

ASYMPTOTIC SHARPNESS OF A BERNSTEIN-TYPE INEQUALITY FOR RATIONAL FUNCTIONS IN H^2

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Abstract

A Bernstein-type inequality in the standard Hardy space H^2 of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, for rational functions in \mathbb{D} having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, $0 < r < 1$, is considered. The asymptotic sharpness is shown as $n \rightarrow \infty$ and $r \rightarrow 1$.

I. Introduction

First we recall the classical Bernstein inequality for polynomials: we denote by \mathcal{P}_n the class of all polynomials with complex coefficients, of degree n : $P = \sum_{k=0}^n a_k z^k$. Let

$$\|P\|_2 = \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{T}} |P(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} = \left(\sum_{k=0}^n |a_k|^2 \right)^{\frac{1}{2}}.$$

The classical inequality

$$\|P'\|_2 \leq n \|P\|_2 \tag{1}$$

is known as Bernstein's inequality. A great number of refinements and generalizations of (1) have been obtained. See [RaSc, Part III] for an extensive study of that subject. The constant n in (1) is obviously sharp (take $P = z^n$).

Now let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional space defined by

$$K_{B_\sigma} = \mathcal{L}in(k_{\lambda_i} : i = 1 \dots n),$$

where σ is a family of distinct elements of \mathbb{D} , and where $k_\lambda = \frac{1}{1-\bar{\lambda}z}$ is the Szegő kernel associated to λ . An obvious modification allows to generalize the definition of K_{B_σ} in the case where the sequence σ admits multiplicities.

Notice that using the scalar product $(\cdot, \cdot)_{H^2}$ on H^2 , an equivalent description of this space is:

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2,$$

where H^2 stands for the standard Hardy space of the unit disc \mathbb{D} ,

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) < \infty \right\},$$

m being the Lebesgue normalized measure on \mathbb{T} . We notice that the case $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ gives $K_{B_\sigma} = \mathcal{P}_n$. The issue of this paper is to generalize classical Bernstein inequality (1) to the spaces K_{B_σ} . Notice that every rational functions with poles outside of $\overline{\mathbb{D}}$ lies in a space K_{B_σ} . It has already been proved in [Z1] that if $r = \max_j |\lambda_j|$, and $f \in K_{B_\sigma}$, then

$$\|f'\|_{H^2} \leq \frac{5}{2} \frac{n}{1-r} \|f\|_{H^2}. \quad (2)$$

In fact, inequality (2) is a partial case ($p = 2$) of the following K. Dyakonov's result [Dya] (which is, in turn, a generalization of M. Levin's inequality [L] corresponding to the case $p = \infty$): for every p , $1 < p \leq \infty$ there exists a constant $c_p > 0$ such that

$$\|f'\|_{H^p} \leq c_p \|B'\|_\infty \|f\|_{H^p}$$

for all $f \in K_B$, where B is a finite Blaschke product (of order n) and $\|\cdot\|_\infty$ means the norm in $L^\infty(\mathbb{T})$. For our partial case, our proof (in [Z1]) is different and the constant is slightly better. We notice that in general, Bernstein type inequalities have already been the subject of a lot of papers. Among others, Chapter 7 of P. Borwein and T. Erdélyi book's [BoEr] is devoted to such inequalities. This is also the case for A. Baranov's papers [B1], [B2] and [B3], and also of R. A. DeVore and G. G. Lorentz's book [DeLo].

Now a natural question arises: is the constant $\frac{5}{2} \frac{n}{1-r}$ in (2) sharp? For the case $r = 0$ (the classical Bernstein case) we know that this is not the case since constant n is sharp. Below we show that the growth order $\frac{n}{1-r}$ when $n \rightarrow \infty$ and $r \rightarrow 1$ is sharp and give loose bounds for the numerical constants arised.

II. The result

Theorem

Let $n \geq 1$, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , and B_σ the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional subspace of H^2 defined by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Let D be the operator of differentiation on $(K_{B_\sigma}, \|\cdot\|_2)$:

$$D : (K_{B_\sigma}, \|\cdot\|_2) \rightarrow (H^2, \|\cdot\|_2)$$

$$f \mapsto f',$$

where $\|f\|_2 = \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{T}} |f(\zeta)|^2 d\zeta \right)^{\frac{1}{2}}$. For $r \in [0, 1)$ and $n \geq 1$, we set

$$C_{n,r} = \sup \left\{ \|D\|_{K_{B_\sigma} \rightarrow H^2} : 1 \leq \#\sigma \leq n, |\lambda| \leq r \forall \lambda \in \sigma \right\}.$$

(i) If $n = 1$ and $\sigma = \{\lambda\}$, we have

$$\|D\|_{K_{B\sigma} \rightarrow H^2} = |\lambda| \left(\frac{1}{1 - |\lambda|^2} \right)^{\frac{1}{2}}.$$

If $n \geq 2$,

$$a(n, r) \frac{n}{1-r} \leq C_{n,r} \leq A(n, r) \frac{n}{1-r},$$

where

$$a(n, r) \geq \frac{1}{1+r} \left(1 + 5r^4 - \frac{4r^4}{n} - \min\left(\frac{3}{4}, \frac{2}{n}\right) \right)^{\frac{1}{2}},$$

and

$$A(n, r) \leq 1 + r + \frac{1}{\sqrt{n}}.$$

(ii) Moreover, the sequence

$$\left(\frac{1}{n} C_{n,r} \right)_{n \geq 1},$$

is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{n} C_{n,r} = \frac{1+r}{1-r},$$

for all $r \in [0, 1)$.

Proof.

Proof of (i). **The case $n = 1$.** In this case, $K_B = \mathbb{C}e_1$, where

$$e_1 = \frac{(1 - |\lambda|^2)^{\frac{1}{2}}}{(1 - \bar{\lambda}z)}, \quad |\lambda| \leq r,$$

(e_1 being of norm 1 in H^2). Calculating,

$$e'_1 = \frac{\bar{\lambda}(1 - |\lambda|^2)^{\frac{1}{2}}}{(1 - \bar{\lambda}z)^2},$$

and

$$\begin{aligned} \|e'_1\|_{H^2} &= |\lambda| (1 - |\lambda|^2)^{\frac{1}{2}} \left\| \frac{1}{(1 - \bar{\lambda}z)^2} \right\|_{H^2} = \\ &= |\lambda| (1 - |\lambda|^2)^{\frac{1}{2}} \left(\sum_{k \geq 0} (k+1) |\lambda|^{2k} \right)^{\frac{1}{2}} = |\lambda| (1 - |\lambda|^2)^{\frac{1}{2}} \frac{1}{(1 - |\lambda|^2)} = |\lambda| \left(\frac{1}{1 - |\lambda|^2} \right)^{\frac{1}{2}}, \end{aligned}$$

we get

$$\|D\|_{K_{B\sigma}} = |\lambda| \left(\frac{1}{1 - |\lambda|^2} \right)^{\frac{1}{2}}.$$

The case $n \geq 2$. First, we prove the left hand side inequality. Let

$$e_n = \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-1}.$$

Then $e_n \in K_{b_r^n}$ and $\|e_n\|_2 = 1$, (see [N1], Malmquist-Walsh Lemma, p.116). Moreover,

$$\begin{aligned} e'_n &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b'_r b_r^{n-2} = \\ &= -\frac{r}{(1-r^2)^{\frac{1}{2}}} b'_r b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b'_r b_r^{n-2}, \end{aligned}$$

since $b'_r = \frac{r^2-1}{(1-rz)^2}$. Then,

$$e'_n = b'_r \left[-\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-2} \right],$$

and

$$\begin{aligned} \|e'_n\|_2^2 &= \frac{1}{2\pi} \int_{\mathbb{T}} |b'_r(w)| |b'_r(w)| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} (b_r(w))^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} (b_r(w))^{n-2} \right|^2 dm(w) = \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} |b'_r(w)| |b'_r(w)| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r(w) + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} \right|^2 dm(w), \end{aligned}$$

which gives, using the variables $u = b_r(w)$,

$$\|e'_n\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |b'_r(b_r(u))| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rb_r(u)} \right|^2 dm(u).$$

But $1-rb_r = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$ and $b'_r \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2}$. This implies

$$\begin{aligned} \|e'_n\|_2^2 &= \frac{1}{2\pi} \int_{\mathbb{T}} \left| \frac{(1-ru)^2}{1-r^2} \right| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-r^2} (1-ru) \right|^2 dm(u) = \\ &= \frac{1}{(1-r^2)^2} \frac{1}{2\pi} \int_{\mathbb{T}} |(1-ru)(-ru + (n-1)(1-ru))|^2 dm(u). \end{aligned}$$

Without loss of generality we can replace r by $-r$, which gives

$$\|e'_n\|_2 = \frac{1}{(1-r^2)} \|\varphi_n\|_2,$$

where $\varphi_n = (1+rz)(rz + (n-1)(1+rz))$. Expanding, we get

$$\varphi_n = (1+rz)(nrz + (n-1)) =$$

$$\begin{aligned}
&= nrz + (n-1) + nr^2z^2 + (n-1)rz = \\
&= (n-1) + (nr + (n-1)r)z + nr^2z^2,
\end{aligned}$$

and

$$\begin{aligned}
\|e'_n\|_2^2 &= \frac{1}{(1-r^2)^2} ((n-1)^2 + (2n-1)^2r^2 + n^2r^4) = \\
&= \frac{n^2}{(1-r^2)^2} \left(1 + 4r^2 + r^4 - \frac{2}{n} - \frac{4r^2}{n} + \frac{1}{n^2} + \frac{r^2}{n^2} \right) = \\
&= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(1 + 4r^2 + r^4 - \frac{2}{n} - \frac{4r^2}{n} + \frac{1+r^2}{n^2} \right) = \\
&= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(1 + 4r^2 + r^4 - 5r^4 + 5r^4 - \frac{4r^4}{n} + \frac{4r^4}{n} - \frac{4r^2}{n} - \frac{2}{n} + \frac{1+r^2}{n^2} \right) = \\
&= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(4r^2(1-r^2) - \frac{4r^2}{n}(1-r^2) + \frac{1+r^2}{n^2} + 1 + 5r^4 - \frac{4r^4}{n} - \frac{2}{n} \right) = \\
&= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(4r^2(1-r^2)\left(1 - \frac{1}{n}\right) + \frac{1+r^2}{n^2} + 1 + 5r^4 - \frac{4r^4}{n} - \frac{2}{n} \right) \geq \\
&\geq \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \begin{cases} 1 + 5r^4 - \frac{4r^4}{n} - \frac{2}{n} & \text{if } n > 2 \\ \frac{1}{4} + 1 + 5r^4 - \frac{4r^4}{2} - \frac{2}{2} & \text{if } n = 2 \end{cases} \geq \\
&\geq \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(1 + 5r^4 - \frac{4r^4}{n} - \min\left(\frac{3}{4}, \frac{2}{n}\right) \right),
\end{aligned}$$

and

$$a(n, r) \geq \frac{1}{1+r} \left(1 + 5r^4 - \frac{4r^4}{n} - \min\left(\frac{3}{4}, \frac{2}{n}\right) \right)^{\frac{1}{2}},$$

which completes the proof of the left hand side inequality .

We show now the right hand side one. Let σ be a sequence in \mathbb{D} such that $1 \leq \#\sigma \leq n$, $|\lambda| \leq r \forall \lambda \in \sigma$. Using [Z1], Proposition 4.1, we have

$$\begin{aligned}
\|D\|_{K_{B_\sigma} \rightarrow H^2} &\leq \frac{1}{1-r} + \frac{1+r}{1-r}(n-1) + \frac{1}{1-r}\sqrt{n-2} = \\
&= \frac{1}{1-r} (1 + (1+r)(n-1) + \sqrt{n-2}) = \\
&= \frac{1}{1-r} (n(1+r) - r + \sqrt{n-2}) = \frac{n}{1-r} \left(1 + r - \frac{r}{n} + \sqrt{\frac{1}{n} - \frac{2}{n^2}} \right) = \\
&\leq \frac{n}{1-r} \left(1 + r + \sqrt{\frac{1}{n}} \right),
\end{aligned}$$

which gives the result.

Proof of (ii). **Step 1.** We first prove the right-hand-side inequality:

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} C_{n,r} \leq \frac{1+r}{1-r},$$

which becomes obvious since

$$\|D\|_{K_{B_\sigma} \rightarrow H^2} \leq \frac{n}{1-r} \left(1+r + \sqrt{\frac{1}{n}} \right).$$

Step 2. We now prove the left-hand-side inequality:

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{n} C_{n,r} \geq \frac{1+r}{1-r}.$$

More precisely, we show that

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \|D\|_{K_{b_r^n} \rightarrow H^2} \geq \frac{1+r}{1-r}.$$

Let $f \in K_{b_r^n}$. Then,

$$\begin{aligned} f' &= (f, e_1)_{H^2} \frac{r}{(1-rz)} e_1 + \sum_{k=2}^n (k-1) (f, e_k)_{H^2} \frac{b_r'}{b_r} e_k + r \sum_{k=2}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k = \\ &= r \sum_{k=1}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k + \frac{1-r^2}{(1-rz)(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} e_k = \\ &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{3}{2}}}{(1-rz)^2(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} = \\ &= -b_r' \left[\frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{z-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right]. \end{aligned}$$

Now using the change of variables $v = b_r(u)$, we get

$$\begin{aligned} \|f'\|_{H^2}^2 &= \int_{\mathbb{T}} |b_r'(u)| |b_r'(u)| \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{u-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right|^2 du = \\ &= \int_{\mathbb{T}} |b_r'(b_r(v))| \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{b_r(v)-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv. \end{aligned}$$

But

$$b_r - r = \frac{r - z - r(1-rz)}{1-rz} = \frac{z(r^2-1)}{1-rz},$$

and

$$b_r' \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2},$$

which gives

$$\begin{aligned}
& \left\| f' \right\|_{H^2}^2 = \\
&= \frac{1}{1-r^2} \int_{\mathbb{T}} |(1-rv)^2| \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{v(r^2-1)} (1-rv) \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv = \\
&= \frac{1}{(1-r^2)^2} \int_{\mathbb{T}} |(1-rv)^2| \left| r \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} - (1-rv) \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-2} \right|^2 dv = \\
&= \frac{1}{(1-r^2)^2} \int_{\mathbb{T}} \left| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k - (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right|^2 dv.
\end{aligned}$$

But

$$\begin{aligned}
\left\| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 &\leq r(1+r) \left(\sum_{k=0}^{n-1} |(f, e_{k+1})_{H^2}|^2 \right)^{1/2} \leq \\
&\leq r(1+r) \|f\|_2,
\end{aligned}$$

and in particular

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 = 0.$$

Now,

$$\begin{aligned}
& (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k = \\
&= (1-2rv+r^2v^2) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k = \\
&= \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k - 2r \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^{k+1} + r^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^{k+2} = \\
&= \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k - 2r \sum_{k=1}^{n-1} k (f, e_{k+1})_{H^2} v^k + r^2 \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^k = \\
&= (f, e_2)_{H^2} + 2(f, e_3)_{H^2} v + \sum_{k=2}^{n-2} [(k+1) (f, e_{k+2})_{H^2} - 2rk (f, e_{k+1})_{H^2} + r^2(k-1) (f, e_k)_{H^2}] v^k + \\
&-2r [(f, e_2)_{H^2} v + (n-1) (f, e_n)_{H^2} v^{n-1}] + r^2 [(n-2) (f, e_{n-1})_{H^2} v^{n-1} + (n-1) (f, e_n)_{H^2} v^n] = \\
&= (f, e_2)_{H^2} + 2[(f, e_3)_{H^2} - r(f, e_2)_{H^2}] v + \\
&+ \sum_{k=2}^{n-2} [(k+1) (f, e_{k+2})_{H^2} - 2rk (f, e_{k+1})_{H^2} + r^2(k-1) (f, e_k)_{H^2}] v^k + \\
&+ [r^2(n-2) (f, e_{n-1})_{H^2} - 2r(n-1) (f, e_n)_{H^2}] v^{n-1} + r^2(n-1) (f, e_n)_{H^2} v^n.
\end{aligned}$$

Since

$$\begin{aligned} & \frac{1}{n} \frac{1}{(1-r^2)} \left[\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2 + \left\| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 \right] \geq \\ & \geq \frac{1}{n} \left\| f' \right\|_{H^2} \geq \\ & \geq \frac{1}{n} \frac{1}{(1-r^2)} \left[\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2 - \left\| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 \right], \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 = 0,$$

we get that

$$\begin{aligned} & \frac{1}{1+r} \lim_{n \rightarrow \infty} \frac{1}{n} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2 \geq \\ & \geq \lim_{n \rightarrow \infty} \frac{1-r}{n} \left\| f' \right\|_{H^2} \geq \\ & \geq \frac{1}{1+r} \lim_{n \rightarrow \infty} \frac{1}{n} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2. \end{aligned}$$

This gives

$$\lim_{n \rightarrow \infty} \frac{1-r}{n} \left\| f' \right\|_{H^2} = \frac{1}{1+r} \lim_{n \rightarrow \infty} \frac{1}{n} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2.$$

But

$$\begin{aligned} & \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 = |(f, e_2)_{H^2}|^2 + 4|(f, e_3)_{H^2} - r(f, e_2)_{H^2}|^2 + \\ & + |r^2(n-2)(f, e_{n-1})_{H^2} - 2r(n-1)(f, e_n)_{H^2}|^2 + r^4(n-1)^2 |(f, e_n)_{H^2}|^2 + \\ & + \sum_{k=2}^{n-2} |(k+1)(f, e_{k+2})_{H^2} - 2rk(f, e_{k+1})_{H^2} + r^2(k-1)(f, e_k)_{H^2}|^2. \end{aligned}$$

Now, let $s = s_n$ be a sequence of even integers such that

- $\lim_{n \rightarrow \infty} s_n = \infty$ and
- $s_n = o(n)$ as $n \rightarrow \infty$.

Then we consider the following function f in K_{b^n} :

$$f = e_n - e_{n-1} + e_{n-2} - e_{n-3} + \dots + (-1)^k e_{n-k} + \dots + e_{n-s} - e_{n-s-1} + e_{n-s-2} =$$

$$= \sum_{k=0}^{s+2} (-1)^k e_{n-k}.$$

With such an f , we get

$$\begin{aligned} & \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 = \\ & = |r^2(n-2) + 2r(n-1)|^2 + r^4(n-1)^2 + \\ & + \sum_{l=2}^{n-2} |(n-l+1) (f, e_{n-l+2})_{H^2} - 2r(n-l) (f, e_{n-l+1})_{H^2} + r^2(n-l-1) (f, e_{n-l})_{H^2}|^2, \end{aligned}$$

setting the change of index $l = n - k$ in the last sum. This finally gives

$$\begin{aligned} & \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 = \\ & = |r^2(n-2) + 2r(n-1)|^2 + r^4(n-1)^2 + \\ & + \sum_{l=2}^{s+1} |(n-l+1) + 2r(n-l) + r^2(n-l-1)|^2 + \\ & + |(n-s-1) + 2r(n-s-2)|^2 + |n-s-2|^2. \end{aligned}$$

And

$$\begin{aligned} & \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq \\ & \geq |r^2(n-2) + 2r(n-1)|^2 + r^4(n-1)^2 + \\ & + s |(n-s) + 2r(n-s-1) + r^2(n-s-2)|^2 + \\ & + |(n-s-1) + 2r(n-s-2)|^2 + |n-s-2|^2. \end{aligned}$$

In particular,

$$\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq s |(n-s) + 2r(n-s-1) + r^2(n-s-2)|^2.$$

Now, since

$$\|f\|_2^2 = s + 3 = s_n + 3,$$

we get

$$\begin{aligned} & \underline{\lim}_{n \rightarrow \infty} \frac{1}{n^2 \|f\|_2^2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq \\ & \geq \underline{\lim}_{n \rightarrow \infty} \frac{1}{n^2 \|f\|_2^2} (\|f\|_2^2 - 3) |(n-s) + 2r(n-s-1) + r^2(n-s-2)|^2. \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \frac{3}{n^2 s_n^2} |(n-s) + 2r(n-s-1) + r^2(n-s-2)|^2 = 0,$$

we get

$$\begin{aligned} & \underline{\lim}_{n \rightarrow \infty} \frac{1}{n^2 \|f\|_2^2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq \\ & \geq \underline{\lim}_{n \rightarrow \infty} \frac{1}{n^2 s_n^2} s_n^2 |(n-s_n) + 2r(n-s_n-1) + r^2(n-s_n-2)|^2 = \\ & = \lim_{n \rightarrow \infty} \frac{1}{n^2} |(n-s_n) + 2r(n-s_n-1) + r^2(n-s_n-2)|^2 = \\ & = \lim_{n \rightarrow \infty} \frac{1}{n^2} |n + 2rn + r^2n|^2 = (1+r)^4. \end{aligned}$$

We can now conclude that

$$\begin{aligned} & \underline{\lim}_{n \rightarrow \infty} \frac{1-r}{n} \|D\|_{K_{b_r^n} \rightarrow H^2} \geq \underline{\lim}_{n \rightarrow \infty} \frac{1-r}{n} \frac{\|f'\|_2}{\|f\|_2} = \\ & = \frac{1}{1+r} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2 \geq \frac{(1+r)^2}{1+r} = 1+r. \end{aligned}$$

Step 3. Conclusion. Using both **Step 1** and **Step 2**, we get

$$\overline{\lim}_{n \rightarrow \infty} \frac{1-r}{n} C_{n,r} = \underline{\lim}_{n \rightarrow \infty} \frac{1-r}{n} C_{n,r} = 1+r,$$

which means that the sequence $(\frac{1}{n} C_{n,r})_{n \geq 1}$ is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{n} C_{n,r} = \frac{1+r}{1-r}. \quad \square$$

Comments

(a) Using both [BoEr] Th. 7.1.7 p. 324, (or equivalently M. Levin's inequality [L]) and complex interpolation, this Bernstein-type inequality can be extended to H^p spaces, $2 \leq p \leq \infty$. This result is of course already known via K. Dyakonov's result [Dya], but our method could give a better numerical constants c_p in the inequality

$$\|f'\|_{H^p} \leq c_p \|B'\|_{\infty} \|f\|_{H^p}.$$

The case $1 \leq p \leq 2$ can be treated using a result of Baranov ($p = 1$, private communication) and still complex interpolation.

(b) In the same spirit, it is also possible to generalize the above Bernstein-type inequality to the same class of rational functions f in \mathbb{D} , replacing the Hardy space H^2 by Besov spaces $B_{2,2}^s$, $s \in \mathbb{R}$, of all holomorphic functions $f = \sum_{k \geq 0} \hat{f}(k) z^k$ in \mathbb{D} satisfying

$$\|f\|_{B_{2,2}^s} := \left(\sum_{k \geq 0} (k+1)^{2s} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty.$$

The same spaces are also known as Sobolev, or Hardy weighted spaces, or Dirichlet weighted spaces. (In particular, the classical Bergman space corresponds to $s = -\frac{1}{2}$ and the classical Dirichlet space corresponds to $s = \frac{1}{2}$). Using the above approach, one can prove the sharpness of the growth order $\frac{n}{1-r}$ in the corresponding Bernstein-type inequality

$$\left\| f' \right\|_{B_{2,2}^s} \leq c_s \frac{n}{1-r} \|f\|_{B_{2,2}^s}, \quad (3)$$

(at least for integers values of s).

(c) One can also prove an inequality

$$\|f\|_{B_{2,2}^s} \leq c'_s \left(\frac{n}{1-r} \right)^s \|f\|_{H^2}, \quad (4)$$

for $s \geq 0$ and the same class of functions, and show the sharpness of the growth order $\left(\frac{n}{1-r}\right)^s$ (at least for integers values of s). An interesting application of this inequality lies in constrained Nevanlinna-Pick interpolation in weighted Hardy and Bergman spaces, see [Z1] and [Z2] for details.

Notice that both Dyn'kin (for example in [Dyn]), and Pekarskii (in [Pe1], [Pe2] and [PeSt]), studied Bernstein-type inequalities for rational functions in Besov and Sobolev spaces. In particular, they applied such inequalities to inverse theorems of rational approximation. Our approach is different and more constructive. We are able to obtain uniform bounds depending on the geometry of poles of order n , which allows us to obtain estimates which are asymptotically sharp.

(d) The above comments can lead to wonder what happens if we replace Besov spaces $B_{2,2}^s$ by other Banach spaces, for example by W , the Wiener algebra of absolutely convergent Taylor series. In this case, we obtain

$$\|f\|_W \leq c(n, r) \|f\|_{H^2} \quad (5)$$

where $c(n, r) \leq c \left(\frac{n^2}{1-r} \right)^{\frac{1}{2}}$ and c is a numerical constant. We suspect that $\left(\frac{n^2}{1-r} \right)^{\frac{1}{2}}$ is the right growth order of $c(n, r)$. An interesting application of this inequality to an estimate of the norm of the resolvent of an $n \times n$ power-bounded matrix T on a Banach space is given in [Z3]. The inequality (5), above, is deeply linked with the inequality

$$\left\| f' \right\|_{H^1} \leq \gamma n \|f\|_{H^\infty}, \quad (6)$$

through Hardy's inequality :

$$\|f\|_W \leq \pi \left\| f' \right\|_{H^1} + |f(0)|,$$

for all $f \in W$, (see N. Nikolski, [N2] p. 370 8.7.4 -(c)).

Inequality (6) is (shown and) used by LeVeque and Trefethen in [LeTr] with $\gamma = 2$, and later by Spijker in [Sp] with $\gamma = 1$ (an improvement) so as to apply it to the Kreiss Matrix Theorem which asserts the uniform equivalence over all $n \times n$ matrices of power boundedness and a certain resolvent estimate. It allows them to show that the ratio of the constants in these two conditions grows linearly with n .

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