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BOUNDARY VALUE PROBLEMS WITH MEASURES FOR ELLIPTIC EQUATIONS WITH SINGULAR POTENTIALS ¹

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Abstract

We study the boundary value problem with Radon measures for nonnegative solutions of $-\Delta u + Vu = 0$ in a bounded smooth domain Ω , when V is a locally bounded nonnegative function. Introducing some specific capacity, we give sufficient conditions on a Radon measure μ on $\partial\Omega$ so that the problem can be solved. We study the reduced measure associated to this equation as well as the boundary trace of positive solutions.

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1 Introduction

Let Ω be a smooth bounded domain of \mathbb{R}^N and V a locally bounded real valued measurable function defined in Ω . The first question we adress is the solvability of the following non-homogeneous Dirichlet problem with a Radon measure for boundary data,

$$\begin{cases} -\Delta u + Vu = 0 & \text{in } \Omega \\ u = \mu & \text{in } \partial\Omega. \end{cases} \quad (1.1)$$

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Let ρ be the first (and positive) eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$. By a solution we mean a function $u \in L^1(\Omega)$, such that $Vu \in L^1_\rho$, which satisfies

$$\int_{\Omega} (-u\Delta\zeta + Vu\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu. \quad (1.2)$$

for any function $\zeta \in C_0^1(\overline{\Omega})$ such that $\Delta\zeta \in L^\infty(\Omega)$. When V is a bounded nonnegative function, it is straightforward that there exist a unique solution. However, it is less obvious to find general conditions which allow the solvability for any $\mu \in \mathfrak{M}(\partial\Omega)$, the set of Radon measures on $\partial\Omega$. In order to avoid difficulties due to Fredholm type obstructions, we shall most often assume that V is nonnegative, in which case there exists at most one solution.

Let us denote by K^Ω the Poisson kernel in Ω and by $\mathbb{K}[\mu]$ the Poisson potential of a measure, that is

$$\mathbb{K}[\mu](x) := \int_{\partial\Omega} K^\Omega(x, y) d\mu(y) \quad \forall x \in \Omega. \quad (1.3)$$

We first observe that, when $V \geq 0$ and the measure μ satisfies

$$\int_{\Omega} \mathbb{K}[|\mu|](x) V(x) \rho(x) dx < \infty, \quad (1.4)$$

then problem (1.1) admits a solution. A Radon measure which satisfies (1.4) is called *an admissible measure* and a measure for which a solution exists is called *a good measure*.

We first consider the *subcritical case* which means that the boundary value is solvable for any $\mu \in \mathfrak{M}(\partial\Omega)$. As a first result, we prove that any measure μ is admissible if V is nonnegative and satisfies

$$\sup_{y \in \partial\Omega} \text{ess} \int_{\Omega} K^\Omega(x, y) V(x) \rho(x) dx < \infty. \quad (1.5)$$

Using estimates on the Poisson kernel, this condition is fulfilled if there exists $M > 0$ such that for any $y \in \partial\Omega$,

$$\int_0^{D(\Omega)} \left(\int_{\Omega \cap B_r(y)} V(x) \rho^2(x) dx \right) \frac{dr}{r^{N+1}} \leq M \quad (1.6)$$

where $D(\Omega) = \text{diam}(\Omega)$. We give also sufficient conditions which ensures that the boundary value problem (1.1) is stable from the weak*-topology of $\mathfrak{M}(\partial\Omega)$ to $L^1(\Omega) \cap L^1_{V\rho}(\Omega)$. One of the sufficient conditions is that $V \geq 0$ satisfies

$$\lim_{\epsilon \rightarrow 0} \int_0^\epsilon \left(\int_{\Omega \cap B_r(y)} V(x) \rho^2(x) dx \right) \frac{dr}{r^{N+1}} = 0, \quad (1.7)$$

uniformly with respect to $y \in \partial\Omega$.

In the *supercritical case* problem (1.1) cannot be solved for any $\mu \in \mathfrak{M}(\partial\Omega)$. In order to characterize positive good measures, we introduce a framework of nonlinear analysis which have been used by Dynkin and Kuznetsov (see [9] and references therein) and Marcus and Véron [16] in their study of the boundary value problems with measures

$$\begin{cases} -\Delta u + |u|^{q-1}u = 0 & \text{in } \Omega \\ u = \mu & \text{in } \partial\Omega, \end{cases} \quad (1.8)$$

where $q > 1$. In these works, positive good measures on $\partial\Omega$ are completely characterized by the $C_{2/q,q'}$ -Bessel in dimension $N-1$ and the following property:

A measure $\mu \in \mathfrak{M}_+(\partial\Omega)$ is good for problem (1.8) if and only if it does charge Borel sets with zero $C_{2/q,q'}$ -capacity, i.e

$$C_{2/q,q'}(E) = 0 \implies \mu(E) = 0 \quad \forall E \subset \partial\Omega, E \text{ Borel.} \quad (1.9)$$

Moreover, any positive good measure is the limit of an increasing sequence $\{\mu_n\}$ of admissible measures which, in this case, are the positive measures belonging to the Besov space $B_{2/q,q'}(\partial\Omega)$. They also characterize removable sets in terms of $C_{2/q,q'}$ -capacity.

In our present work, and always with $V \geq 0$, we use a capacity associated to the Poisson kernel K^Ω and belongs to a class studied by Fuglede [10] [11]. It is defined by

$$C_V(E) = \sup\{\mu(E) : \mu \in \mathfrak{M}_+(\partial\Omega), \mu(E^c) = 0, \|V\mathbb{K}[\mu]\|_{L^1_\rho} \leq 1\}, \quad (1.10)$$

for any Borel set $E \subset \partial\Omega$. Furthermore $C_V(E)$ is equal to the value of its dual expression $C_V^*(E)$ defined by

$$C_V^*(E) = \inf\{\|f\|_{L^\infty} : \check{\mathbb{K}}[f] \geq 1 \text{ on } E\}, \quad (1.11)$$

where

$$\check{\mathbb{K}}[f](y) = \int_\Omega K^\Omega(x,y)f(x)V(x)\rho(x)dx \quad \forall y \in \partial\Omega. \quad (1.12)$$

If E is a compact subset of $\partial\Omega$, this capacity is explicitly given by

$$C_V(E) = C_V^*(E) = \max_{y \in E} \left(\int_\Omega K^\Omega(x,y)V(x)\rho(x)dx \right)^{-1}. \quad (1.13)$$

We denote by Z_V the largest set with zero C_V capacity, i.e.

$$Z_V = \left\{ y \in \partial\Omega : \int_\Omega K^\Omega(x,y)V(x)\rho(x)dx = \infty \right\}, \quad (1.14)$$

and we prove the following.

- 1- If $\{\mu_n\}$ is an increasing sequence of positive good measures which converges to a measure μ in the weak* topology, then μ is a good measure.
- 2- If $\mu \in \mathfrak{M}_+(\partial\Omega)$ satisfies $\mu(Z_V) = 0$, then μ is a good measure.
- 3- A good measure μ vanishes on Z_V if and only if there exists an increasing sequence of positive admissible measures which converges to μ in the weak* topology.

In section 4 we study relaxation phenomenon in replacing (1.1) by the truncated problem

$$\begin{cases} -\Delta u + V_k u = 0 & \text{in } \Omega \\ u = \mu & \text{in } \partial\Omega. \end{cases} \quad (1.15)$$

where $\{V_k\}$ is an increasing sequence of positive bounded functions which converges to V locally uniformly in Ω . We adapt to the linear problem some of the principles of the reduced measure.

This notion is introduced by Brezis, Marcus and Ponce [5] in the study of the nonlinear Poisson equation

$$-\Delta u + g(u) = \mu \quad \text{in } \Omega \quad (1.16)$$

and extended to the Dirichlet problem

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } \Omega \\ u = \mu & \text{in } \partial\Omega, \end{cases} \quad (1.17)$$

by Brezis and Ponce [6]. In our construction, problem (1.15) admits a unique solution u_k . The sequence $\{u_k\}$ decreases and converges to some u which satisfies a relaxed boundary value problem

$$\begin{cases} -\Delta u + Vu = 0 & \text{in } \Omega \\ u = \mu^* & \text{in } \partial\Omega. \end{cases} \quad (1.18)$$

The measure μ^* is called the *reduced measure* associated to μ and V . Note that μ^* is the largest measure for which the problem

$$\begin{cases} -\Delta u + Vu = 0 & \text{in } \Omega \\ u = \nu \leq \mu & \text{in } \partial\Omega. \end{cases} \quad (1.19)$$

admits a solution. This truncation process allows to construct the Poisson kernel K_V^Ω associated to the operator $-\Delta + V$ as being the limit of the decreasing limit of the sequence of kernel functions $\{K_{V_k}^\Omega\}$ associated to $-\Delta + V_k$. The solution $u = u_{\mu^*}$ of (1.18) is expressed by

$$u_{\mu^*}(x) = \int_{\partial\Omega} K_V^\Omega(x, y) d\mu(y) = \int_{\partial\Omega} K_V^\Omega(x, y) d\mu^*(y) \quad \forall x \in \Omega. \quad (1.20)$$

We define the vanishing set of K_V by

$$Z_V^* = \{y \in \partial\Omega : K_V^\Omega(x_0, y) = 0\}, \quad (1.21)$$

for some $x_0 \in \Omega$, and thus for any $x \in \Omega$ by Harnack inequality. We prove

- 1- $Z_V^* \subset Z_V$.
- 2- $\mu^* = \mu \chi_{Z_V^*}$.

A challenging open problem is to give conditions on V which allows $Z_V^* = Z_V$.

The last section is devoted to the construction of the boundary trace of positive solutions of

$$-\Delta u + Vu = 0 \quad \text{in } \Omega, \quad (1.22)$$

assuming $V \geq 0$. Using results of [18], we defined the regular set $\mathcal{R}(u)$ of the boundary trace of u . This set is a relatively open subset of $\partial\Omega$ and the regular part of the boundary trace is represented by a positive Radon measure μ_u on $\mathcal{R}(u)$. In order to study the singular set of the boundary trace $\mathcal{S}(u) := \partial\Omega \setminus \mathcal{R}(u)$, we adapt the sweeping method introduced by Marcus and Véron in [19] for equation

$$-\Delta u + g(u) = 0 \quad \text{in } \Omega. \quad (1.23)$$

If μ is a good positive measure concentrated on $\mathcal{S}(u)$, and u_μ is the unique solution of (1.1) with boundary data μ , we set $v_\mu = \min\{u, u_\mu\}$. Then v_μ is a positive super solution which admits a positive trace $\gamma_u(\mu) \in \mathfrak{M}_+(\partial\Omega)$. The extended boundary trace $Tr^e(u)$ of u is defined by

$$\nu(u)(E) := Tr^e(u)(E) = \sup\{\gamma_u(\mu)(E) : \mu \text{ good}, E \subset \partial\Omega, E \text{ Borel}\}. \quad (1.24)$$

Then $Tr^e(u)$ is a Borel measure on Ω . If we assume moreover that

$$\lim_{\epsilon \rightarrow 0} \int_0^\epsilon \left(\int_{\Omega \cap B_r(y)} V(x) \rho^2(x) dx \right) \frac{dr}{r^{N+1}} = 0 \quad \text{uniformly with respect to } y \in \partial\Omega, \quad (1.25)$$

then $Tr^e(u)$ is a bounded measure and therefore a Radon measure. Finally, if $N = 2$ and (1.25) holds, or if $N = 2$ and there holds

$$\lim_{\epsilon \rightarrow 0} \int_0^\epsilon \left(\int_{\Omega \cap B_r(y)} V(x) (\rho(x) - \epsilon)_+^2 dx \right) \frac{dr}{r^{N+1}} = 0, \quad (1.26)$$

uniformly with respect to $\epsilon \in (0, \epsilon_0]$ and y s.t. $\text{dist}(x, \partial\Omega) = \epsilon$, then $u = u_{\nu(u)}$.

If $V(x) \leq v(\rho(x))$ for some v which satisfies

$$\int_0^1 v(t) t dt < \infty, \quad (1.27)$$

then Marcus and Véron proved in [18] that $u = u_{\nu_u}$. Actually, when V has such a geometric form, the assumptions (1.25)-(1.26) and (1.27) are equivalent.

2 The subcritical case

In the sequel Ω is a bounded smooth domain in \mathbb{R}^N and $V \in L_{loc}^\infty$. We denote by ρ the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$, $\rho > 0$ with the corresponding eigenvalue λ , by $\mathfrak{M}(\partial\Omega)$ the space of bounded Radon measures on $\partial\Omega$ and by $\mathfrak{M}_+(\partial\Omega)$ its positive cone. For any positive Radon measure on $\partial\Omega$, we shall denote by the same symbol the corresponding outer regular bounded Borel measure. Conversely, for any outer regular bounded Borel μ , we denote by the same expression μ the Radon measure defined on $C(\partial\Omega)$ by

$$\zeta \mapsto \mu(\zeta) = \int_{\partial\Omega} \zeta d\mu.$$

If $\mu \in \mathfrak{M}(\partial\Omega)$, we are concerned with the following problem

$$\begin{cases} -\Delta u + Vu = 0 & \text{in } \Omega \\ u = \mu & \text{in } \partial\Omega. \end{cases} \quad (2.1)$$

Definition 2.1 Let $\mu \in \mathfrak{M}(\partial\Omega)$. We say that u is a weak solution of (2.1), if $u \in L^1(\Omega)$, $Vu \in L_\rho^1(\Omega)$ and, for any $\zeta \in C_0^1(\overline{\Omega})$ with $\Delta\zeta \in L^\infty(\Omega)$, there holds

$$\int_{\Omega} (-u\Delta\zeta + Vu\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu. \quad (2.2)$$

In the sequel we put

$$T(\Omega) := \{\zeta \in C_0^1(\overline{\Omega}) \text{ such that } \Delta\zeta \in L^\infty(\Omega)\}.$$

We recall the following estimates obtained by Brezis [4]

Proposition 2.2 *Let $\mu \in L^1(\partial\Omega)$ and u be a weak solution of problem (2.1). Then there holds*

$$\|u\|_{L^1(\Omega)} + \|V_+u\|_{L^1_\rho(\Omega)} \leq \|V_-u\|_{L^1_\rho(\Omega)} + c\|\mu\|_{L^1(\partial\Omega)} \quad (2.3)$$

$$\int_{\Omega} (-|u|\Delta\zeta + V|u|\zeta) dx \leq - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} |\mu| dS \quad (2.4)$$

and

$$\int_{\Omega} (-u_+\Delta\zeta + Vu_+\zeta) dx \leq - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} \mu_+ dS, \quad (2.5)$$

for all $\zeta \in T(\Omega)$, $\zeta \geq 0$.

We denote by $K^\Omega(x, y)$ the Poisson kernel in Ω and by $\mathbb{K}[\mu]$ the Poisson potential of $\mu \in \mathfrak{M}(\partial\Omega)$ defined by

$$\mathbb{K}[\mu](x) = \int_{\partial\Omega} K^\Omega(x, y) d\mu(y) \quad \forall x \in \Omega. \quad (2.6)$$

Definition 2.3 *A measure μ on $\partial\Omega$ is **admissible** if*

$$\int_{\Omega} \mathbb{K}[|\mu|](x) |V(x)| \rho(x) dx < \infty. \quad (2.7)$$

*It is **good** if problem (2.1) admits a weak solution.*

We notice that, if there exists at least one admissible positive measure μ , then

$$\int_{\Omega} V(x) \rho^2(x) dx < \infty. \quad (2.8)$$

Theorem 2.4 *Assume $V \geq 0$, then problem (2.1) admits at most one solution. Furthermore, if μ is admissible, then there exists a unique solution that we denote u_μ .*

Proof. Uniqueness follows from (2.3). For existence we can assume $\mu \geq 0$. For any $k \in \mathbb{N}_*$ set $V_k = \inf\{V, k\}$ and denote by $u := u_k$ the solution of

$$\begin{cases} -\Delta u + V_k(x)u = 0 & \text{in } \Omega \\ u = \mu & \text{on } \partial\Omega. \end{cases} \quad (2.9)$$

Then $0 \leq u_k \leq \mathbb{K}[\mu]$. By the maximum principle, u_k is decreasing and converges to some u , and

$$0 \leq V_k u_k \leq V \mathbb{K}[\mu].$$

Thus, by dominated convergence theorem $V_k u_k \rightarrow Vu$ in L^1_ρ . Setting $\zeta \in T(\Omega)$ and letting k tend to infinity in equality

$$\int_{\Omega} (-u_k \Delta \zeta + V_k u_k \zeta) dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\mu, \quad (2.10)$$

implies that u satisfies (2.2). \square

Remark. If V changes sign, we can put $\tilde{u} = u + \mathbb{K}[\mu]$. Then (2.1) is equivalent to

$$\begin{cases} -\Delta \tilde{u} + V \tilde{u} = V \mathbb{K}[\mu] & \text{in } \Omega \\ \tilde{u} = 0 & \text{in } \partial\Omega. \end{cases} \quad (2.11)$$

This is a Fredholm type problem (at least if the operator $\phi \mapsto R(\phi) := (-\Delta)^{-1}(V\phi)$ is compact in $L^1_\rho(\Omega)$). Existence will be ensured by orthogonality conditions.

If we assume that $V \geq 0$ and

$$\int_{\Omega} K^\Omega(x, y) V(x) \rho(x) dx < \infty, \quad (2.12)$$

for some $y \in \partial\Omega$, then δ_y is admissible. The following result yields to the solvability of (2.1) for any $\mu \in \mathfrak{M}_+(\Omega)$.

Proposition 2.5 *Assume $V \geq 0$ and the integrals (2.12) are bounded uniformly with respect to $y \in \partial\Omega$. Then any measure on $\partial\Omega$ is admissible.*

Proof. If M is the upper bound of these integrals and $\mu \in \mathfrak{M}_+(\partial\Omega)$, we have,

$$\int_{\Omega} \mathbb{K}[\mu](x) V(x) \rho(x) dx = \int_{\partial\Omega} \left(\int_{\Omega} K^\Omega(x, y) V(x) \rho(x) dx \right) d\mu(y) \leq M \mu(\partial\Omega), \quad (2.13)$$

by Fubini's theorem. Thus μ is admissible. \square

Remark. Since the Poisson kernel in Ω satisfies the two-sided estimate

$$c^{-1} \frac{\rho(x)}{|x-y|^N} \leq K^\Omega(x, y) \leq c \frac{\rho(x)}{|x-y|^N} \quad \forall (x, y) \in \Omega \times \partial\Omega, \quad (2.14)$$

for some $c > 0$, assumption (2.12) is equivalent to

$$\int_{\Omega} \frac{V(x) \rho^2(x)}{|x-y|^N} dx < \infty. \quad (2.15)$$

This implies (2.8) in particular. If we set $D_y = \max\{|x-y| : x \in \Omega\}$, then

$$\begin{aligned} \int_{\Omega} \frac{V(x) \rho^2(x)}{|x-y|^N} dx &= \int_0^{D_y} \left(\int_{\{x \in \Omega : |x-y|=r\}} V(x) \rho^2(x) dS_r(x) \right) \frac{dr}{r^N} \\ &= \lim_{\epsilon \rightarrow 0} \left(\left[r^{-N} \int_{\Omega \cap B_r(y)} V(x) \rho^2(x) dx \right]_{\epsilon}^{D_y} + N \int_{\epsilon}^{D_y} \left(\int_{\Omega \cap B_r(y)} V(x) \rho^2(x) dx \right) \frac{dr}{r^{N+1}} \right) \end{aligned}$$

(both quantity may be infinite). Thus, if we assume

$$\int_0^{D_y} \left(\int_{\Omega \cap B_r(y)} V(x) \rho^2(x) dx \right) \frac{dr}{r^{N+1}} < \infty, \quad (2.16)$$

there holds

$$\liminf_{\epsilon \rightarrow 0} \epsilon^{-N} \int_{\Omega \cap B_\epsilon(y)} V(x) \rho^2(x) dS = 0. \quad (2.17)$$

Consequently

$$\int_{\Omega} \frac{V(x) \rho^2(x)}{|x-y|^N} dx = D_y^{-N} \int_{\Omega} V(x) \rho^2(x) dx + N \int_0^{D_y} \left(\int_{\Omega \cap B_r(y)} V(x) \rho^2(x) dx \right) \frac{dr}{r^{N+1}}. \quad (2.18)$$

Therefore (2.12) holds and δ_y is admissible.

As a natural extension of Proposition 2.5, we have the following stability result.

Theorem 2.6 *Assume $V \geq 0$ and*

$$\lim_{\substack{E \text{ Borel} \\ |E| \rightarrow 0}} \int_E K^\Omega(x, y) V(x) \rho(x) dx = 0 \quad \text{uniformly with respect to } y \in \partial\Omega. \quad (2.19)$$

If μ_n is a sequence of positive Radon measures on $\partial\Omega$ converging to μ in the weak topology, then u_{μ_n} converges to u_μ in $L^1(\Omega) \cap L^1_{V\rho}(\Omega)$ and locally uniformly in Ω .*

Proof. We put $u_{\mu_n} := u_n$. By the maximum principle $0 \leq u_n \leq \mathbb{K}[\mu_n]$. Furthermore, it follows from (2.3) that

$$\|u_n\|_{L^1(\Omega)} + \|Vu_n\|_{L^1_\rho(\Omega)} \leq c \|\mu_n\|_{L^1(\partial\Omega)} \leq C. \quad (2.20)$$

Since $-\Delta u_n$ is bounded in $L^1_\rho(\Omega)$, the sequence $\{u_n\}$ is relatively compact in $L^1(\Omega)$ by the regularity theory for elliptic equations. Therefore, there exist a subsequence u_{n_k} and some function $u \in L^1(\Omega)$ with $Vu \in L^1_\rho(\Omega)$ such that u_{n_k} converges to u in $L^1(\Omega)$, almost everywhere on Ω and locally uniformly in Ω since $V \in L^\infty_{loc}(\Omega)$. The main question is to prove the convergence of Vu_{n_k} in $L^1_\rho(\Omega)$. If $E \subset \Omega$ is any Borel set, there holds

$$\begin{aligned} \int_E u_n V(x) \rho(x) dx &\leq \int_E \mathbb{K}[\mu_n] V(x) \rho(x) dx \\ &\leq \int_{\partial\Omega} \left(\int_E K^\Omega(x, y) V(x) \rho(x) dx \right) d\mu_n(y) \\ &\leq M_n \max_{y \in \partial\Omega} \int_E K^\Omega(x, y) V(x) \rho(x) dx, \end{aligned}$$

where $M_n := \mu_n(\partial\Omega)$. Thus

$$\int_E u_n V(x) \rho(x) dx \leq M_n \max_{y \in \partial\Omega} \int_E K^\Omega(x, y) V(x) \rho(x) dx. \quad (2.21)$$

Then, by (2.19),

$$\lim_{|E| \rightarrow 0} \int_E u_n V(x) \rho(x) dx = 0.$$

As a consequence the set of function $\{u_n \rho V\}$ is uniformly integrable. By Vitali's theorem $Vu_{n_k} \rightarrow Vu$ in $L^1_\rho(\Omega)$. Since

$$\int_\Omega (-u_n \Delta \zeta + Vu_n \zeta) dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\mu_n, \quad (2.22)$$

for any $\zeta \in T(\Omega)$, the function u satisfies (2.2). \square

Assumption (2.19) may be difficult to verify and the following result gives an easier formulation.

Proposition 2.7 *Assume $V \geq 0$ satisfies*

$$\lim_{\epsilon \rightarrow 0} \int_0^\epsilon \left(\int_{\Omega \cap B_r(y)} V(x) \rho^2(x) dx \right) \frac{dr}{r^{N+1}} = 0 \quad \text{uniformly with respect to } y \in \partial\Omega. \quad (2.23)$$

Then (2.19) holds.

Proof. If $E \subset \Omega$ is a Borel set and $\delta > 0$, we put $E_\delta = E \cap B_\delta(y)$ and $E_\delta^c = E \setminus E_\delta$. Then

$$\int_E \frac{V(x) \rho^2(x)}{|x-y|^N} dx = \int_{E_\delta} \frac{V(x) \rho^2(x)}{|x-y|^N} dx + \int_{E_\delta^c} \frac{V(x) \rho^2(x)}{|x-y|^N} dx.$$

Clearly

$$\int_{E_\delta^c} \frac{V(x) \rho^2(x)}{|x-y|^N} dx \leq \delta^{-N} \int_E V(x) \rho^2(x) dx. \quad (2.24)$$

Since (2.16) holds for any $y \in \partial\Omega$, (2.18) implies

$$\int_{E_\delta} \frac{V(x) \rho^2(x)}{|x-y|^N} dx = \delta^{-N} \int_{E_\delta} V(x) \rho^2(x) dx + N \int_0^\delta \left(\int_{E \cap B_r(y)} V(x) \rho^2(x) dx \right) \frac{dr}{r^{N+1}}. \quad (2.25)$$

Using (2.23), for any $\epsilon > 0$, there exists $s_0 > 0$ such that for any $s > 0$ and $y \in \partial\Omega$

$$s \leq s_0 \implies N \int_0^s \left(\int_{B_r(y)} V(x) \rho^2(x) dx \right) \frac{dr}{r^{N+1}} \leq \epsilon/2.$$

We fix $\delta = s_0$. Since (2.8) holds,

$$\lim_{\substack{E \text{ Borel} \\ |E| \rightarrow 0}} \int_E V(x) \rho^2(x) dx = 0. \quad (2.26)$$

Then there exists $\eta > 0$ such that for any Borel set $E \subset \Omega$,

$$|E| \leq \eta \implies \int_E V(x) \rho^2(x) dx \leq s_0^N \epsilon/4.$$

Thus

$$\int_E \frac{V(x)\rho^2(x)}{|x-y|^N} dx \leq \epsilon.$$

This implies the claim by (2.14). \square

An assumption which is used in [18, Lemma 7.4] in order to prove the existence of a boundary trace of any positive solution of (1.22) is that there exists some nonnegative measurable function v defined on \mathbb{R}_+ such that

$$|V(x)| \leq v(\rho(x)) \quad \forall x \in \Omega \quad \text{and} \quad \int_0^s tv(t)dt < \infty \quad \forall s > 0. \quad (2.27)$$

In the next result we show that condition (2.27) implies (2.19).

Proposition 2.8 *Assume V satisfies (2.27). Then*

$$\lim_{\substack{E \text{ Borel} \\ |E| \rightarrow 0}} \int_E K^\Omega(x, y) |V(x)| \rho(x) dx = 0 \quad \text{uniformly with respect to } y \in \partial\Omega. \quad (2.28)$$

Proof. Since $\partial\Omega$ is C^2 , there exist $\epsilon_0 > 0$ such that any for any $x \in \Omega$ satisfying $\rho(x) \leq \epsilon_0$, there exists a unique $\sigma(x) \in \partial\Omega$ such that $|x - \sigma(x)| = \rho(x)$. We use (2.23) in Proposition 2.7 under the equivalent form

$$\lim_{\epsilon \rightarrow 0} \int_0^\epsilon \left(\int_{\Omega \cap C_r(y)} |V(x)| \rho^2(x) dx \right) \frac{dr}{r^{N+1}} = 0 \quad \text{uniformly with respect to } y \in \partial\Omega, \quad (2.29)$$

in which we have replaced $B_r(y)$ by the the cylinder $C_r(y) := \{x \in \Omega : \rho(x) < r, |\sigma(x) - y| < r\}$. Then

$$\begin{aligned} \int_0^\epsilon \left(\int_{\Omega \cap C_r(y)} |V(x)| \rho^2(x) dx \right) \frac{dr}{r^{N+1}} &\leq c \int_0^\epsilon \left(\int_0^r v(t) t^2 dt \right) \frac{dr}{r^2} \\ &\leq c \int_0^\epsilon v(t) \left(1 - \frac{t}{\epsilon} \right) t dt \\ &\leq c \int_0^\epsilon v(t) t dt. \end{aligned}$$

Thus (2.23) holds. \square

3 The capacity approach

Throughout this section V is a locally bounded nonnegative and measurable function defined on Ω . We assume that there exists a positive measure μ_0 on $\partial\Omega$ such that

$$\int_\Omega \mathbb{K}[\mu_0] V(x) \rho(x) dx = \mathcal{E}(1, \mu_0) < \infty. \quad (3.1)$$

Definition 3.1 *If $\mu \in \mathfrak{M}_+(\partial\Omega)$ and f is a nonnegative measurable function defined in Ω such that*

$$(x, y) \mapsto \mathbb{K}[\mu](y)f(x)V(x)\rho(x) \in L^1(\Omega \times \partial\Omega; dx \otimes d\mu),$$

we set

$$\mathcal{E}(f, \mu) = \int_{\Omega} \left(\int_{\partial\Omega} K^{\Omega}(x, y) d\mu(y) \right) f(x)V(x)\rho(x) dx. \quad (3.2)$$

If we put

$$\check{\mathbb{K}}_V[f](y) = \int_{\Omega} K^{\Omega}(x, y) f(x)V(x)\rho(x) dx, \quad (3.3)$$

then, by Fubini's theorem, $\check{\mathbb{K}}_V[f] < \infty$, μ -almost everywhere on $\partial\Omega$ and

$$\mathcal{E}(f, \mu) = \int_{\partial\Omega} \left(\int_{\Omega} K^{\Omega}(x, y) f(x)V(x)\rho(x) dx \right) d\mu(y). \quad (3.4)$$

Proposition 3.2 *Let f be fixed. Then*

(a) $y \mapsto \check{\mathbb{K}}_V[f](y)$ *is lower semicontinuous on $\partial\Omega$.*

(b) $\mu \mapsto \mathcal{E}(f, \mu)$ *is lower semicontinuous on $\mathfrak{M}_+(\partial\Omega)$ in the weak*-topology*

Proof. Since $y \mapsto K^{\Omega}(x, y)$ is continuous, statement (a) follows by Fatou's lemma. If μ_n is a sequence in $\mathfrak{M}_+(\partial\Omega)$ converging to some μ in the weak*-topology, then $\mathbb{K}[\mu_n]$ converges to $\mathbb{K}[\mu]$ everywhere in Ω . By Fatou's lemma

$$\mathcal{E}(f, \mu) \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \mathbb{K}[\mu_n](x) f(x)V(x)\rho(x) dx = \liminf_{n \rightarrow \infty} \mathcal{E}(f, \mu_n). \quad \square$$

Notice that if $V\rho f \in L^p(\Omega)$, for $p > N$, then $\mathbb{G}[Vf\rho] \in C^1(\bar{\Omega})$ and

$$\check{\mathbb{K}}[f](y) := \int_{\Omega} K^{\Omega}(x, y) V(x) f(x) \rho(x) dx = -\frac{\partial}{\partial \mathbf{n}} \mathbb{G}[Vf\rho](y). \quad (3.5)$$

This is in particular the case if f has compact support in Ω .

Definition 3.3 *We denote by $\mathfrak{M}^V(\partial\Omega)$ the set of all measures μ on $\partial\Omega$ such that $V\mathbb{K}[\mu] \in L^1_{\rho}(\Omega)$. If μ is such a measure, we denote*

$$\|\mu\|_{\mathfrak{M}^V} = \int_{\Omega} |\mathbb{K}[\mu](x)| V(x)\rho(x) dx = \|V\mathbb{K}[\mu]\|_{L^1_{\rho}}. \quad (3.6)$$

Clearly $\|\cdot\|_{\mathfrak{M}^V}$ is a norm. The space $\mathfrak{M}^V(\partial\Omega)$ is not complete but its positive cone $\mathfrak{M}^V_+(\partial\Omega)$ is complete. If $E \subset \partial\Omega$ is a Borel subset, we put

$$\mathfrak{M}_+(E) = \{\mu \in \mathfrak{M}_+(\partial\Omega) : \mu(E^c) = 0\} \quad \text{and} \quad \mathfrak{M}^V_+(E) = \mathfrak{M}_+(E) \cap \mathfrak{M}^V(\partial\Omega).$$

Definition 3.4 If $E \subset \partial\Omega$ is any Borel subset we set

$$C_V(E) := \sup\{\mu(E) : \mu \in \mathfrak{M}_+^V(E), \|\mu\|_{\mathfrak{M}^V} \leq 1\}. \quad (3.7)$$

We notice that (3.7) is equivalent to

$$C_V(E) := \sup\left\{\frac{\mu(E)}{\|\mu\|_{\mathfrak{M}^V}} : \mu \in \mathfrak{M}_+^V(E)\right\}. \quad (3.8)$$

Proposition 3.5 The set function C_V satisfies.

$$C_V(E) \leq \sup_{y \in E} \left(\int_{\Omega} K^{\Omega}(x, y) V(x) \rho(x) dx \right)^{-1} \quad \forall E \subset \partial\Omega, E \text{ Borel}, \quad (3.9)$$

and equality holds in (3.9) if E is compact. Moreover,

$$C_V(E_1 \cup E_2) = \sup\{C_V(E_1), C_V(E_2)\} \quad \forall E_i \subset \partial\Omega, E_i \text{ Borel}. \quad (3.10)$$

Proof. Notice that $E \mapsto C_V(E)$ is a nondecreasing set function for the inclusion relation and that (3.7) implies

$$\mu(E) \leq C_V(E) \|\mu\|_{\mathfrak{M}^V} \quad \forall \mu \in \mathfrak{M}_+^V(E). \quad (3.11)$$

Let $E \subset \partial\Omega$ be a Borel set and $\mu \in \mathfrak{M}_+(E)$. Then

$$\begin{aligned} \|\mu\|_{\mathfrak{M}^V} &= \int_E \left(\int_{\Omega} K^{\Omega}(x, y) V(x) \rho(x) dx \right) d\mu(y) \\ &\geq \mu(E) \inf_{y \in E} \int_{\Omega} K^{\Omega}(x, y) V(x) \rho(x) dx. \end{aligned}$$

Using (3.7) we derive

$$C_V(E) \leq \sup_{y \in E} \left(\int_{\Omega} K^{\Omega}(x, y) V(x) \rho(x) dx \right)^{-1}. \quad (3.12)$$

If E is compact, there exists $y_0 \in E$ such that

$$\inf_{y \in E} \int_{\Omega} K^{\Omega}(x, y) V(x) \rho(x) dx = \int_{\Omega} K^{\Omega}(x, y_0) V(x) \rho(x) dx,$$

since $y \mapsto \check{\mathbb{K}}[1](y)$ is l.s.c.. Thus

$$\|\delta_{y_0}\|_{\mathfrak{M}^V} = \delta_{y_0}(E) \int_{\Omega} K^{\Omega}(x, y_0) V(x) \rho(x) dx$$

and

$$C_V(E) \geq \frac{\delta_{y_0}(E)}{\|\delta_{y_0}\|_{\mathfrak{M}^V}} = \sup_{y \in E} \left(\int_{\Omega} K^{\Omega}(x, y) V(x) \rho(x) dx \right)^{-1}.$$

Therefore equality holds in (3.9). Identity (3.10) follows (3.9) when there is equality. Moreover it holds if E_1 and E_2 are two arbitrary compact sets. Since C_V is eventually an inner regular

capacity (i.e. $C_V(E) = \sup\{C_V(K) : K \subset E, K \text{ compact}\}$) it holds for any Borel set. However we give below a self-contained proof. If E_1 and E_2 be two disjoint Borel subsets of $\partial\Omega$, for any $\epsilon > 0$ there exists $\mu \in \mathfrak{M}_+^V(E_1 \cup E_2)$ such that

$$\frac{\mu(E_1) + \mu(E_2)}{\|\mu\|_{\mathfrak{M}^V}} \leq C_V(E_1 \cup E_2) \leq \frac{\mu(E_1) + \mu(E_2)}{\|\mu\|_{\mathfrak{M}^V}} + \epsilon.$$

Set $\mu_i = \chi_{E_i} \mu$. Then $\mu_i \in \mathfrak{M}_+^V(E_i)$ and $\|\mu\|_{\mathfrak{M}^V} = \|\mu_1\|_{\mathfrak{M}^V} + \|\mu_2\|_{\mathfrak{M}^V}$. By (3.11)

$$C_V(E_1 \cup E_2) \leq \frac{\|\mu_1\|_{\mathfrak{M}^V}}{\|\mu_1\|_{\mathfrak{M}^V} + \|\mu_2\|_{\mathfrak{M}^V}} C_V(E_1) + \frac{\|\mu_2\|_{\mathfrak{M}^V}}{\|\mu_1\|_{\mathfrak{M}^V} + \|\mu_2\|_{\mathfrak{M}^V}} C_V(E_2) + \epsilon \quad (3.13)$$

This implies that there exists $\theta \in [0, 1]$ such that

$$C_V(E_1 \cup E_2) \leq \theta C_V(E_1) + (1 - \theta) C_V(E_2) \leq \max\{C_V(E_1), C_V(E_2)\}. \quad (3.14)$$

Since $C_V(E_1 \cup E_2) \geq \max\{C_V(E_1), C_V(E_2)\}$ as C_V is increasing,

$$E_1 \cap E_2 = \emptyset \implies C_V(E_1 \cup E_2) = \max\{C_V(E_1), C_V(E_2)\}. \quad (3.15)$$

If $E_1 \cap E_2 \neq \emptyset$, then $E_1 \cup E_2 = E_1 \cup (E_2 \cap E_1^c)$ and therefore

$$C_V(E_1 \cup E_2) = \max\{C_V(E_1), C_V(E_2 \cap E_1^c)\} \leq \max\{C_V(E_1), C_V(E_2)\}.$$

Using again (3.8) we derive (3.10). □

The following set function is the dual expression of $C_V(E)$.

Definition 3.6 For any Borel set $E \subset \partial\Omega$, we set

$$C_V^*(E) := \inf\{\|f\|_{L^\infty} : \mathbb{K}[f](y) \geq 1 \quad \forall y \in E\}. \quad (3.16)$$

The next result is stated in [11, p 922] using minimax theorem and the fact that K^Ω is lower semi continuous in $\Omega \times \partial\Omega$. Although the proof is not explicated, a simple adaptation of the proof of [1, Th 2.5.1] leads to the result.

Proposition 3.7 For any compact set $E \subset \partial\Omega$,

$$C_V(E) = C_V^*(E). \quad (3.17)$$

In the same paper [11], formula (3.9) with equality is claimed (if E is compact).

Theorem 3.8 If $\{\mu_n\}$ is an increasing sequence of good measures converging to some measure μ in the weak* topology, then μ is good.

Proof. We use formulation (4.10). We take for test function the function η solution of

$$\begin{cases} -\Delta\eta = 1 & \text{in } \Omega \\ \eta = 0 & \text{on } \Omega, \end{cases} \quad (3.18)$$

there holds

$$\int_{\Omega} (1+V) u_{\mu_n} \eta dx = - \int_{\partial\Omega} \frac{\partial\eta}{\partial\mathbf{n}} d\mu_n \leq c^{-1} \mu_n(\partial\Omega) \leq c^{-1} \mu(\partial\Omega)$$

where $c > 0$ is such that

$$c^{-1} \geq -\frac{\partial\eta}{\partial\mathbf{n}} \geq c \quad \text{on } \partial\Omega.$$

Since $\{u_{\mu_n}\}$ is increasing and $\eta \leq c\rho$ by Hopf boundary lemma, we can let $n \rightarrow \infty$ by the monotone convergence theorem. If $u := \lim_{n \rightarrow \infty} u_{\mu_n}$, we obtain

$$\int_{\Omega} (1+V) u \eta dx \leq c^{-1} \mu(\partial\Omega).$$

Thus u and $\rho V u$ are in $L^1(\Omega)$. Next, if $\zeta \in C_0^1(\overline{\Omega}) \cap C^{1,1}(\overline{\Omega})$, then $u_{\mu_n} |\Delta\zeta| \leq C u_{\mu_n}$ and $V u_{\mu_n} |\zeta| \leq C V u_{\mu_n} \eta$. Because the sequence $\{u_{\mu_n}\}$ and $\{V u_{\mu_n} \eta\}$ are uniformly integrable, the same holds for $\{u_{\mu_n} \Delta\zeta\}$ and $\{V u_{\mu_n} \zeta\}$. Considering

$$\int_{\Omega} (-u_{\mu_n} \Delta\zeta + V u_{\mu_n} \zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu_n.$$

it follows by Vitali's theorem,

$$\int_{\Omega} (-u \Delta\zeta + V u \zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu.$$

Thus μ is a good measure. □

We define the *singular boundary set* Z_V by

$$Z_V = \left\{ y \in \partial\Omega : \int_{\Omega} K^{\Omega}(x, y) V(x) \rho(x) dx = \infty \right\}. \quad (3.19)$$

Since $\mathbb{K}[1]$ is l.s.c., it is a Borel function and Z_V is a Borel set. The next result characterizes the good measures.

Proposition 3.9 *Let μ be an admissible positive measure. Then $\mu(Z_V) = 0$.*

Proof. If $K \subset Z_V$ is compact, $\mu_K = \chi_K \mu$ is admissible, thus, by Fubini theorem

$$\|\mu_K\|_{\mathfrak{M}^V} = \int_K \left(\int_{\Omega} K^{\Omega}(x, y) V(x) \rho(x) dx \right) d\mu(y) < \infty.$$

Since

$$\int_{\Omega} K^{\Omega}(x, y) V(x) \rho(x) dx \equiv \infty \quad \forall y \in K$$

it follows that $\mu(K) = 0$. This implies $\mu(Z_V) = 0$ by regularity. □

Theorem 3.10 *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$ such that*

$$\mu(Z_V) = 0. \quad (3.20)$$

Then μ is good.

Proof. Since $\check{\mathbb{K}}[1]$ is l.s.c., for any $n \in \mathbb{N}_*$,

$$K_n := \{y \in \partial\Omega : \check{\mathbb{K}}[1](y) \leq n\}$$

is a compact subset of $\partial\Omega$. Furthermore $K_n \cap Z_V = \emptyset$ and $\cup K_n = Z_V^c$. Let $\mu_n = \chi_{K_n} \mu$, then

$$\mathcal{E}(1, \mu_n) = \int_{\Omega} \mathbb{K}[\mu_n] V(x) \rho(x) dx \leq n \mu_n(K_n). \quad (3.21)$$

Therefore μ_n is admissible. By the monotone convergence theorem, $\mu_n \uparrow \chi_{Z_V^c} \mu$ and by Theorem 3.8, $\chi_{Z_V^c} \mu$ is good. Since (5.7) holds, $\chi_{Z_V^c} \mu = \mu$, which ends the proof. \square

The full characterization of the good measures in the general case appears to be difficult without any further assumptions on V . However the following holds

Theorem 3.11 *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$ be a good measure. The following assertions are equivalent:*

(i) $\mu(Z_V) = 0$.

(ii) *There exists an increasing sequence of admissible measures $\{\mu_n\}$ which converges to μ in the weak*-topology.*

Proof. If (i) holds, it follows from the proof of Theorem 3.10 that the sequence $\{\mu_n\}$ increases and converges to μ . If (ii) holds, any admissible measure μ_n vanishes on Z_V by Proposition 3.9. Since $\mu_n \leq \mu$, there exists an increasing sequence of μ -integrable functions h_n such that $\mu_n = h_n \mu$. Then $\mu_n(Z_V)$ increases to $\mu(Z_V)$ by the monotone convergence theorem. The conclusion follows from the fact that $\mu_n(Z_V) = 0$. \square

4 Representation formula and reduced measures

We recall the construction of the Poisson kernel for $-\Delta + V$: if we look for a solution of

$$\begin{cases} -\Delta v + V(x)v = 0 & \text{in } \Omega \\ v = \nu & \text{in } \partial\Omega, \end{cases} \quad (4.1)$$

where $\nu \in \mathfrak{M}(\partial\Omega)$, $V \geq 0$, $V \in L_{loc}^\infty(\Omega)$, we can consider an increasing sequence of smooth domains Ω_n such that $\overline{\Omega}_n \subset \Omega_{n+1}$ and $\cup_n \Omega_n = \cup_n \overline{\Omega}_n = \Omega$. For each of these domains, denote by $K_{V\chi_{\Omega_n}}^\Omega$ the Poisson kernel of $-\Delta + V\chi_{\Omega_n}$ in Ω and by $\mathbb{K}_{V\chi_{\Omega_n}}[\cdot]$ the corresponding operator. We denote by $K_0^\Omega := K_0^\Omega$ the Poisson kernel in Ω and by $\mathbb{K}[\cdot]$ the Poisson operator in Ω . Then the solution $v := v_n$ of

$$\begin{cases} -\Delta v + V\chi_{\Omega_n} v = 0 & \text{in } \Omega \\ v = \nu & \text{in } \partial\Omega, \end{cases} \quad (4.2)$$

is expressed by

$$v_n(x) = \int_{\partial\Omega} K_{V\chi_{\Omega_n}}^\Omega(x, y) d\nu(y) = \mathbb{K}_{V\chi_{\Omega_n}}[\nu](x). \quad (4.3)$$

If G^Ω is the Green kernel of $-\Delta$ in Ω and $\mathbb{G}[\cdot]$ the corresponding Green operator, (4.3) is equivalent to

$$v_n(x) + \int_{\Omega} G^\Omega(x, y)(V\chi_{\Omega_n} v_n)(y) dy = \int_{\partial\Omega} K^\Omega(x, y) d\nu(y), \quad (4.4)$$

equivalently

$$v_n + \mathbb{G}[V\chi_{\Omega_n} v_n] = \mathbb{K}[\nu].$$

Notice that this equality is equivalent to the weak formulation of problem (4.2): for any $\zeta \in T(\Omega)$, there holds

$$\int_{\Omega} (-v_n \Delta \zeta + V\chi_{\Omega_n} v_n \zeta) dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\nu. \quad (4.5)$$

Since $n \mapsto K_{V\chi_{\Omega_n}}^\Omega$ is decreasing, the sequence $\{v_n\}$ inherits this property and there exists

$$\lim_{n \rightarrow \infty} K_{V\chi_{\Omega_n}}^\Omega(x, y) = K_V^\Omega(x, y). \quad (4.6)$$

By the monotone convergence theorem,

$$\lim_{n \rightarrow \infty} v_n(x) = v(x) = \int_{\partial\Omega} K_V^\Omega(x, y) d\nu(y). \quad (4.7)$$

By Fatou's theorem

$$\int_{\Omega} G^\Omega(x, y) V(y) v(y) dy \leq \liminf_{n \rightarrow \infty} \int_{\Omega} G^\Omega(x, y) (V\chi_{\Omega_n} v_n)(y) dy, \quad (4.8)$$

and thus,

$$v(x) + \int_{\Omega} G^\Omega(x, y) V(y) v(y) dy \leq \mathbb{K}[\nu](x) \quad \forall x \in \Omega. \quad (4.9)$$

Now the main question is to know whether v keeps the boundary value ν . Equivalently, whether the equality holds in (4.8) with \lim instead of \liminf , and therefore in (4.9). This question is associated to the notion of reduced measured in the sense of Brezis-Marcus-Ponce: Since $Vv \in L^1_\rho(\Omega)$ and

$$-\Delta v + V(x)v = 0 \quad \text{in } \Omega \quad (4.10)$$

holds, the function $v + \mathbb{G}[Vv]$ is positive and harmonic in Ω . Thus it admits a boundary trace $\nu^* \in \mathfrak{M}_+(\partial\Omega)$ and

$$v + \mathbb{G}[Vv] = \mathbb{K}[\nu^*]. \quad (4.11)$$

Equivalently v satisfies the relaxed problem

$$\begin{cases} -\Delta v + V(x)v = 0 & \text{in } \Omega \\ v = \nu^* & \text{in } \partial\Omega, \end{cases} \quad (4.12)$$

and thus $v = u_{\nu^*}$. Noticed that $\nu^* \leq \nu$ and the mapping $\nu \mapsto \nu^*$ is nondecreasing.

Definition 4.1 *The measure ν^* is the reduced measure associated to ν .*

Proposition 4.2 *There holds $\mathbb{K}_V[\nu] = \mathbb{K}_V[\nu^*]$. Furthermore the reduced measure ν^* is the largest measure for which the following problem*

$$\begin{cases} -\Delta v + V(x)v = 0 & \text{in } \Omega \\ \lambda \in \mathfrak{M}_+(\partial\Omega), \lambda \leq \nu \\ v = \lambda & \text{in } \partial\Omega, \end{cases} \quad (4.13)$$

admits a solution.

Proof. The first assertion follows from the fact that $v = \mathbb{K}_V[\nu]$ by (4.6) and $v = u_{\nu^*} = \mathbb{K}_V[\nu^*]$ by (4.12). It is clear that $\nu^* \leq \nu$ and that the problem (4.13) admits a solution for $\lambda = \nu^*$. If λ is a positive measure smaller than μ , then $\lambda^* \leq \mu^*$. But if there exist some λ such that the problem (4.13) admits a solution, then $\lambda = \lambda^*$. This implies the claim. \square

As a consequence of the characterization of ν^* there holds

Corollary 4.3 *Assume $V \geq 0$ and let $\{V_k\}$ be an increasing sequence of nonnegative bounded measurable functions converging to V a.e. in Ω . Then the solution u_k of*

$$\begin{cases} -\Delta u + V_k u = 0 & \text{in } \Omega \\ u = \nu & \text{in } \partial\Omega, \end{cases} \quad (4.14)$$

converges to u_{ν^} .*

Proof. The previous construction shows that $u_k = \mathbb{K}_{V_k}[\nu]$ decreases to some \tilde{u} which satisfies a relaxed equation, the boundary data of which, $\tilde{\nu}^*$, is the largest measure $\lambda \leq \nu$ for which problem (4.13) admits a solution. Therefore $\tilde{\nu}^* = \nu^*$ and $\tilde{u} = u_{\nu^*}$. Similarly $\{K_{V_k}^\Omega\}$ decreases and converges to K_V^Ω . \square

We define the boundary vanishing set of K_V^Ω by

$$Z_V^* := \{y \in \partial\Omega \mid K_V^\Omega(x, y) = 0\} \quad \text{for some } x \in \Omega. \quad (4.15)$$

Since $V \in L_{loc}^\infty(\Omega)$, Z_V^* is independent of x by Harnack inequality; furthermore it is a Borel set.

Theorem 4.4 *Let $\nu \in \mathfrak{M}_+(\partial\Omega)$.*

(i) If $\nu((Z_V^)^c) = 0$, then $\nu^* = 0$.*

(ii) There always holds $Z_V^ \subset Z_V$.*

Proof. The first assertion is clear since $\nu = \chi_{Z_V^*} \nu + \chi_{(Z_V^*)^c} \nu = \chi_{Z_V^*} \nu$ and, by Proposition 4.2,

$$u_{\nu^*}(x) = \mathbb{K}_V[\nu^*](x) = \int_{Z_V^*} K_V^\Omega(x, y) d\nu(y) = 0 \quad \forall x \in \Omega,$$

by definition of Z_V^* . For proving (ii), we assume that $C_V(Z_V^*) > 0$; there exists $\mu \in \mathfrak{M}_+^V(Z_V^*)$ such that $\mu(Z_V^*) > 0$. Since μ is admissible let u_μ be the solution of (1.1). Then $\mu^* = \mu$, thus $u_\mu = \mathbb{K}^V[\mu]$ and

$$\mathbb{K}^V[\mu](x) = \int_{\partial\Omega} K_V^\Omega(x, y) d\mu(y) = \int_{Z_V^*} K_V^\Omega(x, y) d\mu(y) = 0,$$

contradiction. Thus $C_V(Z_V^*) = 0$. Since (3.9) implies that Z_V is the largest Borel set with zero C_V -capacity, it implies $Z_V^* \subset Z_V$. \square

In order to obtain more precise informations on Z_V^* some minimal regularity assumptions on V are needed. We also recall the following result proved by Ancona [2].

Theorem 4.5 *Assume $V \geq 0$ satisfies $\rho^2 V \in L^\infty(\Omega)$. If for some $y_0 \in \partial\Omega$ and any cone C_{y_0} with vertex y_0 having the property that $\overline{C}_{y_0} \cap B_r(y_0) \subset \Omega \cup \{y_0\}$ for some $r > 0$, there exists $c_1 > 0$ such that*

$$\forall (x, y) \in \Omega \cap B_r(y_0) \times \Omega \cap B_r(y_0), |x - y_0| = |y - y_0| \leq r \implies c^{-1} \leq \frac{V(x)}{V(y)} \leq c_1 \quad (4.16)$$

and

$$\int_0^r V(t\mathbf{n}_{y_0}) t dt = \infty, \quad (4.17)$$

where \mathbf{n}_0 is the normal outward unit vector to $\partial\Omega$ at y_0 , then

$$K_V^\Omega(x, y_0) = 0 \quad \forall x \in \Omega. \quad (4.18)$$

We define the *conical singular boundary set*

$$\tilde{Z}_V = \left\{ y \in \partial\Omega : \int_{\Omega \cap C_y} K^\Omega(x, y) V(x) \rho(x) dx = \infty \text{ for some cone } C_y \Subset \Omega \right\} \quad (4.19)$$

where $C_y \Subset \Omega$ means that there exists $a > 0$ such that $\overline{C}_y \cap B_a(y) \subset \Omega \cup \{y\}$. Clearly $\tilde{Z}_V \subset Z_V$.

Corollary 4.6 *Assume $V \geq 0$ satisfies $\rho^2 V \in L^\infty(\Omega)$ and the conical oscillation condition (4.16) of Theorem 4.5 for any $y \in Z_V$. Then $\tilde{Z}_V = Z_V^*$.*

Proof. We can assume that $y = 0$ and denote $C_y = C$. Since

$$K^\Omega(x, 0) V(x) \rho(x) \leq ca^{-N} V(x) \rho^2(x) \quad \forall x \in \Omega \cap B_a^c,$$

and $V\rho^2 \in L^1(\Omega)$, there holds, using (2.14),

$$\int_{B_a \cap C} V(x) \rho^2(x) \frac{dx}{|x|^N} = \infty.$$

Using spherical coordinates and the fact that $\rho^2(x) \geq c|x|$ in $B_a \cap C_y$,

$$\int_0^a \int_S V(r, \sigma) r d\sigma dr = \infty.$$

where $S = C \cap \partial B_1$. But in $C \cap B_a$ the oscillation condition (4.16) holds. This implies

$$\int_0^a V(r, \sigma) t dt = \infty \quad \forall \sigma \in S. \quad (4.20)$$

Thus $y \in Z_V^*$. \square

5 The boundary trace

5.1 The regular part

In this section, $V \in L_{loc}^\infty(\Omega)$ is nonnegative. If $0 < \epsilon \leq \epsilon_0$, we denote $d(x) = \text{dist}(x, \partial\Omega)$ for $x \in \Omega$, and set $\Omega_\epsilon := \{x \in \Omega : d(x) > \epsilon\}$, $\Omega'_\epsilon = \Omega \setminus \Omega_\epsilon$ and $\Sigma_\epsilon = \partial\Omega_\epsilon$. It is well known that there exists ϵ_0 such that, for any $0 < \epsilon \leq \epsilon_0$ and any $x \in \Omega'_\epsilon$ there exists a unique projection $\sigma(x)$ of x on $\partial\Omega$ and any $x \in \Omega'_\epsilon$ can be written in a unique way under the form

$$x = \sigma(x) - d(x)\mathbf{n}$$

where \mathbf{n} is the outward normal unit vector to $\partial\Omega$ at $\sigma(x)$. The mapping $x \mapsto (d(x), \sigma(x))$ is a C^2 diffeomorphism from Ω'_ϵ to $(0, \epsilon_0] \times \partial\Omega$. We recall the following definition given in [18]. If \mathcal{A} is a Borel subset of $\partial\Omega$, we set $\mathcal{A}_\epsilon = \{x \in \Sigma_\epsilon : \sigma(x) \in \mathcal{A}\}$.

Definition 5.1 *Let \mathcal{A} be a relatively open subset of $\partial\Omega$, $\{\mu_\epsilon\}$ be a set of Radon measures on \mathcal{A}_ϵ ($0 < \epsilon \leq \epsilon_0$) and $\mu \in \mathfrak{M}(\mathcal{A})$. We say that $\mu_\epsilon \rightarrow \mu$ in the weak*-topology if, for any $\zeta \in C_c(\mathcal{A})$,*

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{A}_\epsilon} \zeta(\sigma(x)) d\mu_\epsilon(x) = \int_{\mathcal{A}} \zeta d\mu. \quad (5.1)$$

A function $u \in C(\Omega)$ possesses a boundary trace $\mu \in \mathfrak{M}(\mathcal{A})$ if

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{A}_\epsilon} \zeta(\sigma(x)) u(x) dS(x) = \int_{\mathcal{A}} \zeta d\mu \quad \forall \zeta \in C_c(\mathcal{A}). \quad (5.2)$$

The following result is proved in [18, p 694].

Proposition 5.2 *Let $u \in C(\Omega)$ be a positive solution of*

$$-\Delta u + V(x)u = 0 \quad \text{in } \Omega. \quad (5.3)$$

Assume that, for some $z \in \partial\Omega$, there exists an open neighborhood U of z such that

$$\int_{U \cap \Omega} V u \rho(x) dx < \infty. \quad (5.4)$$

Then $u \in L^1(K \cap \Omega)$ for any compact subset $K \subset G$ and there exists a positive Radon measure μ on $\mathcal{A} = U \cap \partial\Omega$ such that

$$\lim_{\epsilon \rightarrow 0} \int_{U \cap \Sigma_\epsilon} \zeta(\sigma(x)) u(x) dS(x) = \int_{\mathcal{A}} \zeta d\mu \quad \forall \zeta \in C_c(U \cap \Omega). \quad (5.5)$$

Notice that any continuous solution of (5.3) in Ω belongs to $W_{loc}^{2,p}(\Omega)$ for any $(1 \leq p < \infty)$. This previous result yields to a natural definition of the regular boundary points.

Definition 5.3 *Let $u \in C(\Omega)$ be a positive solution of (5.3). A point $z \in \partial\Omega$ is called a regular boundary point for u if there exists an open neighborhood U of z such that (5.31) holds. The set of regular boundary points is a relatively open subset of $\partial\Omega$, denoted by $\mathcal{R}(u)$. The set $\mathcal{S}(u) = \partial\Omega \setminus \mathcal{R}(u)$ is the singular boundary set of u . It is a closed set.*

By Proposition 5.2 and using a partition of unity, we see that there exists a positive Radon measure $\mu := \mu_u$ on $\mathcal{R}(u)$ such that (5.5) holds with U replaced by $\mathcal{R}(u)$. The couple $(\mu_u, \mathcal{S}(u))$ is called the **boundary trace of u** . *The main question of the boundary trace problem is to analyse the behaviour of u near the set $\mathcal{S}(u)$.*

For any positive good measure μ on $\partial\Omega$, we denote by u_μ the solution of (4.1) defined by (4.10)-(4.11).

Proposition 5.4 *Let $u \in C(\Omega) \cap W_{loc}^{2,p}(\Omega)$ for any $(1 \leq p < \infty)$ be a positive solution of (5.3) in Ω with boundary trace $(\mu_u, \mathcal{S}(u))$. Then $u \geq u_{\mu_u}$.*

Proof. Let $G \subset \partial\Omega$ be a relatively open subset such that $\overline{G} \subset \mathcal{R}(u)$ with a C^2 relative boundary $\partial^*G = \overline{G} \setminus G$. There exists an increasing sequence of C^2 domains Ω_n such that $\overline{G} \subset \partial\Omega_n$, $\partial\Omega_n \setminus \overline{G} \subset \Omega$ and $\cup_n \Omega_n = \Omega$. For any n , let $v := v_n$ be the solution of

$$\begin{cases} -\Delta v + Vv = 0 & \text{in } \Omega_n \\ v = \chi_G \mu & \text{in } \partial\Omega_n. \end{cases} \quad (5.6)$$

Let u_n be the restriction of u to Ω_n . Since $u \in C(\Omega)$ and $Vu \in L^1(\Omega_n)$, there also holds $Vu \in L^1(\Omega_n)$ where we have denoted by ρ_n the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega_n)$. Consequently u_n admits a regular boundary trace μ_n on $\partial\Omega_n$ (i.e. $\mathcal{R}(u_n) = \partial\Omega_n$) and u_n is the solution of

$$\begin{cases} -\Delta v + Vv = 0 & \text{in } \Omega_n \\ v = \mu_n & \text{in } \partial\Omega_n. \end{cases} \quad (5.7)$$

Furthermore $\mu_n|_G = \chi_G \mu_u$. It follows from Brezis estimates and in particular (2.5) that $u_n \leq u$ in Ω_n . Since $\Omega_n \subset \Omega_{n+1}$, $v_n \leq v_{n+1}$. Moreover

$$v_n + \mathbb{G}^{\Omega_n}[Vv_n] = \mathbb{K}^{\Omega_n}[\chi_G \mu] \quad \text{in } \Omega_n.$$

Since $\mathbb{K}^{\Omega_n}[\chi_G \mu] \rightarrow \mathbb{K}^\Omega[\chi_G \mu_u]$, and the Green kernels $G^{\Omega_n}(x, y)$ are increasing with n , it follows from monotone convergence that $v_n \uparrow v$ and there holds

$$v + \mathbb{G}^\Omega[Vv] = \mathbb{K}^\Omega[\chi_G \mu_u] \quad \text{in } \Omega.$$

Thus $v = u_{\chi_G \mu_u}$ and $u_{\chi_G \mu_u} \leq u$. We can now replace G by a sequence $\{G_k\}$ of relatively open sets with the same properties as G , $\overline{G}_k \subset G_k$ and $\cup_k G_k = \mathcal{R}(u)$. Then $\{u_{\chi_{G_k} \mu_u}\}$ is increasing and converges to some \tilde{u} . Since

$$u_{\chi_{G_k} \mu_u} + \mathbb{G}^\Omega[Vu_{\chi_{G_k} \mu_u}] = \mathbb{K}^\Omega[\chi_{G_k} \mu_u],$$

and $\mathbb{K}^\Omega[\chi_{G_k} \mu] \uparrow \mathbb{K}^\Omega[\mu_u]$, we derive

$$\tilde{u} + \mathbb{G}^\Omega[V\tilde{u}] = \mathbb{K}^\Omega[\mu_u].$$

This implies that $\tilde{u} = u_{\mu_u} \leq u$. □

5.2 The singular part

The following result is essentially proved in [18, Lemma 2.8].

Proposition 5.5 *Let $u \in C(\Omega)$ for any $(1 \leq p < \infty)$ be a positive solution of (5.3) and suppose that $z \in \mathcal{S}(u)$ and that there exists an open neighborhood U_0 of z such that $u \in L^1(\Omega \cap U_0)$. Then for any open neighborhood U of z , there holds*

$$\lim_{\epsilon \rightarrow 0} \int_{U \cap \Sigma_\epsilon} \zeta(\sigma(x))u(x)dS(x) = \infty. \quad (5.8)$$

As immediate consequences, we have

Corollary 5.6 *Assume u satisfies the regularity assumption of Proposition 5.4. Then for any $z \in \mathcal{S}(u)$ and any open neighborhood U of z , there holds*

$$\limsup_{\epsilon \rightarrow 0} \int_{U \cap \Sigma_\epsilon} \zeta(\sigma(x))u(x)dS(x) = \infty. \quad (5.9)$$

Corollary 5.7 *Assume u satisfies the regularity assumption of Proposition 5.4. If $u \in L^1(\Omega)$, Then for any $z \in \mathcal{S}(u)$ and any open neighborhood U of z , (5.8) holds.*

The two next results give conditions on V which imply that $\mathcal{S}(u) = \emptyset$.

Theorem 5.8 *Assume $N = 2$, V is nonnegative and satisfies (2.19). If u is a positive solution of (5.3), then $\mathcal{R}(u) = \partial\Omega$.*

Proof. We assume that

$$\int_{\Omega} V \rho u dx = \infty. \quad (5.10)$$

If $0 < \epsilon \leq \epsilon_0$, we denote by $(\rho_\epsilon, \lambda_\epsilon)$ are the normalized first eigenfunction and first eigenvalue of $-\Delta$ in $W_0^{1,2}(\Omega_\epsilon)$, then

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} V \rho_\epsilon u dx = \infty. \quad (5.11)$$

Because

$$\int_{\Omega_\epsilon} (\lambda_\epsilon + \rho_\epsilon V) u dx = - \int_{\partial\Omega_\epsilon} \frac{\partial \rho_\epsilon}{\partial \mathbf{n}} u dS,$$

and

$$c^{-1} \leq - \frac{\partial \rho_\epsilon}{\partial \mathbf{n}} \leq c,$$

for some $c > 1$ independent of ϵ , there holds

$$\lim_{\epsilon \rightarrow 0} \int_{\partial\Omega_\epsilon} u dS = \infty. \quad (5.12)$$

Denote by m_ϵ this last integral and set $v_\epsilon = m_\epsilon^{-1}u$ and $\mu_\epsilon = m_\epsilon^{-1}u|_{\partial\Omega_\epsilon}$. Then

$$v_\epsilon + \mathbb{G}^{\Omega_\epsilon}[V v_\epsilon] = \mathbb{K}^{\Omega_\epsilon}[\mu_\epsilon] \quad \text{in } \Omega_\epsilon \quad (5.13)$$

where

$$\mathbb{K}^{\Omega_\epsilon}[\mu_\epsilon](x) = \int_{\partial\Omega_\epsilon} K^{\Omega_\epsilon}(x, y)\mu_\epsilon(y)dS(y) \quad (5.14)$$

is the Poisson potential of μ_ϵ in Ω_ϵ and

$$\mathbb{G}^{\Omega_\epsilon}[Vu](x) = \int_{\Omega_\epsilon} G^{\Omega_\epsilon}(x, y)V(y)u(y)dy,$$

the Green potential of Vu in Ω_ϵ . Furthermore

$$\begin{cases} -\Delta v_\epsilon + Vv_\epsilon = 0 & \text{in } \Omega_\epsilon \\ v_\epsilon = \mu_\epsilon & \text{in } \partial\Omega_\epsilon. \end{cases} \quad (5.15)$$

By Brezis estimates and regularity theory for elliptic equations, $\{\chi_{\Omega_\epsilon} v_\epsilon\}$ is relatively compact in $L^1(\Omega)$ and in the local uniform topology of Ω_ϵ . Up to a subsequence $\{\epsilon_n\}$, μ_{ϵ_n} converges to a probability measure μ on $\partial\Omega$ in the weak*-topology. It is classical that

$$\mathbb{K}^{\Omega_{\epsilon_n}}[\mu_{\epsilon_n}] \rightarrow \mathbb{K}[\mu]$$

locally uniformly in Ω , and $\chi_{\Omega_{\epsilon_n}} v_{\epsilon_n} \rightarrow v$ in the local uniform topology of Ω , and a.e. in Ω . Because $G^{\Omega_\epsilon}(x, y) \uparrow G^\Omega(x, y)$, there holds for any $x \in \Omega$

$$\lim_{n \rightarrow \infty} \chi_{\Omega_{\epsilon_n}}(y)G^{\Omega_{\epsilon_n}}(x, y)V(y)v_{\epsilon_n}(y) = G^\Omega(x, y)V(y)v(y) \quad \text{for almost all } y \in \Omega \quad (5.16)$$

Furthermore $v_{\epsilon_n} \leq \mathbb{K}^{\Omega_{\epsilon_n}}[\mu_{\epsilon_n}]$ reads

$$v_{\epsilon_n}(y) \leq c\rho_{\epsilon_n}(y) \int_{\partial\Omega_n} \frac{\mu_{\epsilon_n}(z)dS(z)}{|y-z|^2}.$$

In order to go to the limit in the expression

$$L_n := \mathbb{G}^{\Omega_{\epsilon_n}}[Vv_{\epsilon_n}](x) = \int_{\Omega} \chi_{\Omega_{\epsilon_n}}(y)G^{\Omega_{\epsilon_n}}(x, y)V(y)v_{\epsilon_n}(y)dy, \quad (5.17)$$

we may assume that $x \in \Omega_{\epsilon_1}$ where $0 < \epsilon_1 \leq \epsilon_0$ is fixed and write $\Omega = \Omega_{\epsilon_1} \cup \Omega'_{\epsilon_1}$ where

$$\Omega'_{\epsilon_1} = \Omega \setminus \Omega_{\epsilon_1} := \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \epsilon_1\}$$

and $L_n = M_n + P_n$ where

$$M_n = \int_{\Omega_{\epsilon_1}} \chi_{\Omega_{\epsilon_n}}(y)G^{\Omega_{\epsilon_n}}(x, y)V(y)v_{\epsilon_n}(y)dy \quad (5.18)$$

and

$$P_n = \int_{\Omega'_{\epsilon_1}} \chi_{\Omega_{\epsilon_n}}(y)G^{\Omega_{\epsilon_n}}(x, y)V(y)v_{\epsilon_n}(y)dy. \quad (5.19)$$

Since

$$\begin{aligned} \chi_{\Omega_{\epsilon_1}}(y)G^{\Omega_{\epsilon_n}}(x, y)V(y)v_{\epsilon_n}(y) &\leq c\chi_{\Omega_{\epsilon_1}}(y)|\ln(|x-y||V(y)v_{\epsilon_n}(y)| \\ &\leq c\|V\|_{L^\infty(\Omega_{\epsilon_1})}\chi_{\Omega_{\epsilon_1}}(y)|\ln(|x-y||v_{\epsilon_n}(y)|), \end{aligned}$$

it follows by the dominated convergence theorem that

$$\lim_{n \rightarrow \infty} M_n = \int_{\Omega_{\epsilon_1}} G^\Omega(x, y) V(y) v(y) dy. \quad (5.20)$$

Let $E \subset \Omega$ be a Borel subset. Then $G^{\Omega_{\epsilon_n}}(x, y) \leq c(x) \rho_{\epsilon_n}(y)$ if $y \in \Omega'_{\epsilon_1}$. By Fubini,

$$\begin{aligned} \int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x, y) V(y) v_{\epsilon_n}(y) dy &\leq cc(x) \int_{\partial \Omega_n} \left(\int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) \frac{\rho_{\epsilon_n}^2(y) V(y)}{|y-z|^2} dy \right) \mu_{\epsilon_n}(z) dS(z) \\ &\leq cc(x) \max_{z \in \partial \Omega_{\epsilon_n}} \int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) \frac{\rho_{\epsilon_n}^2(y) V(y)}{|y-z|^2} dy \end{aligned} \quad (5.21)$$

If $y \in \Omega_{\epsilon_n} \cap E$, there holds $\rho(y) = \rho_{\epsilon_n}(y) + \epsilon_n$. If $z \in \partial \Omega_{\epsilon_n} \cap E$ and we denote by $\sigma(z)$ the projection of z onto $\partial \Omega$, there holds $|y - \sigma(z)| \leq |y - z| + \epsilon_n$. By monotonicity

$$\frac{\rho_{\epsilon_n}(y)}{|y-z|} \leq \frac{\rho_{\epsilon_n}(y) + \epsilon_n}{|y-z| + \epsilon_n} \leq \frac{\rho(y)}{|y-\sigma(z)|}, \quad (5.22)$$

thus

$$\int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x, y) V(y) v_{\epsilon_n}(y) dy \leq cc(x) \max_{z \in \partial \Omega} \int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) \frac{\rho^2(y) V(y)}{|y-z|^2} dy. \quad (5.23)$$

By (2.19) this last integral goes to zero if $|\Omega'_{\epsilon_1} \cap E \cap \Omega_{\epsilon_n}| \rightarrow 0$. Thus by Vitali's theorem, the sequence of functions $\{\chi_{\Omega_{\epsilon_n}}(\cdot) G^{\Omega_{\epsilon_n}}(x, \cdot) V(y) v_{\epsilon_n}(\cdot)\}_{n \in \mathbb{N}}$ is uniformly integrable in y , for any $x \in \Omega$. It implies that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x, y) V(y) v_{\epsilon_n}(y) dy = \int_{\Omega} G^\Omega(x, y) V(y) v(y) dy, \quad (5.24)$$

and there holds $v + \mathbb{G}[Vv] = \mathbb{K}[\mu]$. Since $u = m_\epsilon v_\epsilon$ in Ω and $m_\epsilon \rightarrow \infty$, we get a contradiction since it would imply $u \equiv \infty$. \square

In order to deal with the case $N \geq 3$ we introduce an additional assumption of stability.

Theorem 5.9 *Assume $N \geq 3$. Let $V \in L_{loc}^\infty(\Omega)$, $V \geq 0$ such that*

$$\lim_{\substack{E \text{ Borel} \\ |E| \rightarrow 0}} \int_E V(y) \frac{(\rho(y) - \epsilon)_+^2}{|y-z|^N} dy = 0 \quad \text{uniformly with respect to } z \in \Sigma_\epsilon \text{ and } \epsilon \in (0, \epsilon_0]. \quad (5.25)$$

If u is a positive solution of (5.3), then $\mathcal{R}(u) = \partial \Omega$.

Proof. We proceed as in Theorem 5.8. All the relations (5.10)-(5.20) are valid and (5.21) has to be replaced by

$$\int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x, y) V(y) v_{\epsilon_n}(y) dy \leq cc(x) \max_{z \in \Sigma_{\epsilon_n}} \int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) \frac{\rho_{\epsilon_n}^2(y) V(y)}{|y-z|^{N+1}} dy. \quad (5.26)$$

Since (5.22) is no longer valid, (5.22) is replaced by

$$\int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x, y) V(y) v_{\epsilon_n}(y) dy \leq cc(x) \max_{z \in \Sigma_{\epsilon_n}} \int_E V(y) \frac{(\rho(y) - \epsilon_n)_+^2}{|y - z|^{N+1}} dy. \quad (5.27)$$

By (5.25) the left-hand side of (5.27) goes to zero when $|E| \rightarrow 0$, uniformly with respect to ϵ_n . This implies that (5.29) is still valid and the conclusion of the proof is as in Theorem 5.8. \square

Remark. A simpler statement which implies (5.25) is the following.

$$\lim_{\delta \rightarrow 0} \int_0^\delta \left(\int_{B_r(z)} V(y) (\rho(y) - \epsilon)_+^2 dy \right) \frac{dr}{r^{N+1}} = 0, \quad (5.28)$$

uniformly with respect to $0 < \epsilon \leq \epsilon_0$ and to $z \in \Sigma_\epsilon$. The proof is similar to the one of Proposition 2.7.

Remark. When the function V depends essentially of the distance to $\partial\Omega$ in the sense that

$$|V(x)| \leq v(\rho(x)) \quad \forall x \in \Omega, \quad (5.29)$$

and v satisfies

$$\int_0^a tv(t)dt < \infty, \quad (5.30)$$

Marcus and Véron proved [18, Lemma 7.4] that $\mathcal{R}(u) = \partial\Omega$, for any positive solution u of (5.3). This assumption implies also (5.25). The proof is similar to the one of Proposition 2.8.

5.3 The sweeping method

This method introduced in [21] for analyzing isolated singularities of solutions of semilinear equations has been adapted in [15] and [19] for defining an extended trace of positive solutions of differential inequalities in particular in the super-critical case. Since the boundary trace of a positive solutions of (5.3) is known on $\mathcal{R}(u)$ we shall study the sweeping with measure concentrated on the singular set $\mathcal{S}(u)$

Proposition 5.10 *Let $u \in C(\Omega)$ be a positive solution of (5.3) with singular boundary set $\mathcal{S}(u)$. If $\mu \in \mathfrak{M}_+(\mathcal{S}(u))$ we denote $v_\mu = \inf\{u, u_\mu\}$. Then*

$$-\Delta v_\mu + V(x)v_\mu \geq 0 \quad \text{in } \Omega, \quad (5.31)$$

and v_μ admits a boundary trace $\gamma_u(\mu) \in \mathfrak{M}_+(\mathcal{S}(u))$. The mapping $\mu \mapsto \gamma_u(\mu)$ is nondecreasing and $\gamma_u(\mu) \leq \mu$.

Proof. We know that (5.31) holds. But $Vu_\mu \in L^1_\rho(\Omega) \implies Vv_\mu \in L^1_\rho(\Omega)$, if we set $w := \mathbb{G}[Vv_\mu]$, then $v_\mu + w$ is nonnegative and super-harmonic, thus it admits a boundary trace in $\mathfrak{M}_+(\partial\Omega)$ that we denote by $\gamma_u(\mu)$. Clearly $\gamma_u(\mu) \leq \mu$ since $v_\mu \leq u_\mu$ and $\gamma_u(\mu)$ is nondecreasing with μ as $\mu \mapsto u_\mu$ is. Finally, since v_μ is a supersolution, it is larger than the solution of (5.3) with the same boundary trace $\gamma_u(\mu)$, and there holds

$$u_{\gamma_u(\mu)} \leq v_\mu. \quad (5.32)$$

Proposition 5.11 *Let*

$$\nu_S(u) := \sup\{\gamma_u(\mu) : \mu \in \mathfrak{M}_+(\mathcal{S}(u))\}. \quad (5.33)$$

Then $\nu_S(u)$ is a Borel measure on $\mathcal{S}(u)$.

Proof. We borrow the proof to Marcus-Véron [19], and we naturally extend any positive Radon measure to a positive bounded and regular Borel measure by using the same notation. It is clear that $\nu_S(u) := \nu_S$ is an outer measure in the sense that

$$\nu_S(\emptyset) = 0, \quad \text{and } \nu_S(A) \leq \sum_{k=1}^{\infty} \nu_S(A_k), \quad \text{whenever } A \subset \bigcup_{k=1}^{\infty} A_k. \quad (5.34)$$

Let A and $B \subset \mathcal{S}(u)$ be disjoint Borel subsets. In order to prove that

$$\nu_S(A \cup B) = \nu_S(A) + \nu_S(B), \quad (5.35)$$

we first notice that the relation holds if $\max\{\nu_S(A), \nu_S(B)\} = \infty$. Therefore we assume that $\nu_S(A)$ and $\nu_S(B)$ are finite. For $\varepsilon > 0$ there exist two bounded positive measures μ_1 and μ_2 such that

$$\gamma_u(\mu_1)(A) \leq \nu_S(A) \leq \gamma_u(\mu_1)(A) + \varepsilon/2$$

and

$$\gamma_u(\mu_2)(B) \leq \nu_S(B) \leq \gamma_u(\mu_2)(B) + \varepsilon/2$$

Hence

$$\begin{aligned} \nu_S(A) + \nu_S(B) &\leq \gamma_u(\mu_1)(A) + \gamma_u(\mu_2)(B) + \varepsilon \\ &\leq \gamma_u(\mu_1 + \mu_2)(A \cup B) + \varepsilon \\ &= \gamma_u(\mu_1 + \mu_2)(A \cup B) + \varepsilon \\ &\leq \nu_S(A \cup B) + \varepsilon. \end{aligned}$$

Therefore ν_S is a finitely additive measure. If $\{A_k\}$ ($k \in \mathbb{N}$) is a sequence of disjoint Borel sets and $A = \bigcup A_k$, then

$$\nu_S(A) \geq \nu_S\left(\bigcup_{1 \leq k \leq n} A_k\right) = \sum_{k=1}^n \nu_S(A_k) \implies \nu_S(A) \geq \sum_{k=1}^{\infty} \nu_S(A_k).$$

By (5.34), it implies that ν_S is a countably additive measure. □

Definition 5.12 *The Borel measure $\nu(u)$ defined by*

$$\nu(u)(A) := \nu_S(A \cap \mathcal{S}(u)) + \mu_u(A \cap \mathcal{R}(u)), \quad \forall A \subset \partial\Omega, A \text{ Borel}, \quad (5.36)$$

is called the extended boundary trace of u , denoted by $Tr^e(u)$.

Proposition 5.13 *If $A \subset \mathcal{S}(u)$ is a Borel set, then*

$$\nu_S(A) := \sup\{\gamma_u(\mu)(A) : \mu \in \mathfrak{M}_+(A)\}. \quad (5.37)$$

Proof. If $\lambda, \lambda' \in \mathfrak{M}_+(\mathcal{S}(u))$

$$\inf\{u, u_{\lambda+\lambda'}\} = \inf\{u, u_\lambda + u_{\lambda'}\} \leq \inf\{u, u_\lambda\} + \inf\{u, u_{\lambda'}\}.$$

Since the three above functions admit a boundary trace, it follows that

$$\gamma_u(\lambda + \lambda') \leq \gamma_u(\lambda) + \gamma_u(\lambda').$$

If A is a Borel subset of $\mathcal{S}(u)$, then $\mu = \mu_A + \mu_{A^c}$ where $\mu_A = \chi_E \mu$. Thus

$$\gamma_u(\mu) \leq \gamma_u(\mu_A) + \gamma_u(\mu_{A^c}),$$

and

$$\gamma_u(\mu)(A) \leq \gamma_u(\mu_A)(A) + \gamma_u(\mu_{A^c})(A).$$

Since $\gamma_u(\mu_{A^c}) \leq \mu_{A^c}$ and $\mu_{A^c}(A) = 0$, it follows

$$\gamma_u(\mu)(A) \leq \gamma_u(\mu_A)(A).$$

But $\mu_A \leq \mu$, thus $\gamma_u(\mu_A) \leq \gamma_u(\mu)$ and finally

$$\gamma_u(\mu)(A) = \gamma_u(\mu_A)(A). \quad (5.38)$$

If $\mu \in \mathfrak{M}_+(A)$, $\mu = \mu_A$, thus (5.37) follows. \square

Proposition 5.14 *There always holds*

$$\nu(u)(Z_V^*) = 0, \quad (5.39)$$

where Z_V^* is the vanishing set of $K_V^\Omega(x, \cdot)$ defined by (4.15).

Proof. This follows from the fact that for any $\mu \in \mathfrak{M}_+(\partial\Omega)$ concentrated on Z_V^* , $u_\mu = 0$. Thus $\gamma_u(\mu) = 0$. If μ is a general measure, we can write $\mu = \chi_{Z_V^*} \mu + \chi_{(Z_V^*)^c} \mu$, thus $u_\mu = u_{\chi_{(Z_V^*)^c} \mu}$. Because of (5.32)

$$\gamma_u(\mu)(Z_V^*) = \gamma_u(\chi_{(Z_V^*)^c} \mu)(Z_V^*) \leq (\chi_{(Z_V^*)^c} \mu)(Z_V^*) = 0,$$

thus (5.39) holds. \square

Remark. This process for determining the boundary trace is ineffective if there exist positive solutions u in Ω such that

$$\lim_{d(x) \rightarrow 0} u(x) = \infty.$$

This is the case if $\Omega = B_R$ and $V(x) = c(R - |x|)^{-2}$ ($c > 0$). In this case $K_V^\Omega(x, \cdot) \equiv 0$. For any $a > 0$, there exists a radial solution of

$$-\Delta u + \frac{cu}{(R - |x|)^2} = 0 \quad \text{in } B_R \quad (5.40)$$

under the form

$$u(r) = u_a(r) = a + c \int_0^r s^{1-N} \int_0^s u(t) \frac{t^{N-1} dt}{(R-t)^2}. \quad (5.41)$$

Such a solution is easily obtained by fixed point, $u(0) = a$ and the above formula shows that u_a blows up when $r \uparrow R$. We do not know if there exist non-radial positive solutions of (5.40). More generally, if Ω is a smooth bounded domain, we do not know if there exists a non trivial positive solution of

$$-\Delta u + \frac{c}{d^2(x)} u = 0 \quad \text{in } \Omega. \quad (5.42)$$

Theorem 5.15 *Assume $V \geq 0$ and satisfies (2.19). If u is a positive solution of (5.3), then $Tr^e(u) = \nu(u)$ is a bounded measure.*

Proof. Set $\nu = \nu(u)$ and assume $\nu(\partial\Omega) = \infty$. By dichotomy there exists a decreasing sequence of relatively open domains $D_n \subset \partial\Omega$ such that $\overline{D_n} \subset D_{n-1}$, $\text{diam } D_n = r_n \rightarrow 0$ as $n \rightarrow \infty$, and $\nu(D_n) = \infty$. For each n , there exists a Radon measure $\mu_n \in \mathfrak{M}_+(D_n)$ such that $\gamma_u(\mu_n)(D_n) = n$, and

$$u \geq v_{\mu_n} = \inf\{u, u_{\mu_n}\} \geq u_{\gamma_u(\mu_n)}.$$

Set $m_n = n^{-1}\gamma_u(\mu_n)$, then $m_n \in \mathfrak{M}_+(D_n)$ has total mass 1 and it converges in the weak*-topology to δ_a , where $\{a\} = \cap_n D_n$. By Theorem 2.6, u_{m_n} converges to u_{δ_a} . Since $u \geq nu_{m_n}$, it follows that

$$u \geq \lim_{n \rightarrow \infty} nu_{m_n} = \infty,$$

a contradiction. Thus ν is a bounded Borel measure (and thus outer regular) and it corresponds to a unique Radon measure. \square

Remark. If $N = 2$, it follows from Theorem 5.8 that $u = u_\nu$ and thus the extended boundary trace coincides with the usual boundary trace. The same property holds if $N \geq 3$, if (5.25) holds.

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