

ASYMPTOTIC SHARPNESS OF A BERNSTEIN-TYPE INEQUALITY FOR RATIONAL FUNCTIONS IN H^2

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ABSTRACT. A Bernstein-type inequality in the standard Hardy space H^2 of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, for rational functions in \mathbb{D} having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, $0 < r < 1$, is considered. The asymptotic sharpness is shown as $n \rightarrow \infty$, for every $r \in [0, 1)$.

I. INTRODUCTION

First we recall the classical Bernstein inequality for polynomials: we denote by \mathcal{P}_n the class of all polynomials with complex coefficients, of degree n : $P = \sum_{k=0}^n a_k z^k$. Let

$$\|P\|_2 = \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{T}} |P(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} = \left(\sum_{k=0}^n |a_k|^2 \right)^{\frac{1}{2}}.$$

The classical inequality

$$(1) \quad \|P'\|_2 \leq n \|P\|_2$$

is known as Bernstein's inequality. A great number of refinements and generalizations of (1) have been obtained. See [RaSc, Part III] for an extensive study of that subject. The constant n in (1) is obviously sharp (take $P = z^n$).

Now let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional space defined by

$$K_{B_\sigma} = \mathcal{L}in(k_{\lambda_i} : i = 1, \dots, n),$$

where σ is a family of distinct elements of \mathbb{D} , and where $k_\lambda = \frac{1}{1 - \bar{\lambda}z}$ is the Szegő kernel associated to λ . An obvious modification allows to generalize the definition of K_{B_σ} in the case where the sequence σ admits multiplicities.

Notice that using the scalar product $(\cdot, \cdot)_{H^2}$ on H^2 , an equivalent description of this space is:

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2,$$

where H^2 stands for the standard Hardy space of the unit disc \mathbb{D} ,

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_2^2 = \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) < \infty \right\},$$

m being the Lebesgue normalized measure on \mathbb{T} . We notice that the case $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ gives $K_{B_\sigma} = \mathcal{P}_{n-1}$. The issue of this paper is to generalize classical Bernstein inequality (1) to the spaces K_{B_σ} . Notice that every rational functions with poles outside of \mathbb{D} lies in a space K_{B_σ} . It has already been proved in [Z1] that if $r = \max_j |\lambda_j|$, and $f \in K_{B_\sigma}$, then

$$(2) \quad \|f'\|_2 \leq \frac{5}{2} \frac{n}{1-r} \|f\|_2.$$

In fact, Bernstein-type inequalities for rational functions were the subject of a number of papers and monographs (see, for instance, [L], [BoEr], [DeLo], [B1], [B2], [B3], [B4] and [B5]). Perhaps, the stronger and closer to ours of all known results are due to K. Dyakonov [Dya1]. In particular, it is proved in [Dya1] that the norm $\|D\|_{K_B \rightarrow H^2}$ of the differentiation operator $Df = f'$ on a space K_B satisfies the following double inequality

$$a \|B'\|_\infty \leq \|D\|_{K_B \rightarrow H^2} \leq A \|B'\|_\infty,$$

where $a = \frac{1}{36c}$, $A = \frac{36+c}{2\pi}$ and $c = 2\sqrt{3\pi}$ (as one can check easily (c is not precised in [Dya1])). It implies an inequality of type (2) (with a constant about $\frac{13}{2}$ instead of $\frac{5}{2}$).

Our goal is to find an inequality for $\sup \|D\|_{K_B \rightarrow H^2} = C_{n,r}$ (sup is over all B with given $n = \deg B$ and $r = \max_{\lambda \in \sigma} |\lambda|$), which is asymptotically sharp as $n \rightarrow \infty$. Our result is that there exists a limit $\lim_{n \rightarrow \infty} \frac{C_{n,r}}{n} = \frac{1+r}{1-r}$ for every r , $0 \leq r < 1$. Our method is different from [Dya1] and is based on an elementary Hilbert space construction for an orthonormal basis in K_B .

II. THE RESULT

Theorem. *Let $n \geq 1$, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , and B_σ the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional subspace of H^2 defined by*

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Let D be the operator of differentiation on $(K_{B_\sigma}, \|\cdot\|_2)$:

$$\begin{aligned} D : (K_{B_\sigma}, \|\cdot\|_2) &\rightarrow (H^2, \|\cdot\|_2) \\ f &\mapsto f', \end{aligned}$$

where $\|f\|_2 = \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{T}} |f(\zeta)|^2 d\zeta \right)^{\frac{1}{2}}$. For $r \in [0, 1)$ and $n \geq 1$, we set

$$C_{n,r} = \sup \left\{ \|D\|_{K_{B_\sigma} \rightarrow H^2} : 1 \leq \text{card } \sigma \leq n, |\lambda| \leq r \forall \lambda \in \sigma \right\}.$$

(i) If $n = 1$ and $\sigma = \{\lambda\}$, we have

$$\|D\|_{K_{B_\sigma} \rightarrow H^2} = |\lambda| \left(\frac{1}{1-|\lambda|^2} \right)^{\frac{1}{2}}.$$

If $n \geq 2$,

$$a(n, r) \frac{n}{1-r} \leq C_{n,r} \leq A(n, r) \frac{n}{1-r},$$

where

$$a(n, r) \geq \frac{1}{1+r} \left(1 + 5r^4 - \frac{4r^4}{n} - \min \left(\frac{3}{4}, \frac{2}{n} \right) \right)^{\frac{1}{2}},$$

and

$$A(n, r) \leq 1 + r + \frac{1}{\sqrt{n}}.$$

(ii) Moreover, the sequence

$$\left(\frac{1}{n} C_{n,r} \right)_{n \geq 1},$$

is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{n} C_{n,r} = \frac{1+r}{1-r},$$

for all $r \in [0, 1)$.

Proof. We first prove (i). We suppose that $n = 1$. In this case, $K_B = \mathbb{C}e_1$, where

$$e_1 = \frac{(1 - |\lambda|^2)^{\frac{1}{2}}}{(1 - \bar{\lambda}z)}, \quad |\lambda| \leq r,$$

(e_1 being of norm 1 in H^2). Calculating,

$$e'_1 = \frac{\bar{\lambda}(1 - |\lambda|^2)^{\frac{1}{2}}}{(1 - \bar{\lambda}z)^2},$$

and

$$\begin{aligned} \|e'_1\|_2 &= |\lambda| (1 - |\lambda|^2)^{\frac{1}{2}} \left\| \frac{1}{(1 - \bar{\lambda}z)^2} \right\|_2 = \\ &= |\lambda| (1 - |\lambda|^2)^{\frac{1}{2}} \left(\sum_{k \geq 0} (k+1) |\lambda|^{2k} \right)^{\frac{1}{2}} = |\lambda| (1 - |\lambda|^2)^{\frac{1}{2}} \frac{1}{(1 - |\lambda|^2)} = |\lambda| \left(\frac{1}{1 - |\lambda|^2} \right)^{\frac{1}{2}}, \end{aligned}$$

we get

$$\|D|_{K_{B\sigma}}\| = |\lambda| \left(\frac{1}{1 - |\lambda|^2} \right)^{\frac{1}{2}}.$$

Now, we suppose that $n \geq 2$. First, we prove the left-hand side inequality. Let

$$e_n = \frac{(1 - r^2)^{\frac{1}{2}}}{1 - rz} b_r^{n-1}.$$

Then $e_n \in K_{b_r^n}$ and $\|e_n\|_2 = 1$, (see [N1], Malmquist-Walsh Lemma, p.116). Moreover,

$$\begin{aligned} e'_n &= \frac{r(1 - r^2)^{\frac{1}{2}}}{(1 - rz)^2} b_r^{n-1} + (n-1) \frac{(1 - r^2)^{\frac{1}{2}}}{1 - rz} b'_r b_r^{n-2} = \\ &= -\frac{r}{(1 - r^2)^{\frac{1}{2}}} b'_r b_r^{n-1} + (n-1) \frac{(1 - r^2)^{\frac{1}{2}}}{1 - rz} b'_r b_r^{n-2}, \end{aligned}$$

since $b'_r = \frac{r^2-1}{(1-rz)^2}$. Then,

$$e'_n = b'_r \left[-\frac{r}{(1 - r^2)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1 - r^2)^{\frac{1}{2}}}{1 - rz} b_r^{n-2} \right],$$

and

$$\begin{aligned} \|e'_n\|_2^2 &= \frac{1}{2\pi} \int_{\mathbb{T}} |b'_r(w)| |b'_r(w)| \left| -\frac{r}{(1 - r^2)^{\frac{1}{2}}} (b_r(w))^{n-1} + (n-1) \frac{(1 - r^2)^{\frac{1}{2}}}{1 - rw} (b_r(w))^{n-2} \right|^2 dm(w) = \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} |b'_r(w)| |b'_r(w)| \left| -\frac{r}{(1 - r^2)^{\frac{1}{2}}} b_r(w) + (n-1) \frac{(1 - r^2)^{\frac{1}{2}}}{1 - rw} \right|^2 dm(w), \end{aligned}$$

which gives, using the variables $u = b_r(w)$,

$$\|e'_n\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |b'_r(b_r(u))| \left| -\frac{r}{(1 - r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1 - r^2)^{\frac{1}{2}}}{1 - r b_r(u)} \right|^2 dm(u).$$

But $1 - rb_r = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$ and $b'_r \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2}$. This implies

$$\begin{aligned} \|e'_n\|_2^2 &= \frac{1}{2\pi} \int_{\mathbb{T}} \left| \frac{(1-ru)^2}{1-r^2} \right| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}}u + (n-1)\frac{(1-r^2)^{\frac{1}{2}}}{1-r^2}(1-ru) \right|^2 dm(u) = \\ &= \frac{1}{(1-r^2)^2} \frac{1}{2\pi} \int_{\mathbb{T}} |(1-ru)(-ru + (n-1)(1-ru))|^2 dm(u). \end{aligned}$$

Without loss of generality we can replace r by $-r$, which gives

$$\|e'_n\|_2 = \frac{1}{1-r^2} \|\varphi_n\|_2,$$

where $\varphi_n = (1+rz)(rz + (n-1)(1+rz))$. Expanding, we get

$$\begin{aligned} \varphi_n &= (1+rz)(nrz + (n-1)) = \\ &= nrz + (n-1) + nr^2z^2 + (n-1)rz = \\ &= (n-1) + (nr + (n-1)r)z + nr^2z^2, \end{aligned}$$

and

$$\begin{aligned} \|e'_n\|_2^2 &= \frac{1}{(1-r^2)^2} ((n-1)^2 + (2n-1)^2r^2 + n^2r^4) = \\ &= \frac{n^2}{(1-r^2)^2} \left(1 + 4r^2 + r^4 - \frac{2}{n} - \frac{4r^2}{n} + \frac{1}{n^2} + \frac{r^2}{n^2} \right) = \\ &= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(1 + 4r^2 + r^4 - \frac{2}{n} - \frac{4r^2}{n} + \frac{1+r^2}{n^2} \right) = \\ &= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(1 + 4r^2 + r^4 - 5r^4 + 5r^4 - \frac{4r^4}{n} + \frac{4r^4}{n} - \frac{4r^2}{n} - \frac{2}{n} + \frac{1+r^2}{n^2} \right) = \\ &= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(4r^2(1-r^2) - \frac{4r^2}{n}(1-r^2) + \frac{1+r^2}{n^2} + 1 + 5r^4 - \frac{4r^4}{n} - \frac{2}{n} \right) = \\ &= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(4r^2(1-r^2)\left(1 - \frac{1}{n}\right) + \frac{1+r^2}{n^2} + 1 + 5r^4 - \frac{4r^4}{n} - \frac{2}{n} \right) \geq \\ &\geq \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \begin{cases} 1 + 5r^4 - \frac{4r^4}{n} - \frac{2}{n} & \text{if } n > 2 \\ \frac{1}{4} + 1 + 5r^4 - \frac{4r^4}{2} - \frac{2}{2} & \text{if } n = 2 \end{cases} \geq \\ &\geq \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(1 + 5r^4 - \frac{4r^4}{n} - \min\left(\frac{3}{4}, \frac{2}{n}\right) \right), \end{aligned}$$

and

$$a(n, r) \geq \frac{1}{1+r} \left(1 + 5r^4 - \frac{4r^4}{n} - \min\left(\frac{3}{4}, \frac{2}{n}\right) \right)^{\frac{1}{2}},$$

which completes the proof of the left hand side inequality .

We show now the right hand side one. Let σ be a sequence in \mathbb{D} such that $1 \leq \text{card } \sigma \leq n$, $|\lambda| \leq r \forall \lambda \in \sigma$. Using [Z1], Proposition 4.1, we have

$$\begin{aligned} \|D\|_{K_{B_\sigma} \rightarrow H^2} &\leq \frac{1}{1-r} + \frac{1+r}{1-r}(n-1) + \frac{1}{1-r}\sqrt{n-2} = \\ &= \frac{1}{1-r} (1 + (1+r)(n-1) + \sqrt{n-2}) = \\ &= \frac{1}{1-r} (n(1+r) - r + \sqrt{n-2}) = \frac{n}{1-r} \left(1+r - \frac{r}{n} + \sqrt{\frac{1}{n} - \frac{2}{n^2}} \right) = \\ &\leq \frac{n}{1-r} \left(1+r + \sqrt{\frac{1}{n}} \right), \end{aligned}$$

which gives the result.

Now, we prove (ii). **Step 1.** We first prove the right-hand side inequality:

$$\limsup_{n \rightarrow \infty} \frac{1}{n} C_{n,r} \leq \frac{1+r}{1-r},$$

which becomes obvious since

$$\|D\|_{K_{B_\sigma} \rightarrow H^2} \leq \frac{n}{1-r} \left(1+r + \sqrt{\frac{1}{n}} \right).$$

Step 2. We now prove the left-hand side inequality:

$$\liminf_{n \rightarrow \infty} \frac{1}{n} C_{n,r} \geq \frac{1+r}{1-r}.$$

More precisely, we show that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \|D\|_{K_{b^n} \rightarrow H^2} \geq \frac{1+r}{1-r}.$$

Let $f \in K_{b^n}$. Then,

$$\begin{aligned} f' &= (f, e_1)_{H^2} \frac{r}{(1-rz)} e_1 + \sum_{k=2}^n (k-1) (f, e_k)_{H^2} \frac{b_r'}{b_r} e_k + r \sum_{k=2}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k = \\ &= r \sum_{k=1}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k + \frac{1-r^2}{(1-rz)(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} e_k = \\ &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{3}{2}}}{(1-rz)^2(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1}, \end{aligned}$$

which gives

$$(3) \quad f' = -b_r' \left[\frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{z-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right].$$

Now using the change of variables $v = b_r(u)$, we get

$$\|f'\|_2^2 = \int_{\mathbb{T}} |b_r'(u)| |b_r(u)| \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{u-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right|^2 du =$$

$$= \int_{\mathbb{T}} |b'_r(b_r(v))| \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{b_r(v) - r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv.$$

But

$$b_r - r = \frac{r - z - r(1 - rz)}{1 - rz} = \frac{z(r^2 - 1)}{1 - rz},$$

and

$$b'_r \circ b_r = \frac{r^2 - 1}{(1 - rb_r)^2} = -\frac{(1 - rz)^2}{1 - r^2},$$

which gives

$$\begin{aligned} & \|f'\|_2^2 = \\ &= \frac{1}{1 - r^2} \int_{\mathbb{T}} |(1 - rv)^2| \left| \frac{r}{(1 - r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1 - r^2)^{\frac{1}{2}}}{v(r^2 - 1)} (1 - rv) \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv = \\ &= \frac{1}{(1 - r^2)^2} \int_{\mathbb{T}} |(1 - rv)^2| \left| r \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} - (1 - rv) \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-2} \right|^2 dv, \end{aligned}$$

and

$$(4) \quad \|f'\|_2^2 = \frac{1}{(1 - r^2)^2} \int_{\mathbb{T}} \left| r(1 - rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k - (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right|^2 dv.$$

In particular,

$$\begin{aligned} (5) \quad & \frac{1}{n \|f\|_2} \left[\left\| (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2 + \left\| r(1 - rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 \right] \geq \\ & \geq \frac{1 - r^2}{n} \frac{\|f'\|_2}{\|f\|_2} \geq \\ & \frac{1}{n \|f\|_2} \left[\left\| (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2 - \left\| r(1 - rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 \right]. \end{aligned}$$

Now, we notice that on one hand

$$(6) \quad \left\| r(1 - rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 \leq r(1 + r) \left(\sum_{k=0}^{n-1} |(f, e_{k+1})_{H^2}|^2 \right)^{1/2} \leq r(1 + r) \|f\|_2,$$

and on the other hand,

$$\begin{aligned} & (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k = \\ &= (1 - 2rv + r^2v^2) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k = \\ &= \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k - 2r \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^{k+1} + r^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^{k+2} = \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k - 2r \sum_{k=1}^{n-1} k (f, e_{k+1})_{H^2} v^k + r^2 \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^k = \\
&= (f, e_2)_{H^2} + 2(f, e_3)_{H^2} v + \sum_{k=2}^{n-2} [(k+1) (f, e_{k+2})_{H^2} - 2rk (f, e_{k+1})_{H^2} + r^2(k-1) (f, e_k)_{H^2}] v^k + \\
&\quad - 2r [(f, e_2)_{H^2} v + (n-1) (f, e_n)_{H^2} v^{n-1}] + r^2 [(n-2) (f, e_{n-1})_{H^2} v^{n-1} + (n-1) (f, e_n)_{H^2} v^n],
\end{aligned}$$

which gives

$$\begin{aligned}
(7) \quad &(1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k = (f, e_2)_{H^2} + 2[(f, e_3)_{H^2} - r(f, e_2)_{H^2}] v + \\
&\quad + \sum_{k=2}^{n-2} [(k+1) (f, e_{k+2})_{H^2} - 2rk (f, e_{k+1})_{H^2} + r^2(k-1) (f, e_k)_{H^2}] v^k + \\
&\quad + [r^2(n-2) (f, e_{n-1})_{H^2} - 2r(n-1) (f, e_n)_{H^2}] v^{n-1} + r^2(n-1) (f, e_n)_{H^2} v^n.
\end{aligned}$$

Now, let $s = (s_n)_n$ be a sequence of even integers such that

$$\lim_{n \rightarrow \infty} s_n = \infty \text{ and } s_n = o(n) \text{ as } n \rightarrow \infty.$$

Then we consider the following function f in $K_{b_r^n}$:

$$\begin{aligned}
f &= e_n - e_{n-1} + e_{n-2} - e_{n-3} + \dots + (-1)^k e_{n-k} + \dots + e_{n-s} - e_{n-s-1} + e_{n-s-2} = \\
&= \sum_{k=0}^{s+2} (-1)^k e_{n-k}.
\end{aligned}$$

Using (6) on one hand, we get

$$(8) \quad \lim_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 = 0,$$

and applying (7) on the other hand, we obtain

$$\begin{aligned}
&\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 = |(f, e_2)_{H^2}|^2 + 4|(f, e_3)_{H^2} - r(f, e_2)_{H^2}|^2 + \\
&\quad + |r^2(n-2) (f, e_{n-1})_{H^2} - 2r(n-1) (f, e_n)_{H^2}|^2 + r^4(n-1)^2 |(f, e_n)_{H^2}|^2 + \\
&\quad + \sum_{k=2}^{n-2} |(k+1) (f, e_{k+2})_{H^2} - 2rk (f, e_{k+1})_{H^2} + r^2(k-1) (f, e_k)_{H^2}|^2,
\end{aligned}$$

which gives

$$\begin{aligned}
&\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 = \\
&= |r^2(n-2) + 2r(n-1)|^2 + r^4(n-1)^2 + \\
&\quad + \sum_{l=2}^{n-2} |(n-l+1) (f, e_{n-l+2})_{H^2} - 2r(n-l) (f, e_{n-l+1})_{H^2} + r^2(n-l-1) (f, e_{n-l})_{H^2}|^2,
\end{aligned}$$

setting the change of index $l = n - k$ in the last sum. This finally gives

$$\begin{aligned} & \left\| (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 = \\ & = |r^2(n-2) + 2r(n-1)|^2 + r^4(n-1)^2 + \\ & + \sum_{l=2}^{s+1} |(n-l+1) + 2r(n-l) + r^2(n-l-1)|^2 + \\ & + |(n-s-1) + 2r(n-s-2)|^2 + |n-s-2|^2. \end{aligned}$$

And

$$\begin{aligned} & \left\| (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq \\ & \geq |r^2(n-2) + 2r(n-1)|^2 + r^4(n-1)^2 + \\ & + s |(n-s) + 2r(n-s-1) + r^2(n-s-2)|^2 + \\ & + |(n-s-1) + 2r(n-s-2)|^2 + |n-s-2|^2. \end{aligned}$$

In particular,

$$(9) \quad \left\| (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq s |(n-s) + 2r(n-s-1) + r^2(n-s-2)|^2.$$

Passing after to the limit as $n \rightarrow \infty$ in (5), we obtain (using (8))

$$\begin{aligned} (10) \quad & \frac{1}{1+r} \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2 \geq \\ & \geq \liminf_{n \rightarrow \infty} \frac{1-r \|f'\|_2}{n \|f\|_2} \geq \\ & \frac{1}{1+r} \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2. \end{aligned}$$

This gives

$$(11) \quad \liminf_{n \rightarrow \infty} \frac{1-r \|f'\|_2}{n \|f\|_2} = \frac{1}{1+r} \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2.$$

Now, since

$$\|f\|_2^2 = s_n + 3,$$

using (9) we obtain

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n^2 \|f\|_2^2} \left\| (1 - rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq \\ & \geq \liminf_{n \rightarrow \infty} \frac{1}{n^2 \|f\|_2^2} (\|f\|_2^2 - 3) |(n-s) + 2r(n-s-1) + r^2(n-s-2)|^2. \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \frac{3}{n^2 s_n^2} \left| (n-s) + 2r(n-s-1) + r^2(n-s-2) \right|^2 = 0,$$

we get

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n^2 \|f\|_2^2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq \\ & \geq \liminf_{n \rightarrow \infty} \frac{1}{n^2 s_n^2} s_n^2 \left| (n-s_n) + 2r(n-s_n-1) + r^2(n-s_n-2) \right|^2 = \\ & = \lim_{n \rightarrow \infty} \frac{1}{n^2} \left| (n-s_n) + 2r(n-s_n-1) + r^2(n-s_n-2) \right|^2 = \\ & = \lim_{n \rightarrow \infty} \frac{1}{n^2} |n + 2rn + r^2n|^2 = (1+r)^4. \end{aligned}$$

We can now conclude that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1-r}{n} \|D\|_{K_{b_n^r} \rightarrow H^2} \geq \liminf_{n \rightarrow \infty} \frac{1-r}{n} \frac{\|f'\|_2}{\|f\|_2} = \\ & = \frac{1}{1+r} \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2 \geq \frac{(1+r)^2}{1+r} = 1+r. \end{aligned}$$

Step 3. Conclusion. Using both **Step 1** and **Step 2**, we get

$$\limsup_{n \rightarrow \infty} \frac{1-r}{n} C_{n,r} = \liminf_{n \rightarrow \infty} \frac{1-r}{n} C_{n,r} = 1+r,$$

which means that the sequence $(\frac{1}{n} C_{n,r})_{n \geq 1}$ is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{n} C_{n,r} = \frac{1+r}{1-r}.$$

□

Comments

(a) Bernstein-type inequalities for K_B appeared as early as in 1991. There, the boundedness of $D : (K_B, \|\cdot\|_{H^p}) \rightarrow (H^p, \|\cdot\|_{H^p})$ was covered for the full range $1 \leq p \leq \infty$. In [Dya1], the chief concern of K. Dyakonov was compactness (plus a new, simpler, proof of boundedness). Now, using both [BoEr] Th. 7.1.7 p. 324, (or equivalently M. Levin's inequality [L]) and complex interpolation, we could recover the result of K. Dyakonov for H^p spaces, $2 \leq p \leq \infty$ and our method could give a better numerical constant c_p in the inequality

$$\|f'\|_{H^p} \leq c_p \|B'\|_\infty \|f\|_{H^p}.$$

The case $1 \leq p \leq 2$ can be treated using the partial result of K. Dyakonov ($p = 1$) and still complex interpolation.

(b) In the same spirit, it is also possible to generalize the above Bernstein-type inequality to the same class of rational functions f in \mathbb{D} , replacing the Hardy space H^2 by Besov spaces $B_{2,2}^s$, $s \in \mathbb{R}$, of all holomorphic functions $f = \sum_{k \geq 0} \hat{f}(k) z^k$ in \mathbb{D} satisfying

$$\|f\|_{B_{2,2}^s} := \left(\sum_{k \geq 0} (k+1)^{2s} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty.$$

The same spaces are also known as Dirichlet-Bergman spaces. (In particular, the classical Bergman space corresponds to $s = -\frac{1}{2}$ and the classical Dirichlet space corresponds to $s = \frac{1}{2}$). Using

the above approach, one can prove the sharpness of the growth order $\frac{n}{1-r}$ in the corresponding Bernstein-type inequality

$$(3) \quad \|f'\|_{B_{2,2}^s} \leq c_s \frac{n}{1-r} \|f\|_{B_{2,2}^s},$$

(at least for integers values of s).

(c) One can also prove an inequality

$$(4) \quad \|f\|_{B_{2,2}^s} \leq c'_s \left(\frac{n}{1-r} \right)^s \|f\|_{H^2},$$

for $s \geq 0$ and the same class of functions (essentially, this inequality can be found in [Dya2]), and show the sharpness of the growth order $\left(\frac{n}{1-r}\right)^s$ (at least for integers values of s). An application of this inequality lies in constrained H^∞ interpolation in weighted Hardy and Bergman spaces, see [Z1] and [Z2] for details.

Notice that already E. M. Dyn'kin (in [Dyn]), and A. A. Pekarskii (in [Pe1], [Pe2] and [PeSt]), studied Bernstein-type inequalities for rational functions in Besov and Sobolev spaces. In particular, they applied such inequalities to inverse theorems of rational approximation. Our approach is different and more constructive. We are able to obtain uniform bounds depending on the geometry of poles of order n , which allows us to obtain estimates which are asymptotically sharp.

Also, in paper [Dya3] of K. Dyakonov (see Sections 10, 11 at the end), there are Bernstein-type inequalities involving Besov and Sobolev spaces that contain, as special cases, the earlier version from , Pekarskii's inequalities for rational functions, and much more. K. Dyakonov used those Bernstein-type inequalities to "interpolate", in a sense, between the polynomial and rational inverse approximation theorems (in response to a question raised by E. M. Dyn'kin). Finally, he has recently studied the "reverse Bernstein inequality" in K_B ; this is done in [Dya3].

(d) The above comments can lead to wonder what happens if we replace Besov spaces $B_{2,2}^s$ by other Banach spaces, for example by W , the Wiener algebra of absolutely convergent Taylor series. In this case, we obtain

$$(5) \quad \|f\|_W \leq c(n, r) \|f\|_{H^2}$$

where $c(n, r) \leq c \left(\frac{n^2}{1-r} \right)^{\frac{1}{2}}$ and c is a numerical constant. We suspect that $\left(\frac{n^2}{1-r} \right)^{\frac{1}{2}}$ is the right growth order of $c(n, r)$. An application of this inequality to an estimate of the norm of the resolvent of an $n \times n$ power-bounded matrix T on a Banach space is given in [Z3]. Inequality (5), above, is deeply linked with the inequality

$$(6) \quad \|f'\|_{H^1} \leq \gamma n \|f\|_{H^\infty},$$

through Hardy's inequality :

$$\|f\|_W \leq \pi \|f'\|_{H^1} + |f(0)|,$$

for all $f \in W$, (see [N2] p. 370 8.7.4 -(c)).

Inequality (6) is (shown and) used by R. J. LeVeque and L. N. Trefethen in [LeTr] with $\gamma = 2$, and later by M. N. Spijker in [Sp] with $\gamma = 1$ (an improvement) so as to apply it to the Kreiss Matrix Theorem in which the power boundedness of $n \times n$ matrices is related to a resolvent condition on these matrices.

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