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# INFERENCE IN $\varphi$ -FAMILIES OF DISTRIBUTIONS

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## Abstract

This paper is devoted to the study of the parametric family of multivariate distributions obtained by minimizing a convex functional under linear constraints. Under certain assumptions on the convex functional, it is established that this family admits an affine parametrization, and parametric estimation from an i.i.d. random sample is studied. It is also shown that the members of this family are the limit distributions arising in inference based on empirical likelihood. As a consequence, given a probability measure  $\mu_0$  and an i.i.d. random sample drawn from  $\mu_0$ , nonparametric confidence domains on the generalized moments of  $\mu_0$  are obtained.

*Index Terms* — Parametric statistics, Maximum entropy,  $\varphi$ -divergence, empirical likelihood, generalized moment.

*AMS 2000 Classification:* 62F10, 62G05.

## 1 Introduction

Exponential families of distributions cover a large number of useful distributions and their properties have long been studied. It is well known that an exponential family of distributions may be derived by maximizing the entropy under several moments constraints. The entropy, also called the *relative entropy* or the *Shannon entropy*  $I(\mu)$ , of a probability measure  $\mu$  on a space  $\mathcal{X}$  is defined by

$$I(\mu) = - \int_{\mathcal{X}} \log \frac{d\mu}{d\mu_0}(x) \mu_0(dx),$$

where  $\mu_0$  is a reference measure. In this definition, the entropy may take infinite values when  $\mu$  is not absolutely continuous with respect to  $\mu_0$ .

The negative entropy, i.e.  $-I(\mu)$ , is a convex functional in its argument  $\mu$ . Several types of other (negative) entropy-like convex functionals have been defined and used mainly in the context of linear inverse problems and moments problems (Borwein and Lewis, 1991, 1993a, 1993b; Dacunha-Castelle and Gamboa, 1990; Decarreau *et al*, 1992; Gamboa and Gassiat, 1997). In these problems, the objective is to reconstruct an unknown measure  $\mu$  from the observation  $y$  of *generalized moments* of  $\mu$ , or  $\Phi$ -*moments* of  $\mu$ , i.e., the data  $y$  is related to  $\mu$  by

$$y = \int_{\mathcal{X}} \Phi(x) \mu(dx), \quad (1.1)$$

where  $\Phi$  is a known map from  $\mathcal{X}$  to  $\mathbb{R}^k$ . Recovering the measure  $\mu$  from the data  $y$  is an ill-posed inverse problem in the sense that a solution may not exist for every  $y$  in  $\mathbb{R}^k$  (e.g., in the case of perturbed data), and if a solution exists, it may not be unique nor may it depend continuously on the data. In the field of inverse problems, regularization methods are very popular to cope with these issues. In particular, regularization by entropy amounts at minimizing a negative entropy-like convex functional  $I_\varphi(\mu)$  over all measures  $\mu$  subject to the constraint (1.1). The convex functional  $I_\varphi$  is defined by

$$I_\varphi(\mu) = \int_{\mathcal{X}} \varphi \left( \frac{d\mu}{d\mu_0}(x) \right) \mu_0(dx), \quad (1.2)$$

where  $\varphi$  is a convex function on  $\mathbb{R}$ . Under certain conditions on  $\varphi$  and the data  $y$ , Borwein and Lewis (1991, 1993a, 1993b) have shown that the problem of minimizing  $I_\varphi(\mu)$  subject to the constraint (1.1) admits a unique solution  $\hat{\mu}$  which may be written as

$$\hat{\mu} = \varphi^{*'}(\langle \omega, \Phi(x) \rangle) \mu_0, \quad (1.3)$$

where  $\varphi^{*'}$  is the derivative of the Fenchel-Legendre transform of  $\varphi$ , and where  $\omega$  is a vector of scalar parameters obtained as the unique solution to a dual optimization problem.

The present paper focuses on the family of probability measures which are in the form of (1.3), further referred to as a  $\varphi$ -*family*. These measures also arise as the limit distributions in inference based on empirical likelihood, under certain conditions on the function  $\varphi$  which turn the functional (1.2) into a  $\varphi$ -divergence (Liese

and Vajda, 1987; Keziou, 2003; Broniatowski and Keziou, 2006, Pardo, 2006). To see this, let  $\mu_0$  be a probability measure, and suppose that we are interested in  $\mu_0$  only through its  $\Phi$ -moment  $y_0 = \int_{\mathcal{X}} \Phi(x) \mu_0(dx)$ . Basically, the method of empirical likelihood introduced in Owen (1988, 2001) amounts at minimizing the Kullback-Leibler divergence  $K(\mu; \mathbb{P}_n)$  between the empirical measure  $\mathbb{P}_n$  of the random sample, and a measure  $\mu \ll \mathbb{P}_n$  satisfying the constraints of the model. In this display, the statistic

$$T_n(y) = \inf \left\{ K(\mu; \mathbb{P}_n) : \mu \ll \mathbb{P}_n \text{ and } \int_{\mathcal{X}} \Phi(x) \mu(dx) = y \right\} \quad (1.4)$$

is used to test for  $y_0$  as well as to construct a nonparametric confidence domain on  $y_0$ . Recently, several authors (Keziou, 2003; Broniatowski, 2004; Bertail, 2006; Broniatowski and Keziou, 2006) have proposed to use other convex statistical divergences in the form of (1.2) in lieu of the Kullback-Leibler divergence. This leads to alternative statistics in the form of (1.4) which are intimately related to the  $\varphi$ -family considered herein. Indeed, as exposed further in the paper, for a feasible  $y$ , the infimum in (1.4) is attained by a random discrete measure which converges to a member of the  $\varphi$ -family, i.e., a probability measure in the form of (1.3).

The paper is organized as follows. The  $\varphi$ -family of distributions is introduced in Section 2. In Section 3, we show that the  $\varphi$ -family admits an affine parametrization. Section 4 is devoted to the estimation of the affine parameter of a member of the family from an i.i.d. random sample. In Section 5, we show that the  $\varphi$ -family is the limit family of distributions arising in empirical likelihood. Next, nonparametric confidence domains on the  $\Phi$ -moment of the underlying probability measure are derived. Technical results are postponed in an Appendix, at the end of the paper.

## 2 Notation and definitions

Let  $(\mathcal{X}, \mu_0)$  be a finite measure space, where  $\mathcal{X}$  is a measurable subset of  $\mathbb{R}^d$ . Let  $\Phi_1, \dots, \Phi_k$  be  $k$  functions in  $L_2(\mathcal{X}, \mu_0)$  such that the maps  $1, \Phi_1, \dots, \Phi_k$  are linearly independent. We shall denote by  $\Phi = (\Phi_1, \dots, \Phi_k)$  the map  $\mathcal{X} \rightarrow \mathbb{R}^k$ , and by  $\tilde{\Phi} = (1, \Phi_1, \dots, \Phi_k)$  the map  $\mathcal{X} \rightarrow \mathbb{R}^{k+1}$ . The set of finite measures and probability measures on  $\mathcal{X}$  will be denoted respectively by  $\mathcal{M}(\mathcal{X})$  and  $\mathcal{M}_1^+(\mathcal{X})$ .

Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  be an extended function satisfying the following assumption.

**Assumption 1**

- (i)  $\text{dom}(\varphi) = (0, +\infty)$ ,
- (ii)  $\varphi$  is strictly convex and essentially smooth,
- (iii)  $\lim_{x \rightarrow \infty} \frac{\varphi(x)}{x} = \kappa \in (0, +\infty]$ ,
- (iv)  $\varphi$  is  $\mathcal{C}^2$  on the interior of  $\text{dom}(\varphi)$ .

We recall that a proper convex function  $\varphi$  is said to be *essentially smooth* if it is differentiable on the interior of its domain, supposed non empty, and if  $|\nabla\varphi(x_i)| \rightarrow \infty$  whenever  $x_i$  is a sequence converging to a boundary point of  $\text{dom}(\varphi)$  (Rockafellar, 1970, Chap. 26). Note that since  $\text{dom}(\varphi) = (0, +\infty)$ , we have  $\varphi(x) = +\infty$  for all  $x < 0$ , and that the Fenchel-Legendre transform of  $\varphi$ , further denoted by  $\varphi^*$ , may be written as

$$\varphi^*(u) = \sup_{x \geq 0} \{xu - \varphi(x)\}.$$

From this definition, it follows that  $\varphi^*$  is monotone increasing, so that its derivative  $\varphi^{*'} \geq 0$ . Under conditions (i) and (iii), we have  $\text{dom}(\varphi^*) = (-\infty; \kappa)$ . The essential smoothness of  $\varphi$  implies that  $\varphi^*$  is strictly convex. At last,  $\varphi^{*'}$  is invertible with  $(\varphi^{*'})^{-1} = \varphi'$ .

As explained in the Introduction, the aim of this paper is to study the family of measures minimizing the convex functional  $I_\varphi$  defined in (1.2) under the moments constraints (1.1). Solutions to this problem have been obtained by Borwein and Lewis (1991) (see also Borwein and Lewis, 1993a, 1993b). More precisely, we have the following result.

**Theorem 2.1** *Let  $\varphi$  be a strictly convex function satisfying Assumption 1, and let  $\tilde{y} \in \mathbb{R}^{k+1}$ . Consider the following primal problem:*

$$\begin{aligned} & \text{Minimize} && I_\varphi(\mu) := \int_{\mathcal{X}} \varphi \left( \frac{d\mu}{d\mu_0}(x) \right) \mu_0(dx) \\ & \text{subject to} && \mu \in \mathcal{M}(\mathcal{X}) \quad \mu \ll \mu_0 \\ & && \text{and} \quad \int_{\mathcal{X}} \tilde{\Phi}(x) \mu(dx) = \tilde{y}. \end{aligned}$$

Suppose that there exists at least one solution  $\bar{\mu}$  with  $I_\varphi(\bar{\mu})$  finite. Let  $\bar{u}$  be the unique solution of the dual problem:

$$\begin{aligned} & \text{Maximize} && \langle \bar{y}, u \rangle - \int_{\mathcal{X}} \varphi^* \left( \langle u, \tilde{\Phi}(x) \rangle \right) \mu_0(dx) \\ & \text{subject to} && u \in \mathbb{R}^{k+1}. \end{aligned}$$

Suppose that  $\text{ess sup} \langle \bar{u}, \tilde{\Phi}(x) \rangle < \kappa$ . Then the unique optimal solution of the primal problem is given by

$$\bar{\mu} = \varphi^{*'} \left( \langle \bar{u}, \tilde{\Phi}(x) \rangle \right) \mu_0,$$

with dual attainment.

We are now in a position to define the  $\varphi$ -family of probability measures. To this aim, consider the parametric family  $\tilde{\mathcal{F}}$  of finite measures on  $\mathcal{X}$  defined by

$$\tilde{\mathcal{F}} = \left\{ \tilde{\mu}_{\tilde{\xi}} := \varphi^{*'} \left( \langle \tilde{\xi}, \tilde{\Phi}(x) \rangle \right) \mu_0; \tilde{\xi} \in \tilde{\Xi} \right\}, \quad (2.1)$$

where

$$\tilde{\Xi} = \left\{ \tilde{\xi} \in \mathbb{R}^{k+1} : \text{ess sup} \langle \tilde{\xi}, \tilde{\Phi}(x) \rangle < \kappa \right\}, \quad (2.2)$$

where the essential supremum is taken with respect to  $\mu_0$ . For all  $\tilde{\xi}$  in  $\tilde{\Xi}$ , the Radon-Nikodym derivative of  $\tilde{\mu}_{\tilde{\xi}}$  with respect to  $\mu_0$  is in  $L_\infty(\mathcal{X}, \mu_0)$  by Lemma A.1. Then we define the  $\varphi$ -family  $\mathcal{F}$  as the set of probability measures in  $\tilde{\mathcal{F}}$ , i.e., we set

$$\mathcal{F} = \tilde{\mathcal{F}} \cap \mathcal{M}_1^+(\mathcal{X}). \quad (2.3)$$

Some examples of possible choices for the convex function  $\varphi$  satisfying Assumption 1 are provided below.

**Example 2.1** Consider the function  $\varphi$  defined by

$$\varphi(x) = \begin{cases} x \log(x) - x + 1, & \text{if } x > 0, \\ 1 & \text{if } x = 0, \\ +\infty & \text{if } x < 0. \end{cases}$$

We have  $\text{dom}(\varphi) = (0, +\infty)$  and  $\kappa = +\infty$ . The convex conjugate of  $\varphi$  is given by  $\varphi^*(u) = \exp(u) - 1$  and  $\text{dom}(\varphi^*) = \mathbb{R}$ . Then  $\varphi^{*'}(u) = \exp(u)$  and the family  $\mathcal{F}$  is therefore an exponential family. Also in this case, the functional  $I_\varphi$  corresponds to the Kullback-Leibler divergence when restricted to probability measures arguments.

**Example 2.2** Consider the function  $\varphi$  defined by

$$\varphi(x) = \begin{cases} 2(\sqrt{x} - 1)^2 & \text{if } x \geq 0, \\ +\infty & \text{if } x < 0. \end{cases}$$

We have  $\text{dom}(\varphi) = (0, +\infty)$  and  $\kappa = 2$ . The convex conjugate of  $\varphi$  is given by

$$\varphi^*(u) = \begin{cases} \frac{2u}{2-u} & \text{if } u < 2, \\ +\infty & \text{if } u \geq 2. \end{cases}$$

We have  $\text{dom}(\varphi^*) = (-\infty, 2)$ , and  $\varphi^{*'}(u) = \frac{4}{(2-u)^2}$  on  $(-\infty, 2)$ . When restricted to probability measures arguments,  $I_\varphi$  corresponds to the Hellinger distance.

### 3 Parametrization of $\mathcal{F}$

Consider the set  $\tilde{S}$  of  $\tilde{\Phi}$ -moments of the measures in  $\tilde{\mathcal{F}}$ , i.e.,

$$\tilde{S} = \left\{ \int_{\mathcal{X}} \tilde{\Phi}(x) \tilde{\mu}_\xi(dx) : \xi \in \tilde{\Xi} \right\}. \quad (3.1)$$

**Theorem 3.1** Suppose that Assumption 1 holds. The map  $\tilde{\Psi} : \tilde{\Xi} \rightarrow \tilde{S}$  defined by

$$\tilde{\Psi}(\xi) = \int_{\mathcal{X}} \tilde{\Phi}(x) \tilde{\mu}_\xi(dx)$$

is a diffeomorphism from  $\tilde{\Xi}$  to  $\tilde{S}$ .

**Proof.** Clearly  $\tilde{\Psi}$  is surjective, and differentiable from Lemma A.2. Now we proceed to show that  $\tilde{\Psi}$  is injective. Consider the map  $U : \tilde{\Xi} \rightarrow \mathbb{R}$  defined by

$$U(\xi) = \int_{\mathcal{X}} \varphi^* \left( \langle \xi, \tilde{\Phi}(x) \rangle \right) \mu_0(dx).$$

Note that  $U(\xi)$  is well-defined for all  $\xi \in \tilde{\Xi}$  by Lemma A.1, and differentiable from Lemma A.2. Then the  $\tilde{\Phi}$ -moments of  $\tilde{\mu}_\xi$  are obtained by differentiating  $U$ , i.e., we have

$$\tilde{\Psi}(\xi) = \int_{\mathcal{X}} \tilde{\Phi}(x) \tilde{\mu}_\xi(dx) = \nabla U(\xi).$$

Furthermore, given  $u, v \in \tilde{\xi}$  and  $\alpha \in (0; 1)$ , we have

$$\begin{aligned} U(\alpha u + (1 - \alpha)v) &= \int_{\mathcal{X}} \varphi^* \left( \alpha \langle u, \tilde{\Phi} \rangle + (1 - \alpha) \langle v, \tilde{\Phi} \rangle \right) \mu_0(dx) \\ &< \alpha U(u) + (1 - \alpha)U(v) \end{aligned}$$

since  $\varphi^*$  is strictly convex. Hence,  $U$  is strictly convex. Consequently the gradient map  $\tilde{\xi} \rightarrow \nabla U(\tilde{\xi})$  is injective and so is  $\tilde{\Psi}$ .

There remains to show that  $\tilde{\Psi}^{-1}$  is differentiable. To this aim, consider the map  $H : \tilde{\Xi} \times \tilde{S} \rightarrow \mathbb{R}^{k+1}$  defined by

$$H(\tilde{\xi}; \tilde{y}) = \nabla U(\tilde{\xi}) - \tilde{y},$$

so that  $\tilde{\psi}^{-1}(\tilde{y})$  is the unique solution (in  $\tilde{\xi}$ ) of the equation  $H(\tilde{\xi}, \tilde{y}) = 0$ . Differentiating  $H$  with respect to  $\tilde{\xi}$ , we obtain

$$\frac{\partial}{\partial \tilde{\xi}_i} H_j(\tilde{\xi}; \tilde{y}) = \int_{\mathcal{X}} \tilde{\Phi}_i(x) \tilde{\Phi}_j(x) \varphi^{*''} \left( \langle \tilde{\xi}, \tilde{\Phi}(x) \rangle \right) \mu_0(dx),$$

where  $(H_j)_{j=1, \dots, k+1}$  are the components of  $H$ . Note that, for all  $\tilde{\xi} \in \tilde{\Xi}$ , the integral above is finite since the map  $x \mapsto \varphi^{*''} \left( \langle \tilde{\xi}, \tilde{\Phi}(x) \rangle \right)$  is in  $L_\infty(\mathcal{X}, \mu_0)$  by Lemma A.1, and since the components of  $\tilde{\Phi}$  are in  $L_2(\mathcal{X}, \mu_0)$ . Furthermore,  $\varphi^{*''}$  is strictly positive by the strict convexity of  $\varphi^*$ , so that the matrix  $\left( \frac{\partial}{\partial \tilde{\xi}_i} H_j(\tilde{\xi}; \tilde{y}) \right)_{i,j}$  is the Gram matrix of the scalar products of the maps  $1, \Phi_1, \dots, \Phi_k$  w.r.t. the measure  $\varphi^{*''} \left( \langle \tilde{\xi}, \tilde{\Phi}(x) \rangle \right) \mu_0$ . Since these latter are linearly independent, the above matrix is positive-definite. Consequently, for all  $(\tilde{\xi}, \tilde{y})$ ,  $D_{\tilde{\xi}} H(\tilde{\xi}, \tilde{y})$  is a linear invertible map. The continuity and differentiability of  $\tilde{\Psi}^{-1}$  then follow from the Implicit Function Theorem.  $\square$

Now let

$$\Xi = \{ \tilde{\xi} \in \tilde{\Xi} : \tilde{\mu}_{\tilde{\xi}}(\mathcal{X}) = 1 \}. \quad (3.2)$$

and let  $i_\Xi : \Xi \rightarrow \mathbb{R}^{k+1}$  be the canonical embedding of  $\Xi$  in  $\mathbb{R}^{k+1}$ . Then we may rewrite the family  $\mathcal{F}$  as

$$\mathcal{F} = \left\{ \mu_\xi := \varphi^{*'} \left( \langle i_\Xi(\xi), \tilde{\Phi}(x) \rangle \right) \mu_0 : \xi \in \Xi \right\}. \quad (3.3)$$



Let

$$S = \left\{ \int_{\mathcal{X}} \Phi(x) \mu_{\xi}(dx) : \xi \in \Xi \right\}. \quad (3.4)$$

As an immediate consequence of the Theorem above, we obtain the following result.

**Theorem 3.2** *Suppose that Assumption 1 holds. The map  $\Psi : \Xi \rightarrow S$  defined by*

$$\Psi(\xi) = \int_{\mathcal{X}} \Phi(x) \mu_{\xi}(dx)$$

*is a diffeomorphism from  $\Xi$  to  $S$ .*

We are now in a position to provide an affine parametrization of the family  $\mathcal{F}$ .

**Theorem 3.3** *Suppose that Assumption 1 holds. There exists a unique subset  $\Theta$  of  $\mathbb{R}^k$  diffeomorphic to  $\Xi$  and a unique differentiable map  $g : \Theta \rightarrow \mathbb{R}$  such that*

$$\mathcal{F} = \left\{ \mu_{\theta} := \varphi^{*'}(g(\theta) + \langle \theta, \Phi(x) \rangle) \mu_0; \theta \in \Theta \right\}.$$

**Proof** Let us write  $\tilde{\xi} \in \tilde{\Xi} \subset \mathbb{R}^{k+1}$  as  $\tilde{\xi} = (\alpha, \beta)$ , with  $\alpha \in \mathbb{R}$  and  $\beta \in \mathbb{R}^k$  such that we have

$$\varphi^{*'}(\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle) = \varphi^{*'}(\alpha + \langle \beta, \Phi(x) \rangle).$$

Furthermore, let  $\pi_1$  and  $\pi_2$  be the projections on respectively  $\mathbb{R}$  and  $\mathbb{R}^k$ , i.e.,  $(\alpha, \beta) = (\pi_1(\tilde{\xi}), \pi_2(\tilde{\xi}))$  and let  $F : \pi_1(\tilde{\Xi}) \times \pi_2(\tilde{\Xi}) \rightarrow \mathbb{R} \cup \{+\infty\}$  be the map defined by

$$F(\alpha, \beta) = \int_{\mathcal{X}} \varphi^{*'}(\alpha + \langle \beta, \Phi(x) \rangle) \mu_0(dx) - 1.$$

Note that  $F$  takes infinite values on the complement of  $\tilde{\Xi}$  in  $\pi_1(\tilde{\Xi}) \times \pi_2(\tilde{\Xi})$  and that we have

$$\Xi = \{(\alpha, \beta) : F(\alpha, \beta) = 0\}.$$

First we have

$$\begin{aligned} \frac{\partial}{\partial \alpha} F(\alpha, \beta) &= \int_{\mathcal{X}} \varphi^{*''}(\alpha + \langle \beta, \Phi(x) \rangle) \mu_0(dx) \\ &> 0 \end{aligned}$$

since  $\varphi^*$  is strictly convex. Hence for all  $(\alpha, \beta)$ ,  $D_{\alpha}F(\alpha, \beta)$  is a linear invertible map from  $\pi_1(\tilde{\Xi})$  to itself. Second,  $\Xi$  is connected since  $\Xi$  is homeomorphic to

$S$  by Theorem 3.2 and  $S$  is connected. The existence and uniqueness of the map  $g$  now follows from a global version of the Implicit Function Theorem (see e.g., Dieudonné, 1972, pp. 265-266, or Blot, 1991) and is defined on  $\Theta := \pi_2(\tilde{\Xi})$  which is diffeomorphic to  $\Xi$ .  $\square$

As in the proof of Theorem 3.3, we shall write  $\mathbb{R}^{k+1}$  as  $\mathbb{R} \times \mathbb{R}^k$  and denote by  $\pi_1$  and  $\pi_2$  the projections from  $\mathbb{R}^{k+1}$  on  $\mathbb{R}$  and  $\mathbb{R}^k$ , respectively. Then we have the following diagram:

$$\begin{array}{ccc}
 & \mathcal{F} & \\
 \cong \uparrow & & \\
 \Xi & \xrightarrow{i_{\Xi}} & i_{\Xi}(\Xi) \\
 \cong \downarrow \Psi & & \pi_2 \downarrow \\
 S & \xleftarrow[m]{\cong} & \Theta
 \end{array}$$

where  $i_{\Xi}$  denotes the canonical embedding of  $\Xi$  in  $\mathbb{R}^{k+1}$ , and where  $\cong$  denotes a diffeomorphism. In this diagram, the map  $m$  is a diffeomorphism from  $\Theta$  to  $S$  and is defined by

$$m(\theta) = \int_{\mathcal{X}} \Phi(x) \mu_{\theta}(dx), \quad (3.5)$$

i.e.,  $m$  is the inverse of map of  $\pi_2 \circ i_{\Xi} \circ \Psi^{-1}$ .

## 4 Inference in $\mathcal{F}$

In this section, we consider the estimation of a parameter  $\theta_0 \in \Theta$  based on an i.i.d. random sample  $X_1, \dots, X_n$  drawn from  $\mu_{\theta_0}$ , which may be written as  $\mu_{\theta_0} = \varphi^{*'}(g(\theta_0) + \langle \theta_0, \Phi(x) \rangle) \mu_0$  from Theorem 3.3.

Let us start by drawing some consequences of the results in Section 3. If we denote  $y_{\theta_0}$  the  $\Phi$ -moments of  $\mu_{\theta_0}$ , i.e.,

$$y_{\theta_0} = \int_{\mathcal{X}} \Phi(x) \mu_{\theta_0}(dx).$$

then we have  $\theta_0 = m^{-1}(y_{\theta_0})$ . In practice, though, and depending on the choice of  $\varphi$ , it may be difficult to derive explicit expressions for the maps  $g$  and  $m$ , apart from the special case of an exponential family. However, the results of

Borwein and Lewis (1991, 1993a, 1993b) exposed in Theorem 2.1 provide one with a convenient algorithm to compute the value of  $\theta_0$  given the moment  $y_{\theta_0}$ , without explicit expressions for the maps  $g$  and  $m$ . First of all, we may write  $S = \pi_2 \left( \tilde{S} \cap \{1\} \times \mathbb{R}^k \right)$ . Consider the vector  $\tilde{y}_{\theta_0} = (1, y_{\theta_0})$  in  $\tilde{S}$ . Then  $\tilde{\Psi}^{-1}(\tilde{y}_{\theta_0})$  lies in  $i_{\Xi}(\Xi) \subset \tilde{\Xi}$  so we obtain

$$\theta_0 = (\pi_2 \circ \tilde{\Psi}^{-1})(\tilde{y}_{\theta_0}).$$

Second, from the proof of Theorem 3.1, for all  $\tilde{y}$  in  $\tilde{S}$ ,  $\tilde{\Psi}^{-1}(\tilde{y})$  is the unique solution to the following minimization problem:

$$\begin{aligned} & \text{Minimize} && \int_{\mathcal{X}} \varphi^* \left( \langle u, \tilde{\Phi}(x) \rangle \right) \mu_0(dx) - \langle \tilde{y}, u \rangle \\ & \text{subject to} && u \in \mathbb{R}^{k+1}. \end{aligned}$$

Consequently,  $\theta_0$  may be evaluated by taking the  $k$  last components of the unique minimum over  $\mathbb{R}^{k+1}$  of the map

$$u := (u_0, \dots, u_k) \mapsto \int_{\mathcal{X}} \varphi^* \left( u_0 + \sum_{i=1}^k u_i \Phi_i(x) \right) \mu_0(dx) - \left( u_0 + \sum_{i=1}^k u_i y_{\theta_0, i} \right), \quad (4.1)$$

i.e., letting  $\bar{u} := (\bar{u}_0, \dots, \bar{u}_k)$  be the unique minimum in (4.1), then  $\theta_0 = (\bar{u}_1, \dots, \bar{u}_k)$ . In addition, we also have  $g(\theta_0) = \bar{u}_0$ . Another interest of this procedure is that the map in (4.1) is convex. So evaluating  $\theta_0$  from  $y_{\theta_0}$  requires solving an unconstrained convex minimization problem for which efficient numerical algorithms are available.

These observations lead us to estimate  $\theta_0$  by minimizing the empirical version of (4.1). More precisely, let  $\hat{y}_n$  be the empirical  $\Phi$ -moment of  $\mu_{\theta_0}$  associated with the sample  $X_1, \dots, X_n$ , i.e.,

$$\hat{y}_n = \frac{1}{n} \sum_{i=1}^n \Phi(X_i), \quad (4.2)$$

set  $\tilde{y}_n = (1, \hat{y}_n)$ , and let  $\mathbb{P}_n$  be the empirical measure associated with the random sample. Then we define the estimate  $\hat{\theta}_n$  as a minimizer over  $\mathbb{R}^{k+1}$  of the map

$$u \mapsto \int_{\mathcal{X}} \varphi^* \left( \langle u, \tilde{\Phi}(x) \rangle \right) \mathbb{P}_n(dx) - \langle \tilde{y}_n, u \rangle,$$

which is the empirical version of (4.1). Indeed,  $\hat{\theta}_n$  is an M-estimator, and on the probability event that  $\hat{y}_n$  lies in the set  $S$ , we may write

$$\hat{\theta}_n = m^{-1}(\hat{y}_n). \quad (4.3)$$

Next, by the law of large numbers,  $\hat{y}_n$  belongs to  $S$  for  $n$  large enough, with probability one. Consequently, since  $m$  is a diffeomorphism from  $\Theta$  to  $S$ , it follows that  $\hat{\theta}_n$  converges in probability to  $\theta_0$ , and since  $\hat{y}_n$  is asymptotically normally distributed, it follows that  $\hat{\theta}_n$  is in turn asymptotically normal. Finally, we have the following Theorem.

**Theorem 4.1** *Suppose that Assumption 1 holds. The sequence  $\sqrt{n}(\hat{\theta}_n - \theta_0)$  converges in distribution to a normal distribution with mean 0 and covariance matrix given by*

$$\Sigma = [\gamma(\theta_0)]^{-2} \left[ Cov_{\mu_{\theta_0}^\dagger}(\Phi(X)) \right]^{-1} Cov_{\mu_{\theta_0}}(\Phi(X)) \left[ Cov_{\mu_{\theta_0}^\dagger}(\Phi(X)) \right]^{-1},$$

where

$$\gamma(\theta) = \int_{\mathcal{X}} \varphi^{*''}(g(\theta) + \langle \theta, \Phi(x) \rangle) \mu_0(dx),$$

and where  $\mu_{\theta_0}^\dagger$  is the measure defined by

$$\mu_{\theta_0}^\dagger = \gamma(\theta_0)^{-1} \varphi^{*''}(g(\theta) + \langle \theta, \Phi(x) \rangle) \mu_0.$$

**Proof** Since  $\hat{y}_n$  is asymptotically normal, and since  $m$  is a diffeomorphism, it follows from standard arguments on moment estimators (see e.g. Van der Vaart, 1998, Theo. 4.1., p. 36), that  $\sqrt{n}(\hat{\theta}_n - \theta_0)$  converges to a normal distribution with mean 0 and covariance matrix

$$\Sigma = m'_{\theta_0}{}^{-1} Cov_{\mu_{\theta_0}}(\Phi(X)) (m'_{\theta_0}{}^{-1})^t,$$

where  $m'_{\theta_0}$  is the derivative of  $m$  at  $\theta_0$ . We have

$$\frac{\partial m_j}{\partial \theta_i}(\theta) = \int_{\mathcal{X}} \Phi_j(x) \left( \frac{\partial g}{\partial \theta_i}(\theta) + \Phi_i(x) \right) \varphi^{*''}(g(\theta) + \langle \theta, \Phi(x) \rangle) \mu_0(dx). \quad (4.4)$$

and

$$\frac{\partial g}{\partial \theta_i}(\theta) = - \frac{\int_{\mathcal{X}} \Phi_i(x) \varphi^{*''}(g(\theta) + \langle \theta, \Phi(x) \rangle) \mu_0(dx)}{\int_{\mathcal{X}} \varphi^{*''}(g(\theta) + \langle \theta, \Phi(x) \rangle) \mu_0(dx)} \quad (4.5)$$

since  $\int_{\mathcal{X}} \varphi^{*'}(g(\theta) + \langle \theta, \Phi(x) \rangle) \mu_0(dx) = 1$ . Reporting (4.5) in (4.4) yields the desired result.  $\square$

## 5 Nonparametric inference on the $\Phi$ -moment

Let  $X_1, \dots, X_n$  be an i.i.d. random sample drawn from a probability measure  $\mu_0$  on  $\mathcal{X}$ . Suppose that we are interested in  $\mu_0$  only through its  $\Phi$ -moment  $y_0 = \int_{\mathcal{X}} \Phi(x) \mu_0(dx)$ . As exposed in the Introduction, the method of empirical likelihood (Owen, 1988, 2001) amounts at minimizing the Kullback-Leibler divergence between the empirical measure  $\mathbb{P}_n$  of the random sample, and a measure  $\mu$  satisfying the constraints of the model and absolutely continuous with respect to  $\mathbb{P}_n$ . Replacing the Kullback-Leibler divergence by a  $\varphi$ -divergence provides one with an alternative statistic to test for  $y_0$ , as well as to construct a confidence domain on  $y_0$ .

First of all, let  $\mathbb{P}_n$  be the empirical measure associated with the random sample  $X_1, \dots, X_n$ . Define the functional  $I_\varphi^n(\mu)$  over  $\mathcal{M}(\mathcal{X})$  by

$$I_\varphi^n(\mu) = \int_{\mathcal{X}} \varphi\left(\frac{d\mu}{d\mathbb{P}_n}(x)\right) \mathbb{P}_n(dx),$$

whenever  $\mu \ll \mathbb{P}_n$  and set  $I_\varphi^n(\mu) = +\infty$  otherwise. Observe that if  $I_\varphi^n(\mu)$  is finite then  $\mu$  is a discrete measure concentrated on the  $X_i$ 's. Additional conditions on  $\varphi$  are needed to ensure that  $I_\varphi$  is a divergence between probability measure. More precisely, we shall need the following assumption.

### Assumption 2

- (i)  $\varphi(1) = 0$
- (ii)  $\frac{\varphi(x)}{x} \rightarrow +\infty$  as  $x \rightarrow +\infty$ , i.e.,  $\kappa = +\infty$ .

For all  $y \in S$ , we shall let  $\tilde{y} = (1, y)$ , and we consider the following primal problem:

$$\begin{aligned} & \text{Minimize} && I_\varphi^n(\mu) \\ & \text{subject to} && \mu \in \mathcal{M}(\mathcal{X}), \quad \mu \ll \mathbb{P}_n, \\ & && \text{and} \quad \int_{\mathcal{X}} \tilde{\Phi}(x) \mu(dx) = \tilde{y}. \end{aligned}$$

The dual optimization problem is:

$$\begin{aligned} & \text{Maximize} && \langle \tilde{y}, \tilde{v} \rangle - \int_{\mathcal{X}} \varphi^*(\langle \tilde{v}, \tilde{\Phi}(x) \rangle) \mathbb{P}_n(dx) \\ & \text{subject to} && \tilde{v} \in \mathbb{R}^{k+1}. \end{aligned}$$

Let  $\Omega_n$  be the probability event that a solution to the dual problem exists, solution further denoted by  $\tilde{\xi}_n$ . Then, by Theorem 2.1, on  $\Omega_n$ , the unique primal solution is given by

$$\tilde{\mu}_n = \frac{1}{n} \sum_{i=1}^n \varphi^{*'} \left( \langle \tilde{\xi}_n, \tilde{\Phi}(X_i) \rangle \right) \delta_{X_i}. \quad (5.1)$$

The convergence of  $\tilde{\mu}_n$  may be analysed using known results on M-estimators (see e.g., van de Geer, 2000, Chap. 12, and van der Vaart, 1998, Chap. 5). In essence, the concavity of the objective function in the dual program (i.e., the convexity of the negative objective function) is sufficient to establish the convergence of  $\tilde{\xi}_n$  to  $\tilde{\xi}$  in probability, where  $\tilde{\xi} = \tilde{\Psi}^{-1}(\tilde{y})$ .

More precisely, since  $y \in S$ , we have  $\tilde{y} = \int_{\mathcal{X}} \tilde{\Phi}(x) \tilde{\mu}_{\tilde{\xi}}(dx)$ . Consequently, by the law of large numbers, it follows that  $\mathbb{P}(\Omega_n) \rightarrow 1$  as  $n \rightarrow \infty$ . So on  $\Omega_n$ ,  $\tilde{\xi}_n$  is the point of minimum of the map  $\tilde{v} \mapsto \int_{\mathcal{X}} h_{\tilde{v}}(x) \mathbb{P}_n(dx)$ , where

$$h_{\tilde{v}}(x) = \varphi^* \left( \langle \tilde{v}, \tilde{\Phi}(x) \rangle \right) - \langle \tilde{v}, \tilde{y} \rangle.$$

Since  $\tilde{v} \mapsto h_{\tilde{v}}(x)$  is continuous and convex for  $\mu_0$ -almost every  $x$ , and since by Lemma A.2, for  $\varepsilon > 0$  small enough,

$$\int_{\mathcal{X}} \sup_{\tilde{v} \in B_\varepsilon(\tilde{\xi})} |h_{\tilde{v}}(x)| \tilde{\mu}_{\tilde{\xi}}(dx) < \infty,$$

where  $B_\varepsilon(\tilde{\xi})$  is the Euclidean ball centered at  $\tilde{\xi}$  and of radius  $\varepsilon$ , it follows that

$$\tilde{\xi}_n \rightarrow \tilde{\xi} \quad \text{in probability as } n \rightarrow \infty. \quad (5.2)$$

As a consequence, we obtain the convergence of  $\tilde{\mu}_n$  to the member of the family  $\mathcal{F}$  having  $\Phi$ -moment  $y$ , which is stated below without proof.

**Theorem 5.1** *Suppose that Assumption 1 and Assumption 2 hold. Then for all  $y \in S$ ,  $\tilde{\mu}_n$  converges weakly to the probability measure  $\tilde{\mu}_{\tilde{\xi}}$ , in probability, where  $\tilde{\xi} = \tilde{\Psi}^{-1}(\tilde{y})$ .*

Additionally, since  $\tilde{\xi}_n$  converges in probability to  $\tilde{\xi}$ , by applying Theorem 5.23 in van der Vaart (1998, p. 53), we obtain:

$$\sqrt{n}(\tilde{\xi}_n - \tilde{\xi}) = -V_{\tilde{\xi}}^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \tilde{\Phi}(X_i) \varphi^{*'} \left( \langle \tilde{\xi}, \tilde{\Phi}(X_i) \rangle \right) - \tilde{y} \right] + o_P(1), \quad (5.3)$$

where  $V_{\tilde{\xi}}$  is the matrix defined by

$$V_{\tilde{\xi}} = \left[ \int_{\mathcal{X}} \tilde{\Phi}_i(x) \tilde{\Phi}_j(x) \varphi^{*''}(\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle) \mu_0(dx) \right]_{i,j}. \quad (5.4)$$

Now consider the statistic  $T_n(y)$  defined by

$$T_n(y) = \inf \left\{ I_{\varphi}^n(\mu) : \int_{\mathcal{X}} \tilde{\Phi}(x) \mu(dx) = \tilde{y} \right\}. \quad (5.5)$$

Then we have the following result, which proves that a confidence domain on the  $\Phi$ -moment  $y_0$  and a convergent test for  $y_0$  may be based on the statistic  $T_n(y)$ .

**Theorem 5.2** *Suppose that Assumption 1 and Assumption 2 hold. Suppose in addition that  $\varphi^*$  is  $C^3$  on  $\mathbb{R}$  and that, for all  $j, k, l$ , there exists  $\varepsilon > 0$  such that*

$$\sup_{\tilde{v} \in B_{\varepsilon}(\tilde{\xi})} \left| \varphi^{*''' }(\langle \tilde{v}, \tilde{\Phi}(x) \rangle) \tilde{\Phi}_i(x) \tilde{\Phi}_j(x) \tilde{\Phi}_l(x) \right| \leq h_{jkl}(x)$$

for some  $\mu_0$ -integrable functions  $h_{jkl}$ , and where  $B_{\varepsilon}(\tilde{\xi})$  denotes the ball centered at  $\tilde{\xi}$  and of radius  $\varepsilon$ .

(i) *If  $y \neq y_0$ , then*

$$\sqrt{n}(T_n(y) - I_{\varphi}(y)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2),$$

as  $n \rightarrow \infty$ , where

$$\sigma^2 = \int_{\mathcal{X}} \varphi^{*2}(\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle) \mu_0(dx) - \left[ \int_{\mathcal{X}} \varphi^*(\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle) \mu_0(dx) \right]^2.$$

(ii) *If  $y = y_0$ , then*

$$\frac{2n}{\varphi''(1)} T_n(y) \xrightarrow{\mathcal{D}} \chi^2(k),$$

as  $n \rightarrow \infty$ .

**Proof** By dual attainment, we have

$$T_n(y) = \langle \tilde{\xi}_n, \tilde{y} \rangle - \frac{1}{n} \sum_{i=1}^n \varphi^*(\langle \tilde{\xi}_n, \tilde{\Phi}(X_i) \rangle).$$

Let us start with the following decomposition of the sum in the preceding equation:

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^n \varphi^*(\langle \tilde{\xi}_n, \tilde{\Phi}(X_i) \rangle) &= \frac{1}{n} \sum_{i=1}^n \varphi^*(\langle \tilde{\xi}, \tilde{\Phi}(X_i) \rangle) \\
&+ \frac{1}{n} \sum_{i=1}^n \varphi^{*'}(\langle \tilde{\xi}, \tilde{\Phi}(X_i) \rangle) \langle \tilde{\Phi}(X_i), \tilde{\xi}_n - \tilde{\xi} \rangle \\
&+ \frac{1}{n} \sum_{i=1}^n \varphi^{*''}(\langle \tilde{\xi}, \tilde{\Phi}(X_i) \rangle) \langle \tilde{\Phi}(X_i), \tilde{\xi}_n - \tilde{\xi} \rangle^2 \\
&+ R_n,
\end{aligned}$$

where

$$R_n = \frac{1}{n} \sum_{i=1}^n \varphi^{*'''}(\langle \tilde{\xi} + \alpha_n(\tilde{\xi}_n - \tilde{\xi}), \tilde{\Phi}(X_i) \rangle) \langle \tilde{\Phi}(X_i), \tilde{\xi}_n - \tilde{\xi} \rangle^3,$$

for some  $\alpha_n \in (0; 1)$ . Since the sequence  $\sqrt{n}(\tilde{\xi}_n - \tilde{\xi})$  is uniformly tight, and since for all  $j, k, l$ , the functions  $x \mapsto \sup_{\tilde{v} \in B_\varepsilon(\tilde{\xi})} \left| \varphi^{*'''}(\langle \tilde{v}, \tilde{\Phi}(x) \rangle) \tilde{\Phi}_i(x) \tilde{\Phi}_j(x) \tilde{\Phi}_l(x) \right|$  are dominated by some  $\mu_0$ -integrable functions by assumption, we conclude that

$$nR_n = o_P(1). \quad (5.6)$$

First, suppose that  $y \neq y_0$ . In this case, it suffices to consider the decomposition at the order two. Set

$$\tilde{z}_n = \frac{1}{n} \sum_{i=1}^n \varphi^{*'}(\langle \tilde{\xi}, \tilde{\Phi}(X_i) \rangle) \tilde{\Phi}(X_i)^t.$$

The Central Limit Theorem entails that the sequence  $\sqrt{n}(\tilde{z}_n - \tilde{y})$  is uniformly



tight. Then we may write

$$\begin{aligned}
T_n(y) - I_\varphi(y) &= \langle \tilde{\xi}_n, \tilde{y} \rangle - \frac{1}{n} \sum_{i=1}^n \varphi^*(\langle \tilde{\xi}_n, \tilde{\Phi}(X_i) \rangle) - \langle \tilde{\xi}, \tilde{y} \rangle \\
&\quad + \int_{\mathcal{X}} \varphi^*(\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle) \mu_0(dx) \\
&= \langle \tilde{\xi}_n, \tilde{y} \rangle - \frac{1}{n} \sum_{i=1}^n \varphi^*(\langle \tilde{\xi}, \tilde{\Phi}(X_i) \rangle) - \langle \tilde{z}_n, \tilde{\xi}_n - \tilde{\xi} \rangle - o_P(1/\sqrt{n}) \\
&\quad - \langle \tilde{\xi}, \tilde{y} \rangle + \int_{\mathcal{X}} \varphi^*(\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle) \mu_0(dx) \\
&= \langle \tilde{\xi}_n - \tilde{\xi}, \tilde{y} - \tilde{z}_n \rangle - \frac{1}{n} \sum_{i=1}^n \varphi^*(\langle \tilde{\xi}, \tilde{\Phi}(X_i) \rangle) \\
&\quad + \int_{\mathcal{X}} \varphi^*(\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle) \mu_0(dx).
\end{aligned}$$

But  $\sqrt{n} \langle \tilde{\xi}_n - \tilde{\xi}, \tilde{y} - \tilde{z}_n \rangle \rightarrow 0$  in probability, and so the first statement follows from the Central Limit Theorem.

Second, suppose that  $y = y_0$ . Then  $\tilde{\xi} = \tilde{\xi}_0$ , and for all  $i = 1, \dots, n$ , the following relations hold:

$$\begin{aligned}
\varphi^*(\langle \tilde{\xi}_0, \tilde{\Phi}(X_i) \rangle) &= \varphi^*(\varphi'(1)) = \varphi'(1), \\
\varphi^{*'}(\langle \tilde{\xi}_0, \tilde{\Phi}(X_i) \rangle) &= \varphi^{*'}(\varphi'(1)) = 1, \\
\varphi^{*''}(\langle \tilde{\xi}_0, \tilde{\Phi}(X_i) \rangle) &= \varphi^{*''}(\varphi'(1)) = \frac{1}{\varphi''(1)}.
\end{aligned}$$

Let  $\hat{y}_n = \frac{1}{n} \sum_{i=1}^n \Phi(X_i)$  and set  $\tilde{y}_n = (1, \hat{y}_n)$ . Let  $\bar{V}_n$  be the matrix defined by

$$\bar{V}_n = \frac{1}{n} \sum_{i=1}^n \tilde{\Phi}(X_i) \tilde{\Phi}(X_i)^t.$$

Then we obtain

$$\begin{aligned}
T_n(y) - I_\varphi(y) &= \langle \tilde{\xi}_n, \tilde{y} \rangle - \varphi'(1) - \langle \tilde{y}_n, \tilde{\xi}_n - \tilde{\xi}_0 \rangle - \frac{1}{2\varphi''(1)}(\tilde{\xi}_n - \xi)^t \bar{V}_n(\tilde{\xi}_n - \xi_0) \\
&\quad - o_P(1/n) - \langle \tilde{\xi}_0, \tilde{y} \rangle + \int_{\mathcal{X}} \varphi^*(\langle \tilde{\xi}_0, \tilde{\Phi}(x) \rangle) \mu_0(dx) \\
&= \langle \tilde{\xi}_n - \tilde{\xi}_0, \tilde{y} - \tilde{y}_n \rangle - \frac{1}{2\varphi''(1)}(\tilde{\xi}_n - \xi_0)^t \bar{V}_n(\tilde{\xi}_n - \xi_0) + o_P(1/n),
\end{aligned}$$

since  $\int_{\mathcal{X}} \varphi^*(\langle \tilde{\xi}_0, \tilde{\Phi}(x) \rangle) \mu_0(dx) = \varphi'(1)$ . From (5.3), we have

$$\sqrt{n}(\tilde{\xi}_n - \tilde{\xi}) = -V_{\tilde{\xi}_0}^{-1}(\tilde{y}_n - \tilde{y}_0) + o_P(1),$$

where the matrix  $V_{\tilde{\xi}_0}$  is defined in (5.4). Since  $\bar{V}_n \rightarrow \mathbb{E}[\tilde{\Phi}(X)\tilde{\Phi}(X)^t]$  element-wise as  $n \rightarrow \infty$ , and since  $I_\varphi(y) = 0$  when  $y = y_0$ , we obtain

$$T_n(y) = \frac{\varphi''(1)}{2}(\tilde{y}_n - \tilde{y}_0)^t V_{\tilde{\xi}_0}^{-1}(\tilde{y}_n - y_0) + o_P(1/n). \quad (5.7)$$

Letting  $\Sigma = Cov_{\mu_0}(\Phi(X))$ , we may write

$$\begin{aligned}
V_{\tilde{\xi}_0} &= \mathbb{E}[\tilde{\Phi}(X)\tilde{\Phi}(X)^t] \\
&= \begin{pmatrix} 1 & y_0^t \\ y_0 & \Sigma \end{pmatrix}
\end{aligned}$$

Using the following relation for an invertible matrix defined by block:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix},$$

we obtain the expression of the inverse of  $V_{\tilde{\xi}_0}$ :

$$V_{\tilde{\xi}_0}^{-1} = \begin{pmatrix} 1 + y_0^t \Sigma^{-1} y_0 & -y_0^t \Sigma^{-1} \\ -\Sigma^{-1} y_0 & \Sigma^{-1} \end{pmatrix}. \quad (5.8)$$

Reporting (5.8) in (5.7), and since  $(\tilde{y}_n - \tilde{y}_0) = (0, \hat{y}_n - y_0)$  yields

$$\frac{2n}{\varphi''(1)} T_n(y) = (\hat{y}_n - y_0) \Sigma^{-1} (\hat{y}_n - y_0) + o_P(1),$$

from which the result follows.  $\square$ .

## A Technical Lemma

**Lemma A.1** *Suppose that  $\varphi$  satisfies Assumption 1.*

(i) *For all  $p \in \{0; 1; 2\}$  and for all  $\tilde{\xi} \in \tilde{\Xi}$ , the map  $f_p : \mathcal{X} \rightarrow \mathbb{R}$  defined by*

$$f_p(x) = \varphi^{*(p)}(\langle \tilde{\xi}, \tilde{\phi}(x) \rangle)$$

*is  $\mu_0$ -integrable, where  $\varphi^{*(p)}$  denotes the  $p^{\text{th}}$  derivative of  $\varphi^*$ .*

(ii) *Furthermore, for  $p = 1$  or  $p = 2$ ,  $f_p$  is in  $L_\infty(\mathcal{X}, \mu_0)$ .*

**Proof** Let us start by recalling the properties of  $\varphi^*$ . First, since  $\varphi$  is essentially smooth,  $\varphi^*$  is strictly convex, and since  $\text{dom}(\varphi) = (0, +\infty)$ ,  $\varphi^*$  is monotone increasing. Consequently,  $\varphi^{*'} and  $\varphi^{*''}$  are positive, and additionally,  $\varphi^{*'}$  is monotone increasing. Combination of these facts entails that  $\varphi^{*''}(u) \rightarrow 0$  as  $u \rightarrow -\infty$ . At last,  $\varphi^*(u)/u \rightarrow 0$  as  $u \rightarrow -\infty$  since  $\inf \text{dom}(\varphi) = 0$ .$

Given  $\tilde{\xi} \in \tilde{\Xi}$ , let  $a = \text{ess sup } \langle \tilde{\xi}, \tilde{\Phi}(x) \rangle < \kappa$  by definition of  $\tilde{\Xi}$ .

For  $p = 0$ , since  $\varphi^*(u)/u \rightarrow 0$  as  $u \rightarrow -\infty$ , there exists  $\alpha < 0$  such that  $|\varphi^*(u)| \leq |u|$  whenever  $u \leq \alpha$ . Let

$$A = \{x \in \mathcal{X} : \langle \tilde{\xi}, \tilde{\Phi}(x) \rangle \leq \alpha\}.$$

First, for  $\mu_0$ -a.e.  $x$ , we have

$$|f_0(x)\mathbf{1}_{A^c}(x)| \leq \sup_{[\alpha, a]} |\varphi^*(u)| < \infty,$$

and second

$$|f_0(x)\mathbf{1}_A(x)| \leq \langle |\tilde{\xi}|, |\tilde{\Phi}(x)| \rangle.$$

Since  $\tilde{\Phi}$  is  $\mu_0$ -integrable, and since  $\mu_0$  is finite, we conclude that  $f_0$  is  $\mu_0$ -integrable.

For  $p = 1$ , since  $\varphi^{*'}$  is positive monotone increasing, we have  $0 \leq f_1(x) \leq \varphi^{*'}(a)$   $\mu_0$ -a.e., and so  $f_1$  is in  $L_\infty(\mathcal{X}, \mu_0)$ .

For  $p = 2$ , since  $\varphi^{*''}$  is positive with  $\varphi^{*''}(u) \rightarrow 0$  as  $u \rightarrow -\infty$ , we have  $0 \leq f_2(x) \leq \sup_{u \in (-\infty, a]} \varphi^{*''}(u)$   $\mu_0$ -a.e., so  $f_2$  is in  $L_\infty(\mathcal{X}, \mu_0)$ .  $\square$

**Lemma A.2** For all  $p \in \{0; 1; 2\}$  and for all  $\tilde{\xi} \in \tilde{\Xi}$ , there exists  $\varepsilon > 0$  and a  $\mu_0$ -integrable function  $h$  such that

$$\sup_{\tilde{v} \in B_\varepsilon(\tilde{\xi})} |\varphi^{*(p)}(\langle \tilde{v}, \tilde{\Phi}(x) \rangle)| < h(x),$$

where  $B_\varepsilon(\tilde{\xi})$  is the Euclidean ball centered at  $\tilde{\xi}$  and of radius  $\varepsilon$ . Moreover, for  $p = 1$  or  $p = 2$ ,  $h$  may be taken as a constant function.

**Proof** Choose  $\varepsilon$  small enough such that the ball is included in a cube in turn included in  $\tilde{\Xi}$ , and denote by  $\tilde{v}_i$  the vertices of the cube, for  $i = 1, \dots, 2^{k+1}$ . For all  $\tilde{v} \in \tilde{\Xi}$ , let  $C(\tilde{v}) = \text{ess sup } \langle \tilde{v}, \tilde{\Phi}(x) \rangle$ , which is strictly less than  $\kappa$  by construction. Then, for all  $\tilde{v} \in B_\varepsilon(\tilde{\xi})$ , and for  $\mu_0$ -almost every  $x$ , we have

$$\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle - \varepsilon \|\tilde{\Phi}(x)\| \leq \langle \tilde{v}, \tilde{\Phi}(x) \rangle \leq \max_i C(\tilde{v}_i), \quad (\text{A.1})$$

where  $\|\tilde{\Phi}(x)\|$  denotes the Euclidean norm in  $\mathbb{R}^{k+1}$ , and where the upper inequality follows from the convexity of the cube. Since  $\varphi^*$  is monotone increasing, it follows that

$$\sup_{\tilde{v} \in B_\varepsilon(\tilde{\xi})} |\varphi^*(\langle \tilde{v}, \tilde{\Phi}(x) \rangle)| \leq \max \left\{ \left| \varphi^*(\max_i C(\tilde{v}_i)) \right|; \left| \varphi^*(\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle - \varepsilon \|\tilde{\Phi}(x)\|) \right| \right\},$$

for  $\mu_0$ -a.e.  $x$ . Since  $\mu_0$  is a finite measure, it is sufficient to prove that the second term in the maximum is  $\mu_0$  integrable. As in the proof of Lemma A.1, let  $\alpha \leq 0$  be such that  $|\varphi^*(u)| \leq |u|$  for all  $u \leq \alpha$ , and let

$$A = \{x \in \mathcal{X} : \langle \tilde{\xi}, \tilde{\Phi}(x) \rangle - \varepsilon \|\tilde{\Phi}(x)\| \leq \alpha\}.$$

We have

$$|\varphi^*(\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle - \varepsilon \|\tilde{\Phi}(x)\|)| \mathbf{1}_{A^c}(x) \leq \sup_{[\alpha; \max_i C(\tilde{v}_i)]} |\varphi^*(u)|,$$

and

$$|\varphi^*(\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle - \varepsilon \|\tilde{\Phi}(x)\|)| \mathbf{1}_A(x) \leq |\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle - \varepsilon \|\tilde{\Phi}(x)\||,$$

and  $\int_{\mathcal{X}} |\langle \tilde{\xi}, \tilde{\Phi}(x) \rangle - \varepsilon \|\tilde{\Phi}(x)\|| \mu_0(dx)$  is finite since the components of  $\tilde{\Phi}$  are in  $L_2(\mathcal{X}, \mu_0)$  and since  $\mu_0(A^c) < \infty$ . This proves the result for  $p = 0$ .

For  $p = 1$ , since  $\varphi^{*'} is positive and monotone increasing, the result follows directly from (A.1).$

For  $p = 2$ , the result follows from the fact that  $\varphi^{*''}$  is positive with  $\varphi^{*''}(u) \rightarrow 0$  as  $u \rightarrow -\infty$  and (A.1).  $\square$

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