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# Generalized Percolation Processes Using Pretopology Theory

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**Abstract**— We propose in this paper a generalization of percolation processes in  $\mathbb{Z}^2$  using the pretopology theory. We formalize the notion of neighborhood by extending it to the concept of proximity, expressing different types of connections that may exist between the elements of a population. A modeling and simulation of forest fire using this approach shows the efficiency of this formalism to provide a realistic and powerful modeling of complex systems.

**Index Terms**— Pretopology, percolation processes, complex systems modeling.

## I. INTRODUCTION

Percolation Theory studies the deterministic propagation of a fluid<sup>1</sup> on a random medium [4]. As phase transition phenomenon, an abrupt change in the behavior of the system, observed in percolation processes is a characteristic property of complex systems, percolation theory was successfully applied to model and simulate complex phenomena in statistical physics, economy and recently in social networks. However, percolation processes are based on the neighborhood concept which is not well formalized. It is difficult to express a certain kind of relations between the components of the system such as remote connections, the dynamical evolution of the neighborhood basis, the heterogeneity of the interactions and the hierarchy between the components. For example, to model the spreading of oil in water, of fire through forests or infectious diseases among a population, it is difficult to express the qualitative properties of the system.

In this work, we provide a generalized formulation of percolation processes in  $\mathbb{Z}^2$  using pretopology theory[2]. This can be achieved using the pretopological concepts of pseudoclosure and closed subsets. It is a general formalism that expresses different types of connections that may exist between the components of a system. It provides also theoretical tools to express specific phenomena in distributed systems such as the alliance phenomena, acceptability processes and emergence of collective behavior.

The rest of the paper is organized as follows : in section 2, we present an overview of percolation theory. The section 3 concerns a general presentation of pretopology theory. The dynamical aspect of the structural transformation and the neighborhood notion are formalized in section 4. To illustrate

<sup>1</sup>The word *fluid* is considered in the widest sense of the term(water, information, disease, etc.).

the efficiency of our approach we present, respectively in section 5 and 6, a percolation based modeling of forest fire, expressed on a pretopological space, and its simulation. We conclude, in section 7, with a discussion of the results and future work.

## II. PERCOLATION THEORY

The first percolation model was introduced by Simon Broadbent et John M. Hammersley in 1957, using the example of a porous stone immersed in a bucket of water. This fundamental question was asked : What is the probability that the center of the stone is wetted? Equivalently, what is the probability that an infinite size percolation cluster of pores exists. This probability depends on the porosity of the stone (i.e. the density of pores). [4]

Physical problems are mathematically modeled as a network of points (or vertices) and the connections (or edges) between each two neighbors may be open (allowing the liquid to pass through) with probability  $p$ , or closed with probability  $(1-p)$ , and we assume they are independent. For a given  $p$ , what is the probability that an open path exists from the top to the bottom? Generally the interest concerns the behavior for large  $n$ . As is quite typical, it is actually easier to examine infinite networks than just large ones. In this case the corresponding question is : does there exist an infinite open cluster ? That is, is there a path of connected points of infinite length "through" the network. In this case we may use Kolmogorov's zero-one law to see that, for any given  $p$ , the probability that an infinite cluster exists is either *zero* or *one*. Since this probability is increasing, there must be a critical probability  $p_c$  such that (figure 1) :

$$P(p) \begin{cases} = 0 & \text{if } p < p_c \\ = 1 & \text{if } p > p_c \end{cases}$$

where  $P$  is the percolation probability which indicates the probability of appearance of the giant cluster in the system.

A model where we open and close vertices rather than edges, is called *site percolation* (figure 2 a) while the model described above is more properly called *bond percolation* (figure 2 b). The model where the uncertainty concerns both sites and bonds is called *mixed percolation* (figure 2 c).

If we consider the percolation processes on  $\mathbb{Z}^2$ , for example, we have different models : depending on the chosen neighborhood, on the nature of the studied aspect (dynamic or static

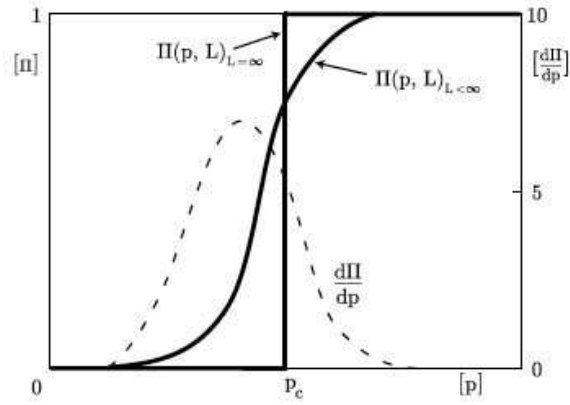


Fig. 1. Phase transition around the critical probability. Source : in [8]

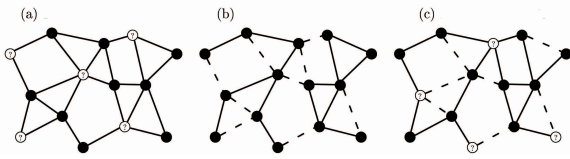


Fig. 2. Basic percolation models

aspect of a system) and on the nature of the studied object (a specific phenomenon or a property of a system). The most used kind of neighborhoods are the *Von Neumann* neighborhood and the *Moore* neighborhood (figure 4). The values of the critical parameters depend on the used neighborhood. We can see in the figure 3 that an infinite cluster (according to a Von Neumann neighborhood) appears abruptly at a critical threshold  $p_c \simeq 0,6$ . This value is different if we consider a Moore neighborhood. In fact, with a Moore neighborhood we obtain a giant connected cluster with less density (*i.e.* probability of active sites).



Fig. 3. Appearance of a giant cluster.

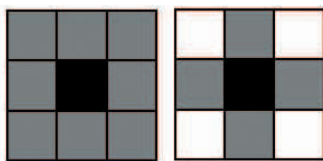


Fig. 4. Moore neighborhood (eight neighbors) and Von Neumann neighborhood (four neighbors).

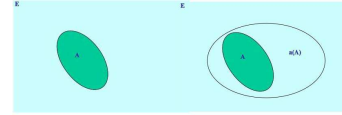


Fig. 5. Pretopological space.

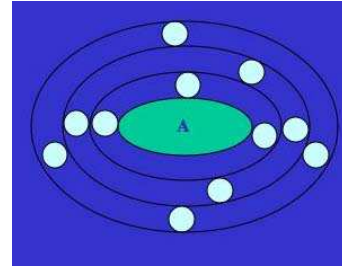


Fig. 6. Dilation process.

### III. PRETOPOLOGY THEORY

Pretopology theory expresses the structural transformation of sets composed of interacting elements. It allows the representation of discrete structures, thanks to a general notion of neighborhood. The pretopological concepts of pseudoclosure and closed subsets developed in [2] allows a mathematical representation of various aspects and phenomena such as constitution of decisive coalition among a population, alliance phenomena, tolerance and acceptability processes and emergence of collective behavior. Even if closure operators have been widely studied in algebra, topology and computer science theory the axiomatics that define them are too limited to express concrete problems.[6]

#### A. Pretopology formalism

Let us consider a non-empty finite set  $E$ , and  $\mathcal{P}(E)$  designates all of the subsets of  $E$ .

##### 1) Pretopological space:

*Definition 1:* A pretopological space is a pair  $(E, a)$  (figure 5) where  $a$  is a map  $a(\cdot) : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$  called pseudo-closure and defined as follows :  $\forall A, A \subseteq E$  the pseudo-closure of  $A$ ,  $a(A) \subseteq E$  such that :

- $a(\emptyset) = \emptyset$  (P<sub>1</sub>)
- $A \subseteq a(A)$  (P<sub>2</sub>)

The pseudo-closure is associated to the *dilation process* (figure 6).

We can also define a pretopological space  $(E, i)$  using the *interior map* (figure 7).  $\forall A, A \subseteq E$  we define the *interior*  $i(A) \subseteq E$  such that :

$$i(A) = [a(A^c)]^c$$

- $\forall A, A \subseteq E, i(A) \subseteq A$

The interior map is associated to the *erosion process* (figure 8).

*Definition 2:* Let  $(E, i, a)$  be a pretopological space.  $\forall A, A \subseteq E$  :

- $A$  is a closed subset if  $a(A) = A$
- $A$  is an open subset if  $i(A) = A$

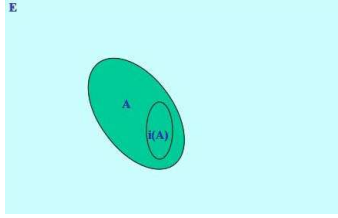


Fig. 7. Interior map.

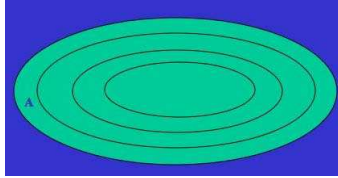


Fig. 8. Erosion process.

## 2) $\mathcal{V}$ Pretopological space :

*Definition 3:* A  $\mathcal{V}$  pretopological space  $(E, i, a)$  is defined by :

- $\forall A, B, A \subseteq E, B \subseteq E$  and  $A \subset B$  then  $a(A) \subset a(B)$
- equivalently  $i(A) \subset i(B)$

## IV. EXPRESSING A PERCOLATION PROCESS ON A PRETOPOLOGICAL SPACE

Using pretopology we can define a general percolation process including the classical ones and allowing new ones. The advantage of this approach is that it allows us to formulate and treat structural dynamical aspects of complex systems in a unified manner. Percolation processes on  $\mathbb{Z}^2$  depend on the neighborhood. Using a neighborhood basis (set of objects of  $\mathcal{V}(x)$ ) expressed in a pretopological space, any neighborhood  $V(x)$  can be constructed. The diversity of the connections between the elements can be expressed by adapting the choice of the pseudoclosure to the studied problem.

### A. Neighborhood sets

Let  $(E, i, a)$  be a  $\mathcal{V}$  pretopological space.  
 $\forall x \in E :$

$$\mathcal{V}(x) = \{V \subset E / x \in i(V)\}$$

$\mathcal{V}(x)$  is a family of neighborhoods of  $x$ .

### B. Pretopological transformation in $\mathbb{Z}^2$

$\forall A, A \subseteq E, a(A) = \{x \in E / \forall V, V \in \mathcal{V}(x), V \cap A \neq \emptyset\}$  If we consider, for example, three different neighborhoods as in figure 9 and construct the associated pseudoclosure the dilation process have different propagation effects (figure 10a, 10b, 10c).

With the concepts of pseudoclosure, interior and neighborhood sets we can express in a general way different types of connections and take into account their evolution over time by changing their neighborhood basis. An example of the efficiency of this approach is presented in the following paragraph and concerns the forest fire modeling. Our choice

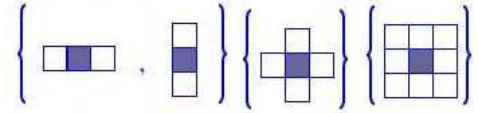


Fig. 9. Examples of neighborhood basis.

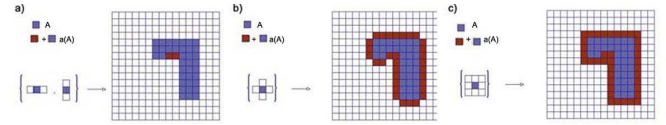


Fig. 10. The dilation process with a user-defined neighborhood (in red) and the pseudo-closure (red &amp; blue).

was motivated by the fact that this phenomenon exhibits a phase transition behavior [3], which is a common property of complex systems, and the fact that several percolation based models were developed to simulate it. We can also show through it, the difficulty to express a certain kind of qualitative factors using the traditional neighborhood definition.

## V. EXAMPLE OF APPLICATION : FOREST FIRE MODELING

### A. Percolation and forest fire modeling

In the case of the forest fire modeling, the density of trees is the most important factor. In fact, fire spreads by ignition of nearby trees, and the density of trees decides how far the fire spreads. If the trees are sparse, the fire will burn out before it gets very far. If the forest density exceeds a certain threshold, the fire will burn straight through the forest.

Several percolation based models were presented to simulate the propagation of fire through forests. In these models, generally, trees are arranged in a rectangular grid where the spread of fire is propagated from a tree to its neighbors, defined in an ad-hoc manner. The basic difference between these model is the neighborhood definition (figure 11). For example, some models using a Von Neumann neighborhood are not realistic because they do not take into account the wind effect. For this reason models using the *anisotropic percolation* or *oriented percolation* were developed in order to provide a more realistic modeling [7]. All these models could be considered as particular cases of a general percolation model if the the neighborhood concept was well formalized. With the mathematical definition of the neighborhood expressed in a pretopological space we are able not only to unify percolation processes formalism but also provide a more efficient modeling.

### B. Percolation based model of forest fire expressed in a pretopological space

Using the pseudo-closure, we express the *propagation* process of the fire according to the direction of the wind, defined using the proximity bases determining the neighborhood  $\mathcal{V}(x)$  (figure 12).

By changing the neighborhood basis we can model the wind direction changes (figure 13). We can also model the **consumed trees** using the *interior* mapping (figure 14).

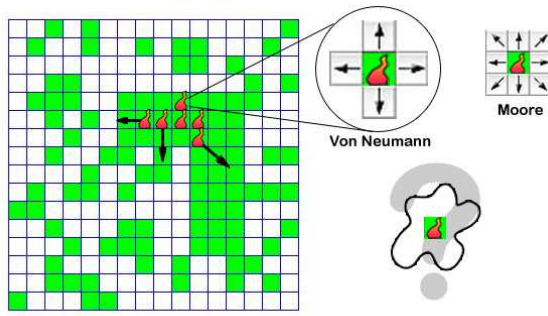


Fig. 11. The neighborhood is the basic difference between various percolation processes.

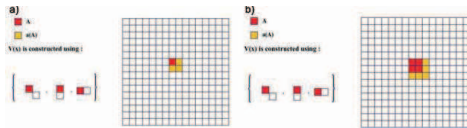


Fig. 12.

The interior map is defined as follows :

$$\forall A, A \subseteq E, i(A) = \{x \in E / \exists V, V \in \mathcal{V}(x), V \subseteq A\}$$

### VI. SIMULATION

A preliminary simulation and implementation of the concept of neighborhood and in a pretopological space was performed using Java (figure 15 and 16). The user can define until five neighborhood bases (to express the direction of the wind or the presence of an obstacle) and specify the number of the iterations. Trees are generated randomly according to fixed

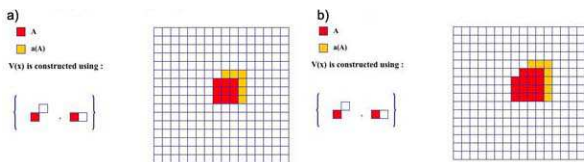


Fig. 13. By changing the neighborhood basis we model the wind direction changes.

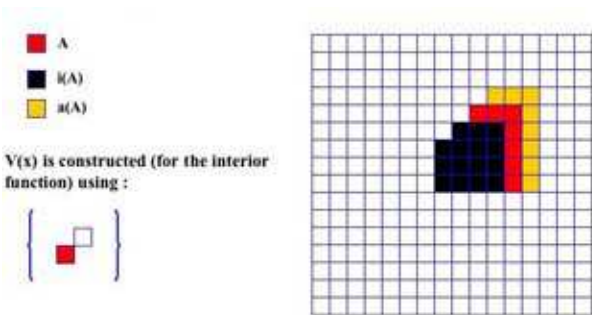


Fig. 14. By applying the interior map we model the consumed trees (inactive sites in the propagation process).

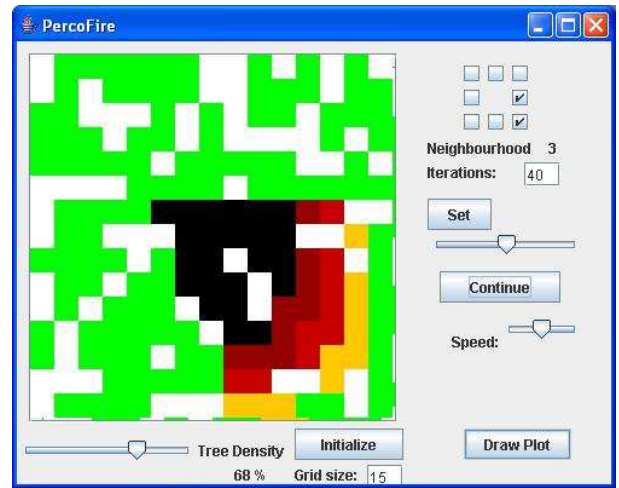


Fig. 15. The propagation process using the successive computation of the pseudoclosure and interior maps.

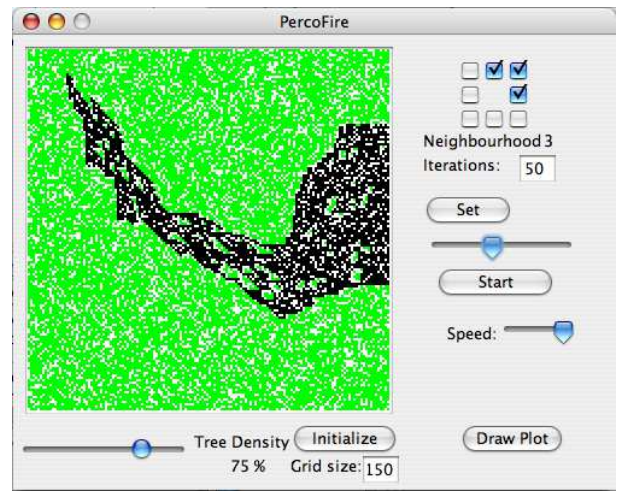


Fig. 16. The effect of the changes of the wind direction on the propagation process.

probability (*i.e.* density) using a slider. We assumed that burning trees will be consumed after three iterations. The results of the simulation are reported in the plot representing the evolution of the consumed trees proportion over time.

### VII. CONCLUSION

We showed that interesting concepts developed in pretopology theory may be applied to formalize the neighborhood notion, to generalize percolation processes in  $\mathbb{Z}^2$  and to enhance the modeling of complex phenomena and distributed systems. In a future work, we aim to provide a general formalism unifying the mathematical description of percolation processes and cellular automata in order to express the heterogeneity of neighborhoods and local dynamics in asynchronous distributed systems.

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