The method of weighted residuals : a general approach to deriving time and frequency-domain numerical methods

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The Method of Weighted Residuals: A General Approach to Deriving Time- and Frequency-Domain Numerical Methods

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The Method of the Weighted Residuals (MWR), sometimes known as the Method of Moments (MoM), has traditionally been applied in the frequency domain and has been shown to be effective and efficient, especially in computing open electromagnetic structure problems. Although it has been extended to the time domain in various forms, it is generally employed to solve integral formulations derived from Maxwell's equations. Therefore, it is often considered to be one type of numerical method that is different from other numerical methods, such as finite-difference methods. However, in this paper we will show that the MWR, or MoM, is not just a method *per se*: it can in fact be a general framework for or approach to unifying or deriving most of the numerical methods developed so far, either in the frequency domain or in the time domain. As a result, all numerical methods can be quite easily understood and can be categorized in one general method, although their conventional derivations may still have their respective advantages. One potential application is that the hybridization of different numerical methods can now be done within a uniform framework. The paper is intended for both beginners and experienced practitioners in the area of numerical electromagnetic modeling.

Keywords: Method of weighted residuals; moment methods; time domain analysis; frequency domain analysis; expansion basis functions; weighting functions; testing functions; FDTD; finite element methods; finite difference frequency domain; method of lines; spectral domain method; mode matching; transmission line matrix method

1. Introduction

Electromagnetic field modeling and simulation have become increasingly popular for accurate design in modern electrical and electronic engineering [1-6], thanks to the drastic advances in computer technology. Such techniques have been applied in many areas, in particular in industrial designs, where the reduction in the number of design cycles or the product's time-to-market has become very critical in a global competitive environment. For instance, in antennas, most of the practical designs are now done with simulations using commercially available software packages, such as *HFSS* by Ansoft [7] and *IE3D* by Zeland [8], before actual prototyping and testing. Due to the fact that analyses based on simple circuit theory are no longer adequate, electromagnetic simulators are increasingly being used to improve design accuracy in high-frequency electronic circuits [4, 6]. In EMC/EMI, electromagnetic field modeling is the norm for assessments and evaluations of shielding effects and emissions [3, 5]. In biomedical engineering, numerical simulations are often required to compute dosimetry in a biological body, since an actual measurement is either difficult or impossible to carry out [3, 5, 9]. For an exhaustive inventory of existing academic or commercial electromagnetic software, the reader may consult [10]. These solvers are mostly based on the use of numerical schemes that will be mentioned in this paper.

In all these applications, various shapes of electromagnetic structures with different material compositions need to be analyzed. In general, analytical methods present the most efficient and desirable solutions. However, they are only applicable to a limited number of canonical structures. To solve for arbitrary structures, numerical methods have to be employed. They usually present approximate solutions with acceptable accuracies.

Many numerical methods have been developed so far [1-6]. These give various approaches to the approximate solutions of the governing electromagnetic-field equations, namely Maxwell's equations. In general, numerical methods can be categorized into two types: frequency-domain and time-domain methods. The frequency-domain methods solve Maxwell's equations in the temporal frequency domain, while the time-domain methods solve the equations in the natural time domain. The frequency-domain meth-

ods include the Finite-Difference Frequency-Domain (FDFD) Method [11], the Finite-Element Method (FEM) [5, 6], and the conventional Method of Moments (MoM) [1]. The time-domain methods include the Finite-Difference Time-Domain (FDTD) method [2], the Transmission-Line-Matrix Method [12, 13], the Time-Domain Finite-Element Method (TD-FEM) ([14, 15] and references therein), and time-domain integral formulations ([16, 17] and references therein).

On the surface, all numerical methods have appeared to be derived from different mathematical bases, and their solution procedures have also appeared to be different from each other. For instance, finite-element methods are based on a variational approach, and finite-difference methods are developed on direct replacement of derivatives with finite differences. Consequently, one may conclude that they are not related to each other. However, this paper intends to show otherwise: all the numerical methods can be generalized or derived with the Method of Weighted Residuals, or Method of Moments. The numerical methods developed so far can therefore be unified in a general framework. As a result, an easy way to understand and apply numerical methods is presented. This also provides some new perspective and possible new ways to develop innovative numerical methods.

The remainder of the paper is organized in the following manner. Section 2 briefly revisits the governing electromagnetic-field equations, Maxwell's equations, since all the numerical methods discussed in this paper directly or indirectly solve Maxwell's equations. Physical explanations of the Fourier transform are then presented to facilitate the understanding of frequency-domain solutions to Maxwell's equations. Section 3 describes the Method of Weighted Residuals (MWR), its solution procedure, and the associated mathematical implications related to numerical issues. These form the framework that generalizes all the numerical methods. Section 4 shows the generalization and derivations of the typical frequency-domain methods within the MWR framework. Section 5 shows the generalization and derivations of the typical time-domain methods. Finally, Section 6 gives the conclusions. The paper is intended for both beginners and experienced practitioners in the area of computational electromagnetics, instead of mathematical theoreticians. Therefore, not much mathematical theory is presented, due to limitations of space. Concepts and ideas are presented in a general form, and many detailed mathematical derivations in strict mathematical terms are omitted, or referred to in previous publications.

2. Maxwell's Equations and their Frequency-Domain Representations

As indicated before, in this section we will briefly describe the time- and frequency-domain Maxwell's equations and their physical interpretations. This is done since all the numerical methods discussed in this paper directly or indirectly solve Maxwell's equations. These equations constitute a useful preamble for the contents of this paper.

2.1 Maxwell's Equations in the Time Domain

Electromagnetic fields are governed by the fundamental Maxwell's equations, which were published at the end of the 19th century [18]. They thus constitute the starting point for finding solutions that should predict fields' behavior.

Mathematically, Maxwell's equations can be expressed in a local form as follows:

$$\begin{aligned}
&\nabla\times\mathbf{E}(\mathbf{r},t)=-\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}, && \text{(Faraday-Maxwell's Law)}\\
&\nabla\times\mathbf{H}(\mathbf{r},t)=\frac{\partial\mathbf{D}(\mathbf{r},t)}{\partial t}+\mathbf{J}(\mathbf{r},t), && \text{(Ampère-Maxwell's Law)}\\
&\nabla\bullet\mathbf{D}(\mathbf{r},t)=\rho(\mathbf{r},t), && \text{(Gauss' Law)}\\
&\nabla\bullet\mathbf{B}(\mathbf{r},t)=0, && \text{(Magnetic Flux Conservation Law)}
\end{aligned} \tag{1}$$

for any observer at rest in a frame of reference who probes, at a given coordinate **r** and at time *t*, field quantities produced by some currents and charges. Here, **E** is the *electrical field intensity*, **D** is the *electric field flux density* or *displacement field*, **H** is the *magnetic field intensity*, and **B** is **the** *magnetic field flux density*. **J** is the *electric surface current density*, and ρ is *electric volume charge density*. In general, these quantities are functions of *time* and the *spatial positions*. The relationships between **E** and **D** and between **H** and **B** are determined by medium properties. For instance, in a lossless, linear, homogeneous, and isotropic medium, they are related through the quantities called "permittivity," ε, and "permeability," μ of the medium:

$$\begin{aligned}
&\mathbf{D}=\varepsilon\mathbf{E},\\
&\mathbf{B}=\mu\mathbf{H}.
\end{aligned} \tag{2}$$

In Equation (1), $\nabla\times$ represents the differential curl operation and $\nabla\bullet$ is the differential divergence operation. The expressions for these operations in different coordinates can be easily found in many electromagnetic textbooks, such as [19]. A close look at the mathematical definition of the above operators shows that Equation (1) is just a translation, at the infinitesimal scale, of the so-called integral form of Maxwell's equations generally introduced in text books. It is recommended that the reader consult [19] for more details.

2.2 Maxwell's Equations in the Frequency Domain

In electrical and electronic engineering, a sinusoidal or time-harmonic signal, as seen on most oscilloscopes, is often dealt with:

$$f(t)=A\cos(\omega t+\varphi). \tag{3}$$

Here, A is the amplitude, ω is the specified angular frequency, and φ is the phase. The above expression can also be expressed as

$$f(t)=A\cos(\omega t+\varphi)=\mathrm{Re}\left[\left(Ae^{j\varphi}\right)e^{j\omega t}\right]=\mathrm{Re}\left(\dot{A}e^{j\omega t}\right), \tag{4}$$

where $\dot{A}=Ae^{j\varphi}$ is a complex number, commonly known as a *phasor* in electrical engineering. $\mathrm{Re}[\cdots]$ is the operation of taking the real part of a complex number.

The above equation indicates that any time-harmonic signal can be expressed as the real part of the product of $e^{j\omega t}$ and a phasor that is formed by the amplitude and phase of the sinusoidal or time-harmonic signal. In other words, a phasor is related to a time-domain signal specified by Equation (4) with a specified angular frequency ω.

Suppose that electromagnetic-field quantities considered now are time-harmonic or sinusoidal in time. That is,

$$\begin{aligned}
\mathbf{D} &= \mathrm{Re}\left(\dot{\mathbf{D}}e^{j\omega t}\right),\\
\mathbf{E} &= \mathrm{Re}\left(\dot{\mathbf{E}}e^{j\omega t}\right),\\
\mathbf{B} &= \mathrm{Re}\left(\dot{\mathbf{B}}e^{j\omega t}\right),\\
\mathbf{H} &= \mathrm{Re}\left(\dot{\mathbf{H}}e^{j\omega t}\right),\\
\mathbf{J} &= \mathrm{Re}\left(\dot{\mathbf{J}}e^{j\omega t}\right),\\
\rho &= \mathrm{Re}\left(\dot{\rho}e^{j\omega t}\right).
\end{aligned} \tag{5}$$

Substitution of the above expressions into Equation (1) then leads to

$$\begin{aligned}
&\mathrm{Re}\left(\nabla\times\dot{\mathbf{E}}\right) = -\mathrm{Re}\left(j\omega\dot{\mathbf{B}}\right),\\
&\mathrm{Re}\left(\nabla\times\dot{\mathbf{H}}\right) = \mathrm{Re}\left(j\omega\dot{\mathbf{D}}+\dot{\mathbf{J}}\right),\\
&\mathrm{Re}\left(\nabla\bullet\dot{\mathbf{D}}\right) = \mathrm{Re}\left(\dot{\rho}\right),\\
&\mathrm{Re}(\nabla\bullet\dot{\mathbf{B}}) = 0.
\end{aligned} \tag{6}$$

By removing the "Re" operation from both sides, the following well-known time-harmonic Maxwell's equations are obtained:

$$\begin{aligned}
&\nabla\times\dot{\mathbf{E}} = -j\omega\dot{\mathbf{B}},\\
&\nabla\times\dot{\mathbf{H}} = j\omega\dot{\mathbf{D}}+\dot{\mathbf{J}},\\
&\nabla\bullet\dot{\mathbf{D}} = \dot{\rho},\\
&\nabla\bullet\dot{\mathbf{B}} = 0.
\end{aligned} \tag{7}$$

The above time-harmonic Maxwell's Equations (7) can also be obtained by applying the Fourier transform to the time-domain Maxwell's Equations (1). Therefore, the *physical implications of the Fourier transform* are now clear: the Fourier transform essentially transforms Maxwell's equations to the equations for time-harmonic solutions, where phasor solutions or complex field quantities give the magnitudes and phases of real-time field components in the sinusoidal form of Equation (3). This concept has presented some difficulty for beginners or students who try to relate phasors to real time-domain signals.

In most cases, the dot sign above each symbol in Equation (7) is normally omitted for convenience in notation. Even with the omission, the quantities in Equation (7) can be easily identified as phasor quantities, because of the appearance of the $j\omega$ term in the equations.

3. The Method of Weighted Residuals (MWR)

The *Method of Weighted Residuals* (MWR) encompasses a general family of approximation methods used for the solution of ordinary and partial differential equations [20]. In computational electromagnetics, the method is often referred to as the Method of Moments (MoM) [1]. An overview of the method can be presented as follows. Consider an equation to be solved as

$$Lu(t,\mathbf{r}) = f(t,\mathbf{r}),\quad \mathbf{r}\in\Omega, \tag{8}$$

where L is a differential or integral operator (e.g., $L = a\dfrac{d}{dt} - \dfrac{d}{dz}$; $L = \dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}-\mu\varepsilon\dfrac{\partial^2}{\partial t^2}$ in electromagnetics); $u(t,\mathbf{r})$ is the unknown function of time, t, and spatial coordinate, $\mathbf{r}$, which is to be solved for; $f(t,\mathbf{r})$ is the known force function; Ω is the domain over which the operator L applies; and $\partial\Omega$ is the boundary of Ω.

The Method of Weighted Residuals (MWR) proceeds with the two-step operations as described below.

3.1 Step 1: Expansions

A set of basis functions $\{\phi_1(t,\mathbf{r}),\ \phi_2(t,\mathbf{r}),...,\phi_N(t,\mathbf{r})\}$ is first chosen. The unknown function or solution, $u(t,\mathbf{r})$, is then approximated or expanded in terms of the basis functions:

$$u(t,\mathbf{r}) \approx U(t,\mathbf{r}) = \sum_{j=1}^{N} c_j\phi_j(t,\mathbf{r}), \tag{9}$$

where the expansion coefficients, c_j, are to be determined. The choice of the $\{\phi_1(t,\mathbf{r}),\ \phi_2(t,\mathbf{r}),...,\phi_N(t,\mathbf{r})\}$ is not random [21]. They are required to form a function space within Ω, or to be in the domain of operator L. This space is called the *trial space*. The approximating function, $U(t,\mathbf{r})$, is called the *trial* function, which has to satisfy boundary conditions. In addition, $L\phi_j(t,\mathbf{r})_{j=1,2,...,N}$ should form a complete set in the domain of operator L.

Once the basis or expansion functions are chosen, one can proceed to the second MWR step, which allows one to determine the unknown expansion coefficients, c_j.

3.2 Step 2: Error Testing or Weighted-Residual Minimization

In most cases, the trial function, $U(t,\mathbf{r})$, is not the same as the exact solution, $u(t,\mathbf{r})$. It will thus introduce an error into the

original Equation (8), called the *residual*, and denoted as R. This can be found as

$$R(t,\mathbf{r}) = LU(t,\mathbf{r}) - f(t,\mathbf{r}). \quad \mathbf{r} \in \Omega \tag{10}$$

The MWR seeks to minimize $R(t,\mathbf{r})$ by forcing it to zero in a weighted-average sense over an entire domain, or, to use a more-accurate term, in the inner-product space. Mathematically, this is expressed as

$$\langle R(t,\mathbf{r}), W_i(t,\mathbf{r}) \rangle = \int_t \int_\Omega R(t,\mathbf{r}) W_i(t,\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega. \tag{11}$$

Here, the $W_i(t,\mathbf{r})_{i=1,2,...,M}$ are a *pre-selected* set of mutually independent weighting, or testing, functions. By introducing Equations (9) and (10) into Equation (11), and using the linear properties of both operator L and the inner product, one can obtain an algebraic-equation system that allows a solution of the expansion coefficients, c_j:

$$\sum_{j=1}^{N} c_j \langle W_i, L\phi_j \rangle = \langle W_i, f \rangle|_{i=1 \text{ to } M}. \tag{12}$$

Note that the number of weighting functions, M, can be larger than the number of basis functions, N. The solution can then be sought by using least-square techniques, such as the pseudo-inverse operator [22].

Like the selection of basis functions, selection of the weighting functions is also not random. The weighting functions have to be in the range of the operator L, or, more generally, in the domain of the adjoint operator [23]. Once they are selected appropriately, together with the right choice of the basis functions, Equation (11) will force residual R go to zero, and the expansion of Equation (9) will converge to the exact solution. This is analogous to the fact that a vector that has zero dot (or inner) products with all of the three base axial vectors, $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$, has to be zero.

Note that normally M (i.e., the number of weighting functions) and N (i.e., the number of basis functions) may need to go to infinity in order to make Equation (9) converge to the exact solution. In practice, since M and N are finite, MWR implementation will usually yield some truncation error.

A few typical choices of weighting functions are often used and have special appeal. For instance,

- $W_i = L\phi_i \Rightarrow$ Least-square procedure,
- $W_i = \phi_i \Rightarrow$ Galerkin's procedure,
- $W_i = \delta(\mathbf{r}_i) \Rightarrow$ Point-matching or collocation procedure,

where $\delta(\mathbf{r}_i)$ is the Dirac impulse function. Note that because of the Dirac function's properties, the point-matching procedure corresponds to enforce the residual of Equation (10) to be zero at discrete coordinates $\mathbf{r}_i$.

In short, two pre-selections must be made to apply MWR. The first is a choice of a trial space with concomitant definition of basis functions $\{\phi_1(t,\mathbf{r}), \phi_2(t,\mathbf{r}),...,\phi_N(t,\mathbf{r})\}$. The second is a selection of weighting or testing functions, $\{W_i(t,\mathbf{r})\}$. Each of these choices is important, and has to meet the conditions as outlined in [21, 23]. Otherwise, the MWR procedure may experience convergence problems.

In the following paragraph, a simple example is shown for a successful application of the MWR.

Consider the simple problem below, which corresponds to a one-dimensional Poisson's equation:

$$\frac{d^2u(x)}{dx^2} + x = 0, \qquad u(x=0) = u(x=1) = 0. \tag{13}$$

The forcing function, $f = -x$, corresponds to a linear electric charge density. The exact solution can be found to be $u(x) = \frac{1}{6}x(1-x^2)$, which is the electrostatic potential produced by the charge density between two grounded metal plates located at $x=0$ and $x=1$.

Now the MWR is applied. In the first step, the following basis functions are chosen:

$$\phi_j(x) = \sin(j\pi x)_{j=1,2,...N}. \tag{14}$$

These all satisfy the boundary condition $u(x=0) = u(x=1) = 0$. The approximating expanded solution can then be written as

$$u(x) \approx U(x) = \sum_{j=0}^{N} c_j \phi_j = \sum_{j=1}^{N} c_j \sin(j\pi x). \tag{15}$$

It satisfies the boundary conditions because each basis function satisfies the boundary condition. From Equation (10), the residual is

$$R(x) = \frac{dU}{dx^2} + x = \frac{d}{dx^2} \sum_{j=1}^{N} c_j \sin(j\pi x) + x. \tag{16}$$

In the second MWR step, the testing or weighting function is chosen to be the same as the expansion basis function:

$$W_i(x) = \sin(i\pi x)_{i=1,2,...M}. \tag{17}$$

Note that this choice corresponds to a Galerkin procedure. The residual is minimized in the weighted-average sense, that is, with Equation (11):

$$\begin{aligned} 0 = \langle R, w_i \rangle &= \int_{x=0}^{x=1} R w_i dx \\ &= \int_{x=0}^{x=1} \left[\frac{d^2}{dx^2} \sum_{m=0}^{M} c_j \sin(j\pi x) + x \right] \sin(i\pi x) dx \\ &= -\frac{(i\pi)^2}{2} c_i + \frac{(-1)^i}{i\pi} = 0 \end{aligned} \tag{18}$$

$$\Rightarrow c_i = (-1)^i \frac{2}{(i\pi)^3} \text{ or } c_j = (-1)^j \frac{2}{(j\pi)^3}.$$

The approximate solution is then given by Equation (15):

$$u(x) \approx U(x) = \sum_{j=1}^{N} (-1)^j \frac{2}{(j\pi)^3} \sin(j\pi x). \tag{19}$$

Figure 1 shows comparisons between the exact solution, $u(x)$, and the approximate solution, $U(x)$, obtained with the MWR method. As can be seen, when the number of expansion terms is as low as $N = 3$, the error with the MWR solution is small but visible. However, when the number of terms reaches $N = 10$, the error is invisible and negligible. Therefore, the MWR technique is effective in obtaining approximate solutions with acceptable accuracies. Note that as indicated before, this is one of the successful examples of the MWR applications. In some cases, also depending on the properties of the operator *L*, errors may be large even with an increased number of basis functions or weighting functions. Therefore, cautions need to be taken to ensure the convergence of the solutions; readers are referred to [21, 23] for more theory and details on the issue.

In the following sections, we will show how the MWR can be applied to derive the numerical methods developed so far for solving electromagnetic problems.

4. MWR Derivations of Frequency-Domain Numerical Methods

Many numerical methods have been developed to solve the frequency-domain Maxwell's equations, Equation (7). These include the Frequency-Domain Finite-Difference Method, the Finite-Element Method, the spectral-domain approach, the Method of Lines, mode matching, and the transverse resonance method. A summary of these methods was well presented in [6]. We will discuss how each of these can be derived with the MWR method in the following paragraphs.

4.1 The Frequency-Domain Finite-Difference Method

Although not as popular as their time-domain counterparts, many forms of the Frequency-Domain Finite-Difference Methods have been developed and applied in the past decades [6, 11]. The common procedure of these methods is to first set up the differential equations that are either the frequency-domain Maxwell's equations themselves, or formulations derived from Maxwell's equations (e.g., Helmholtz's equations). By then replacing differential operators with their finite-difference counterparts, finite-difference frequency-domain formulations are obtained. In this paper, the case presented in [11] is taken as an example. The curl's frequency-domain Maxwell's equations of Equation (7) for a linear isotropic medium are first expressed in the Cartesian coordinates as follows:

$$\begin{aligned}
j\omega\varepsilon_0\varepsilon_r E_x &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial x} - J_x, \\
j\omega\varepsilon_0\varepsilon_r E_y &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - J_y, \\
j\omega\varepsilon_0\varepsilon_r E_z &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_z, \\
j\omega\varepsilon_0\varepsilon_r H_x &= \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}, \\
j\omega\varepsilon_0\varepsilon_r H_y &= \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}, \\
j\omega\varepsilon_0\varepsilon_r E_z &= \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}.
\end{aligned} \tag{20}$$

Now the solution domain is discretized according to Yee's distributed grid [24]. The discrete position is denoted as

$$\left(x = i_x \Delta x,\ y = i_y \Delta y,\ z = i_z \Delta z\right) \triangleq \left(i_x, i_y, i_z\right), \tag{21}$$

and any function, *f*, at the grid position is denoted as

$$f\left(x = i_x \Delta x,\ y = i_y \Delta y,\ z = i_z \Delta z\right) \triangleq f\,|_{i_x, i_y, i_z}. \tag{22}$$

Here Δx, Δy, and Δz are the space increments along the x, y, and z directions, respectively. i_x, i_y, and i_z are the grid-position indices.

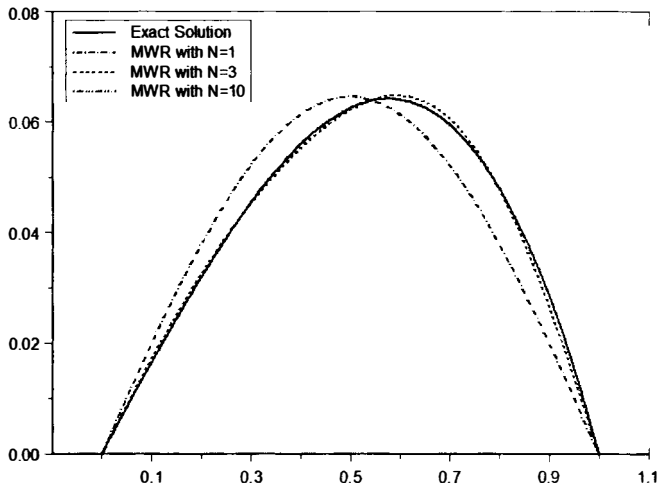

Figure 1. Comparisons between the exact solution and the MWR solutions.

In the conventional FDFD derivation, a Taylor's series expansion is applied [20]. A spatial derivative is approximated by its central finite-difference counterpart, as shown below:

$$\frac{\partial f(\xi)}{\partial \xi} \approx \frac{f\left(\xi+\frac{\Delta\xi}{2}\right)-f\left(\xi-\frac{\Delta\xi}{2}\right)}{\Delta\xi}. \tag{23}$$

The first equation of Equation (20) can them be approximated as

$$\begin{aligned} & j\omega\varepsilon_0\varepsilon_r E_x\big|^{n+1}_{i_x+\frac{1}{2},i_y,i_z} \\ & = \frac{H_z\big|_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z} - H_z\big|_{i_x+\frac{1}{2},i_y-\frac{1}{2},i_z}}{\Delta y} \\ & \quad - \frac{H_y\big|_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}} - H_y\big|_{i_x+\frac{1}{2},i_y,i_z-\frac{1}{2}}}{\Delta z} \\ & \quad - J_x\big|_{i_x+\frac{1}{2},i_y,i_z} = 0, \end{aligned} \tag{24}$$

or

$$\begin{aligned} & j\omega\varepsilon_0\varepsilon_r\Delta y\Delta z E_x\big|^{n+1}_{i_x+\frac{1}{2},i_y,i_z} \\ & \quad -\Delta z H_z\big|_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z} + \Delta z H_z\big|_{i_x+\frac{1}{2},i_y-\frac{1}{2},i_z} \\ & \quad +\Delta y H_y\big|_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}} - \Delta z H_y\big|_{i_x+\frac{1}{2},i_y,i_z-\frac{1}{2}} \\ & \quad = -J_x\big|_{i_x+\frac{1}{2},i_y,i_z}. \end{aligned} \tag{25}$$

By applying the same procedure to the other equations of Equation (20), the finite-difference frequency-domain formulations can be obtained. They are basically a system of linear equations with the field quantities at grid positions being the unknowns to be found.

We will now show that Equations (24) or (25) can also be derived from Equations (7) or (20) by using the MWR technique. To do so, consider a triangle (or rooftop) pulse function, T:

$$T(\xi,\xi_o,\Delta\xi) = \begin{cases} 1-\dfrac{|\xi-\xi_0|}{\Delta\xi} & \text{when } \dfrac{|\xi-\xi_0|}{\Delta\xi} \le 1.0 \\ 0 & \text{when } \dfrac{|\xi-\xi_0|}{\Delta\xi} > 1.0 \end{cases}. \tag{26}$$

The graphical representation of the function is shown in Figure 2. As seen, the function has a duration or width of $2\Delta\xi$, with zero values at both ends. It is centered at ξ_0 with the value of unity. The derivative of T is a double pulse function, as represented by Equation (27) and shown graphically in Figure 3.

$$\frac{d}{d\xi}T(\xi,\xi_0,\Delta\xi) = \begin{Bmatrix} 1 & \text{when } 0 \le (\xi_0-\xi) \le \Delta\xi \\ -1 & \text{when } 0 \le (\xi-\xi_0) \le \Delta\xi \\ 1 & \text{when } |\xi-\xi_0| > \Delta\xi \end{Bmatrix}. \tag{27}$$

Following the MWR method, the field components in Equation (20) are first expanded in terms of the triangular functions in the following manner:

$$\begin{aligned} E_x \approx & \sum_{i_x,i_y,i_z} E_x\big|_{i_x+\frac{1}{2},i_y,i_z} T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right] \\ & T\left(y,i_y\Delta y,\Delta y\right)T\left(z,i_z\Delta z,\Delta z\right) \\ H_y \approx & \sum_{i_x,i_y,i_z} H_y\big|_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}} T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right] \\ & T\left(y,i_y\Delta y,\Delta y\right)T\left[z,\left(i_z+\frac{1}{2}\right)\Delta z,\Delta z\right] \\ H_z \approx & \sum_{i_x,i_y,i_z} H_z\big|_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z} T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right] \\ & T\left[y,\left(i_y+\frac{1}{2}\right)\Delta y,\Delta y\right]T\left(z,i_z\Delta z,\Delta z\right) \end{aligned} \tag{28}$$

where

$$E_x\big|_{i_x+\frac{1}{2},i_y,i_z},$$

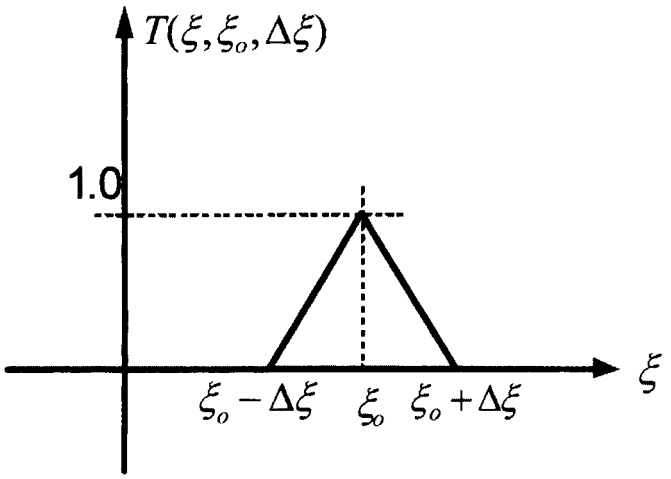

Figure 2. An illustration of the triangle function, T, defined by Equation (26).

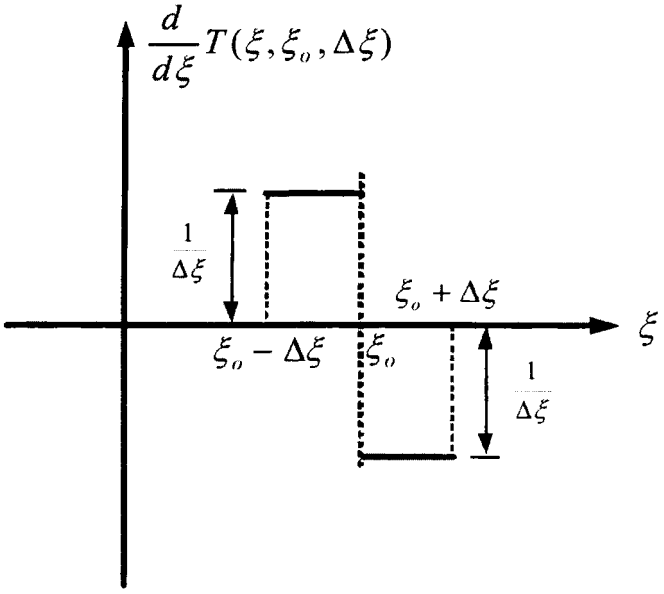

Figure 3. The derivative of the function $T(\xi,\xi_0,\Delta\xi)$.

$H_y|_{i_x+\frac{1}{2}, i_y, i_z+\frac{1}{2}}$,

and

$H_z|_{i_x+\frac{1}{2}, i_y+\frac{1}{2}, i_z}$

are the expansion coefficients. The above expansion functions are substituted into the first equation of Equation (20), and the residual is weighted with the Dirac impulse function,

$$\delta\left[x-\left(i_x+\frac{1}{2}\right)\Delta x\right]\delta\left(y-i_y\Delta y\right)\delta\left(z-i_z\Delta z\right)$$

(point-matching procedure). In other words, Equation (28) is substituted into the first equation of Equation (20), and then the residual minimization is performed with the following inner product:

$$\int_{z=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty}(...)\delta\left[x-\left(i_x+\frac{1}{2}\right)\Delta x\right]\delta\left(y-i_y\Delta y\right)\delta\left(z-i_z\Delta z\right)dxdydz \tag{29}$$

The resulting equation is exactly the same as Equation (24). Similar derivations can be performed for other FDFD equations, and the same conclusion can be obtained. In other words, the FDFD formulations are derivable from the MWR.

Careful examination of Equation (28) leads to the following observations:

1. Due to the fact that the value of the triangular function is unity at its central point, the expansion coefficients happen to be the electric field and magnetic field at grid points. That is, E_x at $x=\left(i_x+\frac{1}{2}\right)\Delta x$, $y=i_y\Delta y$, $z=i_z\Delta z$ is equal to coefficient $E_x|_{i_x+\frac{1}{2}, i_y, i_z}$; H_y at $x=\left(i_x+\frac{1}{2}\right)\Delta x$, $y=i_y\Delta y$, $z=\left(i_z+\frac{1}{2}\right)\Delta z$ is equal to coefficient $H_y|_{i_x+\frac{1}{2}, i_y, i_z+\frac{1}{2}}$; and H_z at $x=\left(i_x+\frac{1}{2}\right)\Delta x$, $y=\left(i_y+\frac{1}{2}\right)\Delta y$, $z=i_z\Delta z$ is equal to coefficient $H_z|_{i_x+\frac{1}{2}, i_y+\frac{1}{2}, i_z}$. This is only applicable to the case studied here, and may not be true for other FDFD-based methods.

2. The field value in between two neighboring grid points is equal to a linear interpolation of the field values at two neighboring grid points.

There are other types of FDFD algorithms. They can all be derived from an MWR procedure in a way similar to that described above, by selecting appropriate basis and weighting functions.

4.2 Frequency-Domain Finite-Element Method (FD-FEM)

The Finite-Element Method was introduced to solve electromagnetic problems as early as 1969 by Silvester [25]. Since then, it has been improved and expanded extensively in its theory and applications [5, 6, 9]. Commercially available software packages, such as *HFSS* and Agilent *EMDS*, are FD-FEM-based electromagnetic simulators [7, 26]. Developments of the FEM methods have mainly been made along two lines: one with direct applications of the MWR method, or the Method of Moments (e.g., [6, 14]); and the other with the variational approach [5, 6, 27]. The former is about derivations with the MWR method, and is therefore conformal with the claim of this paper: they are therefore repeated here. In the following paragraphs, the latter case, namely the variational approach, is discussed.

In the variational approach [27], a functional expression is first derived from the original Equation (8):

$$F(u)=\langle Lu,u\rangle-\langle u,f\rangle-\langle f,u\rangle, \tag{30}$$

where $\langle\bullet\rangle$ indicates an inner product. In the function space of interest in engineering, an inner product is usually defined by the integral

$$\langle u,f\rangle=\int_{\Omega}uf^{*}d\Omega, \tag{31}$$

with the asterisk being the complex conjugate, and Ω being the volume (or surface or line segment) of interest. $F(u)$ thus depends fundamentally on operator L with its boundary conditions, and on the force function, f. It can be shown that u is the solution of the original problem of Equation (8) if it makes F stationary. Except for a few cases, the solution for stationary F cannot be found analytically. u is thus expanded in terms of pre-selected basis functions. By substituting the expanded function into the functional, the later becomes a function of the expansion coefficients. It can then be made stationary by taking the derivatives with respect to the expansion coefficients (variational form), and setting all of them simultaneously to zero. This generates a set of linear equations for the expansion coefficients.

Consider again the one-dimensional Poisson's equation, Equation (13), as an example. Based on the variational theory, the corresponding functional for Poisson's operator accompanied by the specific boundary condition can be found as

$$F\left[u(x)\right]=\int_{x=0}^{1}\left[\frac{d^2u(x)}{dx^2}+2x\right]u(x)dx,$$
$$u(x=0)=u(x=1)=0. \tag{32}$$

It is interesting to note that for electrostatic problems in general, a functional such as Equation (32) is proportional to the energy stored in the structure. In our case, the straightforward relationship of the variational approach with physics can thus be explained: The exact solution, $u(x)$, for Equation (13), which is the potential

distribution, occurs only when the energy stored, given by $F[u(x)]$, is the minimum.

With the variational approach, $u(x)$ is expanded with a set of basis functions, which are again selected as

$$\phi_j(x) = \sin(j\pi x)_{j=1,2,...N}. \tag{33}$$

The expanded function is then

$$u(x) \approx U(x) = \sum_{j=0}^{N} c_j \phi_j = \sum_{j=1}^{N} c_j \sin(j\pi x). \tag{34}$$

Substitution of Equation (34) into Equation (32) gives

$$\begin{aligned} F[u(x)] &\approx \int_{x=0}^{1} \left[\frac{d^2U(x)}{dx^2} + 2x \right] U(x)\,dx \\ &= \int_{x=0}^{1} \left[\frac{d^2}{dx^2} \sum_{j=1}^{N} c_j \sin(j\pi x) + 2x \right] \sum_{j=1}^{N} c_j \sin(j\pi x)\,dx \\ &= F(c_1, c_2, ..., c_N). \end{aligned} \tag{35}$$

To make the above functional stationary, one needs to enforce $\frac{\partial F}{\partial c_i} = 0$, $i = 1,2,...N$, which leads to

$$\left.\frac{\partial F}{\partial c_i}\right|_{i=1,2,...N} \approx \int_{x=0}^{1} \frac{\partial}{\partial c_i} \left\{ \left[\frac{d^2}{dx^2} \sum_{j=1}^{N} c_j \sin(j\pi x) + 2x \right] \sum_{j=1}^{N} c_j \sin(j\pi x) \right\} dx \Bigg|_{i=1,2,...,N} = 0 \tag{36}$$

It is not difficult to shown that Equation (36) leads to the same solution as Equation (19) (which was derived with the MWR method). Consequently, the variational method generates the same linear system of equations as the system produced by the MWR method with Galerkin's procedure. In other words, the variational approach is equivalent to an MWR procedure.

4.3 Method of Lines

The method of lines is a technique that is semi-analytical and suitable for planar transmission-line structures [6, 28]. It solves the frequency-domain wave equations derived from the frequency-domain Maxwell's equations, Equation (7), for the longitudinal field components. In the method, the wave equation for a longitudinal component, φ, to be solved is

$$\begin{aligned} &\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} + k^2 \varphi = 0, \\ &k^2 = \omega^2 \mu_0 \mu_r \varepsilon_0 \varepsilon_r. \end{aligned} \tag{37}$$

By assuming the field variation along the z direction to be $e^{-jk_z z}$, with k_z being the longitudinal propagation constant, Equation (37) can be reduced to

$$\begin{aligned} &\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \left(k^2 - k_z^2\right)\varphi = 0, \\ &k^2 = \omega^2 \mu_0 \mu_r \varepsilon_0 \varepsilon_r. \end{aligned} \tag{38}$$

With the method of lines, the discretization is carried out along the x direction, and the central finite-difference approximation is used to replace $\frac{\partial^2 \varphi}{\partial x^2}$. The result is a system of ordinary differential equations, with the notation of $\varphi(x = i_x \Delta x, y) = \varphi|_{i_x}$:

$$\frac{\varphi|_{i_x+1} + \varphi|_{i_x-1} - 2\varphi|_{i_x}}{(\Delta x)^2} + \frac{d^2 \varphi|_{i_x}}{dy^2} + \left(k^2 - k_z^2\right)\varphi|_{i_x} = 0. \tag{39}$$

By denoting $\Psi = \left[\cdots, \varphi|_{i_x}, \cdots \right]^T$, the above equation can be written in a succinct matrix form:

$$\frac{d^2\Psi}{dy^2} + \left[\left(k^2 - k_z^2\right)\mathbf{I} - \frac{1}{(\Delta x)^2}\mathbf{P} \right]\Psi = 0, \tag{40}$$

where $\mathbf{I}$ is the unit matrix and $\mathbf{P}$ is the coefficient matrix that results from the finite-difference implementation. Equation (40) is then solved analytically, for it is only a one-dimensional differential equation.

Equation (40) can also be derived by the MWR method. To do so, a sub-sectional basis function is defined:

$$\phi = \phi(x, x_j, \Delta x) = \begin{cases} -\frac{15}{8}\left(\frac{x - x_j}{\Delta x}\right)^8 + \frac{19}{4}\left(\frac{x - x_j}{\Delta x}\right)^6 \\ -\frac{23}{8}\left(\frac{x - x_j}{\Delta x}\right)^4 - \left(\frac{x - x_j}{\Delta x}\right)^2 + 1, & \left|\frac{x - x_j}{\Delta x}\right| \le 1.0 \\ 0, & \left|\frac{x - x_j}{\Delta x}\right| > 1.0 \end{cases} \tag{41}$$

First, φ is expanded in terms of the above basis function:

$$\varphi(x, y) \approx \sum_{i_x} \varphi|_{i_x} \phi(x, i_x \Delta x, \Delta x). \tag{42}$$

By substituting this into Equation (38) and weighting the residual with the following Dirac impulse functions,

$$w(x, i_x \Delta x, \Delta x) = \delta(x - i_x \Delta x), \tag{43}$$

the following equation can be obtained:

$$\frac{\varphi|_{i_x+1} + \varphi|_{i_x-1} - 2\varphi|_{i_x}}{(\Delta x)^2} + \left[\frac{d^2}{dy^2} + \left(k^2 - k_z^2\right) \right] \varphi|_{i_x} = 0, \tag{45}$$

which is the same as Equation (39). Therefore, the method of lines can be derived from a point-matching MWR procedure. In addition, from Equation (41), one can see that the $\varphi|_{i_x}$ are basically the field values at the grid point $x = i_x \Delta x$.

4.4 Spectral-Domain Method

The spectral-domain method is a specialized numerical method efficiently designed for transmission-line structures of planar types, such as microstrip lines ([6] and references therein). Starting with Helmholtz's Equation (37) in the frequency domain, which is derived from Maxwell's Equations (7), an integral equation is first developed where the current densities or charges on the metal strips are the unknown quantities to be solved for. The MWR technique is then applied, and a system of linear equations is obtained for the expansion coefficients. The core of the spectral-domain method is that the elements of the coefficient matrix of the linear-equation system are efficiently found through the use of the Green's functions in the spectral (or spatial-frequency) domain, rather than directly in the spatial domain. An excellent description of the derivation of the spectral-domain method from an MWR procedure was presented in [29], except that the MoM name was used there, instead of MWR. Therefore, the spectral-domain method falls within the framework of the MWR.

4.5 Mode Matching

Mode matching is one of the most frequently used methods for solving boundary-value problems of waveguide structures ([6] and references therein). It is generally formulated with the boundary conditions at the interfaces of two junctions or regions. The first step of mode matching is that unknown fields in the individual regions are expanded in terms of their respective modes, which are the solutions to the frequency-domain Maxwell's Equations (7). Expanded field components are then matched at the interfaces of two adjacent regions to form a set of linear equations for the expansion coefficients. Such a process can be considered to be exactly the same as that of the MWR method. The explanations are below.

Suppose that the problem to be solved is

$$\begin{array}{ll} \text{Region \#1} & \text{Region \#2} \\ L_1 u_1 - f_1 = 0 & L_2 u_2 - f_2 = 0, \end{array} \tag{46}$$

$$T_1(u_1) - T_2(u_2) = 0 \text{ at interface } \mathbf{r} = \mathbf{r}_{int}, \tag{47}$$

where L_1 and L_2 are two differential or integral operators applicable to Region #1 and Region #2 only; f_1 and f_2 are two source functions existing in Regions #1 and #2, respectively; and T_1 and T_2 are the linear operators that dictate the interface or boundary conditions for solutions u_1 and u_2 in Regions #1 and #2, respectively. Note that the equations in Equation (46) are Maxwell's equations or their derivatives applicable to Regions #1 and #2. The interface conditions, Equation (47), are essentially the simplified form of Maxwell's equations after they are applied to the interface. Therefore, *Equation (47) is also Maxwell's equation but in a different form applicable to an interface.*

In the method of mode matching, u_1 and u_2 are expanded in terms of the known mode functions that satisfy Equation (46) and boundary conditions in Region #1 and Region #2, respectively:

$$u_1 = \sum_j c_{1j} \phi_{1j}, \tag{48}$$

$$u_2 = \sum_j c_{2j} \phi_{2j}. \tag{49}$$

Substitution of these into the interface conditions of Equation (47) leads to

$$\sum_j c_{1j} T_1(\phi_{1j}) = \sum_j c_{2j} T_2(\phi_{2j}), \tag{50}$$

which is then solved by applying integration over the interface with a particular mode function, ϕ_{1m} and ϕ_{2m}.

$$\begin{aligned} \sum_j c_{1j} \int T_1(\phi_{1j}) \phi_{1m} ds &= \sum_j c_{2j} \int T_2(\phi_{2m}) \phi_{1j} ds, \\ \sum_j c_{1j} \int T_1(\phi_{1j}) \phi_{2m} ds &= \sum_j c_{2j} \int T_2(\phi_{2j}) \phi_{2m} ds. \end{aligned} \tag{51}$$

With different values of m, Equation (51) forms a system of linear equations that can be solved for mode-expansion coefficients c_{1j} and c_{2j}.

A simple example of mode matching is the characterization of a step-change waveguide, shown in Figure 4. The fields in Regions 1 and 2 have to satisfy Maxwell's equations, represented by Equations (46) or (7). Suppose a known TE_{10} wave is incident in Region #1, with $E_y^{inc} = \phi_{11} e^{-\Gamma_{11} z}$ and $H_x^{inc} = -Y_{11} \phi_{11} e^{-\Gamma_{11} z}$ (where ϕ_{11}, Γ_{11}, and Y_{11} are defined below). By the theory of

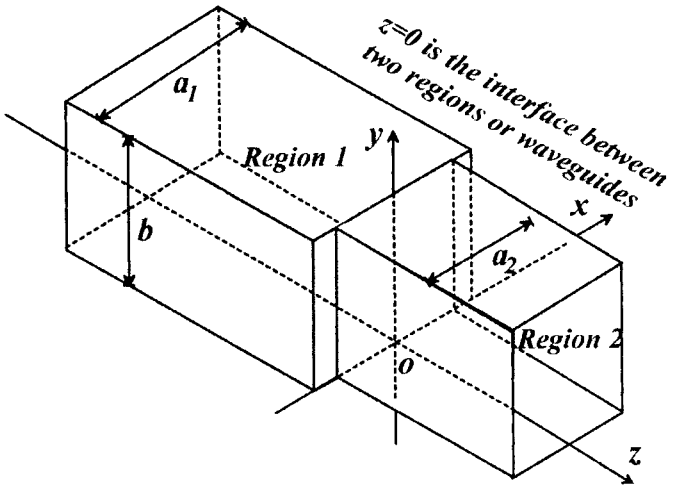

Figure 4. A step discontinuity where the width of a waveguide is changed from a_1 to a_2.

guided waves [30], the field quantities that satisfy Maxwell's equations and boundary conditions in waveguide Regions 1 and 2, respectively, can then be expressed as

$$\begin{aligned} E_{y1} &= E_y^{inc} + E_y^{reflected} \\ &= \left(e^{-\Gamma_{11}z} + c_{11}e^{+\Gamma_{11}z}\right)\phi_{11} + \sum_{j=2}^{\infty} c_{1j}\phi_{1j}e^{+\Gamma_{1j}z}, \\ H_{x1} &= H_x^{inc} + E_x^{reflected} \\ &= -Y_{11}\left(e^{-\Gamma_{11}z} - c_{11}e^{+\Gamma_{11}z}\right)\phi_{11} + \sum_{j=2}^{\infty} c_{1j}Y_{1j}\phi_{1j}e^{+\Gamma_{1j}z}, \end{aligned} \tag{52}$$

for

$$-\frac{a_1}{2} \le x \le \frac{a_1}{2},\ 0 \le y \le b \text{ and } z \le 0,$$

and

$$\begin{aligned} E_{y2} &= \sum_{j=1}^{\infty} c_{2j}\phi_{2j}e^{-\Gamma_{2j}z} \\ H_{x2} &= -\sum_{j=1}^{\infty} c_{2j}Y_{2j}\phi_{2j}e^{-\Gamma_{2j}z} \end{aligned} \tag{53}$$

for

$$-\frac{a_2}{2} \le x \le \frac{a_2}{2},\ 0 \le y \le b \text{ and } z > 0.$$

Here, $\phi_{lj} = \sqrt{\frac{2}{a_l}}\sin\left(\frac{j\pi x}{a_l}\right)e^{\pm\Gamma_{lj}z}$ is the field-distribution function of mode j in region l $(l=1,2)$. Γ_{lj} is the known propagation constant of mode j in region l, while Y_{lj} is the known wave impedance of mode j.

The interface condition requires that at $z=0$,

$$E_{y1} = E_{y2} \text{ and } H_{x1} = H_{x2}\ . \tag{54}$$

That is,

$$\begin{aligned} (1+c_{11})\phi_{11} + \sum_{j=2}^{\infty} c_{1j}\phi_{1j} &= \sum_{j=1}^{\infty} c_{2j}\phi_{2j}, \\ -Y_{11}(1-c_{11})\phi_{11} + \sum_{j=2}^{\infty} c_{1j}Y_{1j}\phi_{1j} &= -\sum_{j=1}^{\infty} c_{2j}Y_{2j}\phi_{2j}, \end{aligned} \tag{55}$$

which corresponds to Equation (50).

By multiplying the above equations with the mode function $\phi_{lj} = \sqrt{\frac{2}{a_l}}\sin\left(\frac{j\pi x}{a_l}\right)$ and then integrating over the interface, one can obtain

$$\begin{aligned} &(1+c_{11})\int_{x=-\frac{a_1}{2}}^{+\frac{a_1}{2}} \phi_{11}\phi_{1m}dx + \sum_{j=2}^{\infty} c_{1j}\int_{x=-\frac{a_1}{2}}^{+\frac{a_1}{2}} \phi_{1j}\phi_{1m}dx \\ &\quad = \sum_{j=1}^{\infty} c_{2j}\int_{x=-\frac{a_2}{2}}^{+\frac{a_2}{2}} \phi_{2j}\phi_{1m}dx, \\ &-Y_{11}(1-c_{11})\int_{x=-\frac{a_1}{2}}^{+\frac{a_1}{2}} \phi_{11}\phi_{1m}dx + \sum_{j=2}^{\infty} c_{1j}Y_{1j}\int_{x=-\frac{a_1}{2}}^{+\frac{a_1}{2}} \phi_{1j}\phi_{1m}dx \\ &\quad = -\sum_{j=1}^{\infty} c_{2j}Y_{2j}\int_{x=-\frac{a_2}{2}}^{+\frac{a_2}{2}} \phi_{2j}\phi_{1m}dx, \end{aligned} \tag{56}$$

and

$$\begin{aligned} &(1+c_{11})\int_{x=-\frac{a_2}{2}}^{+\frac{a_2}{2}} \phi_{11}\phi_{2m}dx + \sum_{j=2}^{\infty} c_{1j}\int_{x=-\frac{a_2}{2}}^{+\frac{a_2}{2}} \phi_{1j}\phi_{2m}dx \\ &\quad = \sum_{j=1}^{\infty} c_{2j}\int_{x=-\frac{a_2}{2}}^{+\frac{a_2}{2}} \phi_{2j}\phi_{2m}dx, \\ &-Y_{11}(1-c_{11})\int_{x=-\frac{a_2}{2}}^{+\frac{a_2}{2}} \phi_{11}\phi_{2m}dx + \sum_{j=2}^{\infty} c_{1j}Y_{1j}\int_{x=-\frac{a_2}{2}}^{+\frac{a_2}{2}} \phi_{1j}\phi_{2m}dx \\ &\quad = -\sum_{j=1}^{\infty} c_{2j}Y_{2j}\int_{x=-\frac{a_2}{2}}^{+\frac{a_2}{2}} \phi_{2j}\phi_{2m}dx, \end{aligned} \tag{57}$$

where $m = 1,2,...\infty$. The above equations, corresponding to Equation (51), are the linear equation system that allows for the solutions of c_{lj}. As a result, the field solutions are found.

One can now show that the above process can be identified as the MWR process in the following manner. The equation to be solved is the interface condition, Equations (47) or (54), which is the simplified form of Maxwell's equations applied to an interface, while the field components have to be conditioned with Equation (46). The unknown field components are then expanded with Equations (48) and (49), or Equations (52) or (53) in our example. They are substituted into Equations (47) or (54), and the residual is tested with both $\int(...)\phi_{1j}\delta(\mathbf{r}-\mathbf{r}_{int})dv$ and $\int(...)\phi_{2j}\delta(\mathbf{r}-\mathbf{r}_{int})dv$. The result is Equation (51), or Equations (56) and (57) in our example. Consequently, mode matching can be considered to be an application of the MWR method with a point-matching procedure.

4.6 Other Frequency-Domain Methods

There are some other frequency-domain methods. In a process similar to what has been described above, they can also be derived or generalized with the MWR method. However, due to limitations of space, they are not presented here.

5. MWR Derivations of Time-domain Numerical Methods

Many time-domain methods have been developed that solve Maxwell's equations, Equation (1), or formulations derived from

them directly in the time domain [1-6]. The solution process then involves four dimensions: three spatial coordinates and one temporal coordinate. Most of the time-domain numerical methods can be categorized into three types: (1) the finite-difference-based time-domain methods, (2) the finite-element time-domain method, and (3) the time-domain integral-integration method. These have become powerful techniques in analyzing electromagnetic structures that include antennas, RF/microwave circuits, and electromagnetic capability/electromagnetic interference apparatus. An increasing number of simulators based on these techniques are being developed commercially, e.g., *XFDTD* by Remcom [31], *Microwave Studio* by CST [32], *CONCERTO FDTD* by Vector Fields [33], and *SEMCAD X* by Schmid and Partner Engineering [34]. We will show how these three types of methods can be derived from or generalized with the MWR in the following paragraphs.

5.1 The Finite-Difference-Based Time-Domain Methods

The finite-difference-based time-domain methods solve Maxwell's equations, Equation (1), directly, by replacing differential operations with their finite-difference correspondences. Consider a simple medium of permeability μ and permittivity ε. Maxwell's Equations (1) in Cartesian coordinates can be expressed with Equation (2) as

$$\begin{aligned}
\frac{\partial E_x}{\partial t} &= \frac{1}{\varepsilon_0\varepsilon_r}\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}-J_x\right),\\
\frac{\partial E_y}{\partial t} &= \frac{1}{\varepsilon_0\varepsilon_r}\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}-J_y\right),\\
\frac{\partial E_z}{\partial t} &= \frac{1}{\varepsilon_0\varepsilon_r}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}-J_z\right),\\
\frac{\partial H_x}{\partial t} &= \frac{1}{\mu_0\mu_r}\left(\frac{\partial E_y}{\partial z}-\frac{\partial E_z}{\partial y}\right),\\
\frac{\partial H_y}{\partial t} &= \frac{1}{\mu_0\mu_r}\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right),\\
\frac{\partial H_z}{\partial t} &= \frac{1}{\mu_0\mu_r}\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}\right).
\end{aligned} \tag{58}$$

Among various numerical schemes that solve the above equations, the most well known are the conventional Finite-Difference Time-Domain (FDTD) Method using Yee's scheme [2, 24], the Transmission-Line-Matrix (TLM) Method [12, 13, 35], the Multi-Resolution Time-Domain (MRTD) Method [36], the Pseudo-Spectral Time-Domain (PSTD) Method [37]; and unconditionally stable methods using the Crank-Nicolson scheme [38], alternating-direction implicit (ADI) schemes [39, 40], or weighted Laguerre polynomials [41]. In the following sections, the MWR derivations of these methods are shown or explained.

5.1.1 MWR Derivation of the Conventional Finite-Difference Time-domain (FDTD) Method

As shown in the last section, frequency-domain finite-difference algorithms can be made equivalent to expansion and residual minimization with the MWR method. In a similar way, it can be shown that the finite-difference-based time-domain methods are also derivable from an MWR procedure. The difference is that the field quantities are now also expanded with basis functions in time, and the residual minimization is also carried out in time with temporal weighting functions. For instance, consider the first equation of Equation (58). One can use the triangular function defined by Equation (26) and illustrated in Figure 2 to expand electromagnetic-field components in the spatial x, y, and z directions and in time t:

$$\begin{aligned}
E_x &\approx \sum_{i_x,i_y,i_z,n} E_x\Big|^n_{i_x+\frac{1}{2},i_y,i_z}\, T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right]\\
&\qquad T\left(y,i_y\Delta y,\Delta y\right)T\left(z,i_z\Delta z,\Delta z\right)T\left(t,n\Delta t,\Delta t\right),\\
E_y &\approx \sum_{i_x,i_y,i_z,n} E_y\Big|^n_{i_x,i_y+\frac{1}{2},i_z}\, T\left(x,i_x\Delta x,\Delta x\right)\\
&\qquad T\left[y,\left(i_y+\frac{1}{2}\right)\Delta y,\Delta y\right]T\left(z,i_z\Delta z,\Delta z\right)T\left(t,n\Delta t,\Delta t\right),\\
E_z &\approx \sum_{i_x,i_y,i_z,n} E_z\Big|^n_{i_x,i_y,i_z+\frac{1}{2}}\, T\left(x,i_x\Delta x,\Delta x\right)\\
&\qquad T\left(y,i_y\Delta y,\Delta y\right)T\left[z,\left(i_z+\frac{1}{2}\right)\Delta z,\Delta z\right]T\left(t,n\Delta t,\Delta t\right),
\end{aligned} \tag{59}$$

$$\begin{aligned}
H_x &\approx \sum_{i_x,i_y,i_z,n} H_y\Big|^{n+\frac{1}{2}}_{i_x,i_y+\frac{1}{2},i_z+\frac{1}{2}}\, T\left(x,i_x\Delta x,\Delta x\right)\\
&\quad T\left[y,\left(i_y+\frac{1}{2}\right)\Delta y,\Delta y\right]T\left[z,\left(i_z+\frac{1}{2}\right)\Delta z,\Delta z\right]T\left[t,\left(n+\frac{1}{2}\right)\Delta t,\Delta t\right],\\
H_y &\approx \sum_{i_x,i_y,i_z,n} H_y\Big|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}}\, T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right]\\
&\qquad T\left(y,i_y\Delta y,\Delta y\right)T\left[z,\left(i_z+\frac{1}{2}\right)\Delta z,\Delta z\right]T\left[t,\left(n+\frac{1}{2}\right)\Delta t,\Delta t\right],\\
H_z &\approx \sum_{i_x,i_y,i_z,n} H_z\Big|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}\, T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right]\\
&\qquad T\left[y,\left(i_y+\frac{1}{2}\right)\Delta y,\Delta y\right]T\left(z,i_z\Delta z,\Delta z\right)T\left[t,\left(n+\frac{1}{2}\right)\Delta t,\Delta t\right].
\end{aligned}$$

Then, by substituting Equation (59) into the first equation of Equation (58) and performing the residual minimization with the inner product of

$$\int_{t=-\infty}^{\infty}\int_{z=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}(\,)\delta\left[x-\left(i_x+\frac{1}{2}\right)\Delta x\right]\delta\left(y-i_y\Delta y\right)$$
$$\delta\left(z-i_z\Delta z\right)\delta\left[t-\left(n+\frac{1}{2}\right)\Delta t\right]dxdydzdt\,,$$

one can obtain the following equation:

$$\varepsilon\frac{E_x\big|_{i_x+\frac{1}{2},i_y,i_z}^{n+1}-E_x\big|_{i_x+\frac{1}{2},i_y,i_z}^{n}}{\Delta t}$$
$$=\frac{H_z\big|_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}^{n+\frac{1}{2}}-H_z\big|_{i_x+\frac{1}{2},i_y-\frac{1}{2},i_z}^{n+\frac{1}{2}}}{\Delta y}$$
$$-\frac{H_y\big|_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}}^{n+\frac{1}{2}}-H_y\big|_{i_x+\frac{1}{2},i_y,i_z-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z}-J_x\big|_{i_x+\frac{1}{2},i_y,i_z}^{n+\frac{1}{2}}=0 \tag{60}$$

At the same time, if one follows the conventional FDTD method by replacing the differential operators with their central finite-difference counterparts [2, 3], one can obtain the conventional FDTD formulation that is exactly the same as Equation (60). For the remaining five equations of Equation (59), the same result can be obtained by following a similar derivation process. As a result, we conclude that the conventional FDTD method is derivable from an MWR procedure.

Although the MWR derivation of the FDTD method may seem more tedious than the direct Yee's derivation, it does offer several advantages in terms of understanding the FDTD method. Based on Equation (59), one can see the following:

1. The quantities computed in the conventional FDTD of Yee's scheme are essentially the expansion coefficients of the approximate field solutions.

2. Due to the fact that the value of the triangular function is unity at its central point, the expansion coefficients happen to be the electric field and magnetic field at the grid points that are the central points of the triangular basis functions. For instance, $\tilde{E}_x$ at

$$x=\left(i_x+\frac{1}{2}\right)\Delta x,\ y=i_y\Delta y,\ z=i_z\Delta z,\ t=\left(n+\frac{1}{2}\right)\Delta t$$

is equal to coefficient $E_x\big|_{i_x+\frac{1}{2},i_y,i_z}^{n}$ and the same for other expansion coefficients. However, this is only applicable to the FDTD case studied here, and may not be true for other FDFD-based methods.

3. The basis functions are the sub-sectional shifted triangular functions that have central points at the specific grid points. However, these grid points (or central points) associated with field quantities are all a half step away from each other in space. For instance, the central point for E_x is

$$x=\left(i_x+\frac{1}{2}\right)\Delta x,\ y=i_y\Delta y,\ z=i_z\Delta z$$

and that for H_z is

$$x=\left(i_x+\frac{1}{2}\right)\Delta x,\ y=\left(i_y+\frac{1}{2}\right)\Delta y,\ z=i_z\Delta z\,;$$

they are $\frac{1}{2}\Delta y$ away from each other in the y direction. On the other hand, in time, the central points for all three electrical field components are the same, at $t=n\Delta t$, whereas the central points for all three magnetic field components are the same at $t=\left(n+\frac{1}{2}\right)\Delta t$. However, in between the electric and magnetic field components, there is a shift of half a time step, $\frac{1}{2}\Delta t$, for the central points of the basis functions.

4. Although the central points of the field components are defined at different grid positions, the overall approximated solution, Equation (59), for each field component is defined continuously across the whole solution domain (rather than only at the discrete points). For instance, E_x is expanded with the sub-sectional triangular functions centered spatially at the grid points $x=\left(i_x+\frac{1}{2}\right)\Delta x,\ y=i_x\Delta y,\ z=i_z\Delta z$, and temporally at $t=n\Delta t$. However, it has continuous values in the whole solution domain, in both space and time.

5. Because of the use of the sub-sectional triangular functions, the field value in between two neighboring grid points (i.e., the central points of the sub-sectional triangular functions) is equal to a linear interpolation of the field values at two neighboring grid points.

Other details about deriving the finite-difference-based time-domain methods from the MWR procedure have been presented in [42] by Chen and Luo.

Note that there is the extended variation of the FDTD method, the Finite Integral Technique (FIT) [43]. By following a similar process, it is not difficult to show that the Finite Integral Technique is derivable from an MWR procedure. In addition, it is worth mentioning that derivation of the FDTD with the Method of Moments or MWR was also reported in [44]. Although the basis and weighting functions used there differed from those in this paper, [44] reconfirmed what was presented in this paper: the FDTD method can be derived from an MWR procedure.

5.1.2 MWR Derivation of the MRTD Method

The MRTD method was first used for solving electromagnetic structure problems in [36]. The basic development procedure follows that of the MWR or Method of Moments. Similarly to the MWR derivation of the conventional FDTD method in time, the triangular function defined by Equation (26) is used as the basis function, and a Dirac delta function is used as the weighting func-

tion for the residual minimization. However, unlike the derivation of the FDTD method in space, the basis and weighting functions are the wavelet functions.

A wavelet is a specially designed mathematical function: it can divide a continuous-solution function into different frequency components, and then match each component with a resolution offered by a particular wavelet [45]. In general, the wavelets can be developed with scaled and translated copies (known as "daughter wavelets") of a finite-length or fast-decaying oscillating waveform (known as the "mother wavelet"). The scaling wavelets are used to model smoothly-varying components of a solution function, while the translated or mother wavelets are used to capture strongly varying components, or even singularities of the solution function. When the wavelets are applied to electromagnetic modeling, this means that the field expansions can have two terms, one with scaling wavelets and another with translating (or mother) wavelets.

For an easy explanation, let's consider the first of Maxwell's Equations (59) with $\mathbf{J} = 0$ in Cartesian coordinates:

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right). \tag{61}$$

Also, translated wavelets or high-resolution wavelets are only required and applied in the y direction. Then, the three field components for Equation (61) are expanded as follows:

$$\begin{aligned} E_x \approx \tilde{E}_x = \sum_{i_x,i_y,i_z,n=-\infty}^{+\infty} & \left\{ E_{\phi_x}|^{n}_{i_x+\frac{1}{2},i_y,i_z}\, \phi\left(y, i_y\Delta y, \Delta y\right) \right. \\ & \left. + E_{\psi_x}|^{n}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}\, \psi\left[y, \left(i_y+\frac{1}{2}\right)\Delta y, \Delta y\right] \right\} \\ & \phi\left[x, \left(i_x+\frac{1}{2}\right)\Delta x, \Delta x\right] \phi\left(z, i_z\Delta z, \Delta z\right) T\left(t, n\Delta t, \Delta t\right) \end{aligned}$$

$$\begin{aligned} H_y \approx \tilde{H}_y = \sum_{i_x,i_y,i_z,n=-\infty}^{+\infty} & \left\{ H_{\phi_y}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}}\, \phi\left(y, i_y\Delta y, \Delta y\right) \right. \\ & \left. + H_{\psi_y}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z+\frac{1}{2}}\, \psi\left[y, \left(i_y+\frac{1}{2}\right)\Delta y, \Delta y\right] \right\} \\ & \phi\left[x, \left(i_x+\frac{1}{2}\right)\Delta x, \Delta x\right] \phi\left[z, (i_z+\frac{1}{2})\Delta z, \Delta z\right] T\left[t, \left(n+\frac{1}{2}\right)\Delta t, \Delta t\right] \end{aligned}$$

$$\begin{aligned} H_z \approx \tilde{H}_z = \sum_{i_x,i_y,i_z,n=-\infty}^{+\infty} & \left\{ H_{\phi_z}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}\, \phi\left[y, \left(i_y+\frac{1}{2}\right)\Delta y, \Delta y\right] \right. \\ & \left. + H_{\psi_z}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y,i_z}\, \psi\left(y, i_y\Delta y, \Delta y\right) \right\} \phi\left[x, \left(i_x+\frac{1}{2}\right)\Delta x, \Delta x\right] \\ & \phi\left(z, i_z\Delta z, \Delta z\right) T\left[t, \left(n+\frac{1}{2}\right)\Delta t, \Delta t\right] \end{aligned} \tag{62}$$

where $\phi\left(\xi, \xi_0, \Delta\xi\right) = \phi\left(\frac{\xi - \xi_0}{\Delta\xi}\right)$ is a scaling wavelet function such as the spline Battle-Lemarie scaling function [36] for modeling spatially smoothly varying components of the fields, $\psi\left(\xi, \xi_0, \Delta\xi\right) = \psi\left(\frac{\xi - \xi_0}{\Delta\xi}\right)$ is a wavelet function such as the cubic Battle-Lemarie function [36] for modeling spatially fast-changing components of the fields, and $T\left(t, t_0, \Delta t\right)$ is the triangular function defined by Equation (26) for expanding field components in time.

$E_{\phi_x}|^{n}_{i_x+\frac{1}{2},i_y,i_z}$,

$E_{\psi_x}|^{n}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}$,

$H_{\phi_y}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}}$,

$H_{\psi_y}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z+\frac{1}{2}}$,

$H_{\phi_z}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}$,

and

$H_{\psi_z}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y,i_z}$

are the expansion coefficients.

By substituting Equation (62) into Maxwell's Equation (61), and performing the residual minimization with the same scaling and wavelet functions as the weighting functions in space and the Dirac impulse function as the weighting functions in time,

$$\begin{aligned} & \phi\left[x, \left(i_x+\frac{1}{2}\right)\Delta x, \Delta x\right] \phi\left(y, i_y\Delta y, \Delta y\right) \phi\left(z, i_z\Delta z, \Delta z\right) \delta\left[t - \left(n+\frac{1}{2}\right)\Delta t\right] \\ & \psi\left[x, \left(i_x+\frac{1}{2}\right)\Delta x, \Delta x\right] \psi\left[y, \left(i_y+\frac{1}{2}\right)\Delta y, \Delta y\right] \\ & \psi\left(z, i_z\Delta z, \Delta z\right) \delta\left[t - \left(n+\frac{1}{2}\right)\Delta t\right], \end{aligned} \tag{63}$$

one can obtain

$$\begin{aligned} & \frac{\varepsilon}{\Delta t}\left(E_{\varphi_x}|^{n+1}_{i_x+\frac{1}{2},i_y,i_z} - E_{\varphi_x}|^{n}_{i_x+\frac{1}{2},i_y,i_z}\right) \\ & = \frac{1}{\Delta y}\sum_i a(i) H_{\varphi_z}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y+i+\frac{1}{2},i_z} + \frac{1}{\Delta y}\sum_i c(i) H_{\psi_z}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y+i,i_z} \\ & - \frac{1}{\Delta z}\sum_i a(i) H_{\varphi_y}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y,i_z+i+\frac{1}{2}} \frac{\varepsilon}{\Delta t}\left(E_{\psi_x}|^{n+1}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z} - E_{\psi_x}|^{n}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}\right) \\ & = \frac{1}{\Delta y}\sum_i c(i) H_{\varphi_z}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y+i+\frac{1}{2},i_z} + \frac{1}{\Delta y}\sum_i b(i) H_{\psi_z}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y+i+1,i_z} \\ & - \frac{1}{\Delta z}\sum_i a(i) H_{\psi_y}|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z+i+\frac{1}{2}}, \end{aligned} \tag{64}$$

where $a(i)$, $b(i)$, and $c(i)$ are constant coefficients that are obtained through the residual minimization process with the weighting functions of Equation (63). For instance, if the spline Battle-Lemarie function is used as the scaling function and the cubic Battle-Lemarie function is used as the wavelet function for expansion and residual minimization in space, $a(i)$, $b(i)$, and $c(i)$ can be found as presented in Table 1. The equations in

Equation (64) are exactly the same as the MRTD Equations (21) and (22) of [36]. The derivations of other MRTD equations can be performed in a very similar manner. As a result, the MRTD method is shown to be derivable from an MWR procedure.

It should be pointed out here that unlike the conventional FDTD method, the expansion coefficients in the MRTD method, e.g.,

$$E_{\phi_x}\Big|^{n}_{i_x+\frac{1}{2},\, i_y,\, i_z},$$

$$E_{\psi_x}\Big|^{n}_{i_x+\frac{1}{2},\, i_y+\frac{1}{2},\, i_z},$$

$$H_{\phi_y}\Big|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},\, i_y,\, i_z+\frac{1}{2}},$$

$$H_{\psi_y}\Big|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},\, i_y+\frac{1}{2},\, i_z+\frac{1}{2}},$$

$$H_{\phi_z}\Big|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},\, i_y+\frac{1}{2},\, i_z},$$

and

$$H_{\psi_z}\Big|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},\, i_y,\, i_z},$$

are not equal to the field values at the central points of the wavelet functions. This is because the values of the wavelet functions are not unity at the central points.

5.1.3 MWR Derivation of Pseudo-Spectral Time-Domain (PSTD) Method

The pseudo-spectral time-domain method is a method where the Fourier transform (or another transform) is applied in the spatial domain to resolve the field components [37]. It can be considered to be a high-order FDTD technique, where the field components at a grid point are related the field quantities at all the grid points in a computational domain. In the following sections, we will show how it is derived from an MWR procedure.

Again, for simplicity, consider Equation (61), one of Maxwell's equations. By following the MWR procedure, we can expand the field quantities in the following manner:

Table 1. The coefficients $a(i)$, $b(i)$, and $c(i)$ found for Equation (64).

i	$a(i)$	$b(i)$	$c(i)$
0	1.2918462	2.4725388	0.0
1	−0.1560761	0.9562282	−0.0465973
2	0.0596391	0.1660587	0.054539
3	−0.0293099	0.0939244	−0.0369996
4	0.0153716	0.0031413	0.0205745
5	−0.0081892	0.0134936	−0.0111530
6	0.0043788	−0.0028589	0.0059769
7	−0.0023433	0.0027788	−0.0032026
8	0.0012542	−0.0011295	0.001714
9			−0.0009177

$$E_x \approx \frac{1}{N_x N_y N_z} \sum_{l_x,l_y,l_z,n} \left[\sum_{i_x,i_y,i_z} E_x\Big|^{n}_{i_x\, i_y\, i_z} e^{-j\left(\frac{2\pi}{N_x} l_x i_x + \frac{2\pi}{N_y} l_y i_y + \frac{2\pi}{N_z} l_z i_z\right)} \right] e^{j\left(\frac{2\pi}{N_x\Delta x} l_x x + \frac{2\pi}{N_y\Delta y} l_y y + \frac{2\pi}{N_z\Delta z} l_z z\right)} T\left(t, n\Delta t, \Delta t\right),$$

$$H_y \approx \frac{1}{N_x N_y N_z} \sum_{l_x,l_y,l_z,n} \left[\sum_{i_x,i_y,i_z} H_y\Big|^{n+\frac{1}{2}}_{i_x\, i_y\, i_z} e^{-j\left(\frac{2\pi}{N_x} l_x i_x + \frac{2\pi}{N_y} l_y i_y + \frac{2\pi}{N_z} l_z i_z\right)} \right] e^{j\left(\frac{2\pi}{N_x\Delta x} l_x x + \frac{2\pi}{N_y\Delta y} l_y y + \frac{2\pi}{N_z\Delta z} l_z z\right)} T\left[t, \left(n+\frac{1}{2}\right)\Delta t, \Delta t\right],$$

$$H_z \approx \frac{1}{N_x N_y N_z} \sum_{l_x,l_y,l_z,n} \left[\sum_{i_x,i_y,i_z} H_z\Big|^{n+\frac{1}{2}}_{i_x\, i_y\, i_z} e^{-j\left(\frac{2\pi}{N_x} l_x i_x + \frac{2\pi}{N_y} l_y i_y + \frac{2\pi}{N_z} l_z i_z\right)} \right] e^{j\left(\frac{2\pi}{N_x\Delta x} l_x x + \frac{2\pi}{N_y\Delta y} l_y y + \frac{2\pi}{N_z\Delta z} l_z z\right)} T\left[t, \left(n+\frac{1}{2}\right)\Delta t, \Delta t\right], \quad (65)$$

where N_x, N_y, and N_z are the total number of the discrete grid points pre-selected in the x, y, and z directions, respectively. $E_x\Big|^{n}_{i_x\, i_y\, i_z}$, $H_y\Big|^{n+\frac{1}{2}}_{i_x\, i_y\, i_z}$, and $H_z\Big|^{n+\frac{1}{2}}_{i_x\, i_y\, i_z}$ are the expansion coefficients.

From Equation (65), one can see that the basis functions in space are like discrete Fourier transform functions:

$$\left[\sum_{i_x,i_y,i_z} E_x\Big|^{n}_{i_x\, i_y\, i_z} e^{-j\left(\frac{2\pi}{N_x} l_x i_x + \frac{2\pi}{N_y} l_y i_y + \frac{2\pi}{N_z} l_z i_z\right)} \right] e^{j\left(\frac{2\pi}{N_x\Delta x} l_x x + \frac{2\pi}{N_y\Delta y} l_y y + \frac{2\pi}{N_z\Delta z} l_z z\right)} T\left(t, n\Delta t, \Delta t\right),$$

$$\left[\sum_{i_x,i_y,i_z} H_y\Big|_{i_x\ i_y\ i_z}^{n+\frac{1}{2}} e^{-j\left(\frac{2\pi}{N_x}l_x i_x+\frac{2\pi}{N_y}l_y i_y+\frac{2\pi}{N_z}l_z i_z\right)}\right]$$

$$e^{j\left(\frac{2\pi}{N_x\Delta x}l_x x+\frac{2\pi}{N_y\Delta y}l_y y+\frac{2\pi}{N_z\Delta z}l_z z\right)} T\left[t,\left(n+\frac{1}{2}\right)\Delta t,\Delta t\right],$$

$$\left[\sum_{i_x,i_y,i_z} H_z\Big|_{i_x\ i_y\ i_z}^{n+\frac{1}{2}} e^{-j\left(\frac{2\pi}{N_x}l_x i_x+\frac{2\pi}{N_y}l_y i_y+\frac{2\pi}{N_z}l_z i_z\right)}\right]$$

$$e^{j\left(\frac{2\pi}{N_x\Delta x}l_x x+\frac{2\pi}{N_y\Delta y}l_y y+\frac{2\pi}{N_z\Delta z}l_z z\right)} T\left[t,\left(n+\frac{1}{2}\right)\Delta t,\Delta t\right]. \tag{66}$$

However, in time the triangular function defined by Equation (26) is still used as the basis function.

By substituting the approximated expansion of Equation (65) into Equation (61) and then performing the residual minimization with the Dirac impulse function,

$$\delta\left(y-i_y\Delta y\right)\delta\left(z-i_z\Delta z\right)\delta\left[t-\left(n+\frac{1}{2}\right)\Delta t\right],$$

as the weighting function, one can obtain

$$\varepsilon\frac{E_x\Big|_{i_x,i_y,i_z}^{n+1}-E_x\Big|_{i_x,i_y,i_z}^{n}}{\Delta t}$$

$$= FFT^{-1}\left[jk_y FFT\left(H_z\Big|_{i_x,i_y,i_z}^{n+\frac{1}{2}}\right)\right]-FFT^{-1}\left[jk_z FFT\left(H_y\Big|_{i_x,i_y,i_z}^{n+\frac{1}{2}}\right)\right]$$

$$-J_x\Big|_{i_x,i_y,i_z}^{n+\frac{1}{2}}$$

$$=0 \tag{67}$$

where FFT and FFT^{-1} represent the fast Fourier transform and its inverse operation.

Equation (67) is exactly the same equation as that presented in [37] for the original PSTD development. The same conclusion can be obtained for the other Maxwell's equations. As a result, we conclude that the PSTD method, although appearing unrelated to the MWR, is derivable from an MWR procedure.

By examining the expansion of Equation (65) and using the orthogonality of the exponential functions, it is not difficult to see that like the conventional FDTD method, the expansion coefficients also happen to be the field values at the grid points. For instance, $E_x\Big|_{i_x\ i_y\ i_z}^{n}$ is equal to E_x at the grid point of $x=i_x\Delta x$, $y=i_y\Delta y$, $z=i_z\Delta z$ and $t=n\Delta t$, while $H_y\Big|_{i_x\ i_y\ i_z}^{n+\frac{1}{2}}$ is equal to H_y at the same grid point but at $t=\left(n+\frac{1}{2}\right)\Delta t$. In other words, all these special grid points for electric and magnetic fields are collocated at the same grid points; but in time, there is still a difference of half-a-step between electric and magnetic fields.

5.1.4 MWR Derivation of Crank-Nicolson (CN) and the ADI- FDTD Methods

The CN FDTD is one of the implicit FDTD methods that are unconditionally stable [38]. Although it is not widely used due to its low computational efficiency, attempts have been made to make its computations more practical [46, 47]. In spite of the fact that the CN-FDTD method was derived traditionally by using the finite-difference and averaging schemes, it is shown below that it is derivable with MWR.

Similar to the MWR procedure described before, consider again Equation (61), one of Maxwell's equations. First, the field quantities are expanded with the triangle function as defined by Equation (26):

$$E_x \approx \sum_{i_x,i_y,i_z,n} E_x\Big|_{i_x+\frac{1}{2},i_y,i_z}^{n} T[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x]$$

$$T\left(y,i_y\Delta y,\Delta y\right)T\left(z,i_z\Delta z,\Delta z\right)T\left(t,n\Delta t,\Delta t\right),$$

$$H_y \approx \sum_{i_x,i_y,i_z,n} H_y\Big|_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}}^{n} T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right]$$

$$T\left(y,i_y\Delta y,\Delta y\right)T\left[z,(i_z+\frac{1}{2})\Delta z,\Delta z\right]T\left(t,n\Delta t,\Delta t\right),$$

$$H_z \approx \sum_{i_x,i_y,i_z,n} H_z\Big|_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}^{n} T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right]$$

$$T\left[y,\left(i_y+\frac{1}{2}\right)\Delta y,\Delta y\right]T\left(z,i_z\Delta z,\Delta z\right)T\left(t,n\Delta t,\Delta t\right) \tag{68}$$

where $E_x\Big|_{i_x\ i_y\ i_z}^{n}$, $H_y\Big|_{i_x\ i_y\ i_z}^{n}$, and $H_z\Big|_{i_x\ i_y\ i_z}^{n}$ are the expansion coefficients. Again, due to the use of the triangular function as basis functions, these expansion coefficients happen to be the field values at the central points of the triangular functions.

The above expansions are the same as those presented in Equation (59) for the MWR derivation of the conventional FDTD method, except that the central points of the triangular basis function in time, i.e., $T\left(t,n\Delta t,\Delta t\right)$, have no half-step displacements for both electric and magnetic fields.

Next, the expansion of Equation (68) is substituted into Maxwell's Equation (61) and the residual minimization is performed with the following Dirac impulse function as the weighting function:

$$\delta\left[x-\left(i_x+\frac{1}{2}\right)\Delta x\right]\delta\left(y-i_y\Delta y\right)\delta\left(z-i_z\Delta z\right)\delta\left[t-\left(n+\frac{1}{2}\right)\Delta t\right] \tag{69}$$

Hence, one obtains

$$\varepsilon\frac{E_x|^{n+1}_{i_x+\frac{1}{2},i_y,i_z}-E_x|^{n}_{i_x+\frac{1}{2},i_y,i_z}}{\Delta t}$$
$$=\frac{1}{2}\left(\frac{H_z|^{n+1}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}-H_z|^{n+1}_{i_x+\frac{1}{2},i_y-\frac{1}{2},i_z}}{\Delta y}\right.$$
$$\left.+\frac{H_z|^{n}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}-H_z|^{n}_{i_x+\frac{1}{2},i_y-\frac{1}{2},i_z}}{\Delta y}\right)$$
$$-\frac{1}{2}\left(\frac{H_y|^{n+1}_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}}-H_y|^{n+1}_{i_x+\frac{1}{2},i_y,i_z-\frac{1}{2}}}{\Delta z}\right.$$
$$\left.+\frac{H_y|^{n}_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}}-H_y|^{n}_{i_x+\frac{1}{2},i_y,i_z-\frac{1}{2}}}{\Delta z}\right)-J_x|^{n+\frac{1}{2}}_{i_x+\frac{1}{2},i_y,i_z}. \quad (70)$$

The above equation is exactly the same as the CN-FDTD equation presented in [38]. The same conclusion can be found for the other Maxwell's equations. We therefore conclude that the CN-FDTD is derivable from the MWR.

Equation (70) and other CN-FDTD equations are implicit: their related computations are CPU-time consuming because at each step, iterative methods are needed to solve for the expansion coefficients. To overcome the inefficiency problem, a modified and approximated version of the CN-FDTD was proposed, the alternating-implicit-direction (ADI) FDTD [39, 40]. The fundamentals of both methods are the same, except that the expansion coefficients obtained by the ADI-FDTD method can be seen as first iteration solutions of the CN-FDTD [47]. In other words, the ADI-FDTD method approximates expansion coefficients. As a result, their computation is much faster. Since the CN-FDTD is derivable from the MWR, the ADI-FDTD is derivable from the MWR, but with improved computational efficiency for the approximated expansion coefficients.

5.1.5 MWR Derivation of the Unconditionally Stable FDTD Method with Laguerre Polynomials

The FDTD method proposed in [41] is a very interesting technique where solutions are always stable. Unlike the conventional FDTD methods described before, the weighted Laguerre polynomials are directly employed as the expansion and weighting functions *in time*, rather than the triangle function defined by Equation (26). The expansion functions and testing functions *in space* are the same as those employed for the conventional FDTD method of Yee's scheme. For instance, consider Equation (61) again. E_x, H_y, and H_z are expanded as follows:

$$E_x=\sum_{i_x,i_y,i_z}\sum_{p=1}^{N_L}E_x|^{p}_{i_x+\frac{1}{2},i_y,i_z}\,T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right]$$
$$T\left(y,i_y\Delta y,\Delta y\right)T\left(z,i_z\Delta z,\Delta z\right)\varphi_p\left(\bar{t}\right),$$

$$H_y=\sum_{i_x,i_y,i_z}\sum_{p=1}^{N_L}H_y|^{p}_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}}\,T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right]$$
$$T\left(y,i_y\Delta y,\Delta y\right)T\left[z,(i_z+\frac{1}{2})\Delta z,\Delta z\right]\varphi_p\left(\bar{t}\right),$$

$$H_z=\sum_{i_x,i_y,i_z}\sum_{p=1}^{N_L}H_z|^{p}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}\,T\left[x,\left(i_x+\frac{1}{2}\right)\Delta x,\Delta x\right]$$
$$T\left[y,\left(i_y+\frac{1}{2}\right)\Delta y,\Delta y\right]T\left(z,i_z\Delta z,\Delta z\right)\varphi_p\left(\bar{t}\right), \quad (71)$$

where $\varphi_p\left(\bar{t}\right)$ is the weighted Laguerre polynomial of order p that is continuous in the whole time domain. $\bar{t}=st$, with s being the time scaling factor, N_L is a finite number determined by the bandwidth of the signal to be simulated [40], and $E_x|^{p}_{i_x+\frac{1}{2},i_y,i_z}$, $H_y|^{p}_{i_x+\frac{1}{2},i_y,i_z+\frac{1}{2}}$, and $H_z|^{p}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}$ are the expansion coefficients. Unlike those coefficients with the conventional FDTD, they are not equal to the field quantities at the grid points due to the use of the weighted Laguerre polynomial.

Substitution of Equation (19) into the first equation of Equation (58) and residual-minimization with

$$\delta\left[x-\left(i_x+\frac{1}{2}\right)\Delta x\right]\delta\left(y-i_y\Delta y\right)\delta\left(z-i_z\Delta z\right)\varphi_p\left(\bar{t}\right)$$

as the weighting function yield

$$E_x|^{p}_{i_x+\frac{1}{2},i_y,i_z}=\bar{C}^{E}_{y}|_{i_x+\frac{1}{2},i_y,i_z}\left(H_z|^{p}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}-H_z|^{p}_{i_x+\frac{1}{2},i_y-\frac{1}{2},i_z}\right)$$
$$-\bar{C}^{E}_{z}|_{i_x+\frac{1}{2},i_y,i_z}\left(H_z|^{p}_{i_x+\frac{1}{2},i_y+\frac{1}{2},i_z}-H_z|^{p}_{i_x+\frac{1}{2},i_y-\frac{1}{2},i_z i_x+\frac{1}{2},i_y-\frac{1}{2},i_z}\right)$$
$$-\frac{2}{s\varepsilon}J_x|^{p}_{i_x+\frac{1}{2},i_y,i_z}-2\sum_{k=0}^{p-1}E_x|^{k}_{i_x+\frac{1}{2},i_y,i_z}. \quad (72)$$

The above equation is exactly the same as that presented in [41]. For the other Maxwell's equations in Equation (58), the same conclusion can be reached. Therefore, the unconditionally stable FDTD scheme with Laguerre polynomials is derivable completely from the MWR.

5.1.6 MWR Derivation of the Transmission-Line-Matrix (TLM) Method

The Transmission-Line-Matrix (TLM) Method is another powerful time-domain numerical technique. It utilizes the analogy between the voltage/current on a specifically designed transmission-line network and electric/magnetic fields in space. It can

therefore be shown to be equivalent to a finite-difference time-domain formulation that condenses electric and magnetic field components at the same grid points [48, 49]. Since the FDTD-based methods are derivable from an MWR procedure, naturally, the TLM method can be derived from MWR. Nevertheless, the MWR derivations of the TLM method are quite involved, and readers are referred to [50], where a systematic way of deriving the TLM formulations using MWR or MoM was presented. It was found that *in space*, the field quantities are expanded with products of one-dimensional rectangular pulse functions and two-dimensional triangular basis functions. Residual minimization is weighted with a rectangle and a Dirac delta pulse function. However, like the derivation of the FDTD method, *in time*, the field quantities are still expanded with the triangle function and weighted with the Dirac function.

5.1.7 Summary of the Results

To provide a clear picture of what has been described about finite-difference based time-domain methods, Table 2 summarizes the MWR interpretations. As can be seen from Table 2, although presented in different forms, the finite-difference-based time-domain methods fall under the framework of the Method of Weighted Residuals (MWR) or Method of Moments (MoM). In other words, they can all be derived from MWR procedures, with use of different basis functions for expansion and weighting functions for residual minimization.

5.2 The Finite-Element Time-Domain (FEM-TD) Method

Various FEM-TD methods have been developed, and different formulations have been presented [14, 15]. Some of them are based directly on Maxwell's equations and others on wave equations. In the derivations known so far, either an integral equation or a functional is first established. The MWR procedure of expansion and residual minimization is then applied in *space* (note that as described before, making the functional stationary is equivalent to Galerkin's method). In the *time* domain, a finite-differencing operator is used to replace temporal derivatives. For instance, as described in [14], the second-order derivative in *time* was replaced with the central finite-difference operator (i.e., Equation (26) of [13]). However, as we demonstrated in the previous section, both the first-order and second-order finite-differences can be derived from the MWR techniques. As a result, the FEM-TD formulations are derivable from the MWR technique.

To be more specific, consider the example presented in [14], where the following wave equation (i.e., Equation (22) in [14]) is to be solved:

$$\nabla\times\frac{1}{\mu_r}\nabla\times\mathbf{E}-\nabla\left(\frac{1}{\mu_r\varepsilon_r}\nabla\bullet\varepsilon_r\mathbf{E}\right)+\varepsilon_0\varepsilon_r\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}=-\mu_0\frac{\partial\mathbf{J}}{\partial t}\ \text{in}\ \Omega, \quad (73)$$

where $\mu=\mu_0\mu_r$ is the permeability of the medium, with μ_0 being the permeability of vacuum and μ_r being the relative permeability; $\varepsilon=\varepsilon_0\varepsilon_r$ is the permittivity, with ε_0 being the permittivity of vacuum and ε_r being the relative permittivity;

The electric field, **E**, is now expanded with the following basis functions:

$$\mathbf{E}(\mathbf{r},t)\approx\sum_{i=1}^{N}\mathbf{E}_i(t)\phi_i(\mathbf{r})=\sum_{i=1}^{N}\sum_{n}\mathbf{E}_i^n h(t,n\Delta t,\Delta t)\phi_i(\mathbf{r}), \quad (74)$$

where

$$\mathbf{E}_i(t)=\sum_{n}\mathbf{E}_i^n h(t,n\Delta t,\Delta t)$$

with

$h(t,n\Delta t,\Delta t)=\phi(t,n\Delta t,\Delta t)$ being defined by Equation (41) and Δt being the time step. $\mathbf{E}_i^n$ is the vectorized expansion coefficient, and $\phi_i(\mathbf{r})$ is the spatial basis function used for carrying out the finite-element approximations in the electric and magnetic grids. Basically, this can be the shape function in a two-dimensional triangular patch element or a three-dimensional tetrahedral volume element.

By substituting Equation (74) into Equation (73) and performing the residual minimization with $\phi_j(\mathbf{r})\delta(t-n\Delta t)$ as the weighting function, after some mathematical manipulations one obtains

Table 2. A summary of the expansion and weighting functions used in the methods discussed.

	Basis Functions for Expansion		Weighting Functions for Residual Minimization	
	In Space	**In Time**	**In Space**	**In Time**
Yee's FDTD	Triangle	Triangle	Dirac Impulse	Dirac Impulse
CN-FDTD	Triangle	Triangle	Dirac Impulse	Dirac Impulse
ADI-FDTD	Triangle	Triangle	Dirac Impulse	Dirac Impulse
TLM	1D Rectangle and 2D Triangle	Triangle	Rectangle and Dirac Impulse	Dirac Impulse
PSTD	FFT	Triangle	Dirac Impulse	Dirac Impulse
MRTD	Scaling/Wavelets	Triangle	Scaling/Wavelets	Dirac Impulse
FDTD with Laguerre Polynomials	Triangle	Weighted Laguerre polynomial	Dirac Impulse	Weighted Laguerre polynomial

$$\varepsilon_r \frac{\mu_0\varepsilon_0}{(\Delta t)^2}\left(\mathbf{E}_j^{n+1}+\mathbf{E}_j^{n-1}-2\mathbf{E}_j^{n}\right)$$
$$=\sum_{i=1}^{N}\int_{\Omega}\left\{\nabla\phi_j\times\left(\nabla\phi_i\times\mathbf{E}_i^{n}\right)-\mathbf{E}_i^{n}\bullet\left[\frac{1}{\mu_r\varepsilon_r}\nabla\left(\varepsilon_r\phi\right)\nabla\phi_j\right]\right\}d\Omega$$
$$-\mu_0\int_{\Omega}\phi_i\frac{\partial\mathbf{J}}{\partial t}d\Omega-\int_{S}\left[\mathbf{a}_n\times\left(\frac{1}{\mu_r}\nabla\times\mathbf{E}\right)+\mathbf{a}_n\frac{1}{\mu_r\varepsilon_r}\nabla\bullet\left(\varepsilon_r\mathbf{E}\right)\right]\phi_j ds \tag{75}$$

where S is a surface that encloses Ω and $\mathbf{a}_n$ is the outward normal unit vector of S.

The relationship of Equation (75) is exactly the same as the FEM-TD Equation (27) of [14] (which was derived with the use of finite difference in time). Therefore, the FEM-TD method has been shown to be derivable from the MWR in this case. This conclusion can be easily extended to other forms of TD-FEM formulations, such as the formulation presented in [15].

It is worth mentioning that the choice of basis and weighting functions does not have to be the choice described above. For instance, in [51] the decaying weighted Laguerre polynomials were used as both the basis and weighting functions in time to develop an unconditionally stable FEM-TD method. As a result, the TD-FEM method is no longer limited by the time step, but by the inherent accuracy of the method.

5.3 The Time-Domain Integral Equation (TDIE) Methods

TDIE methods have been of interest to the electromagnetic modeling community for more than 30 years. They are based on the solutions to integral equations derived from Maxwell's equations via Green's functions. The solutions were obtained with the use of an MWR procedure [17]. Therefore, TDIE methods naturally fall into the framework of the MWR. For instance, in [17], the various terms can be identified as follows:

Equation to be solved:

$$\frac{\mathbf{E}^i(\mathbf{r},t)}{\eta_0}$$
$$=-c\nabla\iint_S\frac{\int_0^{t-R/c}\nabla'\bullet\mathbf{J}(\mathbf{r}',\tau)\,d\tau}{4\pi R}dS'+\frac{1}{c}\iint_S\frac{\frac{\partial}{\partial\tau}\left[\mathbf{J}(\mathbf{r}',\tau)|_{\tau=t-R/c}\right]}{4\pi R}dS' \tag{75}$$

Unknown quantity to be found: $\mathbf{J}(\mathbf{r},t)$

Basis functions: divergence-conforming basis function $\mathbf{S}_n(\mathbf{r})$ in space as defined in [52] and the Knab's bandlimited interpolation function in time:

$$T(t)=\frac{\sin(s\omega_0 t)}{s\omega_0 t}\frac{\sin\left[a\sqrt{\left(\frac{t}{N\Delta t}\right)^2-1}\right]}{\sinh(a)\sqrt{\left(\frac{t}{N\Delta t}\right)^2-1}}. \tag{76}$$

Weighting functions: divergence-conforming basis function $\mathbf{S}_n(\mathbf{r})$ in space as defined in [52] and the Dirac impulse function $\delta(t-n\Delta t)$ in time.

In the above equations, $\mathbf{E}^i(\mathbf{r},t)$ is the known incident electric field, $\mathbf{r}'$ is the integration variable over the surface S, c is the speed of light, η_0 is the wave impedance, N is an integer called the approximate prolate spheroidal wave function (APSWF) width parameter, $a=\pi N(s-1)/s$ is the time-bandwidth product, s is the over-sampling factor, and ω_0 is the highest frequency in the band of interest. Note that Equation (76) is a decaying function to control the late-time instability that has crippled the application of TDIE methods.

Other TDIE methods have been proposed. Although they solve integral equations and use different basis and weighting functions, they can also be shown to derive from the MWR procedure.

6. Conclusion

Many numerical methods have been developed in the past four decades to solve Maxwell's electromagnetic equations, which form the foundation of electrical and electronic engineering theory. Traditionally, these methods have been developed and presented through different mathematical procedures. Hence, they have appeared to be unrelated to each other. For instance, the Finite-Element Method is based on numerical variational solutions of integral equations or a functional for a generalized wave equation. Finite-difference methods are based on direct replacement of derivatives with finite-difference schemes. The Spectral-Domain Method is obtained with applications of the Fourier transform in the spatial domain. In this paper, we have shown a way to present numerical methods: they can be generalized or derived from an MWR procedure. Differences among methods reside in different choices of expansion and testing functions. Therefore, numerical methods are related and unified through the framework of the MWR.

The significance of the work presented is two-fold: (1) hybridization of different numerical methods can now be done relatively easily, as they have a common ground based on MWR formulations; and (2) new time-domain methods, particularly effective and efficient for specific structures, can now be developed with new expansions and testing functions.

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8. References

1. R. Harrington, *Field Computation by Moment Methods*, New York, IEEE Press, 1993.

2. A. Taflove and S. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method, Third Edition*, Norwood, MA, Artech House, 2005.

3. A. Taflove (ed.), *Advances in Computational Electrodynamics: The Finite-Difference Time-Domain Method*, Norwood, MA, Artech House, 1998.

4. T. Itoh (ed.), *Time-Domain Methods for Microwave Structures: Analysis and Design*, New York, IEEE Press, 1998.

5. J. Jin, *The Finite Element Method in Electromagnetics, Second Edition*, New York, Wiley-IEEE Press, 2002.

6. T. Itoh (ed.), *Numerical Techniques for Microwave and Millimeter-Wave Passive Structures*, New York, John Wiley & Sons, 1989.

7. http://www.ansoft.com/products/hf/hfss/ viewed on July 2, 2007.

8. http://www.zeland.com/, viewed on July 2, 2007.

9. J. Jin, *Electromagnetic Analysis and Design in Magnetic Resonance Imaging*, Boca Raton, FL, CRC Press, 1998.

10. http://emclab.mst.edu/csoft.html, viewed on May 13, 2008.

11. T. Weiland, "Three Dimensional Resonator Mode Computation by Finite Difference Method," *IEEE Transactions on Magnetics*, **21**, 6, 1985, pp. 2340-2343.

12. W. J. R. Hoefer, "The Transmission-Line Matrix Method-Theory and Applications," *IEEE Transactions on Microwave and Theory*, **33**, 10, October 1985, pp. 882-893.

13. C. Christopoulos, *The Transmission-Line Modeling Method*, New York, IEEE Press/Oxford University Press, 1995.

14. J. Lee, R. Lee and A. Cangellaris, "Time-Domain Finite-Element Methods," *IEEE Transactions on Antennas and Propagation*, **AP-45**, 3, March 1997, pp. 430-442.

15. Z. Lou and J. Jin, "A New Explicit Time-Domain Finite-Element Method Based on Element-Level Decomposition," *IEEE Transactions on Antennas and Propagation*, **AP-54**, 10, October 2006, pp. 2990-2999.

16. D. Weile, G. Pisharody, N.-W. Chen, B. Shanker, and E. Michielssen, "A Novel Scheme for the Solution of the Time-Domain Integral Equations of Electromagnetics," *IEEE Transactions on Antennas and Propagation*, **AP-52**, 1, January 2004, pp. 283-285.

17. G. Pisharody and D. Weile, "Robust Solution of Time-Domain Integral Equations Using Loop-Tree Decomposition and Band Limited Extrapolation," *IEEE Transactions on Antennas and Propagation*, **AP-53**, 6, June 2005, pp. 2089-2098.

18. J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Oxford, Clarendon Press, 1873.

19. Matthew Sadiku, *Elements of Electromagnetics*, Oxford, Oxford University Press, 2006.

20. P. Chong, *Approximate Solution Methods in Engineering Mechanics*, Lavoisier, 2003.

21. T. K. Sarkar, A. R. Djordjevic, and E. Arvas, "On the Choice of Expansion and Weighting Functions in the Numerical Solution of Operator Equations," *IEEE Transactions on Antennas and Propagation*, **AP-33**, 9, September 1985, pp. 988-996.

22. M. M. Ney, "Method of Moments as Applied to Electromagnetic Problems," in R. Sorrentino (ed.), *Numerical Methods for Passive Microwave and Millimeter Wave Structures*, New York, IEEE Press, 1989, pp. 271-279.

23. T. K. Sarkar, "A Note on the Choice Weighting Functions in the Method of Moments," *IEEE Transactions on Antennas and Propagation*, **AP-33**, 4, April 1985, pp. 436-41.

24. K. Yee, "Numerical Solution of Initial Boundary Value Problems Involving Maxwell's Equations in Isotropic Media," *IEEE Transactions on Antennas and Propagation*, **AP-14**, 1966, pp. 302-307.

25. P. Silvester, "A General High-Order Finite Element Wave Guide Analysis Program," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-17**, April, 1969, pp. 204-210.

26. http://eesof.tm.agilent.com/products/emds_main.html, viewed on July 2, 2007.

27. S. Mikhlin, *Variational Methods in Mathematical Physics*, New York, Macmillan, 1964.

28. A. Dreher and T. Rother, "New Aspects of the Method of Lines," *IEEE Microwave and Guided Wave Letters*, **5**, 11, November 1995, pp. 408-410.

29. D. Davidson and J. Aberle, "An Introduction to Spectral Domain Method-of-Moments Formulations," *IEEE Transactions on Antennas and Propagation*, **AP-46**, 3, June 2004, pp. 11-19.

30. R. E. Collin, *Field Theory of Guided Waves*, New York, IEEE Press, 1990.

31. http://www.remcom.com/xfdtd/overview/xfdtd-overview.html, viewed on July 2, 2007.

32. http://www.cst.com/Content/Products/MWS/Overview.aspx, viewed on July 2, 2007.

33. http://www.vectorfields.com/content/view/132/0/, viewed on July 2, 2007.

34. http://www.semcad.com/simulation/features/, viewed on July 2, 2007.

35. S. Le Maguer, A. Péden, D. Bourreau and M.M Ney, "Split-Step TLM (SS-TLM): A New Scheme for Accelerating Electromagnetic Field Simulation," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-52**, 4, April 2004, pp. 1182-1190.

36. M. Krumpholz and L. Katehi, "MRTD: New Time-Domain Schemes Based on Multiresolution Analysis," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-44**, 4, April 1996, pp. 555-571.

37. Q. Liu, "The Pseudospectral Time-Domain (PSTD) Method: A New Algorithm for Solutions of Maxwell's Equations," 1997 IEEE International Symposium on Antennas and Propagation *Digest*, July 13-18, 1997, **1**, pp. 122-125.

38. T. Namiki, "A New FDTD Algorithm Based on Alternating Direction Implicit Method," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-47**, October 1999, pp. 2003-2007.

39. F. Zheng, Z. Chen and J. Zhang, "Toward the Development of a Three-Dimensional Unconditionally Stable Finite-Difference Time-Domain Method," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-48**, 9, September 2000, pp.1550-1558.

40. G. Sun and C. W. Trueman, "Unconditionally Stable Crank-Nicolson Scheme for Solving the Two-Dimensional Maxwell's Equations," *IEE Electronics Letters*, **39**, April 2003, pp. 595-597.

41. Y. Chung, T. Sarkar, B. Jung, and M. Salazar-Palma, "An Unconditionally Stable Scheme for the Finite-Difference Time-Domain Method," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-51**, 3, March 2003, pp. 697-704.

42. Z. Chen and S. Luo, "Generalization of the Finite-Difference-Based Time-Domain Methods Using the Method Moments," *IEEE Transactions on Antennas and Propagation*, **AP-54**, 9, September 2006, pp. 2515-2524.

43. J. M. Dohlus, P. Hahne, X. Du, B. Wagner, T. Weiland, and S. G. Wipf, "Using the Maxwell Grid Equations to Solve Large Problems," *IEEE Transactions on Magnetics*, **29**, 2, March 1993, pp. 1914-1917.

44. M. Krumpholz, C. Huber, and P. Russer, "A Field Theoretical Comparison of FDTD and TLM," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-43**, 8, August 1995, pp. 1935-1950.

45. C. K. Chui, *An Introduction to Wavelets*, San Diego, Academic Press, Inc., 1992.

46. G. Sun and C. W. Trueman, "Approximate Crank-Nicolson Schemes for the 2-D Finite-Difference Time-Domain Method for TE/sub z/ Waves," *IEEE Transactions on Antennas and Propagation*, **AP-52**, 11, November 2004, pp. 2963-2972.

47. S. Wang, F. Teixeira and J. Chen, "An Iterative ADI-FDTD with Reduced Splitting Errors," *IEEE Microwave and Wireless Components Letters*, **15**, 2, February 2005, pp. 92-94.

48. Z. Chen, M. Ney and W. Hoefer, "A New Finite-Difference Time-Domain Formulation and its Equivalence with the TLM Symmetrical Condensed Node," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-39**, 12, December 1991, pp. 2160-2169.

49. H. Jin and R. Vahldieck, "Direct Derivation of TLM Symmetrical Condensed Node and Hybrid Symmetrical Condensed Node from Maxwell's Equations Using Centered Differencing and Averaging," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-42**, 12, December 1994, pp. 2554-2562.

50. M. Krumpholz and P. Russer, "A Field Theoretical Derivation of TLM," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-42**, 9, April 1994, pp. 1660-1668.

51. Y. Chung, T. Sarkar, S. Liorento-Romano and M. Salarzar-Palma, "Finite-Element Time-Domain Method Using Laguerre Polynomial," 2003 IEEE International Microwave Symposium *Digest*, June 8-13, 2003, Philadelphia, Pennsylvania, **2**, pp. 981-984.

52. R. D. Graglia, D. R. Wilton, and A. F. Peterson, "Higher Order Interpolatory Vector Bases for Computational Electromagnetics," *IEEE Transactions on Antennas and Propagation*, **AP-45**, 3, March 1997, pp. 329-342.