

and thanks to (63), this decomposition easily allows to conclude that

$$\begin{aligned} & \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_M)] \\ & \leq c_{x,\psi}^1 \left\{ T_1^{\gamma' - \gamma} \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_M)] + \|\psi - P_N \psi\|_{\mathcal{B}_{\gamma'}} + N^{-2\lambda} \right\}. \end{aligned}$$

Let $T_1^* > 0$ and $M_0 \in \mathbb{N}$ such that $c_{x,\psi}^1 (T_1^*)^{\gamma - \gamma'} = \frac{1}{2}$ and $(0, T_1^*) \cap \Pi^{M_0} \neq \emptyset$. The time T_1 is now fixed in $(0, T_1^*) \cap \Pi^{M_0}$ so as to retrieve, for any $M \geq M_0$,

$$\mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_M)] \leq 2c_{x,\psi}^1 \left\{ \|\psi - P_N \psi\|_{\mathcal{B}_{\gamma'}} + N^{-2\lambda} \right\}.$$

It is readily checked that the same reasoning (with the same constants) holds on any interval $\llbracket kT_1, (k+1)T_1 \wedge 1 \rrbracket_M$ and leads to

$$\mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket kT_1, (k+1)T_1 \wedge 1 \rrbracket_M)] \leq 2c_{x,\psi}^1 \left\{ \|y_{kT_1}^M - y_{kT_1}^{M,N}\|_{\mathcal{B}_{\gamma'}} + N^{-2\lambda} \right\}.$$

As T_1 only depends on x and ψ , it follows from a standard patching argument that

$$\mathcal{N}[y^M - y^{M,N}; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_M; \mathcal{B}_{\gamma'})] \leq c_{x,\psi} N^{-2\lambda},$$

which, together with the results of Corollary 4.6 and Proposition 4.7, completes the proof of (15). \square

5. APPENDIX A

Let us go back here to the technical proofs that have been left in abeyance in Section 4.

Proof of Lemma 4.11. For sake of clarity, we write here x for x^{2^M} and y for $y^{M,N}$. One has

$$\begin{aligned} (\hat{\delta}J^{M,N})_{tus} &= X_{tu}^{x,i} P_N \delta(f_i(y))_{us} - X_{tu}^{x,i} (\delta x^j)_{us} P_N (f'_i(y_s) \cdot P_N f_j(y_s)) \\ &\quad + X_{tu}^{xx,ij} P_N \delta(f'_i(y) \cdot P_N f_j(y))_{us}, \end{aligned} \quad (64)$$

which easily entails

$$(\hat{\delta}J^{M,N})_{tus} = I_{tus}^{M,N} + II_{tus}^{M,N} + III_{tus}^{M,N} + IV_{tus}^{M,N}, \quad (65)$$

with

$$\begin{aligned} I_{tus}^{M,N} &:= X_{tu}^{x,i} P_N \left(\int_0^1 dr f'_i(y_s + r(\delta y)_{us}) \cdot K_{us}^{M,N} \right), \\ II_{tus}^{M,N} &:= X_{tu}^{x,i} P_N \left(\int_0^1 dr f'_i(y_s + r(\delta y)_{us}) \cdot [a_{us} y_s + X_{us}^{ax,j} P_N f_j(y_s)] \right), \\ III_{tus}^{M,N} &:= X_{tu}^{x,i} P_N \left(\int_0^1 dr [f'_i(y_s + r(\delta y)_{us}) - f'_i(y_s)] (\delta x^j)_{us} \cdot P_N f_j(y_s) \right), \\ IV_{tus}^{M,N} &:= X_{tu}^{xx,ij} P_N \delta(f'_i(y) \cdot P_N f_j(y))_{us}. \end{aligned}$$

First, $\|I_{tus}^{M,N}\|_{\mathcal{B}} \leq c_x |t-u|^\gamma \|K_{us}^{M,N}\|_{\mathcal{B}}$, and

$$\begin{aligned} \|II_{tus}^{M,N}\|_{\mathcal{B}} &\leq c_x |t-u|^\gamma \left\{ |u-s|^{\gamma'} \|y_s\|_{\mathcal{B}_{\gamma'}} + |u-s|^{\gamma+1/2} \|f_i(y_s)\|_{\mathcal{B}_{1/2}} \right\} \\ &\leq c_x |t-s|^{\gamma+\gamma'} \left\{ 1 + \|y_s\|_{\mathcal{B}_{\gamma'}} \right\}. \end{aligned}$$

Then

$$\begin{aligned} \|III_{tus}^{M,N}\|_{\mathcal{B}} &\leq c_x |t-s|^{2\gamma} \|(\delta y)_{us}\|_{\mathcal{B}} \|P_N f_i(y_s)\|_{L^\infty} \\ &\leq c_x |t-s|^{2\gamma} \left\{ \|(\hat{\delta}y)_{us}\|_{\mathcal{B}} + |u-s|^{\gamma'} \|y_s\|_{\mathcal{B}_{\gamma'}} \right\} \left\{ 1 + \|(P_N - \text{Id})f_i(y_s)\|_{L^\infty} \right\} \end{aligned}$$

and

$$\begin{aligned} \|(P_N - \text{Id})f_i(y_s)\|_{L^\infty} &\leq c \|(P_N - \text{Id})f_i(y_s)\|_{\mathcal{B}_{1/2}} \leq \frac{c}{N^{2\gamma'-1}} \|f_i(y_s)\|_{\mathcal{B}_{\gamma'}} \\ &\leq \frac{c}{N^{2\gamma'-1}} \left\{ 1 + \|y_s\|_{\mathcal{B}_{\gamma'}}^2 \right\}. \end{aligned} \quad (66)$$

Finally,

$$\begin{aligned} \|IV_{tus}^{M,N}\|_{\mathcal{B}} &\leq c_x |t-u|^{2\gamma} \|(\delta y)_{us}\|_{\mathcal{B}} \left\{ 1 + \|P_N f_j(y_s)\|_{L^\infty} \right\} \\ &\leq c_x |t-u|^{2\gamma} \|(\delta y)_{us}\|_{\mathcal{B}} \left\{ 1 + \frac{\|y_s\|_{\mathcal{B}_{\gamma'}}^2}{N^{2\gamma'-1}} \right\}. \end{aligned}$$

Going back to (65), those estimates yield (60). To get (61), we resort to the decomposition (64) and observe that (for instance)

$$\begin{aligned} \|X_{tu}^{x,i} P_N f_i(y_u)\|_{\mathcal{B}_{\gamma'}} &\leq c_x |t-u|^{\gamma-(\gamma'-\frac{1}{2})} \|f_i(y_u)\|_{\mathcal{B}_{1/2}} \\ &\leq c_x |t-s|^{\gamma-(\gamma'-\frac{1}{2})} \left\{ 1 + \|y_u\|_{\mathcal{B}_{\gamma'}} \right\}, \end{aligned}$$

and for any $\eta \in (\gamma' - \gamma, \frac{1}{2})$,

$$\begin{aligned} & \|X_{tu}^{x,i}(\delta x^j)_{us} P_N(f'_i(y_s) \cdot P_N f_j(y_s))\|_{\mathcal{B}_{\gamma'}} \\ & \leq c_x |t - u|^{\gamma - (\gamma' - \eta)} |u - s|^\gamma \|f'_i(y_s) \cdot P_N f_j(y_s)\|_{\mathcal{B}_\eta} \\ & \leq c_x |t - s|^{2\gamma - (\gamma' - \eta)} \left\{ 1 + \|y_s\|_{\mathcal{B}_{\gamma'}} \right\} \left\{ 1 + \|P_N f_j(y_s)\|_{L^\infty} \right\}, \end{aligned} \quad (67)$$

where, to get the last estimate, we have used the property (21). Together with (66), this leads to (61). \square

Proof of Lemma 4.12. Observe first that $\hat{\delta} J^M$ can be decomposed as in (64) or as in (65), by suppressing in both expressions the projection operator P_N . In order to estimate $\|\hat{\delta}(J^M - J^{M,N})_{tus}\|_{\mathcal{B}}$, we rely on the decomposition (65) and its equivalent for J^M , with I^M instead of $I^{M,N}$, etc. Write for instance

$$\begin{aligned} & I_{tus}^M - I_{tus}^{M,N} \\ & = X_{tu}^{x,i} \left(\int_0^1 dr [f'_i(y_s^M + r(\delta y^M)_{us}) - f'_i(y_s^{M,N} + r(\delta y^{M,N})_{us})] \cdot K_{us}^M \right) \\ & \quad + X_{tu}^{x,i} \left(\int_0^1 dr f'_i(y_s^{M,N} + r(\delta y^{M,N})_{us}) \cdot [K_{us}^M - K_{us}^{M,N}] \right) \\ & \quad + X_{tu}^{x,i} (\text{Id} - P_N) \left(\int_0^1 dr f'_i(y_s^{M,N} + r(\delta y^{M,N})_{us}) \cdot K_{us}^{M,N} \right) =: I_{tus}^{(1)} + I_{tus}^{(2)} + I_{tus}^{(3)}. \end{aligned}$$

Owing to the uniform estimate (54) and the continuous inclusion $\mathcal{B}_{\gamma'} \subset L^\infty$, one has first

$$\begin{aligned} \|I_{tus}^{(1)}\|_{\mathcal{B}} & \leq c_{x,\psi} |t - s|^{3\gamma} \left\{ \|y_s^M - y_s^{M,N}\|_{L^\infty} + \|y_u^M - y_u^{M,N}\|_{L^\infty} \right\} \\ & \leq c_{x,\psi} |t - s|^{3\gamma} \mathcal{N}[y^M - y^{M,N}; \mathcal{Q}(\llbracket s, t \rrbracket_M)]. \end{aligned}$$

Then clearly $\|I_{tus}^{(2)}\|_{\mathcal{B}} \leq c_x |t - s|^{3\gamma} \mathcal{N}[K^M - K^{M,N}; \mathcal{C}_2^{2\gamma}(\llbracket s, t \rrbracket_M; \mathcal{B})]$ and

$$\begin{aligned} \|I_{tus}^{(3)}\|_{\mathcal{B}} & \leq c_x |t - u|^{\gamma - \lambda} \|(\text{Id} - P_N) \left(\int_0^1 dr f'_i(y_s^{M,N} + r(\delta y^{M,N})_{us}) \cdot K_{us}^{M,N} \right)\|_{\mathcal{B}_{-\lambda}} \\ & \leq c_x |t - u|^{\gamma - \lambda} N^{-2\lambda} \|K_{us}^{M,N}\|_{\mathcal{B}} \leq c_{x,\psi} |t - s|^{3\gamma - \lambda}, \end{aligned}$$

where, for the last estimate, we have used the uniform control (55). The other terms I, II, III of (65) can be handled with similar arguments. Let us only elaborate on the estimate of $\|X_{tu}^{xx,ij} P_N(f'_i(y_u^{M,N}) \cdot (\text{Id} - P_N) \delta f_j(y^{M,N})_{us})\|_{\mathcal{B}}$, which may be a little bit more tricky. Indeed, one must here appeal to the property (22) to get

$$\begin{aligned} & \|X_{tu}^{xx,ij} P_N(f'_i(y_u^{M,N}) \cdot (\text{Id} - P_N) \delta f_j(y^{M,N})_{us})\|_{\mathcal{B}} \\ & \leq c_x |t - u|^{2\gamma - \lambda} \|f'_i(y_u^{M,N}) \cdot (\text{Id} - P_N) \delta f_j(y^{M,N})_{us}\|_{\mathcal{B}_{-\lambda}} \\ & \leq c_x |t - u|^{2\gamma - \lambda} \|f'_i(y_u^{M,N})\|_{\mathcal{B}_{\gamma'}} \|(\text{Id} - P_N) \delta f_j(y^{M,N})_{us}\|_{\mathcal{B}_{-\lambda}} \\ & \leq c_{x,\psi} |t - u|^{2\gamma - \lambda} N^{-2\lambda} \|\delta f_j(y^{M,N})_{us}\|_{\mathcal{B}} \leq c_{x,\psi} |t - s|^{3\gamma - \lambda} N^{-2\lambda}. \end{aligned}$$

As far as $\|\hat{\delta}(J^M - J^{M,N})_{tus}\|_{\mathcal{B}_{\gamma'}}$ is concerned, one can start from the decomposition (64) and observe for instance that

$$\begin{aligned} & \|X_{tu}^{x,i} f_i(y_u^M) - X_{tu}^{x,i} P_N f_i(y_u^{M,N})\|_{\mathcal{B}_{\gamma'}} \\ & \leq \|X_{tu}^{x,i} [f_i(y_u^M) - f_i(y_u^{M,N})]\|_{\mathcal{B}_{\gamma'}} + \|X_{tu}^{x,i} (\text{Id} - P_N) f_i(y_u^{M,N})\|_{\mathcal{B}_{\gamma'}} \\ & \leq c_x |t - u|^\gamma \left\| \int_0^1 dr f'_i(y_u^{M,N} + r(y_u^M - y_u^{M,N})) \cdot (y_u^M - y_u^{M,N}) \right\|_{\mathcal{B}_{\gamma'}} \\ & \quad + c_x |t - u|^{\gamma-\lambda} \|(\text{Id} - P_N) f_i(y_u^{M,N})\|_{\mathcal{B}_{\gamma'-\lambda}} \\ & \leq c_{x,\psi} \left\{ |t - s|^\gamma \|y_u^M - y_u^{M,N}\|_{\mathcal{B}_{\gamma'}} + |t - s|^{\gamma-\lambda} N^{-2\lambda} \right\}. \end{aligned}$$

The other terms stemming from (64) can be estimated along the same lines. \square

6. APPENDIX B: IMPLEMENTATION

We would like to conclude by insisting on the simplicity of the two algorithms (10) and (11) as far as implementation is concerned. To this end, we focus on the case $x = X$ is a fBm with Hurst index $H \in (1/3, 1)$ and x^M is its linear interpolation. As pointed out in Section 2, we know that x^M satisfies Assumption $(\mathbf{X2})_\gamma$ (and accordingly Assumption $(\mathbf{X1})_\gamma$) for any $\gamma \in (\frac{1}{3}, H)$.

6.1. Young case ($H > 1/2$). The objective here is to implement the Euler scheme (10). Remember that we have fixed a basis (e_n) of $L^2(0, 1)$ made of eigenvectors of Δ . By setting $Y_{t_k}^{M,M,l} = \langle Y_{t_k}^{M,M}, e_l \rangle$, one has, for any $l \in \{1, \dots, M\}$,

$$Y_{t_{k+1}}^{M,M,l} = e^{-\lambda_l/M} Y_{t_k}^{M,M,l} + \frac{M}{\lambda_l} \{1 - e^{-\lambda_l/M}\} \sum_{i=1}^m (\delta X^i)_{t_{k+1}t_k} \langle f_i(Y_{t_k}^{M,M}), e_l \rangle. \quad (68)$$

The following Matlab code is a possible implementation of this iterative procedure, for which we have taken $m = 1$ and

$$\psi(\xi) = \frac{1}{2} \sin(\pi\xi) + \frac{3}{5} \sin(3\pi\xi) \quad (\xi \in [0, 1]), \quad f_k(x) = \frac{k \cdot (1 - x)}{1 + x^2} \quad (x \in \mathbb{R}). \quad (69)$$

The parameter k is meant to vary so as to observe the influence of the perturbation. The procedure more precisely simulate the evolution in time of the functional-valued process $Y^{M,M}$. At each step, the Fourier coefficients $\langle f_i(Y_{t_k}^{M,M}), e_l \rangle$ are computed by means of the discrete sinus transform function **dst** (and its inverse **idst**), according to the approximation formula

$$\langle f_i(y_{t_k}^{M,N}), e_l \rangle = \int_0^1 d\xi f_i(y_t^{M,N}(\xi)) e_l(\xi) \approx \frac{1}{N} \sum_{n=0}^N f_i\left(y_{t_k}^{M,N}\left(\frac{n}{N}\right)\right) e_l\left(\frac{n}{N}\right).$$

As for the fBm increments, they are computed via (an appropriately rescaled version of) the Matlab-function **wfbm**, which lean on the decomposition of the process in a wavelet basis, following the method proposed by Abry and Sellan in [1]. Let us finally point out that the action of the semigroup is likely to be qualified by turning the heat semigroup S^Δ into $S_t = S_{\kappa t}^\Delta$, for some parameter κ . The theoretical study contained in Section 3 remains valid for the modified system, of course.

```

function [l]=eigval(N)
l=[]; for i=1:N, l(i)=(pi*i)^2;end

function [S]=semigr(M,N,l,kappa)
S=[]; for i=1:N, v(i)=exp(-l(i)^2/(kappa*M));end

function=simulyoung(H,M,N,k,kappa)
l=eigval(N);S=semigr(M,N,l,kappa);
X=(1/M)^H*wfbm(H,M+1);
A=[1/2,0,3/5,zeros(1,N-3)];
u=zeros(1,N);fy=zeros(1,N);
for i=1:M
    u=dst(A(i,:));fy=0.5*idst(k*(1-u)./(1+u.^2));
    A(i+1,:)=S.*A(i,:)+((kappa./l).*(1-S))*M*(X(i+1)-X(i)).*fy;
end
E=[]; for j=1:M+1, E(j,:)=dst(A(j,:));end
plot(linspace(0,1,N+2),[0,dst([1/2,0,3/5,zeros(1,N-3)],0)]);
F(1)=getframe; for p=1:M
    plot(linspace(0,1,N+2),[0,E(p+1,:),0]);
    hold off;
    F(p+1)=getframe;end
movie(F,1,2)
    
```

Figure 6.1 corresponds to simulations of the process $t \mapsto Y_t^{M,N}(\frac{1}{2})$ for different values of the parameter k ($k = 1, 5, 20, 50$), with the same realization of a fBm with Hurst index $H = 0.6$. The above-described parameter κ has been taken equal to $\kappa = 100$.

6.2. Rough case ($H \in (1/3, 1/2]$). By projecting $Y^{M,N}$ onto e_l , one retrieves here the iterative procedure:

$$\begin{aligned}
 Y_{t_{k+1}}^{M,N,l} &= e^{-\lambda_l/2^M} Y_{t_k}^{M,N,l} + \frac{2^M}{\lambda_l} \left\{ 1 - e^{-\lambda_l/2^M} \right\} \sum_{i=1}^m (\delta X^i)_{t_{k+1}t_k} \left\langle f_i(Y_{t_k}^{M,M}), e_l \right\rangle \\
 &+ (2^M)^2 \sum_{i,j=1}^m (\delta X^i)_{t_{k+1}t_k} (\delta X^j)_{t_{k+1}t_k} \left(\int_{t_k}^{t_{k+1}} e^{-\lambda_l(t_{k+1}-u)} du (u - t_k) \right) \\
 &\left\langle P_N f_j(Y_{t_k}^{M,N}) \cdot f'_i(Y_{t_k}^{M,N}), e_l \right\rangle. \quad (70)
 \end{aligned}$$

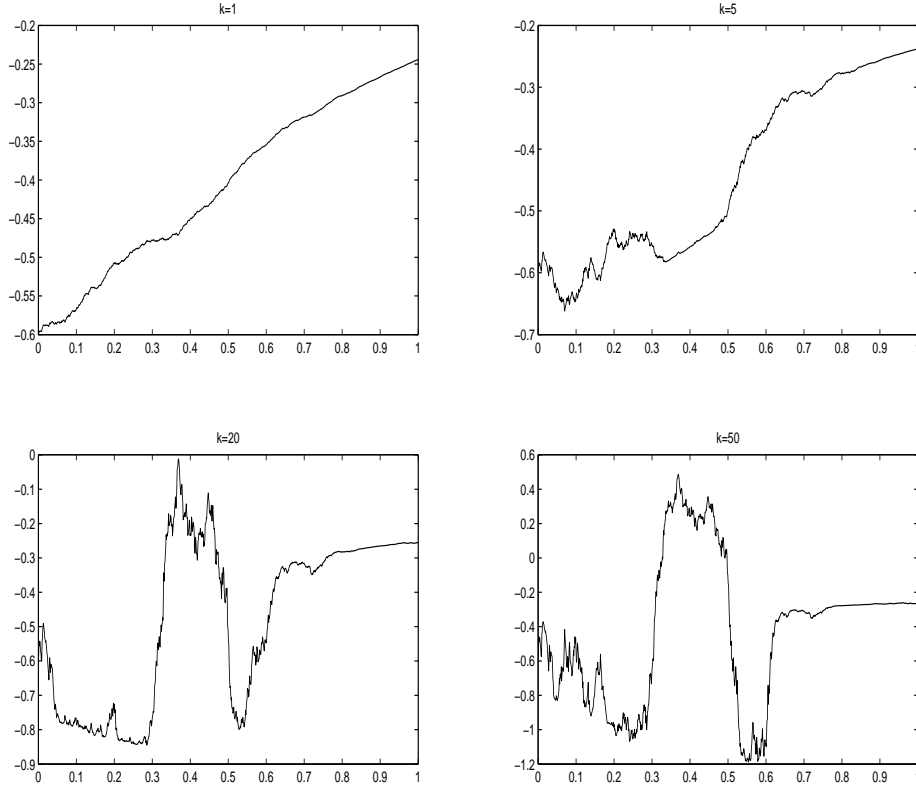


FIGURE 1. Influence of the perturbation term through the observation of the path $t \mapsto Y_t^{M,N}(\frac{1}{2})$, for different values of the parameter k in (69) ($k = 1, 5, 20, 50$). Here, $M = N = 1000$, $H = 0.6$.

The computation of the Fourier coefficients $\langle f_i(Y_{t_k}^{M,N}), e_l \rangle$ can be implemented with the discrete sinus transform, as in the Young case. As for the computation of

$$\langle P_N f_j(Y_{t_k}^{M,N}) \cdot f'_i(Y_{t_k}^{M,N}), e_l \rangle,$$

it can be achieved with the same idea, starting from the approximation:

$$\begin{aligned} & \langle P_N f_j(y_{t_k}^{M,N}) \cdot f'_i(y_{t_k}^{M,N}), e_l \rangle \\ & \approx \frac{1}{N^2} \sum_{n=0}^N \sum_{p=0}^N \sum_{m=0}^N e_l \left(\frac{n}{N} \right) e_p \left(\frac{n}{N} \right) e_p \left(\frac{m}{N} \right) f'_i \left(y_{t_k}^{M,N} \left(\frac{n}{N} \right) \right) f_j \left(y_{t_k}^{M,N} \left(\frac{m}{N} \right) \right). \end{aligned}$$

Those considerations easily lead to the construction of an algorithm for (70).

REFERENCES

- [1] P. Abry and F. Sellan. The wavelet-based synthesis for fractional Brownian motion proposed by F. Sellan and Y. Meyer: remarks and fast implementation. *Appl. Comput. Harmon. Anal.*, 3(4):377–383, 1996.

- [2] R. A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.
- [3] M. Caruana and P. Friz. Partial differential equations driven by rough paths. *J. Differential Equations*, 247(1):140–173, 2009.
- [4] M. Caruana, P. Friz, and H. Oberhauser. A (rough) pathwise approach to a class of non-linear stochastic partial differential equations. *To appear in Ann. Inst. H. Poincaré Probab. Statist.*
- [5] L. Coutin and Z. Qian. Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Related Fields*, 122(1):108–140, 2002.
- [6] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
- [7] Georg Denk, David Meintrup, and Stefan Schäffler. Transient noise simulation: modeling and simulation of $1/f$ -noise. In *Modeling, simulation, and optimization of integrated circuits (Oberwolfach, 2001)*, volume 146 of *Internat. Ser. Numer. Math.*, pages 251–267. Birkhäuser, Basel, 2003.
- [8] A. Deya. A discrete approach to Rough Parabolic Equations. *ArXiv e-prints*, 2010.
- [9] A. Deya, M. Gubinelli, and S. Tindel. Non-linear rough heat equations. *To appear in Probab. Theory Related Fields*.
- [10] A. Deya, A. Neuenkirch, and S. Tindel. A Milstein-type scheme without Levy area terms for SDEs driven by fractional Brownian motion. *To appear in Ann. Inst. H. Poincaré Probab. Statist.*
- [11] J. Diehl and P. Friz. Backward stochastic differential equations with rough drivers. *ArXiv e-prints*, August 2010.
- [12] P. Friz and N. Victoir. *Multidimensional dimensional processes seen as rough paths*. Cambridge University Press, 2010.
- [13] Peter Friz and Nicolas Victoir. Differential equations driven by Gaussian signals. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(2):369–413, 2010.
- [14] M. Gubinelli. Controlling rough paths. *J. Funct. Anal.*, 216(1):86–140, 2004.
- [15] M. Gubinelli and S. Tindel. Rough evolution equations. *Ann. Probab.*, 38(1):1–75, 2010.
- [16] I. Gyöngy. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II. *Potential Anal.*, 11(1):1–37, 1999.
- [17] E. Hausenblas. Approximation for semilinear stochastic evolution equations. *Potential Anal.*, 18(2):141–186, 2003.
- [18] Yaozhong Hu and David Nualart. Stochastic heat equation driven by fractional noise and local time. *Probab. Theory Related Fields*, 143(1-2):285–328, 2009.
- [19] A. Jentzen and P. E. Kloeden. The numerical approximation of stochastic partial differential equations. *Milan J. Math.*, 77:205–244, 2009.
- [20] A. Jentzen and P. E. Kloeden. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 465(2102):649–667, 2009.
- [21] A. Jentzen, P. E. Kloeden, and G. Winkel. Efficient simulation of non-linear parabolic spdes with additive noise. *To appear in The Annals of Applied Probability*.
- [22] S. C. Kou. Stochastic modeling in nanoscale biophysics: subdiffusion within proteins. *Ann. Appl. Stat.*, 2(2):501–535, 2008.
- [23] N. V. Krylov. An analytic approach to SPDEs. In *Stochastic partial differential equations: six perspectives*, volume 64 of *Math. Surveys Monogr.*, pages 185–242. Amer. Math. Soc., Providence, RI, 1999.
- [24] David Nualart and Pierre-A. Vuillermot. Variational solutions for partial differential equations driven by a fractional noise. *J. Funct. Anal.*, 232(2):390–454, 2006.
- [25] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [26] L. C. G. Rogers. Arbitrage with fractional Brownian motion. *Math. Finance*, 7(1):95–105, 1997.
- [27] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3 of *de Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1996.
- [28] M. Sanz-Solé and P. A. Vuillermot. Mild solutions for a class of fractional SPDEs and their sample paths. *J. Evol. Equ.*, 9(2):235–265, 2009.

- [29] S. Tindel, C. A. Tudor, and F. Viens. Stochastic evolution equations with fractional Brownian motion. *Probab. Theory Related Fields*, 127(2):186–204, 2003.

AURÉLIEN DEYA, INSTITUT ÉLIE CARTAN NANCY, UNIVERSITÉ DE NANCY 1, B.P. 239, 54506
VANDŒUVRE-LÈS-NANCY CEDEX, FRANCE
E-mail address: `deya@iecn.u-nancy.fr`